Recall that given a (periodic) function $f : T \to C$, its Fourier series is given by
\[ f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}, \] where
\[ \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx. \]

1. Consider the Heat equation on an infinitely long rectangular plate, namely
\[ u_t = u_{xx} + u_{yy}, \quad \text{for } t > 0, \quad -1 < x < 1, \quad 0 < y, \]
with boundary conditions (BCs)
\[ u(t, \pm 1, y) = u(t, x, \infty) = 0 \quad \text{and} \quad u(t, x, 0) = f(x), \]
where $f(x)$ is a given temperature at the short, nearby edge of the plate. Assume that the data $f$ is symmetric in $x$:
\[ f(-x) = f(x). \]

(a) Argue informally or physically that as $t \to \infty$, the solution approaches a steady state $U(x,y)$, and write down the PDE and boundary conditions for the steady state; sketch the domain and label the BCs.

(b) Use separation of variables, $U(x,y) = F(x)G(y)$, to solve the steady equation and BCs, noting that $F(x) = \cos(kx)$ satisfies the BCs for appropriate $k$.

(c) Find the steady solution for any boundary data of the form
\[ f(x) = \sum_{k=1}^{N} a_k \cos\left(\frac{(2k-1)\pi}{2}x\right). \]

Controversy arose when Fourier took a limit as $N \to \infty$.

2. We construct a $C^\infty$ “bump function” on $B_1(0) \subset \mathbb{R}^d$, and corresponding kernel, as follows:

(a) Show that the function $h(x) = e^{-1/x} \chi_{(0,\infty)}(x)$ is a monotone, $C^\infty$ bounded function on $\mathbb{R}$. What does this tell us about the function $\phi(x) = h(1-x^2) = e^{1/(1-x^2)} \chi_{(-1,1)}$?

(b) Use $\phi$ to define $\psi : \mathbb{R}^d \to \mathbb{R}$ which is non-negative, $C^\infty$, supported on $B_1(0)$ and satisfies $\int_{\mathbb{R}^d} \psi = 1$. [Hint: use radial symmetry but NOT polar decomposition!]

(c) Now for $\delta > 0$, define $K_\delta(x) = \delta^d \psi(x/\delta)$, and show that $K_\delta$ is a (strong) approximate identity.

3. Show that the Heat kernel in $\mathbb{R}^d$,
\[ H_\delta(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}, \]
is a strong approximate identity with parameter $\delta = \sqrt{t}$. Also verify that for $t > 0$, it satisfies the heat equation, $u_t = \Delta u$. 
4. The Dirichlet kernel is given by

\[ D_N(x) = \frac{\sin \left( \left( N + \frac{1}{2} \right) x \right)}{\sin \left( \frac{x}{2} \right)}, \quad \text{for} \quad x \in \mathbb{T}. \]

(a) Show that \( D_N \) satisfies two properties of an approximation of the identity, but not the third.

(b) Compute \( D_N \ast f \) and interpret it in terms of the Fourier series of \( f \).

Hint: show that

\[ D_N(x) = \frac{1}{2\pi} \sum_{k=-N}^{N} e^{ikx} = \frac{1}{2\pi} \left( 1 + \sum_{k=1}^{N} \cos(kx) \right), \]

by trig identities or by manipulating a finite geometric series. It may also be helpful to plot \( D_N \).

5. The Fejér kernel is given by the Cesàro sum of Dirichlet kernels, namely

\[ F_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} D_N(x). \]

(a) Show that, for \( x \neq 0 \),

\[ F_N(x) = \frac{1}{2\pi N} \frac{\sin^2 \left( \frac{Nx}{2} \right)}{\sin^2 \left( \frac{x}{2} \right)} = \frac{1}{2\pi} \frac{1 - \cos(Nx)}{1 - \cos(x)}. \]

(b) Show that \( F_N \) is an approximation of the identity.

(c) Compute \( F_N \ast f \) and interpret it as the Cesàro sum of Fourier series.

6. Show that the following three definitions of continuity at \( x \in X \) of a map \( f : X \to Y \) of metric spaces are equivalent:

- Given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( d_X(x, z) < \delta \), then \( d_Y(f(x), f(z)) < \epsilon \);
- For any sequence \( x_n \to x \), we have \( f(x_n) \to f(x) \);
- For any open set \( G \subset Y \) containing \( f(x) \), the pre-image \( f^{-1}(G) \subset X \) contains an open neighborhood of \( x \).

7. Carry out the construction of the completion of a metric space \( X \):

(a) Define the relation \( \sim \) of the set \( \mathbf{C} = \{ \langle x^k \rangle \} \) of Cauchy sequences in \( X \) by

\[ \{x^k\} \sim \{y^k\} \iff \lim_{k \to \infty} d(x^k, y^k) = 0, \]

and check it’s an equivalence relation. Set \( \bar{X} = \mathbf{C}/\sim \), the set of equivalence classes.

(b) Define \( \bar{d} \) on \( \bar{X} \) by

\[ \bar{d}(\langle x^k \rangle, \langle y^k \rangle) = \lim_{k \to \infty} d(x^k, y^k), \]

and show that \( \bar{d} \) is well-defined and a metric.
(c) Define the map \( \iota : X \to \bar{X} \) by \( \iota(x) = \langle x \rangle \), the constant sequence, and show that \( \iota \) is an isometry. Moreover, the image \( \iota(X) \) is dense in \( \bar{X} \).

(d) Use a diagonal argument to show that \( \bar{X} \) is complete.

You may skip some details as appropriate, but try to avoid reference to the text!

8. Show that Bolzano-Weierstrass (every bounded sequence has a convergent subsequence) implies Heine-Borel (a set is sequentially compact iff it is closed and bounded) in \( \mathbb{R}^d \) (\( d \) finite!).

9. Prove Urysohn’s lemma: this states that if \( F \subset G \subset \mathbb{R}^d \), with \( F \) closed and \( G \) open, then there exists a continuous separating function \( f : \mathbb{R}^d \to [0, 1] \), which satisfies

\[
f(x) = 1, \quad x \in E, \quad \text{and} \quad f(x) = 0, \quad x \notin G.
\]

[Hint: consider \( f(x) = d(x, K_1)/(d(x, K_1) + d(x, K_2)) \) for appropriate sets \( K_i \).]

10. A series \( \sum x_j \) in a normed vector space space \( X \) is absolutely convergent if the real series \( \sum \|x_j\| \) converges in \( \mathbb{R} \). Show that \( X \) is a Banach space if and only if every absolutely convergent series \( \{x_j\} \) converges in \( X \).
11. Recall that in class we defined the surface measure on the unit sphere to be the unique measure \( \Theta \) such that Lebesgue measure \( m = \rho \times \Theta \), where \( \rho = r^{d-1} dr \) on \((0, \infty)\). We calculate the surface area \( \omega_d \) of the \( d-1 \) sphere \( S^{d-1} \subset \mathbb{R}^d \), as follows:

(a) For each positive integer \( d \) and \( a > 0 \), define \( I_d(a) = \int_{\mathbb{R}^d} e^{-a|x|^2} \, dx \). Use Fubini’s theorem to show that for each \( d \), we have \( I_d(a) = (I_1(a))^d \).

(b) Use polar coordinates to calculate \( I_2(a) \), and conclude that for each \( d \), we have \( I_d(a) = (\pi/a)^{d/2} \).

(c) By using spherical coordinates on \( \mathbb{R}^d \), derive the equality \( I_d(1) = \Theta(S^{d-1}) \int_0^\infty g(r) \, dr \) for appropriate function \( g(r) \), and by expressing \( \int g \) in terms of the gamma function

\[
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt, \quad z > 0,
\]

conclude that \( \omega_d = \Theta(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2) \).

12. Recall the heat kernel

\[
K_h(x) = \frac{1}{\sqrt{\pi h}} e^{-x^2/h}
\]

is an approximation of the identity on \( \mathbb{R} \) and that \( K_h \ast f \to f \) uniformly for any \( f \) supported in \([-R, R] \). For such \( f \), we can write

\[
K_h \ast f(x) = \frac{1}{\sqrt{\pi h}} \int_{-R}^R f(u) e^{-(x-u)^2/h} \, du.
\]

Use the approximation \( e^t \approx \sum_{k=0}^N t^k/k! \), uniform on compact sets, to approximate \( K_h \ast f \) by a polynomial of degree \( 2N \). Now given fixed interval \([a, b] \), choose \( R \) such that \([a, b] \subset (-R, R) \) and find a continuous \( \tilde{f} : [-R, R] \) which extends \( f \). Then the restriction of \( K_h \ast \tilde{f} \) to \([a, b] \) is a uniform polynomial approximation of \( f \) on \([a, b] \). This is essentially Weierstrass’ proof of the polynomial approximation theorem.

13. Prove that the set \( \mathcal{L}_K \) of Lipschitz continuous functions on \([a, b] \) with Lipschitz constant less than or equal to \( K \) is compact in \( C([a, b]) \).

14. Let \( w : [0, 1] \to \mathbb{R} \) be a non-negative, continuous function. Define the weighted supnorm \( \| \cdot \|_w \) on \( C([0, 1]) \) by \( \| f \|_w = \sup_x |f(x) w(x)| \). If \( w(x) > 0 \) for \( x \in [0, 1] \), show that \( \| \cdot \|_w \) is equivalent to the usual supnorm on \( C([0, 1]) \), that is, there exist positive constants \( c \) and \( C \) such that

\[
c \| f \| \leq \| f \|_w \leq C \| f \|, \quad \text{for all} \quad f \in C([0, 1]),
\]

so the corresponding topologies are the same. On the other hand, if \( w(x) = x \), show that \( \| \cdot \|_w \) is not equivalent to the usual supnorm. Is \( C([0, 1]) \) with norm \( \| \cdot \|_w \) with \( w(x) = x \) a Banach space? Prove or give a counter-example.

15. Suppose that the sequence \( (f_n) \subset C([0, 1]) \) is equicontinuous and \( f_n \to f \) pointwise. Show that \( f \) is continuous.

16. We briefly analyze the Babylonian (or Greek?!?) method for approximating square roots. Given a guess \( x_k \) for \( \sqrt{a} \), if \( x_k \) under- (respectively over-) estimates \( \sqrt{a} \), then \( a/x_k \) over-
(resp. under-) estimates it; it follows that the average of these should be a better estimate. This yields the iteration

\[ x_{k+1} = Tx_k, \quad \text{where} \quad Tx = \frac{1}{2}(x + \frac{a}{x}), \]

and \( T \) is regarded as a map on \((0, \infty)\).

(a) Apply Newton’s method to the function \( f(x) = x^2 - a \), to get the same iteration, and show directly that there is a unique fixed point.

(b) Show that \( T \) is contractive on \([k\sqrt{a}, \infty)\) if and only if \( k > 1/\sqrt{3} \), and calculate the corresponding contraction constant \( \rho \).

(c) Show that for any \( x > 0 \), \( Tx - \sqrt{a} \geq 0 \), so we can take \( k = 1 \), for which \( \rho \) is minimized. Note that \( Tx_1 - Tx_2 = o(x_1 - x_2) \) near \( \sqrt{a} \), so we expect superlinear convergence.

(d) Setting \( e_k = x_k - \sqrt{a} \), show by induction that

\[ \frac{e_{k+1}}{2\sqrt{a}} \leq \left( \frac{e_1}{2\sqrt{a}} \right)^{2^k}. \]

This is quadratic convergence, typical for (convergent) Newton methods.

17. Consider the (nonlinear) second order boundary value problem,

\[ -u'' + \lambda \sin u = f(x), \quad x \in (a, b), \]
\[ u(a) = 0, \quad u(b) = 0, \]

where \( f \in C([a, b]) \) is given, and we wish to find \( u = u(x) \). Integrate twice and use the boundary conditions to reformulate this as a nonlinear integral equation,

\[ u(x) = T_n(u) + T_l(f), \]

where \( T_n \) and \( T_l \) are integral operators. Beginning with \( u_0 = 0 \), write out the first two iterates of a sequence of approximations. Also show that this iteration converges uniformly, provided \( \lambda \) is small enough.
18. Let $X$ and $Y$ be normed linear spaces, and denote the space of bounded maps from $X$ to $Y$ by $B(X,Y)$. Show that $B(X,Y)$ is a normed linear space and that if $Y$ is complete, the so is $B(X,Y)$.

19. Prove Young’s inequality: for $a, b > 0$,
\[ ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{provided } \frac{1}{p} + \frac{1}{q} = 1, \]
and use this to reprove Hölder’s inequality.
[Hint: For Young, use Jensen’s inequality with $\log t$; then choose $a = f(x)/A, b = g(x)/B$ for appropriate scalings.]


23. Show that a linear functional on a vector space $X$ is bounded iff its kernel is closed.

24. Let $X$ be a Banach space and let $T \in B(X,X)$.

(a) If $I$ is the identity operator and $\|I - T\| < 1$, show that $T$ is invertible (use a Neumann series).

(b) If $T$ is invertible and $\|S - T\| < 1/\|T^{-1}\|$, show that $S$ is invertible. Thus the set of invertible operators is open in $B(X,X)$.

25. We define quotient spaces as follows. If $M$ is a closed subspace of a vector space $X$, say $x \sim y$ iff $x - y \in M$. Then $\sim$ is an equivalence relation and we denote the equivalence of $x$ by $x + M$ and the set of these by $X/M = \{ x + M \mid x \in X \}$.

(a) Define linear operations on $X/M$ and norm $\|x + M\| = \inf\{\|x + y\| \mid y \in M\}$, so that $X/M$ becomes a normed vector space. Moreover, if $X$ is Banach, so is $X/M$.

(b) Given $\epsilon > 0$, there exists $x \in X$ with $\|x\| = 1$, such that $\|x + M\| \geq 1 - \epsilon$. Also, the projection $\pi : X \to X/M$ given by $\pi(x) = x + M$ has norm 1.

(c) If $\|\cdot\|$ is a seminorm (ie $\|x\| = 0$ for $x \neq 0$ is allowed), then this construction for $M = \{ x \in X \mid \|x\| = 0 \}$ turns $X/M$ into a normed vector space.

26. For a convex set $K \subset X$, we defined the Minkowski gauge function by
\[ p_K(x) = \inf \{ r \geq 0 \mid x \in rK \}. \]
Show that $p_K$ satisfies the following conditions:

(a) $p_K(tx) = tp_K(x)$ for any $t \geq 0$;

(b) $p_K(\alpha x + (1 - \alpha)y) \leq \alpha p_K(x) + (1 - \alpha)p_K(y)$ for any $\alpha \in [0,1]$;

(c) $p_K(x) \leq 1$ for $x \in K$ and $p_K(x) \geq 1$ for $x \notin K$.

27. Prove that given a subspace $Z$ of a normed vector space $X$ and $y \in X$ with $\text{dist}(y,Z) = \delta$, there exists $\Lambda \in X^*$ satisfying
\[ \|\Lambda\| \leq 1, \quad \Lambda(y) = \delta, \quad \text{and} \quad \Lambda(z) = 0 \quad \text{for all} \quad z \in Z. \]
28. Let $X$ be a vector space and $P \subset X$ such that: (i) if $x, y \in P$ then $x + y \in P$; (ii) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$; and (iii) if $x \in P$ and $-x \in P$, then $x = 0$. Check that the relation $\leq$ defined by $x \leq y$ iff $y - x \in P$ defines a partial ordering on $X$. Next, prove the Krein Extension Theorem: Suppose that $M$ is a subspace of $X$, such that for each $x \in X$, there is a $y \in M$ satisfying $x \leq y$, and $f$ is a linear functional on $M$ such that $f(x) \geq 0$ for all $x \in P$, and $f|_M = f$. [Hint: consider $p(x) = \inf \{ f(y) \mid y \in M, \ x \leq y \}$.]

29. (a) Show that a (complex) normed vector space is an inner product space if and only if the parallelogram law holds:

$$\|a + b\|^2 + \|a - b\|^2 = 2\|a\|^2 + 2\|b\|^2.$$  

[Hint: you need to find both real and imaginary parts of $(x, y)$.]

(b) Show that $L^p[0, 1]$ can be realized as a Hilbert space only if $p = 2$.


31. Given an independent set $U = \{u_\alpha\}$ in a separable Hilbert space, use the Gram-Schmidt procedure to show that there is an orthonormal basis $V$ such that $\text{Span}(U) = \text{Span}(V)$.


35. Let $H = L^2([\pi, \pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle, with inner product

$$(F, G) = \frac{1}{2\pi} \int_{-\pi}^\pi F(e^{i\theta}) G(e^{i\theta}) \, d\theta.$$  

Using the mapping

$$\phi : \mathbb{R} \to S^1 \text{ given by } \phi(x) = \frac{i - x}{i + x}$$  

of $\mathbb{R}$ to the unit circle, show that:

(a) The map $U : H \to L^2(\mathbb{R})$ given by

$$U(F) = f, \text{ where } f(x) = \frac{1}{\sqrt{\pi(i + x)}} F \circ \phi(x)$$  

is unitary.

(b) As a result,

$$\left\{ \frac{1}{\sqrt{\pi}} \left( \frac{i - x}{i + x} \right)^n \frac{1}{i + x} \right\}_{n \in \mathbb{Z}}$$  

is an orthonormal basis for $L^2(\mathbb{R})$.

36. Show that any integrable function $f : \mathbb{R} \to \mathbb{R}$ which is polynomially bounded (that is, there is a polynomial $p(x)$ such that $|f(x)| \leq p(x)$ for all $x$) defines a tempered distribution. Does $e^{ax}$ define a tempered distribution? Why or why not?
37. Consider the function \( f(x) = \log |x|, \; x \in \mathbb{R} \). Show that this defines a distribution and that its distributional derivative is \( \text{p.v.}(1/x) \).

38. Suppose that \( f \) are \( g \) are continuous on \((a,b)\), with \( Df = g \) as distributions. Show that for every \( x \in (a,b) \),

\[
\frac{f(x+h) - f(x)}{h} \to g(x) \quad \text{as} \quad h \to 0,
\]

as follows. Pick a bump function \( \psi(x) \in C^\infty_c \) which satisfies \( \int \psi = 1 \) and scale to get approximate identities \( \psi_\eta \) (i.e. \( \psi_\eta \to \delta \) as \( \eta \to 0 \)). For fixed \( x_0 \in (a,b) \), consider the test function

\[
\varphi_{h,\eta}(x) = \int_{-\infty}^{x} \psi_\eta(x_0 + h - y) - \psi_\eta(x_0 - y) \, dy,
\]

apply \( Df = g \), and take appropriate limits.

39. (a) Find distributional derivatives of all orders of the function

\[
g_p(x) = \begin{cases} 
  x^p, & x \geq 0, \\
  0, & x < 0,
\end{cases}
\]

where \( p \) is a positive integer.

(b) Find the Fourier transforms of the Heaviside function \( H \) and Dirac mass \( \delta \), as tempered distributions.

(c) Use (a) and (b) to find the Fourier transform of the functions \( g_p \).

40. Use the Fourier transform to (formally) solve the heat equation,

\[
u_t = \epsilon \Delta u, \quad u(x, 0) = f(x),
\]

where \( \Delta \) is the Laplacian and \( x \in \mathbb{R}^d \). You should derive an integral expression for the solution \( u(x, t) \) in terms of the Fourier components \( \hat{f}(k) \) of the data. Don’t worry about convergence issues!