IWASAWA INVARIANTS OF GALOIS DEFORMATIONS

TOM WESTON

ABSTRACT. Fix a residual ordinary representation $\bar{\rho} : G_F \to \mathrm{GL}_n(k)$ of the absolute Galois group of a number field F. Generalizing work of Greenberg–Vatsal and Emerton–Pollack–Weston, we show that the Iwasawa invariants of Selmer groups of deformations of $\bar{\rho}$ depends only on $\bar{\rho}$ and the ramification of the deformation.

Let p be an odd prime and let K be a finite extension of $\mathbf{Q}_p$ with residue field k. Consider a continuous representation

$$\bar{\rho} : G_F \to \mathrm{GL}_n(k)$$

of the absolute Galois group of a number field F. Assume that $\bar{\rho}$ is nearly ordinary in the sense that for any place v dividing p the image of a decomposition group at v lies in some Borel subgroup B_v of GL_n. In this paper we show, under appropriate hypotheses, that the Iwasawa invariants of the Selmer group of a nearly ordinary deformation of $\bar{\rho}$ depend only on $\bar{\rho}$ and the tame ramification of the deformation.

For simplicity, we specialize now to the case $F = \mathbf{Q}$. Assume that $\bar{\rho}$ satisfies the conditions of [11, Section 7] which guarantee that it has a reasonable deformation theory (see Section 3.1) and that $\bar{\rho}$ and its Cartier dual have trivial $G_{\mathbf{Q}}$-invariants. Let $\mathcal{H}$ denote the set of nearly ordinary (with respect to the B_v) deformations of $\bar{\rho}$ to continuous finitely ramified representations

$$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathcal{O}_\rho)$$

satisfying a certain ramification condition (see Section 3.2); here $\mathcal{O}_\rho$ is the ring of integers of some finite totally ramified extension of K depending on ρ.

Fix $\rho \in \mathcal{H}$. In Section 2 we define a variant $\mathrm{Sel}_{\mathcal{W}}(\mathbf{Q}_\infty, \rho)$ of Greenberg's Selmer group of ρ over the cyclotomic $\mathbf{Z}_p$-extension $\mathbf{Q}_\infty$ of $\mathbf{Q}$; it is conjecturally related to the p-adic L-function of ρ. Let $\mu(\rho)$ and $\lambda(\rho)$ denote the Iwasawa invariants of $\mathrm{Sel}_{\mathcal{W}}(\mathbf{Q}_\infty, \rho)$; recall that $\lambda(\rho)$ is simply the $\mathcal{O}$-corank of $\mathrm{Sel}_{\mathcal{W}}(\mathbf{Q}_\infty, \rho)$. For a place v of $\mathbf{Q}$ set

$$\delta_v(\rho) = \sum_{w \mid v} \mathrm{mult}_\omega(\bar{\rho}|_{I_w}) - \mathrm{mult}_\omega(\rho \otimes K|_{I_w});$$

here the sum runs over the places of $\mathbf{Q}_\infty$ dividing v and $\mathrm{mult}_\omega(\cdot)$ is the multiplicity of the Teichmüller character in the given representation of G_w/I_w. Note that $\delta_v(\rho)$ depends only on $\bar{\rho}$ and the restriction of ρ to I_v; in particular, it vanishes if ρ is unramified at v.

Our main result is the following theorem on the variation of these Iwasawa invariants over $\mathcal{H}$.

Theorem 1. *If $\mu(\rho_0) = 0$ for some $\rho_0 \in \mathcal{H}$, then $\mu(\rho) = 0$ for all $\rho \in \mathcal{H}$. If this is the case, then the difference*

$$\lambda(\rho) - \sum_{v \nmid p} \delta_v(\rho) \tag{1}$$

is independent of $\rho \in \mathcal{H}$.

In particular, Theorem 1 implies that the values $\lambda(\rho)$ for all $\rho \in \mathcal{H}$ can be easily determined from the knowledge of $\lambda(\rho_0)$ for a single ρ_0 with $\mu(\rho_0) = 0$. The question of which λ-invariants can occur in $\mathcal{H}$ is intimately related to the questions of level raising and lowering for $\bar{\rho}$. For example, if level lowering holds for $\bar{\rho}$, then the difference (1) equals the minimal λ-invariant in the family $\mathcal{H}$.

Of course, although conjecturally $\mathcal{H}$ is quite large, this is known in very few cases. The case of $F = \mathbf{Q}$ and $n = 2$ is studied via Hida theory in [3]. The work of [4] and [8] provides an additional case where we can apply our results. Specifically, let F be totally real and let $\bar{\rho} : G_F \to \mathrm{GL}_3(k)$ be the trace-zero adjoint of the two-dimensional residual representation attached to some p-ordinary Hilbert modular form f_0 over F. Assume that:

- $\bar{\rho}$ is absolutely irreducible when restricted to $F(\sqrt{(-1)^{(p-1)/2}p})$;
- $F \cap \mathbf{Q}(\mu_p) = \mathbf{Q}$;
- $(\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)^\times$ has no p-torsion;
- $F/\mathbf{Q}$ is unramified at p.

Write $\mathcal{H}_0$ for the set of trace-zero adjoints of Galois representations associated to p-ordinary Hilbert modular forms f which are congruent to f_0 in k. It is known by [4] and [8] that $\mathcal{H}$ contains all but finitely many elements of $\mathcal{H}_0$. In this case our Selmer group coincides with Greenberg's ordinary Selmer group, so that we obtain the following result on the classical Iwasawa invariants of such forms; we write $\mu(\mathrm{ad}^0 f)$ and $\lambda(\mathrm{ad}^0 f)$ for the Iwasawa invariants of the trace-zero adjoint Selmer group of $f \in \mathcal{H}_0$.

Theorem 2. *If $\mu(\mathrm{ad}^0 f_0) = 0$, then $\mu(\mathrm{ad}^0 f) = 0$ for all $f \in \mathcal{H}_0$. If this is the case, then the difference*

$$\lambda(\mathrm{ad}^0 f) - \sum_{v \nmid p} \delta_v(\mathrm{ad}^0 f)$$

is independent of $f \in \mathcal{H}_0$, with possibly finitely many exceptions.

We remark that the above application illustrates that if $\bar{\rho}$ takes values in some algebraic subgroup G of GL_n, then it may be useful to consider only G-valued deformations of $\bar{\rho}$. Nevertheless, although this is a useful motivating philosophy, it does not add any generality to our results, so we will not pursue it further.

The origin of this work is the paper [7] of Greenberg and Vatsal, in which analogous results were obtained for Tate modules of elliptic curves. These results were extended to arbitrary modular forms over $\mathbf{Q}$ in [3]. The methods given in this paper are a generalization of those of [3] to a more general context.

The first issue we must confront for this generalization is that variation of Hodge–Tate weights causes Greenberg's Selmer groups of ordinary representations to behave poorly in families. In Section 1 we introduce and study Selmer groups with weights, which are essentially Greenberg's Selmer group with extra twists to facilitate interpolation over families.

Distinct deformations of a fixed residual representation may have very different Selmer groups. However, in favorable situations it is possible to visualize these differences in the cohomology of the residual representation. This is done via the theory of residual Selmer groups developed in Section 2. We also discuss the connections with level lowering and level raising, focusing on the less conjectural case of GL_2. We obtain Theorem 1 in Section 3 by applying the preceding results to the study of Iwasawa invariants of nearly ordinary deformations of fixed residual representations as in [11].

This work grew out of the paper [3]; it is a pleasure to thank Matthew Emerton and Robert Pollack for many helpful discussions on this material. The author would also like to thank Ralph Greenberg, Paul Gunnells, Chris Skinner and Eric Sommers for their help with this project, and the referee for many thoughtful suggestions.

1. Selmer groups

Throughout this paper p denotes an odd prime.

1.1. **Local Galois representations.** Let K and L denote finite extensions of $\mathbf{Q}_p$ and let G_L denote the absolute Galois group of L. A *nearly ordinary G_L-representation* over K is a finite-dimensional K-vector space V endowed with a continuous K-linear action of G_L and a choice of a G_L-stable complete flag

$$0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^n = V.$$

Let $\chi_i : G_L \to K^\times$ denote the character by which G_L acts on V^i/V^{i-1}. If V is a Hodge–Tate representation of G_L, then the restriction of χ_i to the inertia group of L must be the product of a character of finite order and some integer power ε^{m_i} of the cyclotomic character. In this case we call $m_1, \ldots, m_n$ the *Hodge–Tate weights* of V.

We say that a nearly ordinary G_L-representation V is *ordinary* if it is Hodge–Tate and if

$$m_1 \geq m_2 \geq \cdots \geq m_n.$$

(Note that as we have defined it the property of being ordinary depends on the choice of complete flag and not merely on V.) It is well known (see [10]) that ordinary representations are always potentially semistable. In fact, the converse is essentially true as well.

Lemma 1.1. *If V is nearly ordinary and potentially semistable, then there exists a complete G_L-stable flag*

$$0 = \tilde{V}^0 \subsetneq \tilde{V}^1 \subsetneq \cdots \subsetneq \tilde{V}^n = V$$

with respect to which V is ordinary.

Proof. Let

$$0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^n = V$$

be the given G_L-stable complete flag. V is Hodge–Tate since it is potentially semistable; let $m_1, \ldots, m_n \in \mathbf{Z}$ denote the Hodge–Tate weights as before. Let i be the least index such that $m_i < m_{i+1}$ and let W denote the two-dimensional G_L-representation V^{i+1}/V^{i-1}. Choosing a basis x, y for W with $x \in V^i/V^{i-1}$, the representation of G_L on W has the form

$$\begin{pmatrix} \varepsilon^{m_i} & * \\ 0 & \varepsilon^{m_{i+1}} \end{pmatrix}$$

and is potentially semistable (as this property is preserved under passage to subquotients). Thus the $*$ above may be regarded as an element of $H^1_g(\mathbf{Q}_p, K(m_i - m_{i+1}))$, in the notation of [1]. This group is trivial since $m_i - m_{i+1} < 0$, so that in fact the representation of G_L on V^{i+1}/V^{i-1} can be conjugated to be diagonal. Choosing a vector $y' \in V^{i+1}$ which is an $\varepsilon^{m_{i+1}}$-eigenvector in W, it follows that the complete flag

$$\tilde{V}^j = \begin{cases} V^j & j < i \\ V^{i-1} + K \cdot y' & j = i \\ V^{i+1} & j = i+1 \end{cases}$$

of V^{i+1} gives V^{i+1} the structure of ordinary G_L-representation. Continuing in this way yields the desired ordinary flag for V. □

1.2. **Global Galois representations.** Let F be a finite extension of $\mathbf{Q}$ and let F_∞ denote the cyclotomic $\mathbf{Z}_p$-extension of F. A *nearly ordinary Galois representation* over a finite extension K of $\mathbf{Q}_p$ is a finite-dimensional K-vector space equipped with a K-linear action of the absolute Galois group G_F such that V is equipped with the structure

$$0 = V_v^0 \subsetneq V_v^1 \subsetneq \cdots \subsetneq V_v^n = V$$

of nearly ordinary G_v-representation for each place v of F dividing p.

Let V be a nearly ordinary Galois representation of dimension n. For a real place v of F, let $d_v^-(V)$ denote the K-dimension of the subspace of V on which complex conjugation at v acts by -1. A set $\mathcal{W}$ of *Selmer weights* for V is a choice of integers $c_v(V)$, $0 \le c_v(V) \le n$, for each v dividing p such that

$$\sum_{v|p} [F_v : \mathbf{Q}_p] \cdot c_v(V) = \sum_{v \text{ real}} d_v^-(V) + \sum_{v \text{ complex}} n. \tag{2}$$

Set

$$V_v^{\mathcal{W}} = V_v^{n - c_v(V)}$$

for each place v dividing p.

Example 1.2. Let F be totally real and let V be the two-dimensional representation associated to a p-ordinary Hilbert modular form f of parallel weight $(k, k, \ldots, k)$ with $k \ge 2$. Then V has a natural structure of nearly ordinary Galois representation with Hodge–Tate weights 0 and $k-1$ for each v dividing p. We have $d_v^-(V) = 1$ for every archimedean v, so that we can take $c_v(V) = 1$ for all v. If $\operatorname{ad}^0 V$ denotes the trace zero adjoint of V, then $\operatorname{ad}^0 V$ is nearly ordinary with Hodge–Tate weights $1-k, 0, k-1$. We have $d_v^-(\operatorname{ad}^0 V) = 2$ for each v, so that we can take $c_v(\operatorname{ad}^0 V) = 2$ for all v.

Example 1.3. Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(K)$ be associated to a cohomological cuspidal representation of GL_n of highest weight $(a_1, a_2, \ldots, a_n)$ with $a_1 \ge a_2 \ge \cdots \ge a_n$ and let V denote the associated n-dimensional Galois representation. If ρ is nearly ordinary, then V has a natural structure of nearly ordinary Galois representation; the Hodge–Tate weights are conjectured to be

$$a_1 + n - 1, a_2 + n - 2, a_3 + n - 3, \ldots, a_n.$$

Example 1.4. Let $\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_4(K)$ be associated to a p-ordinary cohomological cuspidal representation of GSp_4 of highest weight $(a, b; a+b)$ with $a \ge b \ge 0$ as in [12] and let V denote the associated four-dimensional Galois representation; it has a natural structure of nearly ordinary Galois representation with Hodge–Tate weights

$$a+b+3, a+2, b+1, 0.$$

We have $d_v^-(V) = 2$ for the unique archimedean place of $\mathbf{Q}$, so that we must take $c_p(V) = 2$.

1.3. **Selmer groups.** Let V be a nearly ordinary Galois representation as above. Let $\mathcal{O}$ denote the ring of integers of K. Fix a G_F-stable $\mathcal{O}$-lattice T in V and set $A = V/T$; we call A a *torsion quotient* of V, although in general A is not uniquely determined by V. We say that a finite set of places Σ of F is *sufficiently large* for A if it contains all archimedean places, all places dividing p, and all places at which A is ramified.

Fix Selmer weights $\mathcal{W}$ for V (assuming they exist). For any place w of F_∞ set

$$H^1_{s,\mathcal{W}}(F_{\infty,w}, A) = \begin{cases} H^1(G_w, A) & w \nmid p; \\ \operatorname{im} H^1(G_w, A) \to H^1(I_w, A/A_w^{\mathcal{W}}) & w \mid p; \end{cases}$$

where $A_w^{\mathcal{W}}$ is the image of $V_v^{\mathcal{W}}$ in A with v the restriction of w to F. We define the *Selmer group* of A with weights $\mathcal{W}$ by

$$\begin{aligned} \operatorname{Sel}_{\mathcal{W}}(F_\infty, A) &= \ker H^1(F_\infty, A) \to \prod_w H^1_{s,\mathcal{W}}(F_{\infty,w}, A) \\ &= \ker H^1(F_\Sigma/F_\infty, A) \to \prod_{w \mid v \in \Sigma} H^1_{s,\mathcal{W}}(F_{\infty,w}, A) \end{aligned} \tag{3}$$

for any finite set of places Σ of F which is sufficiently large for A.

By [5, Propositions 1 and 2] we have

$$\operatorname{corank}_\Lambda H^1_{s,\mathcal{W}}(F_{\infty,w}, A) = \begin{cases} 0 & w \nmid p; \\ [F_v : \mathbf{Q}_p] \cdot c(V) & w \mid v \mid p. \end{cases}$$

By [5, Proposition 3] and (2) we also have

$$\begin{aligned} \operatorname{corank}_\Lambda H^1(F_\Sigma/F_\infty, A) &= \sum_{v \text{ real}} d_v^-(V) + \sum_{v \text{ complex}} n + \operatorname{corank}_\Lambda H^2(F_\Sigma/F_\infty, A) \\ &= \sum_{v \mid p} [F_v : \mathbf{Q}_p] \cdot c_v(V) + \operatorname{corank}_\Lambda H^2(F_\Sigma/F_\infty, A) \end{aligned}$$

for any finite set of places Σ sufficiently large for A; here $\Lambda = \mathcal{O}[[\operatorname{Gal}(F_\infty/F)]]$ is the Iwasawa algebra. By (3) (together with [5, Proposition 6]) we thus have the following result.

Proposition 1.5. $\operatorname{Sel}_{\mathcal{W}}(F_\infty, A)$ *is a cofinitely generated* Λ*-module of* Λ*-corank at least* $\operatorname{corank}_\Lambda H^2(F_\Sigma/F_\infty, A)$.

In fact, it is conjectured [5] that the error term above vanishes.

Conjecture 1.6 (Greenberg)**.** *For A as above* $H^2(F_\Sigma/F_\infty, A) = 0$.

It thus seems reasonable to adapt standard conjectures on Selmer groups as follows.

Conjecture 1.7. *The Selmer group* $\operatorname{Sel}_{\mathcal{W}}(F_\infty, A)$ *is* Λ*-cotorsion for any Selmer weights* $\mathcal{W}$.

In fact, we will see below that the latter conjecture implies the former.

Our Selmer groups above are closely related to those introduced by Greenberg. For simplicity, assume that $F = \mathbf{Q}$. In this case, the only allowable Selmer weights have $c_p(V) = d_\infty^-(V)$ with ∞ denoting the real place of $\mathbf{Q}$. Then for any m such that

$$m_{n-c_p(V)+1} \leq m < m_{n-c_p(V)},$$

we have

$$\mathrm{Sel}_{\mathcal{W}}(\mathbf{Q}_\infty, A) = \mathrm{Sel}(\mathbf{Q}_\infty, A(m) \otimes \omega^{-m})$$

where the latter Selmer group is as in [5, Section 7], $A(m) = A \otimes \varepsilon^m$ is the twist of A by m powers of the cyclotomic character and ω is the Teichmüller character. Note that $A(m) \otimes \omega^{-m} \cong A$ as $G_{\mathbf{Q}_\infty}$-modules; the twist above is simply shifting the Hodge filtration. (Note that this twist is normalized so that the L-function of $V(m) \otimes \omega^{-m}$ has 0 as a critical value.) Similar relationships exist for other base fields F, although our notion of Selmer group is more general in that it allows different twists at different places dividing p.

1.4. **Structure of Selmer groups.** Let A be a torsion quotient of a nearly ordinary Galois representation V with Selmer weights $\mathcal{W}$ as before. Greenberg [6, 7] has obtained powerful results on the structure of Selmer groups under the assumption that they are Λ-cotorsion. In this section we adapt his techniques to Selmer groups with weights. The essential idea is to replace A by a twist which is well-behaved. Let

$$\kappa = \omega^{-1}\varepsilon : G_F \to 1 + p\mathbf{Z}_p$$

denote the character giving the isomorphism $\mathrm{Gal}(F_\infty/F) \cong 1+p\mathbf{Z}_p$. For any $t \in \mathbf{Z}_p$ we set $A_t = A \otimes \kappa^t$ and $A_{t,w}^{\mathcal{W}} = A_w^{\mathcal{W}} \otimes \kappa^t$ for any place w of F_∞ dividing p. Note that A_t is isomorphic to A as a G_{F_∞}-module. Let A_t^* denote the Cartier dual $\mathrm{Hom}_{\mathcal{O}}(A_t, K/\mathcal{O}(1))$; it is a free $\mathcal{O}$-module of finite rank.

Proposition 1.8. *Let A be a torsion quotient of a nearly ordinary Galois representation V with Selmer weights $\mathcal{W}$. Assume that $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ is Λ-cotorsion and that $H^0(F_\infty, A^* \otimes K/\mathcal{O})$ is finite.*

(1) *The sequence*

$$0 \to \mathrm{Sel}_{\mathcal{W}}(F_\infty, A) \to H^1(F_\Sigma/F_\infty, A) \to \prod_{w|v\in\Sigma} H^1_{s,\mathcal{W}}(F_{\infty,w}, A) \to 0$$

is exact for any finite set of places Σ of F sufficiently large for A.

(2) *Assume that $H^0(F, A^* \otimes K/\mathcal{O}) = 0$. Then $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ has no proper Λ-submodules of finite index.*

Proof. The sequence in (1) is exact by definition except for the surjectivity on the right. In fact $H^1_{s,\mathcal{W}}(F_{\infty,w}, A)$ is divisible for each w since G_w has p-cohomological dimension one, so to prove (1) it suffices to show that the the cokernel of the map

$$\gamma : H^1(F_\Sigma/F_\infty, A) \to \prod_{w|v\in\Sigma} H^1_{s,\mathcal{W}}(F_{\infty,w}, A)$$

is finite.

Let F_n denote the unique subfield of F_∞ of degree p^n over F. For a place w of F_n and $t \in \mathbf{Z}_p$ set

$$H^1_s(F_{n,w}, A_t) = \begin{cases} H^1(I_w, A_t)^{G_w/I_w} & w \nmid p \\ \operatorname{im} H^1(G_w, A_t) \to H^1(G_w, A_t/A^{\mathcal{W}}_{t,w}) & w \mid p. \end{cases}$$

(The condition at places dividing p is slightly stronger than we usually impose so that we may apply the appropriate duality results below.) We claim that to prove the surjectivity of γ it suffices to show that for some t the maps

$$\gamma_{n,t} : H^1(F_\Sigma/F_n, A_t) \to \prod_{w|v\in\Sigma} H^1_s(F_{n,w}, A_t)$$

have finite cokernel bounded independent of n. Indeed, taking the limit in n we then find that

$$\gamma_t : H^1(F_\Sigma/F_\infty, A_t) \to \prod_{w|v\in\Sigma} H^1_s(F_{\infty,w}, A_t)$$

has finite cokernel; here $H^1_s(F_{\infty,w}, A_t)$ is defined analogously to $H^1_s(F_{n,w}, A_t)$ above. Since $A_t \cong A$ over F_∞, one sees immediately that the cokernel of γ is a quotient of the cokernel of γ_t, so that this does indeed suffice.

We now select an appropriate value of t. We claim that for all but finitely many $t \in \mathbf{Z}_p$ we have:

(i) $\ker \gamma_{n,t}$ is finite for all n;
(ii) $H^0(F_{n,w}, A_t)$ and $H^0(F_{n,w}, A^*_t)$ are finite for all $w \mid v \in \Sigma$, $v \nmid p\infty$, and all n;
(iii) $H^0(F_{n,w}, A_t/A^{\mathcal{W}}_{t,w})$ and $H^0(F_{n,w}, (A_t/A^{\mathcal{W}}_{t,w})^*)$ are finite for all $w \mid p$ and all n;
(iv) $H^0(F_{n,w}, (A^{\mathcal{W}}_{t,w})^*)$ is finite for all $w \mid p$ and all n;

This is clear for the latter three conditions as $\kappa|_{G_w}$ has infinite order for each of the finitely many non-archimedean places of F_∞ lying over places in Σ. For the first condition, consider the restriction map

$$\ker \gamma_{n,t} \to \operatorname{Sel}_{\mathcal{W}}(F_\infty, A_t)^{\Gamma^{p^n}}$$

where Γ denotes $\operatorname{Gal}(F_\infty/F)$. The kernel of this map lies in $H^1(F_\infty/F_n, A_t^{G_{F_\infty}})$, which has the same corank as $H^0(F_n, A_t)$ and thus is finite for almost all t. Since $\operatorname{Sel}_{\mathcal{W}}(F_\infty, A)$ is Λ-cotorsion by assumption, $\operatorname{Sel}_{\mathcal{W}}(F_\infty, A_t)^{\Gamma^{p^n}}$ is finite for all n for almost all t; any t satisfying the latter two conditions satisfies (i).

Fix such a t. By (ii) and local duality we have that $H^2(F_{n,w}, A_t)$ is finite for all w dividing $v \in \Sigma$, $w \nmid p$, and all n. By (ii) and the local Euler characteristic formula, it follows that $H^1(F_{n,w}, A_t)$ is finite for all such w and all n. By an analogous argument using (iii) one sees that $H^1(F_{n,w}, A_t/A^{\mathcal{W}}_t)$ has corank $[F_{n,w} : \mathbf{Q}_p]c_v(V) = [F_v : \mathbf{Q}_p]c_v(V)p^n$ for all w dividing v dividing p. By (iv) and another application of local duality this implies that $H^1_s(F_{n,w}, A_t)$ has corank $[F_v : \mathbf{Q}_p]c_v(V)p^n$ as well.

The Poitou–Tate global duality sequence yields an exact sequence

$$0 \to \ker \gamma_{n,t} \to H^1(F_\Sigma/F_n, A_t) \xrightarrow{\gamma_{n,t}} \prod_{w|v\in\Sigma} H^1_s(F_{n,w}, A_t) \to H_{1,n} \to H_{2,n} \to 0$$

where $H_{1,n}$ is dual to a subgroup of $H^1(F_\Sigma/F_n, A_t^*)$ and $H_{2,n}$ is a subgroup of $H^2(F_\Sigma/F_n, A_t)$. By the global Euler characteristic formula we have

$$\operatorname{corank}_{\mathcal{O}} H^1(F_\Sigma/F_n, A_t) = p^n \cdot W + \operatorname{corank}_{\mathcal{O}} H^2(F_\Sigma/F_n, A_t)$$

where

$$W := \sum_{v \text{ real}} d_v^-(V) + \sum_{v \text{ complex}} n = \sum_{v|p} [F_v : \mathbf{Q}_p] \cdot c_v(V).$$

By our local computations above we see that the target of $\gamma_{n,t}$ has corank $W \cdot p^n$ as well. Since $\ker \gamma_{n,t}$ is finite by (i), it follows that $H^1(F_\Sigma/F_n, A_t)$ has corank $[F : \mathbf{Q}]c(V)p^n$ and that $H^2(F_\Sigma/F_n, A_t)$ is finite. In fact, $H^2(F_\Sigma/F_\infty, A_t)$ must therefore vanish since by [5, Proposition 4] it is cofree over Λ. Thus $H_{2,n}$ vanishes, so that $H_{1,n} = \operatorname{coker} \gamma_{n,t}$ is finite as well.

In particular, $\operatorname{coker} \gamma_{n,t}$ must be dual to a subgroup of $H^1(F_\Sigma/F_n, A_t^*)_{\text{tors}}$. This latter group is simply the kernel of

$$H^1(F_\Sigma/F_n, A_t^*) \to H^1(F_\Sigma/F_n, A_t^* \otimes K)$$

and thus equals the image of $H^0(F_n, A_t^* \otimes K/\mathcal{O})$. Thus for any n we can identify the dual of $\operatorname{coker} \gamma_{n,t}$ with a subgroup of $H^0(F_\infty, A_t^* \otimes K/\mathcal{O}) = H^0(F_\infty, A^* \otimes K/\mathcal{O})$. This latter group is finite by assumption, so that $\operatorname{coker} \gamma_{n,t}$ is bounded independent of n; the first part of the proposition follows.

We turn now to (2). Let t be as above, subject to the additional hypothesis that

$$H^1(F_\Sigma/F_\infty, A_t)/H^1(F_\Sigma/F_\infty, A_t)_{\Lambda-\text{div}}$$

has finite Γ-covariants. (This is certainly possible since

$$H^1(F_\Sigma/F_\infty, A)/H^1(F_\Sigma/F_\infty, A)_{\Lambda-\text{div}}$$

is visibly Λ-cotorsion.) Since the cokernel of the injection

$$\ker \gamma_t \hookrightarrow \operatorname{Sel}_{\mathcal{W}}(F_\infty, A)$$

is divisible, to show that $\operatorname{Sel}_{\mathcal{W}}(F_\infty, A)$ has no proper Λ-submodules of finite index it suffices to show the same for $\ker \gamma_t$. In fact, by the structure theory of Λ-modules for this it suffices to show that $(\ker \gamma_t)_\Gamma = 0$.

Since Γ is pro-p, the assumption that $H^0(F, A^* \otimes K/\mathcal{O}) = 0$ implies that

$$H^0(F_\infty, A^* \otimes K/\mathcal{O}) = H^0(F_\infty, A_t^* \otimes K/\mathcal{O}) = 0.$$

In particular, the group $H_{1,0}$ above in fact vanishes, so that the map

$$H^1(F_\Sigma/F, A_t) \to \prod_{w|v\in\Sigma} H_s^1(F_w, A_t)$$

is surjective. It follows from this and the fact that Γ has cohomological dimension one that the map

$$H^1(F_\Sigma/F_\infty, A_t)^\Gamma \to \prod_{w|v\in\Sigma} H_s^1(F_{\infty,w}, A_t)^\Gamma$$

is surjective as well. This in turn implies that the map

$$(\ker \gamma_t)_\Gamma \to H^1(F_\Sigma/F_\infty, A_t)_\Gamma$$

is injective. The latter group is finite by our last assumption on t. However, $H^1(F_\Sigma/F_\infty, A_t)$ has no proper Λ-submodules of finite index by [5, Proposition 5]. Thus $H^1(F_\Sigma/F_\infty, A_t)_\Gamma$ must in fact vanish, so that $(\ker \gamma_t)_\Gamma$ vanishes as well, as desired. □

1.5. **Iwasawa invariants.** Fix a uniformizer π of $\mathcal{O}$ and set $k = \mathcal{O}/\pi$. Assume now that A is a torsion quotient of a nearly ordinary Galois representation V with weights $\mathcal{W}$; that $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ is Λ-cotorsion; and that $H^0(F, A^* \otimes K/\mathcal{O}) = 0$. We define $\mu(A, \mathcal{W})$ by

$$\mu(A,\mathcal{W}) := \sum_{i=0}^{\infty} \mathrm{corank}_{\Lambda/\pi}\left(\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)[\pi^{i+1}]/\,\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)[\pi^i]\right);$$

this sum is finite since $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ is Λ-cotorsion. (Note that $\mu(A, \mathcal{W}) = 0$ if and only if $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ is cofinitely generated over $\mathcal{O}$.) We define $\lambda(A, \mathcal{W})$ to be the $\mathcal{O}$-corank of $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$. Proposition 1.8 yields the following result.

Corollary 1.9. *Let A be a torsion quotient of a nearly ordinary Galois representation V with weights $\mathcal{W}$ such that $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ is Λ-cotorsion and $H^0(F, A^* \otimes K/\mathcal{O}) = 0$. Then $\mu(A, \mathcal{W}) = 0$ if and only if $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)[\pi]$ is finite-dimensional. If $\mu(A, \mathcal{W}) = 0$, then $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ is a divisible $\mathcal{O}$-module and*

$$\lambda(A, \mathcal{W}) = \dim_k \mathrm{Sel}_{\mathcal{W}}(F_\infty, A)[\pi].$$

Proof. The first statement is immediate from the definition. If this is the case, then it follows from (2) of Proposition 1.8 that $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ is divisible: indeed, if $\mu(A, \mathcal{W}) = 0$ then the maximal divisible subgroup of $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ has finite index, so that by the proposition it must coincide with $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$. By definition we thus have

$$\mathrm{Sel}_{\mathcal{W}}(F_\infty, A) \cong (K/\mathcal{O})^{\lambda(A,\mathcal{W})};$$

the corollary follows. □

2. Residual Selmer groups

Corollary 1.9 suggests that it should be possible to study the Iwasawa invariants of a torsion quotient A by studying a corresponding residual Selmer group. However, this Selmer group must depend not only on the π-torsion of A but also on the ramification of A. In this section we model the different possible Selmer groups using a version of Mazur's notion of a finite/singular structure.

2.1. **Residual Galois representations.** Let F be a number field as before and let k be a finite field of odd characteristic p. A *nearly ordinary Galois representation* over k consists of a finite-dimensional k-vector space Δ endowed with a k-linear action of G_F and choices of G_v-stable complete flags

$$0 = \Delta_v^0 \subsetneq \Delta_v^1 \subsetneq \cdots \subsetneq \Delta_v^n = \Delta$$

for each place v dividing p. For a real place v of F let $d_v^-(\Delta)$ denote the K-dimension of the subspace of V on which complex conjugation at v acts by -1. A set of *Selmer weights* $\mathcal{W}$ for Δ is a choice of integers $c_v(\Delta)$, $0 \le c_v(\Delta) \le n$, for each v dividing p such that

$$\sum_{v \text{ real}} d_v^-(\Delta) + \sum_{v \text{ complex}} n = \sum_{v|p} [F_v : \mathbf{Q}_p] \cdot c_v(\Delta).$$

Set $\Delta_v^{\mathcal{W}} = \Delta_v^{n - c_v(\Delta)}$ for each place v dividing p.

2.2. **Finite/singular structures.** Let Δ be a nearly ordinary Galois representation over k with weights $\mathcal{W}$.

Definition 2.1. A *finite/singular* structure $\mathcal{S}$ on Δ with weights $\mathcal{W}$ is a choice of k-subspaces

$$H^1_{f,\mathcal{S}}(F_{\infty,w},\Delta) \subseteq H^1(G_w,\Delta)$$

for every place w of F_∞ subject to the restrictions:

(1) $H^1_{f,\mathcal{S}}(F_{\infty,w},\Delta) = \ker H^1(G_w,\Delta) \to H^1(I_w,\Delta_w^{\mathcal{W}})$ for any place w dividing p;
(2) $H^1_{f,\mathcal{S}}(F_{\infty,w},\Delta) = 0$ for almost all w;
(3) $H^1_{f,\mathcal{S}}(F_{\infty,w},\Delta)$ and $H^1_{f,\mathcal{S}}(F_{\infty,w'},\Delta)$ coincide under the canonical isomorphism $H^1(G_w,\Delta) \cong H^1(G_{w'},\Delta)$ for any places w and w' dividing the same place $v \nmid p$ of F.

(Note that $H^1(G_w,\Delta) = 0$ for all archimedean places w, so that we may safely ignore archimedean places below.)

Fix a finite/singular structure $\mathcal{S}$ on Δ. We say that a finite set Σ of places of F is *sufficiently large* for $\mathcal{S}$ if it contains all places at which Δ is ramified, all archimedean places, all places dividing p, and all places v such that $H^1_{f,\mathcal{S}}(F_{\infty,w},\Delta)$ is non-zero for some w dividing v. For a place v of F we set

$$\delta_{\mathcal{S},v}(\Delta) = \sum_{w|v} \dim_k H^1_{f,\mathcal{S}}(F_{\infty,w},\Delta)$$

and for a place w of F_∞ we set

$$H^1_{s,\mathcal{S}}(F_{\infty,w},\Delta) = H^1(G_w,\Delta)/H^1_{f,\mathcal{S}}(F_{\infty,w},\Delta).$$

(Note that $H^1(G_w,\Delta)$ is always finite dimensional over k, so that $\delta_{\mathcal{S},v}(\Delta)$ is finite.) We define the *$\mathcal{S}$-Selmer group* of Δ by

$$\begin{aligned} \mathrm{Sel}_{\mathcal{S}}(F_\infty,\Delta) &= \ker H^1(F_\infty,\Delta) \to \prod_w H^1_{s,\mathcal{S}}(F_{\infty,w},\Delta) \\ &= \ker H^1(F_\Sigma/F_\infty,\Delta) \to \prod_{w|v\in\Sigma} H^1_{s,\mathcal{S}}(F_{\infty,w},\Delta) \end{aligned} \tag{4}$$

for any finite set of places Σ of F which is sufficiently large for $\mathcal{S}$. (We consider the weights $\mathcal{W}$ to be implicit in the finite/singular structure $\mathcal{S}$ and thus omit them from the notation.)

We say that $\mu(\Delta) = 0$ if $H^1(F_\Sigma/F_\infty,\Delta)$ is finite dimensional; this is independent of the choice of sufficiently large Σ. If this is the case, then $\mathrm{Sel}_{\mathcal{S}}(F_\infty,\Delta)$ is finite dimensional for any $\mathcal{S}$, and we set

$$\lambda_{\mathcal{S}}(\Delta) = \dim_k \mathrm{Sel}_{\mathcal{S}}(F_\infty,\Delta).$$

Example 2.2. The *minimal structure* $\mathcal{S}_{\min}$ is given by

$$H^1_{f,\mathcal{S}_{\min}}(F_{\infty,w},\Delta) = 0$$

for w not dividing p. (The condition at w dividing p is fixed by definition.) The corresponding *minimal Selmer group* $\mathrm{Sel}_{\mathcal{S}_{\min}}(F_\infty,\Delta)$ is contained in every other Selmer group of Δ.

Example 2.3. Let K be a finite extension of $\mathbf{Q}_p$ with residue field k; we write $\mathcal{O}$ for the ring of integers and π for a fixed choice of uniformizer. Let A be a torsion quotient of a nearly ordinary Galois representation V over K with weights $\mathcal{W}$. The π-torsion $A[\pi]$ inherits an obvious structure of nearly ordinary Galois representation over k with weights $\mathcal{W}$. We define a structure $\mathcal{S}(A)$ on $A[\pi]$ by setting

$$H^1_{f,\mathcal{S}(A)}(F_{\infty,w}, A[\pi]) = \ker H^1(G_w, A[\pi]) \to H^1_{s,\mathcal{W}}(F_{\infty,w}, A)$$

for every place w of F_∞. Note that for w not dividing p we have

$$H^1_{f,\mathcal{S}(A)}(F_{\infty,w}, A[\pi]) = \operatorname{im} A^{G_w}/\pi \hookrightarrow H^1(G_w, A[\pi]).$$

In order to prove that $\mathcal{S}(A)$ is a finite/singular structure on $A[\pi]$ we need a simple yet crucial lemma on local Galois invariants.

Lemma 2.4. *Let A be a torsion quotient of a nearly ordinary Galois representation V over K. Let v be a place of F not dividing p such that V and $A[\pi]$ have the same Artin conductor at v. Then A^{G_w} is $\mathcal{O}$-divisible for every place w of F_∞ dividing v.*

Proof. Fix a place w of F_∞ dividing v. As the Swan conductor is invariant under reduction (see [9] for example), it follows from the hypothesis and the definition of the Artin conductor that

$$\dim_K V^{I_w} = \dim_k A[\pi]^{I_w}.$$

(Note that $I_w = I_v$ since F_∞/F is unramified at v.) Since A^{I_w} has $\mathcal{O}$-corank equal to $\dim_K V^{I_w}$, we conclude that A^{I_w} is $\mathcal{O}$-divisible. The G_w/I_w-invariants of an $\mathcal{O}$-divisible module are still $\mathcal{O}$-divisible, so that the lemma follows from this. □

Proposition 2.5. *Let A be a torsion quotient of a nearly ordinary Galois representation V over K. If $H^0(I_w, A/A_w^{\mathcal{W}})$ is $\mathcal{O}$-divisible for each place w dividing p, then the structure $\mathcal{S}(A)$ is a finite/singular structure on $A[\pi]$.*

The divisibility hypothesis above appears to be essential to our method. In the case that V arises from a modular form f, it corresponds to the assumption that f is a twist of an ordinary modular form (in the usual sense that the p^{th} Fourier coefficient is prime to π) by a power of the Teichmüller character.

Proof. It follows from Lemma 2.4 that $H^1_{f,\mathcal{S}(A)}(F_{\infty,w}, A[\pi]) = 0$ for almost all w. The coincidence of $H^1_{f,\mathcal{S}(A)}(F_{\infty,w}, A[\pi])$ and $H^1_{f,\mathcal{S}(A)}(F_{\infty,w'}, A[\pi])$ for w and w' dividing $v \nmid p$ is immediate from the definition. Finally, the verification of the conditions at places w dividing p is a simple diagram chase using the fact that the divisibility of $H^0(I_w, A/A_w^{\mathcal{W}})$ implies that

$$H^1(I_w, A/A_w^{\mathcal{W}})[\pi] \to H^1(I_w, A/A_w^{\mathcal{W}})$$

is injective □

Corollary 2.6. *Let A be a torsion quotient of a Galois representation V over K such that $H^0(I_w, A/A_w^{\mathcal{W}})$ is $\mathcal{O}$-divisible for each place w dividing p. If $H^0(F_\infty, A)$ is $\mathcal{O}$-divisible, then the natural map*

$$\mathrm{Sel}_{\mathcal{S}(A)}(F_\infty, A[\pi]) \to \mathrm{Sel}_{\mathcal{W}}(F_\infty, A)[\pi]$$

is an isomorphism. In particular, $\mu(A, \mathcal{W}) = 0$ if and only if $\mu(A[\pi]) = 0$. If this is the case, then $\lambda(A, \mathcal{W}) = \lambda_{\mathcal{S}(A)}(A[\pi])$.

Proof. The identification of Selmer groups is immediate from the injectivity of

$$H^1(F_\infty, A[\pi]) \to H^1(F_\infty, A)$$

and the definition of $\mathcal{S}(A)$. The relation between the Iwasawa invariants now follows from the definitions and Corollary 1.9. □

In particular, the corollary implies that knowledge of the residual representation $A[\pi]$ and finite/singular structure $\mathcal{S}(A)$ (which depends only on $A[\pi]$ and the ramification of A) determines the Iwasawa invariants of $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$. In the next section we will use Proposition 1.8 to give a more precise description of this relation.

We remark that one can consider induced structures as above for lifts of Δ to representations over more general complete local noetherian rings with residue field k. In particular, one can then compare the structures induced from the ring and its quotients. For example, one can show that induced structures are constant on families with constant ramification in an appropriate sense. (In the case of Hida families, by this we mean a branch of the Hida family with all crossing points removed.) We will not pursue this point of view any further here.

2.3. **Variation of structure.** Fix a nearly ordinary Galois representation Δ over k with weights $\mathcal{W}$. Let K be a finite extension of $\mathbf{Q}_p$ with residue field k, ring of integers $\mathcal{O}$ and uniformizer π. A *lift* of Δ over K is a torsion quotient A of a nearly ordinary Galois representation V over K such that $H^0(I_w, A/A_w^{\mathcal{W}})$ is $\mathcal{O}$-divisible for all w dividing p, together with an isomorphism $\Delta \cong A[\pi]$ of nearly ordinary Galois representations. We say that a finite/singular structure $\mathcal{S}$ on Δ is *induced* if there is a lift A of Δ such that the isomorphism $\Delta \cong A[\pi]$ identifies $\mathcal{S}$ with $\mathcal{S}(A)$. We say that $\mathcal{S}$ is *properly induced* if such an A can be chosen so that $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ is Λ-cotorsion.

The key result for the analysis of residual Selmer groups is the following proposition.

Proposition 2.7. *Assume that $\mu(\Delta) = 0$ and $H^0(F, \Delta) = H^0(F, \Delta^*) = 0$. If $\mathcal{S}$ is properly induced, then the sequence*

$$0 \to \mathrm{Sel}_{\mathcal{S}}(F_\infty, \Delta) \to H^1(F_\Sigma/F_\infty, \Delta) \to \prod_{w|v\in\Sigma} H^1_{s,\mathcal{S}}(F_{\infty,w}, \Delta) \to 0$$

is exact for any finite set of places Σ of F sufficiently large for $\mathcal{S}$.

Proof. Fix a proper lift A of Δ which identifies $\mathcal{S}(A)$ with $\mathcal{S}$. Note that the hypotheses imply that

$$H^0(F, A) = H^0(F, A^* \otimes K/\mathcal{O}) = 0.$$

Consider the exact sequence

$$0 \to \mathrm{Sel}_{\mathcal{W}}(F_\infty, A) \to H^1(F_\Sigma/F_\infty, A) \to \prod_{w|v\in\Sigma} H^1_{s,\mathcal{W}}(F_{\infty,w}, A) \to 0$$

of Proposition 1.8. Since $\mathrm{Sel}_{\mathcal{W}}(F_\infty, A)$ is $\mathcal{O}$-divisible by Corollary 1.9, the π-torsion of this sequence is an exact sequence

$$0 \to \mathrm{Sel}_{\mathcal{W}}(F_\infty, A)[\pi] \to H^1(F_\Sigma/F_\infty, A)[\pi] \to \prod_{w|v\in\Sigma} H^1_{s,\mathcal{W}}(F_{\infty,w}, A)[\pi] \to 0.$$

Since $H^0(F_\infty, A) = 0$ is divisible, it follows easily from Corollary 2.6 and the definitions that this sequence identifies with the desired sequence. □

The above proof rests entirely on the crutch of a proper lift of Δ. We do not know how to approach this problem purely via the residual representation Δ.

Corollary 2.8. *Assume that $\mu(\Delta) = 0$ and $H^0(F, \Delta) = H^0(F, \Delta^*) = 0$. The quantity*

$$\lambda_{\mathcal{S}}(\Delta) - \sum_{v \nmid p} \delta_{\mathcal{S},v}(\Delta)$$

is independent of the choice of properly induced structure $\mathcal{S}$.

Proof. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be properly induced structures on Δ and choose Σ which is sufficiently large for both. Then it follows from Proposition 2.7 and the agreement of the local conditions at places dividing p that

$$\lambda_{\mathcal{S}_1}(\Delta) + \sum_{\substack{w|v \in \Sigma \\ v \nmid p}} \dim H^1_{s,\mathcal{S}_1}(F_{\infty,w}, \Delta) = \lambda_{\mathcal{S}_2}(\Delta) + \sum_{\substack{w|v \in \Sigma \\ v \nmid p}} \dim H^1_{s,\mathcal{S}_2}(F_{\infty,w}, \Delta).$$

Since

$$\delta_{\mathcal{S}_i,v}(\Delta) = \sum_{w|v} \dim H^1(F_{\infty,w}, \Delta) - \dim H^1_{s,\mathcal{S}_i}(F_{\infty,w}, \Delta)$$

the corollary follows from this. □

Note that when the minimal structure $\mathcal{S}_{\min}$ itself is properly induced the above difference is simply $\lambda_{\mathcal{S}_{\min}}(\Delta)$. The question of whether or not the minimal structure $\mathcal{S}_{\min}$ is induced is intimately related to level lowering. More precisely, we have the following result, which follows immediately from Lemma 2.4.

Proposition 2.9. *Let A be a lift of Δ over K. If the Artin conductor of Δ equals the Artin conductor of V, then the isomorphism $\Delta \cong A[\pi]$ identifies $\mathcal{S}_{\min}$ with $\mathcal{S}(A)$.*

2.4. **Existence of structures.** Proposition 2.7 gives very precise control over properly induced structures. It thus becomes an interesting question to determine which structures actually occur in this way; that is, for which structures $\mathcal{S}$ on Δ do there exist torsion quotients A of nearly ordinary Galois representations V with Λ-cotorsion Selmer group and an isomorphism $\Delta \cong A[\pi]$ identifying $\mathcal{S}$ with $\mathcal{S}(A)$?

The local part of this question is not difficult: given Δ, a place w of F_∞ dividing a place v of F, and a subspace

$$H \subseteq H^1(G_w, \Delta)$$

it is a straightforward calculation to determine if there is a G_v-representation A lifting Δ such that

$$H = \operatorname{im} A^{G_w}/\pi \to H^1(G_w, \Delta).$$

Much more difficult is the amalgamation of this local information into a global structure. This latter question is intimately connected with level raising in the sense of [2]. To illustrate this connection we give the following result for two-dimensional modular representations of tame level one; one can prove similar results for higher levels, but we focus on this case for simplicity.

Let Δ be an absolutely irreducible nearly ordinary two-dimensional modular Galois representation over $\mathbf{Q}$ unramified away from p. Assume also that the G_p-representation Δ/Δ^1_p is unramified. Let $\mathcal{S}$ be a finite/singular structure on Δ. Consider the two conditions:

(1) $H^1_{f,\mathcal{S}}(\mathbf{Q}_{\infty,w},\Delta) = 0$ for any place $w \nmid p$ dividing a prime $\ell \not\equiv 1 \pmod p$ such that
$$\Delta|_{G_\ell} \cong \begin{pmatrix} \omega & * \\ 0 & \chi \end{pmatrix}$$
with ω the Teichmüller character and $\chi \neq 1$. (Note that $\Delta|_{G_\ell}$ is unramified by assumption and thus will be split in this case unless $\chi = \omega$.)

(2) $\dim H^1_{f,\mathcal{S}}(\mathbf{Q}_{\infty,w},\Delta) \neq 1$ for any place w dividing a prime $\ell \equiv 1 \pmod p$ such that $\Delta|_{G_\ell}$ is trivial.

Proposition 2.10. *Let $\mathcal{S}$ be a finite/singular structure on Δ. If $\mathcal{S}$ is induced, then (1) holds. If (1) and (2) hold, then $\mathcal{S}$ is properly induced.*

It is an interesting question as to whether or not condition (1) alone suffices to determine if $\mathcal{S}$ is induced. This reduces to the question of whether or not for a prime $\ell \equiv 1 \pmod p$ such that $\Delta|_{G_\ell}$ is trivial one can find modular lifts of Δ which are special (of level ℓ) at ℓ and with the unramified line lifting an arbitrary choice of line in Δ. It is not clear to this author if the techniques of [2] can be modified to answer this question.

In the proof we will frequently use the fact that $H^1(G_w,\Delta)$ (resp. $H^1(G_w,A)[\pi]$) has k-dimension equal to the multiplicity of ω in the inertia coinvariants Δ_{I_w} (resp. V_{I_w}); see [7, Section 2].

Proof. Suppose first that $\mathcal{S}$ is induced and let w be a place dividing a prime ℓ with $\Delta|_{G_\ell}$ as in (1). Let A be a torsion quotient of a nearly ordinary Galois representation V lifting $\mathcal{S}$ and consider the corresponding exact sequence
$$0 \to A^{G_w}/\pi \to H^1(G_w,\Delta) \to H^1(G_w,A)[\pi] \to 0.$$
If A is unramified at ℓ, then A^{G_w} is divisible, so that we must have $H^1_{f,\mathcal{S}}(\mathbf{Q}_{\infty,w},\Delta) = 0$. If A is ramified at ℓ, then since Δ is unramified at ℓ yet $\ell \not\equiv 1 \pmod p$ and $\chi \neq 1$, we must have $\chi = \omega^2$. (See [2, Section 1].) But then $\ell \not\equiv -1 \pmod p$ (for that would force $\chi = 1$) so that V must be special at ℓ: that is, the Galois action on V is given by
$$\begin{pmatrix} \omega^2 & * \\ 0 & \omega \end{pmatrix}$$
with $*$ ramified but trivial modulo π. Then both $H^1(G_w,\Delta)$ and $H^1(G_w,A)[\pi]$ are one-dimensional, so that we still have $H^1_{f,\mathcal{S}}(\mathbf{Q}_{\infty,w},\Delta) = 0$.

Next suppose that (1) and (2) are satisfied and fix a prime ℓ different from p. We claim that there exists a two-dimensional K-vector space V_ℓ with a K-linear action of G_ℓ such that
$$\sum_{w|\ell} \dim_k H^1(G_w,\Delta) - \dim_K H^1(G_w,V_\ell) = \delta_{\mathcal{S},\ell}(\Delta).$$
Indeed, this is straightforward from the discussion of [2, Section 1]. Specifically, we may take V_ℓ unramified if $\delta_{\mathcal{S},\ell}(\Delta) = 0$. If $\delta_{\mathcal{S},\ell}(\Delta) \neq 0$ and $\ell \not\equiv 1 \pmod p$, then by (1) we must have
$$\Delta|_{G_\ell} \cong \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}.$$
In this case one simply takes V_ℓ to be the special representation
$$\begin{pmatrix} \varepsilon & * \\ 0 & 1 \end{pmatrix}$$

with $*$ trivial modulo π. If $\delta_{\mathcal{S},\ell}(\Delta) \neq 0$ and $\ell \equiv 1 \pmod{p}$, then one may take V_ℓ to be a ramified principal series lifting of Δ.

After possibly enlarging K, we may now apply [2, Theorem 1] to obtain a torsion quotient A of a (necessarily nearly ordinary) modular representation V together with an isomorphism $\Delta \cong A[\pi]$ and isomorphisms $V|_{I_\ell} \cong V_\ell$ for every $\ell \neq p$. Since $\dim_K H^1(G_w, V)$ depends only on Δ and $V|_{I_\ell}$, we have

$$\sum_{w|\ell} \dim_k H^1(G_w, \Delta) - \dim_K H^1(G_w, V) = \delta_{\mathcal{S},\ell}(\Delta)$$

for all places w dividing $\ell \neq p$. Under the assumption (2) this is in fact enough to determine $H^1_{f,\mathcal{S}}(\mathbf{Q}_{\infty,w}, \Delta)$, so that we must have $\mathcal{S}(A) = \mathcal{S}$. Finally, since V is modular it is known by work of Kato that $\mathrm{Sel}_{\mathcal{W}}(\mathbf{Q}_\infty, A)$ is Λ-cotorsion, so that $\mathcal{S}$ is properly induced, as desired. □

3. Galois deformations

3.1. **Deformations.** Fix a finite extension K_0 of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_0$ and residue field k. Consider the algebraic group GL_n over $\mathcal{O}_0$. For each place v of F dividing p fix a Borel subgroup B_v of GL_n. Let

$$\bar{\rho} : G_F \to \mathrm{GL}_n(k)$$

be a continuous residual Galois representation. We assume:

(1) the centralizer of the image of $\bar{\rho}$ consists only of scalars;
(2) $\bar{\rho}(G_v)$ lies in $\mathrm{B}_v(k)$ for every place v dividing p;
(3) $\bar{\rho}$ is *regular*: for any place v dividing p the invariants $H^0(G_v, \mathfrak{gl}_n/\mathfrak{B}_v)$ vanish. (Here $\mathfrak{gl}_n$ and $\mathfrak{B}_v$ are the Lie algebras of GL_n and B_v endowed with adjoint G_v-action.)

(See [11, Section 6.1] for a discussion of the last condition. It is satisfied in particular if the diagonal characters in the representation $G_v \to \mathrm{B}_v(k)$ are distinct.)

A *nearly ordinary lift* ρ of $\bar{\rho}$ to a local $\mathcal{O}_0$-algebra $\mathcal{O}$ with residue field k is a continuous representation

$$\rho : G_F \to \mathrm{GL}_n(\mathcal{O}),$$

ramified at finitely many places, such that the composition

$$G_F \xrightarrow{\rho} \mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(k)$$

is equal to $\bar{\rho}$ and such that for each place v dividing p there is

$$g_v \in \widehat{\mathrm{GL}}_n(\mathcal{O}) := \ker \mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(k)$$

such that

$$g_v \cdot \rho(G_v) \cdot g_v^{-1} \subseteq \mathrm{B}_v(\mathcal{O}).$$

We consider two such liftings equivalent if one can be conjugated to the other via some element of $\widehat{\mathrm{GL}}_n(\mathcal{O})$; a *nearly ordinary deformation* of $\bar{\rho}$ is an equivalence class of liftings.

For a local $\mathcal{O}_0$-algebra $\mathcal{O}$ with residue field k we write $\mathcal{H}(\mathcal{O})$ for the set of nearly ordinary deformations of $\bar{\rho}$ to $\mathcal{O}$. If Σ is a finite set of places sufficiently large for $\bar{\rho}$, we write $\mathcal{H}_\Sigma(\mathcal{O})$ for the subset of $\mathcal{H}(\mathcal{O})$ of deformations unramified away from Σ. By [11, Proposition 6.2] the functor $\mathcal{H}_\Sigma$ is representable: there is a local $\mathcal{O}_0$-algebra R^{no}_Σ with residue field k such that there is a bijection between $\mathcal{H}_\Sigma(\mathcal{O})$ and $\mathrm{Hom}(R^{\mathrm{no}}_\Sigma, \mathcal{O})$ for any $\mathcal{O}$ as above. (We note that $\mathrm{Hom}(R^{\mathrm{no}}_\Sigma, \mathcal{O})$ is computed in the category of inverse limits of artinian local $\mathcal{O}_0$-algebras with residue field k; such morphisms are required to be local homomorphisms inducing the identity map on k.)

3.2. **Selmer groups.** Let $\Delta(\bar{\rho})$ denote the representation space for $\bar{\rho}$. We endow $\Delta(\bar{\rho})$ with the structure of nearly ordinary Galois representation by letting

$$0 = \Delta_v^0 \subsetneq \Delta_v^1 \subsetneq \cdots \subsetneq \Delta_v^n = \Delta(\bar{\rho})$$

be the complete flag associated to the Borel subgroup B_v for each v dividing p; since $\bar{\rho}(G_v) \subseteq \mathrm{B}_v(k)$, this flag is indeed G_v-stable. Fix also Selmer weights $\mathcal{W}$ for $\Delta(\bar{\rho})$, assuming they exist.

Let $\mathcal{O}$ be the ring of integers of a finite totally ramified extension K of K_0 and fix $\rho \in \mathcal{H}(\mathcal{O})$. Let T_ρ denote the representation space of ρ; it is a free $\mathcal{O}$-module of rank n. The isomorphism class of T_ρ depends only on the deformation class of ρ, so that we may speak of the Galois representation associated to ρ. Set $V_\rho = T_\rho \otimes_{\mathcal{O}} K$ and $A_\rho = V_\rho / T_\rho$.

We claim that we may endow V_ρ with a canonical structure of nearly ordinary Galois representation. Indeed, let v be a place dividing p and fix $g_v \in \widehat{\mathrm{GL}}_n(\mathcal{O})$ such that

$$g_v \cdot \rho(G_v) \cdot g_v^{-1} \subseteq \mathrm{B}_v(\mathcal{O}).$$

Define a complete flag

$$0 = V_{\rho,v}^0 \subsetneq V_{\rho,v}^1 \subsetneq \cdots \subsetneq V_{\rho,v}^n = V_\rho$$

as the g_v^{-1} conjugate of the complete flag associated to the Borel subgroup B_v. It is clear that this flag is G_v-stable. It is also independent of the choice of g_v as above: by [11, Claim of p. 49] (using the regularity assumption) any other $g_v' \in \widehat{\mathrm{GL}}_n(\mathcal{O})$ such that

$$g_v' \cdot \rho(G_v) \cdot g_v'^{-1} \subseteq \mathrm{B}_v(\mathcal{O})$$

must differ from g_v by an element of

$$\widehat{\mathrm{B}}_v(\mathcal{O}) := \ker \mathrm{B}_v(\mathcal{O}) \to \mathrm{B}_v(k)$$

and thus yields the same complete flag. We always regard V_ρ as a nearly ordinary Galois representation via these choices of complete flags. Note that $\mathcal{W}$ is a set of Selmer weights for V_ρ.

Define the Selmer group $\mathrm{Sel}_{\mathcal{W}}(F_\infty, \rho)$ by

$$\mathrm{Sel}_{\mathcal{W}}(F_\infty, \rho) := \mathrm{Sel}_{\mathcal{W}}(F_\infty, A_\rho).$$

We write $\mu(\rho, \mathcal{W})$ and $\lambda(\rho, \mathcal{W})$ for the Iwasawa invariants of $\mathrm{Sel}_{\mathcal{W}}(F_\infty, \rho)$.

Let $\rho_1 \in \mathcal{H}(\mathcal{O}_1)$ and $\rho_2 \in \mathcal{H}(\mathcal{O}_2)$ be two deformations as above. We say that $\rho_1 \in \mathcal{H}(\mathcal{O}_1)$ and $\rho_2 \in \mathcal{H}(\mathcal{O}_2)$ are *stably equivalent* if there is a finite totally ramified extension K_3 of K_0, with ring of integers $\mathcal{O}_3$, and injective morphisms $\sigma_1 : \mathcal{O}_1 \to \mathcal{O}_3$ and $\sigma_2 : \mathcal{O}_2 \to \mathcal{O}_3$ such that $\sigma_1 \circ \rho_1$ and $\sigma_2 \circ \rho_2$ are equivalent deformations over $\mathcal{O}_3$. We simply write $\mathcal{H}$ for the set of stable equivalence classes of nearly ordinary deformations of $\bar{\rho}$ such that $\mathrm{Sel}_{\mathcal{W}}(F_\infty, \rho)$ is Λ-cotorsion and with the additional property that $H^0(F_{\infty,w}, A_\rho / A_{\rho,w}^{\mathcal{W}})$ is $\mathcal{O}$-divisible for each w dividing p. Note that as defined above $\mu(\rho, \mathcal{W})$ and $\lambda(\rho, \mathcal{W})$ depend only on the stable equivalence class of ρ

Theorem 3.1. *Assume that* $H^0(F, \Delta(\bar{\rho})) = H^0(F, \Delta(\bar{\rho})^*) = 0$. *If* $\mu(\rho_0, \mathcal{W}) = 0$ *for some* $\rho_0 \in \mathcal{H}$, *then* $\mu(\rho, \mathcal{W}) = 0$ *for all* $\rho \in \mathcal{H}$.

We then say simply that $\mu(\bar{\rho}, \mathcal{W}) = 0$.

Proof. This is immediate from Corollary 2.6. □

Theorem 3.2. *Assume that* $H^0(F, \Delta(\bar{\rho})) = H^0(F, \Delta(\bar{\rho})^*) = 0$ *and* $\mu(\bar{\rho}, \mathcal{W}) = 0$. *The quantity*

$$\lambda(\rho, \mathcal{W}) - \sum_{w \nmid p} \dim_k A_\rho^{G_w}/\pi$$

is independent of the choice of $\rho \in \mathcal{H}$.

Proof. This is immediate from Corollaries 2.6 and 2.8. □

Note that for a place w of F_∞ dividing a place $v \nmid p$ of F, by [7, Section 2] the dimension $\dim_k A_\rho^{G_w}/\pi$ is equal to the difference between the number of occurrences of the Teichmüller character in the G_w/I_w-representations $\Delta(\rho)_{I_w}$ and $A_{\rho, I_w}[\pi]$. Theorem 1 is thus an immediate consequence of Theorems 3.1 and 3.2.

References

[1] Spencer Bloch and Kazuya Kato, *L-functions and Tamagawa numbers of motives*, in: *The Grothendieck Festschrift*, Vol. I, Birkhäuser, Boston, 1990, 333-400.

[2] Fred Diamond and Richard Taylor, *Lifting modular mod l representations*, Duke Math. J. **74** (1994), 253–269.

[3] Matthew Emerton, Robert Pollack and Tom Weston, *Variation of Iwasawa invariants in Hida families*, preprint, 2004.

[4] Kazuhiro Fujiwara, *Deformation rings and Hecke algebras in the totally real case*, preprint.

[5] Ralph Greenberg, *Iwasawa theory for p-adic representations*, in: *Algebraic number theory*, Adv. Stud. Pure Math. **17**, Academic Press, Boston, 1989, 97–137.

[6] Ralph Greenberg, *Iwasawa theory for elliptic curves*, in: *Arithmetic theory of elliptic curves (Cetraro, 1997)*, Lecture Notes in Math. **1716**, Springer, Berlin, 1999, 51–144.

[7] Ralph Greenberg and Vinayak Vatsal, *On the Iwasawa invariants of elliptic curves*, Invent. Math. **142** (2000), 17–63.

[8] Haruzo Hida, *Adjoint Selmer groups as Iwasawa modules*, Israel J. of Math. **120** (2000), 361–427.

[9] Ron Livne, *On the conductors of mod ℓ Galois representations coming from modular forms*, J. Number Theory **31** (1989), 133–141.

[10] Bernadette Perrin-Riou, *Représentations p-adiques ordinaires*, in: *Périodes p-adiques (Bures-sur-Yvette, 1988)*, Asterisque **223** (1994), 185–200.

[11] Jacques Tilouine, *Deformations of Galois representations and Hecke algebras*, Narosa Publishing House, Delhi, 1996.

[12] Jacques Tilouine and Eric Urban, *Several variable p-adic families of Siegel-Hilbert cusp eigensystems and their Galois representations*, Ann. Sci. École Norm. Sup. **32** (1999), 499–574.

E-mail address: `weston@math.umass.edu`

Department of Mathematics and Statistics, University of Massachusetts Amherst