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An Elementary Proof of the Strong Law of Large Numbers

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Summary. In the following note we present a proof for the strong law of large numbers which is not only elementary, in the sense that it does not use Kolmogorov's inequality, but it is also more applicable because we only require the random variables to be pairwise independent. An extension to separable Banach space-valued r-dimensional arrays of random vectors is also discussed. For the weak law of large numbers concerning pairwise independent random variables, which follows from our result, see Theorem 5.2.2 in Chung [1].

Theorem 1. *Let $\{X_n\}$ be a sequence of pairwise independent, identically distributed random variables. Let $S_n = \sum_{i=1}^{n} X_i$. Then*

$$E|X_1| < \infty \Rightarrow \lim_{n \to \infty} \frac{S_n}{n} = EX_1 \qquad \text{a.s.}$$

Proof. Since $\{X_n^+\}$ and $\{X_n^-\}$ satisfy the assumptions of the theorem and $X_i = X_i^+ - X_i^-$, without loss of generality we can assume that $X_i \geqq 0$. Let $Y_i = X_i I \{X_i \leqq i\}$ with I the indicator function and $S_n^* = \sum_{i=1}^{n} Y_i$. Now for $\varepsilon > 0$ let $k_n = [\alpha^n]$, $\alpha > 1$ and use Chebyšhev's inequality to obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left\{\left|\frac{S_{k_n}^* - ES_{k_n}^*}{k_n}\right| > \varepsilon\right\} &\leqq c \sum_{n=1}^{\infty} \frac{\operatorname{Var} S_{k_n}^*}{k_n^2} = c \sum_{n=1}^{\infty} \frac{1}{k_n^2} \sum_{i=1}^{k_n} \operatorname{Var} Y_i \\
&\leqq c \sum_{i=1}^{\infty} \frac{EY_i^2}{i^2} = c \sum_{i=1}^{\infty} \frac{1}{i^2} \int_0^i x^2\, dF(x) \\
&= c \sum_{i=1}^{\infty} \frac{1}{i^2} \left(\sum_{k=0}^{i-1} \int_k^{k+1} x^2\, dF(x) \right) \\
&\leqq c \sum_{k=0}^{\infty} \frac{1}{k+1} \int_k^{k+1} x^2\, dF(x) \qquad (1) \\
&\leqq c \sum_{k=0}^{\infty} \int_k^{k+1} x\, dF(x) \\
&= cEX_1 < \infty,
\end{aligned}$$

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where $F(x)$ is the distribution of X_1 and c is an unimportant positive constant which is allowed to change. We also have

$$EX_1 = \lim_{n\to\infty} \int_0^n x\,dF(x) = \lim_{n\to\infty} EY_n = \lim_{n\to\infty} \frac{ES^*_{k_n}}{k_n}. \tag{2}$$

Therefore by the Borel-Cantelli Lemma

$$\lim_{n\to\infty} \frac{S^*_{k_n}}{k_n} = EX_1 \quad \text{a.s.} \tag{3}$$

Also

$$\begin{aligned}\sum_{n=1}^{\infty} P\{Y_n \neq X_n\} &= \sum_{n=1}^{\infty} P\{X_n > n\} = \sum_{n=1}^{\infty} \int_n^{\infty} dF(x) = \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} \int_i^{i+1} dF(x) \\ &= \sum_{i=1}^{\infty} i \int_i^{i+1} dF(x) \leqq \sum_{i=1}^{\infty} \int_i^{i+1} x\,dF(x) \\ &\leqq EX_1 < \infty.\end{aligned} \tag{4}$$

Hence by the Borel-Cantelli Lemma $X_n \neq Y_n$ only finitely many times. Consequently

$$\lim_{n\to\infty} \frac{S_{k_n}}{k_n} = EX_1 \quad \text{a.s.} \tag{5}$$

Now from monotonicity of S_n we can conclude that

$$\frac{1}{\alpha}(EX_1) \leqq \varliminf_{n\to\infty} \frac{S_n}{n} \leqq \varlimsup_{n\to\infty} \frac{S_n}{n} \leqq \alpha(EX_1) \quad \text{a.s.} \tag{6}$$

for every $\alpha > 1$ which gives us the desired result.

Theorem 2. *Let $\{X_{mn}\}$ be a double sequence of pairwise independent, identically distributed random variables. Let $S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}$. Then*

$$E(|X_{11}| \log^+ |X_{11}|) < \infty \Rightarrow \lim_{(m,n)\to\infty} \frac{S_{mn}}{mn} = EX_{11} \quad \text{a.s.}$$

Proof. The result follows immediately from Theorem 1 if we fix either m or n and let the other one go to infinity. Hence we consider the case when both m and n tend to infinity. We shall follow the proof of Theorem 1. Without loss of generality assume $X_{ij} \geqq 0$. Define new double sequences $Y_{ij} = X_{ij} I\{X_{ij} \leqq ij\}$ and $S^*_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} Y_{ij}$. If we let d_k to be the number of divisors of k i.e. the cardinality of $\{(i,j): ij = k\}$, and $F(x)$ be the distribution of X_{11}, then (4) adapted to this case becomes,

$$
\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\{Y_{ij} \neq X_{ij}\} &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\{X_{ij} > ij\} = \sum_{k=1}^{\infty} d_k P\{X_{11} > k\} \\
&= \sum_{k=1}^{\infty} d_k \int_k^{\infty} dF(x) = \sum_{i=1}^{\infty}\left(\sum_{k=1}^{i} d_k\right)\int_i^{i+1} dF(x) \\
&\leqq c \sum_{i=1}^{\infty} i \log i \int_i^{i+1} dF(x) \\
&\leqq c E(X_{11} \log^+ X_{11}) < \infty,
\end{aligned} \tag{7}
$$

where we use the fact that $\sum_{k=1}^{n} d_k =$(number of positive integer lattice points "under" hyperbola $xy=n) = O(n \log n)$. Now if we let $k_m = [\alpha^m]$, $l_n = [\alpha^n]$ and $\alpha > 1$. Then the right hand side modified version of (1) gives us

$$
\begin{aligned}
c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{EY_{ij}^2}{(ij)^2} &\leqq c \sum_{k=1}^{\infty} \frac{d_k}{k^2} \int_0^k x^2\, dF(x) \\
&\leqq c \sum_{i=0}^{\infty}\left(\sum_{k=i+1}^{\infty} \frac{d_k}{k^2}\right)\int_i^{i+1} x^2\, dF(x) \\
&\leqq c \int_1^{\infty} x \log x\, dF(x) = c E(X_{11} \log^+ X_{11}) < \infty;
\end{aligned} \tag{8}
$$

where we use $\sum_{k=i+1}^{\infty} \frac{d_k}{k^2} = O\left(\frac{\log i}{i+1}\right)$ which follows easily by summation by part. The rest of the proof follows similarly.

Remark 1. Theorem 2 is called the strond law of large numbers for 2-dimensional arrays of random variables. The generalization to r-dimensional array of random variables is immediate. The sufficient condition becomes $E(|X|\,(\log^+ |X|)^{r-1}) < \infty$. For a martingale approach for i.i.d. random variables see Smythe [4].

Remark 2. Once we show the strong law for real-valued random variables, the generalization of the strong law for separable B-space-valued r-dimensional array of random vectors follows easily. Simply follow the proof of the strong law of large numbers given in Padgett [3] pp. 42–44, with appropriate modifications. The sufficient condition becomes $E(\|X\|\,(\log^+ \|X\|)^{r-1}) < \infty$, where $\|\ \|$ is the norm in the Banach space.

Remark 3. The converse to the above theorems in the Chung sense also follows easily (see Chung [1], Theorem 5.4.2 and Theorem 4.2.5), provided that we use a better estimate for $\sum_{k=1}^{n} d_k$, namely $\sum_{k=1}^{n} d_k \sim n \log n$. For the latter see Hardy [2]. See also Smythe [4].

References

1. Chung, K.L.: A course in probability theory, 2nd ed. New York–London: Academic Press 1974
2. Hardy, G.H., Wright, E.M.: An introduction to the theory of numbers, 4th ed. Oxford: Clarendon 1960
3. Padgett, W.G., Taylor, R.L.: Lecture notes in Mathematics **360**. Berlin-Heidelberg-New York: Springer 1973
4. Smythe, R.T.: Strong laws of large numbers for *r*-dimensional arrays of random variables. Ann. of Probab. **1**, 1, 164–170 (1973)

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