GEOMETRIC EULER SYSTEMS FOR LOCALLY ISOTROPIC MOTIVES

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Let T be a Galois stable lattice in an irreducible ℓ-adic Galois representation of a number field F. When T is motivic of non-negative weight, conjectures of Bloch and Kato on L-functions predict that the Selmer group $\mathcal{S}(F, T \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ is finite. In this paper we give a geometric interpretation of this conjectural finiteness in the case that T is *locally isotropic of weight zero.* (We say that T is locally isotropic if the set of elements of $\mathrm{Gal}(\bar{F}/F)$ which fix some non-zero vector of T contains an open set.) This link with geometry is provided by a theory of *geometric Euler systems*: we formulate a conjecture on the existence of geometric Euler systems in motivic cohomology, and we show that the existence of a geometric Euler system (for the Cartier dual of T) implies the finiteness of $\mathcal{S}(F, T \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ precisely when T is locally isotropic.

As an application we prove the following result.

Theorem 1. *Let f be a classical newform of weight $k \geq 2$, level N, and arbitrary character. Let K be a finite extension of $\mathbf{Q}$ containing the Fourier coefficients of f and fix a prime λ of K dividing the rational prime ℓ. Let $T_{f,\lambda}$ be a Galois stable lattice in the λ-adic representation $\rho_{f,\lambda}$ associated to f by Deligne. Assume that f is not of CM-type; that f is special or supercuspidal at all p dividing N; and that ℓ does not divide N. Then $\mathcal{S}(\mathbf{Q}, \mathrm{End}^0\, T_{f,\lambda} \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ is finite, where $\mathrm{End}^0\, T_{f,\lambda}$ is the space of trace-zero endomorphisms of $T_{f,\lambda}$.*

We remark that a much more precise version of Theorem 1 (giving the order of the Selmer group rather than merely its finiteness) has been obtained in [2] by Diamond, Flach and Guo. They use different methods which do not require our assumptions at p dividing N; instead they require that $\ell > k$ and that the residual representation $\bar{\rho}_{f,\lambda}$ is absolutely irreducible when restricted to $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$, with E the quadratic extension of $\mathbf{Q}$ generated by the square root of $(-1)^{(\ell-1)/2}\ell$. Recently Mark Kisin has also obtained results similar to ours in some cases with ℓ dividing N.

We note the following immediate corollary of our theorem.

Corollary. *Let f be a cuspidal Hecke eigenform of weight $k \geq 2$ and level 1 and let K be a finite extension of $\mathbf{Q}$ containing the Fourier coefficients of f. Then $\mathcal{S}(\mathbf{Q}, \mathrm{End}^0\, T_{f,\lambda} \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ is finite for all primes λ of K.*

Our methods also work over function fields of characteristic different from ℓ. In particular, we have the following result; see Section 5.3 for precise definitions.

Theorem 2. *Let $\mathbf{F}$ be a finite field of characteristic different from ℓ. Let π be a non-CM automorphic representation of $\mathrm{GL}(2)$ of the adeles of $\mathbf{F}(t)$ of weight 2, squarefree level, and trivial character. Let K be a finite extension of $\mathbf{Q}_\ell$ over which one can define the ℓ-adic representation ρ_π associated to π and let T_π be a Galois*

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stable lattice in the representation space of ρ_π. *Then* $\mathcal{S}(\mathbf{F}(t), \operatorname{End}^0 T_\pi \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ *is finite.*

Before we discuss the contents of this paper in more detail we review some related notions. Let T be a finite free $\mathbf{Z}_\ell$-module with a continuous action of the absolute Galois group of a number field F. The Selmer group $\mathcal{S}(F, T \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ of T is the subgroup of the Galois cohomology group $H^1(F, T \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ of elements satisfying certain local conditions at every place of F. A standard approach to bounding such a Selmer group is Kolyvagin's method of arithmetic Euler systems. Roughly speaking, an arithmetic Euler system for the Cartier dual T^* of T consists of a twisted norm compatible collection of classes $c_{F'} \in H^1(F', T^*)$ for a family of abelian extensions F'/F. One descend these classes via Kolyvagin's derivative construction to classes in $H^1(F, T^*/\ell^n T^*)$ for arbitrarily large n. These derived classes have tightly controlled ramification, and in some cases one can use them together with cohomological bounds and duality theorems to bound $\mathcal{S}(F, T \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$. This mechanism is rather delicate; it has been worked out independently by Kato [11], Perrin-Riou [16], and Rubin [17]. When T corresponds to a motive of strictly positive weight there is also a conjectural framework connecting the existence of arithmetic Euler systems to p-adic L-functions. See [10] or [17, Chapter 9] for details.

Kolyvagin's methods were applied in a different setting by Flach in [3] and [4]; he used a geometric construction to exhibit directly classes in $H^1(\mathbf{Q}, \operatorname{End}^0 T_f^*)$ for f a newform of weight 2 and trivial character. He showed that these classes behave like the derived classes of an arithmetic Euler system and thus obtained a bound on the exponent of $\mathcal{S}(\mathbf{Q}, T \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$. These results were generalized to certain higher weight modular forms in [24].

In this paper we fit Flach's work into a general setting of geometric Euler systems. This geometric theory is strikingly different from the arithmetic theory. The fundamental difference is that the existence of useful cohomology classes in $H^1(F, T^*)$ (rather than in $H^1(F, T^*/\ell^n T^*)$ for large n) in the geometric case forces T to be locally isotropic. The arithmetic theory is poorly suited to locally isotropic representations (see the discussion after Proposition 2.3), so that one can regard the geometric theory as filling in this gap in the arithmetic theory. In addition, the basic mechanism in the geometric case is vastly simpler than in the arithmetic case and requires no additional hypotheses beyond the assumption of local isotropy. Finally, one expects that geometric Euler systems on motivic representations should come from fairly simple collections of geometric data on the corresponding motive. This allows for a straightforward and approachable set of conjectures. On the other hand, at this point the geometric theory only allows one to bound the exponent, rather than the order, of the Selmer group.

We now review the contents of the paper. As we have said, one expects that geometric Euler systems can be used to prove the finiteness of $\mathcal{S}(F, T \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ when T is locally isotropic. We prove this in Section 2. (In fact, our methods yield a bound on the exponent of the Selmer group, but we have not attempted to make it explicit.) One of the key ingredients is the cohomological bound of Proposition 2.1, which is a generalization of a result of Rubin. To make the ideas behind this result more clear we present them in a general setting in Section 1.

The remainder of the paper is concerned with the case that T is motivic. We review results on motivic cohomology and regulator maps in Section 3. We state our conjectures and their consequences for the existence of geometric Euler systems in Section 4. In Section 5 we reconsider the case of adjoint motives, and we construct geometric Euler systems for adjoint representations of modular forms as described above.

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Notation. Throughout this paper we fix a prime ℓ and a finite extension K of $\mathbf{Q}_\ell$. We write $\mathcal{O}$ for the ring of integers of K and λ for a fixed uniformizer. By the exponent of an $\mathcal{O}$-module T we mean the least $n \geq 0$ such that $\lambda^n T = 0$. If T is a free $\mathcal{O}$-module, we write T_K (resp. T_n, resp. T_∞) for $T \otimes_{\mathcal{O}} K$ (resp. $T/\lambda^n T$, resp. $T \otimes_{\mathcal{O}} K/\mathcal{O}$). All group actions on such a T are assumed to be continuous and $\mathcal{O}$-linear; in particular, the isomorphism $T_n \cong T_\infty[\lambda^n]$ respects any such action. If T has an action of the absolute Galois group of a field (of characteristic different from ℓ) we let $T(i)$ denote the i-fold Tate twist of T.

By a local field (resp. global field) we mean a finite extension of $\mathbf{Q}_p$ or $\mathbf{F}_p((t))$ (resp. $\mathbf{Q}$ or $\mathbf{F}_p(t)$) for some prime p. In the function field case we always assume that $p \neq \ell$. For a place v of a global field F, we write $\mathrm{Fr}(v)$ for a choice of geometric Frobenius element in the absolute Galois group of F.

1. Cohomological bounds

1.1. **Restricted cohomology.** Let $\rho : G \to \mathrm{Aut}_{\mathcal{O}} T$ be a continuous representation of a topological group G on a free $\mathcal{O}$-module T of finite rank. We say that $g \in G$ is *ρ-isotropic* if $\dim_K T_K^{g=1} > 0$ (or equivalently, if $\dim_K (g-1)T_K < \dim_K T_K$). If Γ is any subset of G, we define the *Γ-restricted cohomology group*

$$H^1_\Gamma(G,T) = \ker\Big(H^1(G,T) \to \prod_{g\in\Gamma} H^1(\langle g\rangle, T)\Big)$$

where $\langle g\rangle$ denotes the subgroup of G generated by g. (Here and throughout the paper all group cohomology is defined with continuous cocycles.) Note that for a surjection $\tilde{G} \twoheadrightarrow G$ with kernel Γ one has $H^1_\Gamma(\tilde{G},T) = H^1(G,T)$. In the next section we will show that one can obtain approximations to this fact when restricting with respect to certain non-trivial cosets of Γ.

1.2. **Cohomology of $\mathcal{O}$-modules.** By a *projective group G* we will always mean an inverse system $\{G_n\}_{n\geq 1}$ of finite groups. We write G_∞ for the inverse limit of the G_n; we regard G_∞ as a topological group with the inverse limit topology. As an example, if T is a free $\mathcal{O}$-module of finite rank, we define a projective group $\mathrm{GL}(T)$ by setting $\mathrm{GL}(T)_n = \mathrm{Aut}_{\mathcal{O}} T_n$. A *representation* $\rho : G \to \mathrm{GL}(T)$ of a projective group G on T is simply an inverse system $\{\rho_n : G_n \to \mathrm{GL}(T)_n\}_{n\geq 1}$ of group homomorphisms. We often simply write $\rho : G_\infty \to \mathrm{Aut}_{\mathcal{O}} T$ for the inverse limit of the ρ_n; we say that ρ is *irreducible* if G_∞ acts irreducibly on T_K via ρ.

Proposition 1.1. *Let $1 \to \Gamma \to \tilde{G} \to G \to 1$ be an exact sequence of projective groups and let $\rho : G \to \mathrm{GL}(T)$ be an irreducible representation. Let $\gamma = \{\gamma_n\} \in G_\infty$ be ρ-isotropic and for each n fix $\tilde{\gamma}_n \in \tilde{G}_n$ mapping to γ_n. Assume that the exponents of the groups $H^1(G_n, T_n)$ are bounded independent of n. Then the exponents of the groups $H^1_{\tilde{\gamma}_n\Gamma_n}(\tilde{G}_n, T_n)$ are bounded independent of n.*

Proof. Let $c : \tilde{G}_n \to T_n$ be a cocycle in $H^1_{\tilde{\gamma}_n\Gamma_n}(\tilde{G}_n, T_n)$. By definition

$$c(\tilde{\gamma}_n g) \in (\tilde{\gamma}_n g - 1)T_n = (\gamma_n - 1)T_n \tag{1.1}$$

for any $g \in \Gamma_n$. Taking $g = 1$ shows that $c(\tilde{\gamma}_n) \in (\gamma_n - 1)T_n$. Using this and expanding out (1.1) via the cocycle relation, one finds that $\gamma_n c(g) \in (\gamma_n - 1)T_n$. It follows that $c(g) \in (\gamma_n - 1)T_n$ for any $g \in \Gamma_n$.

The restriction of c to Γ_n is a G_n-equivariant homomorphism, so that $c(\Gamma_n)$ generates a G_n-stable submodule of $(\gamma_n - 1)T_n$. Since γ is ρ-isotropic it follows from Lemma 1.2 below with $T' = (\gamma - 1)T$ that there is an m, independent of n and c, such that $\lambda^m c(\Gamma_n) = 0$. Therefore, $\lambda^m H^1_{\tilde{\gamma}_n\Gamma_n}(\tilde{G}_n, T_n)$ lies in $H^1(G_n, T_n)$ via inflation, and the proposition now follows from the boundedness of the latter groups. □

Lemma 1.2. *Let $\rho : G \to \mathrm{GL}(T)$ be irreducible and let T' be an $\mathcal{O}$-submodule of T with $T'_K \neq T_K$. For any n, let M_n denote the maximal G_n-stable submodule of T_n contained in the image of T'. Then the exponent of M_n is bounded independent of n.*

Proof. For $t \in T$ we let $v(t)$ be the least integer n such that $t \in \lambda^n T$. We claim that to prove the lemma it suffices to prove that there exists an $m \geq 0$ such that

$$\mathcal{O}[G_\infty]t \supseteq \lambda^{v(t)+m}T \tag{1.2}$$

for all $t \in T$. Indeed, assuming this, let $t \in T$ map to M_n. Since M_n is G_n-stable, $\mathcal{O}[G_\infty]t$ must also map to M_n; by (1.2) and the definition of M_n, we conclude that $\lambda^{v(t)+m}T$ lies in $T' + \lambda^n T$. On the other hand, by hypothesis T' does not contain $\lambda^a T$ for any a. It follows that we must have $v(t) + m \geq n$; thus $v(t) \geq n - m$, so that λ^m kills M_n for any n.

By scaling, to prove (1.2) we may assume that $v(t) = 0$, so that $t \in T - \lambda T$. Define

$$B_n = \{t \in T - \lambda T; \mathcal{O}[G_\infty]t \supseteq \lambda^n T\}.$$

We have $T - \lambda T = \cup_{n \geq 0} B_n$ since G_∞ acts irreducibly on T_K. We will show that each B_n is open; the claim then follows from the compactness of $T - \lambda T$. To show that B_n is open, we show that given $t \in B_n$, any $t' \in t + \lambda^{n+1}T$ lies in B_n as well. Indeed, since $t \in B_n$ and $t' - t \in \lambda^{n+1}T$, there is $\sigma \in \mathcal{O}[G_\infty]$ such that $\lambda\sigma t = t' - t$. Thus $(1 + \lambda\sigma)t = t'$. As λ is topologically nilpotent we can choose $\tau \in \mathcal{O}[G_\infty]$ with $\tau(1 + \lambda\sigma)$ arbitrarily close to 1. Since $\mathcal{O}[G_\infty]t'$ contains a neighborhood of the origin in T, we can in fact choose τ so that $(\tau(1 + \lambda\sigma) - 1)t \in \mathcal{O}[G_\infty]t'$. Thus $\tau t' - t \in \mathcal{O}[G_\infty]t'$, so that $t \in \mathcal{O}[G_\infty]t'$. It follows that $t' \in B_n$, as claimed. □

1.3. **Locally isotropic representations.** Let $\rho : G \to \mathrm{GL}(T)$ be an irreducible representation of a projective group G on a free $\mathcal{O}$-module T. We say that $g, g' \in G_\infty$ are *congruent at level n* if g and g' map to the same element of G_n. A ρ-isotropic $g \in G_\infty$ is said to be *minimal* if there is an m such that $\dim_K T_K^{g=1} = \dim_K T_K^{g'=1}$ for all g' congruent to g at level m. We say that ρ is *locally isotropic* if G has a minimal ρ-isotropic element.

The key property of minimal elements is contained in the next lemma. Note that $T^{g=1}$ is an $\mathcal{O}$-module direct summand of T for any $g \in G_\infty$ since the action of g on T is $\mathcal{O}$-linear.

Lemma 1.3. *Let $g \in G_\infty$ be ρ-isotropic. Then the cokernel of the map*

$$T^{g'=1} \to T_n^{g'=1} = T_n^{g=1}$$

is bounded independent of n and g' congruent to g at level n if and only if g is minimal.

Proof. Set $r = \dim_K T_K^{g=1}$. We assume first that g is minimal. Fix an m such that λ^m kills the torsion submodule of $T/(g-1)T$. It then follows from the isomorphism

$$T_n^{g=1}/\operatorname{im}(T^{g=1} \to T_n^{g=1}) \cong (T/(g-1)T)[\lambda^n]$$

that

$$\lambda^m T_n^{g=1} \subseteq \operatorname{im}(T^{g=1} \to T_n^{g=1}) \tag{1.3}$$

for all n.

Let $g' \in G_\infty$ be congruent to g at some level $n > m$ by the definition of minimality it suffices to consider the case $\dim_K T_K^{g'=1} = r$. Choose an $\mathcal{O}$-basis $t'_1, \dots, t'_r$ of $T^{g'=1}$. Note that $t'_1, \dots, t'_r$ are linearly independent modulo λ since $T^{g'=1}$ is an $\mathcal{O}$-module direct summand of T.

Each t'_i maps to $T_n^{g=1}$ since g' is congruent to g at level n. Thus by (1.3) we can choose $t_{i,0} \in T^{g=1}$ with

$$\lambda^m t'_i \equiv t_{i,0} \pmod{\lambda^n}$$

for each i. In particular, $t_{i,0} \in \lambda^m T$; cancelling a factor of λ^m, we obtain $t_1, \dots, t_r \in T^{g=1}$ with

$$t'_i \equiv t_i \pmod{\lambda^{n-m}}. \tag{1.4}$$

The t'_i are linearly independent modulo λ, so the t_i are as well; since $T^{g=1}$ has rank r, by Nakayama's lemma it follows that $t_1, \dots, t_r$ is a basis of $T^{g=1}$. As $\lambda^m t_i$ lies in the image of $T^{g'=1}$ in $T_n^{g=1}$, we have thus shown that

$$\lambda^m \operatorname{im}(T^{g=1} \to T_n^{g=1}) \subseteq \operatorname{im}(T^{g'=1} \to T_n^{g=1}).$$

Combined with (1.3), we conclude that

$$\lambda^{2m} T_n^{g=1} \subseteq \operatorname{im}(T^{g'=1} \to T_n^{g=1})$$

for all sufficiently large n. This proves the first direction of the lemma.

For the converse, let g be ρ-isotropic but not minimal. By definition, for any n we may choose g' congruent to g at level n but with $\dim T_K^{g'=1} = s < r$. Then $T_n^{g=1}$ contains a copy of $(\mathcal{O}/\lambda^n)^r$, while the image of $T^{g'=1}$ contains only a copy of $(\mathcal{O}/\lambda^n)^s$. The lemma follows. □

2. Geometric Euler systems

2.1. **Local conditions.** Let F be a local field with residue field k and let T be a free $\mathcal{O}$-module of finite rank endowed with an action of the absolute Galois group of F. We always assume that F does not have characteristic ℓ. We say that T is *unramified* if the inertia group I of F acts trivially on T; if T is unramified, we say that it is *pure of weight w* if all the eigenvalues for the action of a geometric Frobenius $\mathrm{Fr}(k)$ on T are algebraic with absolute value $(\#k)^{w/2}$ under any embedding $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$.

We define the *finite/singular* exact sequence

$$0 \to H^1_f(F,T) \to H^1(F,T) \to H^1_s(F,T) \to 0$$

as in [17, Section 1.3]; we use the crystalline definition of [1, Section 3] in the case char $F = 0$ and char $k = \ell$. If T is unramified and char $k \neq \ell$, then we recall that

$$H^1_f(F,T) = H^1(k,T); \tag{2.1}$$

$$H^1_s(F,T) = H^1(I,T)^{\mathrm{Fr}(k)=1} \cong T(-1)^{\mathrm{Fr}(k)=1}; \tag{2.2}$$

and the finite/singular exact sequence identifies with the inflation-restriction sequence. (The last isomorphism in (2.2) follows from the fact that the maximal pro-ℓ quotient of I is isomorphic to $\mathbf{Z}_\ell(1)$ as a $\mathrm{Gal}(\bar{k}/k)$-module.) Thus if T is pure of weight w, then $H^1_s(F_v,T)$ is finite unless $w=-2$. There are of course analogous definitions for the T_n, $n \leq \infty$, and the analogues of (2.1) and (2.2) still hold for T_n.

2.2. **Selmer groups.** We now fix a global field F and a free $\mathcal{O}$-module T of finite rank endowed with an action of the absolute Galois group of F. (As always we assume that F does not have characteristic ℓ.) We assume that T is unramified at almost all places of F. Although it is not essential to the method, we also assume that T is crystalline at all v dividing ℓ. For any n let $F(T_n)$ denote the smallest Galois extension of F such that the Galois action on T_n factors through $\mathrm{Gal}(F(T_n)/F)$. We define the *Galois group of* T to be the projective group G_T with $G_{T,n} = \mathrm{Gal}(F(T_n)/F)$; it is equipped with a natural representation $\rho_T : G_T \to \mathrm{GL}(T)$. More generally, if E/F is a finite Galois extension we set $G_{T/E,n} = \mathrm{Gal}(E(T_n)/F)$ and let $\rho_{T/E} : G_{T/E} \to \mathrm{GL}(T)$ denote the natural representation. For $\gamma \in G_{T/E,\infty}$, we let $\mathcal{P}_\gamma(G_{T/E},n)$ denote the set of places of F, unramified in $E(T_n)/F$ and prime to ℓ, with Frobenius conjugate to γ on $E(T_n)$.

For a finite set of places P we define the *P-Selmer group*

$$\mathcal{S}^P(F,T) = \ker\Big(H^1(F,T) \to \prod_{v\notin P} H^1_s(F_v,T)\Big).$$

Set $\mathcal{S}(F,T) = \mathcal{S}^{\emptyset}(F,T)$ and define the *restricted P-Selmer group*

$$\mathcal{S}_P(F,T) = \ker\Big(\mathcal{S}(F,T) \to \prod_{v\in P} H^1_f(F_v,T)\Big). \tag{2.3}$$

As before there are analogous definitions for the T_n, $n \leq \infty$.

Proposition 2.1. *Let T be as above and assume that ρ_T is irreducible. Fix a finite Galois extension E/F and a $\rho_{T/E}$-isotropic $\gamma \in G_{T/E,\infty}$. Then one can choose finite subsets P_n of $\mathcal{P}_\gamma(G_{T/E},n)$ such that the exponent of $\mathcal{S}_{P_n}(F,T_n)$ is bounded independent of n.*

Proof. The Selmer group $\mathcal{S}(F,T_n)$ is finite for each n by [17, Proposition B.2.7]. We may therefore choose finite Galois extensions $E_n/E(T_n)$ such that $E_n \subseteq E_{n+1}$ and

$$\mathcal{S}(F,T_n) \subseteq H^1(E_n/F,T_n).$$

Let $\tilde{G}$ be the projective group with $\tilde{G}_n = \mathrm{Gal}(E_n/F)$. If v is a place of F, unramified on T and in E/F, then by (2.1) and (2.3) we have

$$\mathcal{S}_{\{v\}}(F,T_n) \subseteq H^1_{\{\mathrm{Fr}(v)\}}\big(\tilde{G}_n,T_n\big), \tag{2.4}$$

in the notation of Section 1.1; here $\mathrm{Fr}(v) \in \tilde{G}_n$ is any choice of Frobenius at v.

The exponent of $H^1(G_{T/E,n},T_n)$ is bounded independent of n by [17, Theorem C.1.1]; thus Proposition 1.1 shows that the exponent of

$$H^1_{\tilde{\gamma}_n\,\mathrm{Gal}(E_n/E(T_n))}\big(\tilde{G}_n,T_n\big)$$

is bounded independent of n, where $\tilde{\gamma}_n$ is a fixed lift to $\tilde{G}_n$ of the image of γ in $G_{T/E,n}$. Fix n and for each $g \in \mathrm{Gal}(E_n/E(T_n))$ fix a place v_g of F, unramified on T and in E/F, with $\mathrm{Fr}(v_g)$ conjugate to $\tilde{\gamma}_n g$ in $\mathrm{Gal}(E_n/F)$. Then the set P_n of these v_g lies in $\mathcal{P}_\gamma(G_{T/E}, n)$ and by (2.4) we have

$$\mathcal{S}_{P_n}(F, T_n) \subseteq H^1_{\tilde{\gamma}_n \,\mathrm{Gal}(E_n/E(T_n))}\big(\tilde{G}_n, T_n\big).$$

The proposition follows. □

2.3. **Locally isotropic Galois representations.** We say that an irreducible Galois representation T is *locally isotropic* if there is some finite Galois extension E/F such that $\rho_{T/E} : G_{T/E} \to \mathrm{GL}(T)$ is locally isotropic. We call any such E an *isotropy field* for T. Note that if T is locally isotropic, then so is any twist of T by a character of finite order. If $T^\vee = \mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O})$ is the contragredient of T, then $G_T = G_{T^\vee}$ (so that $\mathcal{P}_\gamma(G_{T/E}, n) = \mathcal{P}_\gamma(G_{T^\vee/E}, n)$ for any $\gamma \in G_{T/E,\infty}$ and any n) and T is locally isotropic if and only if $T^\vee$ is.

If T is locally isotropic, then the set of places v with $\mathrm{Fr}(v)$ isotropic has positive density. Thus if T is pure of weight w (in the sense that it is pure of weight w locally at almost all places of F), then it must have weight 0. Furthermore, if $T^* = T^\vee(1)$ denotes the *Cartier dual* of T, then $H^1_s(F_v, T^*)$ is infinite for any v with $\mathrm{Fr}(v)$ isotropic. Nevertheless, we do have the following fundamental result.

Lemma 2.2. *Assume that ρ_T is locally isotropic. Let E be an isotropy field for T and let γ be minimal $\rho_{T^\vee/E}$-isotropic. Then the exponents of the cokernels of the maps*

$$H^1_s(F_v, T^*) \to H^1_s(F_v, T^*_n)$$

are bounded independent of n and $v \in \mathcal{P}_\gamma(G_{T/E}, n)$.

Proof. For $v \in \mathcal{P}_\gamma(G_{T/E}, n)$ the map above can be rewritten as

$$(T^\vee)^{\mathrm{Fr}(v)=1} \to (T^\vee_n)^{\mathrm{Fr}(v)=1} = (T^\vee_n)^{\gamma=1}$$

by (2.2). The lemma thus follows from Lemma 1.3 and the fact that $\mathcal{P}_\gamma(G_{T/E}, n) = \mathcal{P}_\gamma(G_{T^\vee/E}, n)$. □

The simplest examples of locally isotropic Galois representation are adjoint representations: for H an arbitrary Galois representation, the trace-zero adjoint $T = \mathrm{End}^0_{\mathcal{O}} H$ of H is locally isotropic with isotropy field F (at least when it is irreducible). We shall investigate this example in more detail in Section 5. In this case bounds on the Selmer group of T have applications to the deformation theory of $H/\lambda H$; see [24] for details. More generally, locally isotropic representations of algebraic groups can be used to generate many locally isotropic Galois representations; see [25] for examples of this construction.

An especially interesting example related to an orthogonal group occurs in the cohomology of Hilbert modular surfaces. Let $F = \mathbf{Q}$ and fix a real quadratic extension $E/\mathbf{Q}$. Let f be a cuspidal Hilbert modular eigenform for E of weight $(2, 2)$; assume that f is not the base change of a form over $\mathbf{Q}$. For sufficiently large $\mathcal{O}$ one can associate to f a free $\mathcal{O}$-module H_f of rank two with an action of $\mathrm{Gal}(\bar{E}/E)$; see [21, Theorem 2]. The determinant of H_f is the product of the cyclotomic character and a character θ_f of finite order. Let $\bar{H}_f$ be the conjugate of H_f and set $T_f = H_f \otimes_{\mathcal{O}} \bar{H}_f(-1)$. Then T_f descends to an irreducible representation of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$; in fact, it occurs in $H^2_{\text{ét}}\big(\bar{X}, \mathcal{O}(1)\big)$ for an appropriate Hilbert modular surface X over $\mathbf{Q}$. If $\mathfrak{P} = p\mathcal{O}_E$ is inert in $E/\mathbf{Q}$, then by [8] the action of $\mathrm{Fr}(p)$ on T_f has matrix

$$\begin{pmatrix} \alpha_{\mathfrak{P}} & & & \\ & 0 & \alpha_{\mathfrak{P}} & \\ & \beta_{\mathfrak{P}} & 0 & \\ & & & \beta_{\mathfrak{P}} \end{pmatrix}$$

where $\alpha_{\mathfrak{P}}\beta_{\mathfrak{P}} = \theta_f(\mathrm{Fr}(\mathfrak{P}))$. As the eigenvalues of this matrix are $\alpha_{\mathfrak{P}}, \beta_{\mathfrak{P}}, \pm\sqrt{\alpha_{\mathfrak{P}}\beta_{\mathfrak{P}}}$, we conclude that $\mathrm{Fr}(p)$ is isotropic on T_f so long as $\theta_f(\mathrm{Fr}(\mathfrak{P})) = 1$. It is now not difficult to see that T_f is locally isotropic with isotropy field $E(\theta_f)$.

2.4. **Duality.** Let T be a Galois representation as in Section 2.2; we do not yet assume that T is irreducible or locally isotropic. Fix a Galois extension E/F and $\gamma \in G_{T/E,\infty}$. We say that the Cartier dual T^* admits a *geometric Euler system at* γ if there is an m such that the cokernel of the map

$$\mathcal{S}^{\{v\}}(F, T^*) \to H^1_s(F_v, T^*) \tag{2.5}$$

is bounded independent of $v \in \mathcal{P}_\gamma(G_{T/E}, m)$.

Proposition 2.3. *Let T be a locally isotropic Galois representation. Let E be an isotropy field for T and let γ be minimal $\rho_{T^\vee/E}$-isotropic. If T^* admits a geometric Euler system at γ, then $\mathcal{S}(F, T_\infty)$ is finite.*

Proof. For any n and any set of places P we have the local/global duality exact sequence (see [17, Theorem 1.7.3])

$$\mathcal{S}^P(F, T_n^*) \to \bigoplus_{v \in P} H^1_s(F_v, T_n^*) \to \mathcal{S}(F, T_n)^\vee \to \mathcal{S}_P(F, T_n)^\vee \to 0. \tag{2.6}$$

We apply this for varying n with $P = P_n$ given by Proposition 2.1 for γ; in particular the $\mathcal{S}_{P_n}(F, T_n)$ are bounded independent of n. Since $P_n \subseteq \mathcal{P}_\gamma(G_{T/E}, n)$, the exponent of the cokernel of the first map in (2.6) is bounded independent of n by the definition of a geometric Euler system and Lemma 2.2. Thus the exponent of $\mathcal{S}(F, T_n)$ is bounded independent of n by (2.6). Every element of $\mathcal{S}(F, T_\infty)$ lies in $\mathcal{S}(F, T_n)$ for some n, so that this implies that $\mathcal{S}(F, T_\infty)$ has finite exponent. It is also co-finitely generated by [17, Proposition B.2.7], so that it must now be finite. □

Note that it is essential for the above proof that there exist minimal isotropic elements. Indeed, the bounds on $\mathcal{S}_{P_n}(F, T_n)$ require γ to be isotropic, and Lemma 2.2 (which relies crucially on Lemma 1.3) then requires γ to be minimal. This is why the notion of a geometric Euler system is only useful for locally isotropic representations.

The arithmetic theories also require the use of an isotropic element γ; see [11, Section 0.6] and [17, Section 2.2]. In fact, they require that $\dim_K T_K^{\gamma=1} = 1$. Thus one can not apply the arithmetic theory in any case where $\dim_K T_K^{\gamma=1} > 1$ for all γ. (A simple example of such a T is the adjoint of a representation of rank at least three.) In particular, the arithmetic theory is not applicable to many locally isotropic representations.

3. Algebraic cycles

3.1. **Local conditions on motivic cohomology.** Let R be a discrete valuation ring with fraction field F and residue field k. Let X be a proper, smooth variety over F. For an integer d, consider the complex

$$\bigoplus_{x\in X^{d-1}} K_2k(x) \to \bigoplus_{x\in X^d} k(x)^\times \to \bigoplus_{x\in X^{d+1}} \mathbf{Z}. \tag{3.1}$$

Here X^i denotes the set of points of codimension i on the scheme X, the first map is the tame symbol and the second (which is more important to us) is the divisor map. We define the *motivic cohomology* group $H^{2d+1}_{\mathcal{M}}\big(X,\mathbf{Z}(d{+}1)\big)$ to be the cohomology of (3.1). Elements are represented by formal sums $\sum(Z_i, f_i)$ of pairs of codimension d cycles Z_i on X and non-zero rational functions f_i on Z_i such that $\sum \operatorname{div}_{Z_i} f_i = 0$ as a Weil divisor on X. (We note that this definition agrees after tensoring with $\mathbf{Q}$ with the usual definitions of motivic cohomology via K-theory or higher Chow groups; see [9, Section 6], for example.)

[19, Theorem 1.1.6] defines a canonical $\mathbf{Q}$-subspace

$$H^{2d+1}_{\mathcal{M}/R}\big(h(X), d{+}1\big) \hookrightarrow H^{2d+1}_{\mathcal{M}}\big(X,\mathbf{Z}(d{+}1)\big)\otimes_{\mathbf{Z}}\mathbf{Q} \tag{3.2}$$

via de Jong's theory of alterations. We use this to define local conditions in motivic cohomology as follows. Let $H^{2d+1}_{\mathcal{M},s}\big(X,\mathbf{Z}(d{+}1)\big)$ be the image of $H^{2d+1}_{\mathcal{M}}\big(X,\mathbf{Z}(d{+}1)\big)$ in the cokernel of (3.2). There is then a natural surjection

$$\operatorname{div}_k : H^{2d+1}_{\mathcal{M}}\big(X,\mathbf{Z}(d{+}1)\big) \to H^{2d+1}_{\mathcal{M},s}\big(X,\mathbf{Z}(d{+}1)\big).$$

We define $H^{2d+1}_{\mathcal{M},f}\big(X,\mathbf{Z}(d{+}1)\big)$ to be the kernel of div_k.

Alternately, if $\mathfrak{X}\to \operatorname{Spec} R$ is a proper smooth model of $X\to\operatorname{Spec} F$ with special fiber X_k, the localization map in K-theory yields a map (see [24, Section 3.1])

$$\operatorname{div}'_k : H^{2d+1}_{\mathcal{M}}\big(X,\mathbf{Z}(d{+}1)\big) \to A^dX_k$$

sending a pair (Z,f) to the divisor of f on the special fiber of the scheme theoretic closure of Z in $\mathfrak{X}$. Here A^dX_k is the Chow group of codimension d cycles X_k. Then div_k factors through div'_k and induces an isomorphism

$$H^{2d+1}_{\mathcal{M},s}\big(X,\mathbf{Z}(d{+}1)\big) \cong \operatorname{im}\operatorname{div}'_k/(\operatorname{im}\operatorname{div}'_k)_{\mathrm{tors}}.$$

In particular, this gives a geometric method to check local conditions in the case of good reduction.

3.2. **Regulators.** We now assume that F is a local or global field of characteristic different from ℓ. For a proper, smooth variety X over F, an integer d, and sufficiently large r, one can define a *regulator map*

$$\ell^r\mathcal{R}_{X,d} : H^{2d+1}_{\mathcal{M}}\big(X,\mathbf{Z}(d{+}1)\big) \to H^1\big(F, H^{2d}_{\text{ét}}(\bar{X},\mathbf{Z}_\ell(d{+}1))\big).$$

(Here $\bar{X}$ is the base change of X to a separable closure of F.) The case where r can be taken to be zero is considered in [24, Section 2.2]. There are three additional difficulties in the general case: the presence of torsion in $H^{2d+1}_{\text{ét}}\big(\bar{X},\mathbf{Z}_\ell(d+1)\big)$; the existence of denominators in the Chern character; and the failure of purity in étale cohomology. All three difficulties can be resolved by taking r large enough; see [22, Theorem 3.5] for the needed purity results. We omit the details.

Let $r(X,d)$ be the least value of r such that $\ell^r\mathcal{R}_{X,d}$ is defined. The next result shows that the regulator map respects the local conditions on the source and the target.

Proposition 3.1. *Let F be a local field with residue field k. Set $T = H^{2d}_{\acute{e}t}\big(\bar{X}, \mathbf{Z}_\ell(d{+}1)\big)$ and fix $r \geq r(X,d)$. Assume that either X admits a proper, smooth model over* $\operatorname{Spec}\mathcal{O}_F$ *or* $\operatorname{char} k \neq \ell$. *Then $\ell^r \mathcal{R}_{X,d}$ maps $H^{2d+1}_{\mathcal{M},f}\big(X, \mathbf{Z}(d{+}1)\big)$ to $H^1_f(F,T)$. In fact, if* $\operatorname{char} k \neq \ell$ *and $\mathfrak{X} \to \operatorname{Spec}\mathcal{O}_F$ is smooth, then there is a commutative diagram*

$$\begin{array}{ccccc} H^{2d+1}_{\mathcal{M}}\big(X, \mathbf{Z}(d{+}1)\big) & \xrightarrow{\operatorname{div}'_k} & A^d X_k & \xrightarrow{\ell^r c_v} & H^{2d}_{\acute{e}t}\big(\bar{X}_k, \mathbf{Z}_\ell(d)\big)^{\operatorname{Fr}(k)=1} \\ \Big\downarrow{\scriptstyle \ell^r \mathcal{R}_{X,d}} & & & & \Big\downarrow{\scriptstyle \simeq} \\ H^1(F,T) & \longrightarrow & H^1_s(F,T) & \xrightarrow{\simeq} & T(-1)^{\operatorname{Fr}(k)=1} \end{array}$$

Here c_v is the cycle class map and the isomorphism on the right is smooth base change.

Proof. When $\operatorname{char} k \neq \ell$ the first statement is [13, Theorem B]. The case $\operatorname{char} k = \ell$ is [14, Theorem 3.1]. The existence of the commutative diagram is proven for for $r(X,d) = 0$ in [24, Theorem 3.1.1]; the proof there is easily adapted for $r > 0$ as well. □

3.3. **Motivic Selmer groups.** We now assume that F is a global field; as always, we assume that F does not have characteristic ℓ. Let X be a proper, smooth variety over F. Let P be a set of places of F containing all places of residue characteristic ℓ at which X has bad reduction. For any d, we define the *motivic P-Selmer group* $\mathcal{S}^P_{\mathcal{M}}\big(H^{2d+1}(X), \mathbf{Z}(d{+}1)\big)$ as the kernel of the map

$$\bigoplus_{v \notin P} \operatorname{div}_v : H^{2d+1}_{\mathcal{M}}\big(X, \mathbf{Z}(d{+}1)\big) \to \bigoplus_{v \notin P} H^{2d+1}_{\mathcal{M},s}\big(X_{F_v}, \mathbf{Z}(d{+}1)\big).$$

Here div_v is the composition of restriction from X to X_{F_v} with div_{k_v}. We record the following consequence of Proposition 3.1.

Corollary 3.2. *Let P be a finite set of places as above. Then*

$$\ell^r \mathcal{R}_{X,d}\Big(\mathcal{S}^P_{\mathcal{M}}\big(H^{2d+1}(X), \mathbf{Z}(d{+}1)\big)\Big) \subseteq \mathcal{S}^P\big(F, H^{2d}_{\acute{e}t}(\bar{X}, \mathbf{Z}_\ell(d{+}1))\big)$$

for $r \geq r(X,d)$.

4. Conjectures

4.1. **Statements.** Let X be a proper, smooth variety over a global field F of characteristic different from ℓ. Let S be an open subscheme of $\operatorname{Spec}\mathcal{O}_F$ and let $\mathfrak{X} \to S$ be proper and smooth with generic fiber $X \to \operatorname{Spec} F$. Let P_0 denote the complement of S in the set of places of F. Our basic conjecture is the following.

Conjecture 4.1. *Let Z be an algebraic cycle on a smooth fiber X_v of $\mathfrak{X} \to S$. Then a non-zero integer multiple of Z is homologically equivalent (on the geometric fiber $\bar{X}_v$) to an algebraic cycle which is trivial in the Chow group of $\mathfrak{X}$.*

For our applications we formulate a uniform combination of Conjecture 4.1 with the conjecture of Tate on algebraic cycles. We first need to introduce some terminology. Let T be a free $\mathcal{O}$-module with an action of the absolute Galois group of F. We say that T is *pre-motivic* for (X,d) if there is an $\mathcal{O}$-linear map with finite cokernel $h_T : H^d_{\text{ét}}(\bar{X}, \mathcal{O}) \to T$ compatible with Galois actions. Note that T is then pure of weight d.

Let T be pre-motivic for (X,d) and fix a finite Galois extension E/F. Let $G_{T/E}$ be the Galois group of T over E with representation $\rho_{T/E} : G_{T/E} \to \operatorname{GL}(T)$. In this context, for $\gamma \in G_{T/E,\infty}$ we restrict the set $\mathcal{P}_\gamma(G_{T/E}, n)$ to contain only places in S. One checks immediately that this restriction does not affect any of our previous results.

Conjecture 4.2. *Let T be pre-motivic for $(X, 2d)$. Fix a finite Galois extension E/F and $\gamma \in G_{T/E,\infty}$. Then there is an m such that the cokernel of the composition*

$$(4.1)\quad \mathcal{S}_{\mathcal{M}}^{P_0\cup\{v\}}\big(H^{2d+1}(X), \mathbf{Z}(d{+}1)\big) \otimes_{\mathbf{Z}} \mathcal{O} \xrightarrow{\mathrm{div}'_v} A^d X_v \otimes_{\mathbf{Z}} \mathcal{O} \xrightarrow{c_v} H^{2d}_{\acute{e}t}\big(\bar{X}_v, \mathcal{O}(d)\big)^{\mathrm{Fr}(v)=1} \xrightarrow{h_T} T(d)^{\mathrm{Fr}(v)=1}$$

is bounded independent of $v \in \mathcal{P}_\gamma(G_{T/E}, m)$.

The surjectivity of (4.1) after tensoring with K is implied by Conjecture 4.1 and the Tate conjecture for the fiber X_v. To see this it suffices to show that $(c{\circ}\mathrm{div}'_v){\otimes}K$ is surjective for any $v \notin P_0$. Fix such a v and fix $t \in H^{2d}_{\text{ét}}(\bar{X}_v, K(d))^{\mathrm{Fr}(v)=1}$. By Tate's conjecture and Conjecture 4.1 there is a codimension d K-cycle Z on X_v, trivial in $A^{d+1}\mathfrak{X} \otimes K$, with $c_v(Z) = t$. By the definition of the Chow group there is therefore a codimension d K-cycle Y on $\mathfrak{X}$ and a rational function f on Y with divisor Z; that is, f has trivial divisor on the generic fiber of X and

$$\mathrm{div}'_w(Y, f) = \begin{cases} 0 & w \notin P_0 \cup \{v\}; \\ Z & w = v. \end{cases}$$

Thus (Y, f) lies in $\mathcal{S}_{\mathcal{M}}^{P_0\cup\{v\}}\big(H^{2d+1}(X), \mathbf{Z}(d+1)\big)$ and $c \circ \mathrm{div}'_v(Y, f) = t$, so that $(c \circ \mathrm{div}'_v) \otimes K$ is surjective.

The basic motivation for these conjectures comes from the conjectures of Beilinson (as extended to incomplete L-functions by Deligne) and Bloch-Kato. Indeed, assume that char $F = 0$ and let T be a motive occurring in $H^{2d+1}(X)$ with $\mathbf{Q}_\ell$-realization T_ℓ. Deligne's generalization of Beilinson's conjectures (see [18, Conjeture 4.2]) and the functional equation predict that

$$\mathrm{ord}_{s=d}\, L_P(T, s) = \mathrm{rank}_{\mathbf{Z}}\, \mathcal{S}_{\mathcal{M}}^{P}\big(T, \mathbf{Z}(d{+}1)\big)$$

for any set of places P; here $L_P(T, s)$ is the L-function of T with Euler factors at P removed and $\mathcal{S}_{\mathcal{M}}^{P}\big(T, \mathbf{Z}(d{+}1)\big)$ is the P-integral motivic cohomology of T. Comparing this equality with $P = \emptyset$ and $P = \{v\}$, one expects that

$$(4.2)\qquad \dim T_\ell(d)^{\mathrm{Fr}(v)=1} = \mathrm{rank}\, \mathcal{S}_{\mathcal{M}}^{\{v\}}\big(T, \mathbf{Z}(d{+}1)\big) - \mathrm{rank}\, \mathcal{S}_{\mathcal{M}}\big(T, \mathbf{Z}(d{+}1)\big).$$

On the other hand, localization and the cycle class map yield a map

$$(4.3)\qquad \mathcal{S}_{\mathcal{M}}^{\{v\}}\big(T, \mathbf{Z}(d{+}1)\big) \otimes \mathbf{Q}_\ell \to A^d T_v \otimes \mathbf{Q}_\ell \to T_\ell(d)^{\mathrm{Fr}(v)=1}$$

which is trivial on $\mathcal{S}_{\mathcal{M}}\big(T, \mathbf{Z}(d{+}1)\big)$. By (4.2) one is then naturally led to hope that (4.3) is surjective; this is conjectured (in a slightly different form) in [1, Conjecture 5.3]. Conjecture 4.2 is nothing more than a uniform version of this; we have avoided passing to the motivic cohomology of T for simplicity. For more discussion along these lines see [5] and [12].

4.2. **Evidence.** Conjecture 4.2 is known in a few cases. It is virtually trivial in the case $d = 0$ (so that $T = \mathcal{O}$): for any v we can choose $\varpi_v \in \mathcal{O}_F$ which is a unit away from v but has $\mathrm{ord}_v\, \varpi_v = h$ with h the class number of $\mathcal{O}_F$. The elements $(X, \varpi_v) \in H^1_{\mathcal{M}}\big(X, \mathbf{Z}(1)\big)$ prove the conjecture for any γ in this case.

We will analyze Conjecture 4.2 more carefully in the case of adjoint motives in Section 5. We use methods of Mildenhall and Flach as in [24] to verify the conjecture when X is a self-product of a Kuga-Sato variety (resp. Drinfeld modular curve) and T is the adjoint representation attached to certain classical modular forms (resp. Drinfeld modular forms). An interesting variation of these ideas is provided by [15], where Conjecture 4.2 is proven for the Fermat quartic surface $z_0^4 + z_3^4 = z_1^4 + z_2^4$ over $\mathbf{Q}$.

4.3. **Consequences.**

Proposition 4.3. *Let X be a proper, smooth variety over F and let T be pre-motivic for $(X, 2d)$. Fix a Galois extension E/F and $\gamma \in G_{T/E,\infty}$. Assume that Conjecture 4.2 holds for some proper,smooth model $\mathfrak{X} \to S$, T, and γ. Further assume that $H^1_s(F_w, T(d{+}1))$ is finite for $w \in P_0$. Then $T(d{+}1)$ admits a geometric Euler system at γ.*

Proof. For $r \geq r(X,d)$ and any $v \in S$ the composition of the regulator $\ell^r \mathcal{R}_{X,d}$ with h_T induces a map

$$\mathcal{R}_{T,v} : \mathcal{S}_{\mathcal{M}}^{P_0 \cup \{v\}}\big(H^{2d+1}(X), \mathbf{Z}(d{+}1)\big) \otimes_{\mathbf{Z}} \mathcal{O} \to \mathcal{S}^{P_0 \cup \{v\}}\big(F, T(d{+}1)\big).$$

If X_v is smooth and v does not divide ℓ, then ℓ^r times (4.1) factors through $\mathcal{R}_{T,v}$ and the map

$$\mathcal{S}^{P_0 \cup \{v\}}\big(F, T(d{+}1)\big) \to H^1_s\big(F_v, T(d{+}1)\big) \tag{4.4}$$

by Proposition 3.1. It thus follows from Conjecture 4.2 that for some m the cokernel of (4.4) is bounded independent of $v \in \mathcal{P}_\gamma(G_{T/E}, m)$. The proposition follows from this and the finiteness of $H^1_s\big(F_w, T(d{+}1)\big)$ for $w \in P_0$. □

Corollary 4.4. *Let T be a locally isotropic Galois representation. Fix an isotropy field E and a minimal $\rho_{T^\vee/E}$-isotropic $\gamma \in G_{T/E,\infty}$. Assume that there is an integer d such that $T^*(-d{-}1)$ is pre-motivic for some $(X, 2d)$ and such that Conjecture 4.2 holds for some proper, smooth model $\mathfrak{X} \to S$, $T^*(-d{-}1)$, and γ. Assume also that $H^1_s(F_w, T^*)$ is finite for $w \in P_0$. Then $\mathcal{S}(F, T_\infty)$ is finite.*

Proof. This is immediate from Propositions 2.3 and 4.3. □

5. Adjoint motives

5.1. **Basic properties.** Let S be an open subscheme of $\operatorname{Spec} \mathcal{O}_F$ for a global field F of characteristic different from ℓ. Let $\mathfrak{X} \to S$ be smooth and proper with generic fiber $X \to \operatorname{Spec} F$ of dimension d. Let H be a pre-motivic Galois representation for (X, d_0); we further assume that H_K is actually a direct summand of $H^{d_0}_{\text{ét}}(\bar{X}, K)$. Let $r = \operatorname{rank}_{\mathcal{O}} H$ and set $T = \operatorname{End}^0_{\mathcal{O}} H$. Note that the existence of the Galois equivariant trace pairing $T \otimes T \to \mathcal{O}$ implies that $T^* \cong T(1)$.

Lemma 5.1. *$T(-d)$ is pre-motivic for $(X \times X, 2d)$.*

Proof. It follows from the Künneth formula and Poincaré duality that $T_K(-d)$ is a direct summand of $H^{2d}_{\text{ét}}(\bar{X} \times \bar{X}, K)$. In particular, there is a projection

$$h : H^{2d}_{\text{ét}}(\bar{X} \times \bar{X}, K) \twoheadrightarrow T_K(-d).$$

The image of $H^{2d}_{\text{ét}}(\bar{X} \times \bar{X}, \mathcal{O})$ under h must be commensurable with $T(-d)$, so that some multiple of h will send $H^{2d}_{\text{ét}}(\bar{X} \times \bar{X}, \mathcal{O})$ to a finite index submodule of $T(-d)$. This is the statement of the lemma. □

Let G_H and G_T be the Galois groups of H and T, respectively; we simply write $\rho : G_H \to \mathrm{GL}(H)$ and $\mathrm{ad}^0\rho : G_T \to \mathrm{GL}(T)$ for the natural representations. Note that there is a natural surjection of projective groups $\nu : G_H \to G_T$; the kernel of ν_∞ (which we usually write simply as ν) consists precisely of those elements of $G_{H,\infty}$ which map to scalars under ρ.

Lemma 5.2. *Assume that T is irreducible. Then T is locally isotropic and F is an isotropy field for T. If $\gamma \in G_{H,\infty}$ is such that $\rho(\gamma)$ has distinct eigenvalues, then $\nu(\gamma)$ is semisimple and minimal $\mathrm{ad}^0\rho$-isotropic.*

Proof. For $g \in G_{H,\infty}$, the eigenvalues of g on $\mathrm{End}_K H_K$ are the ratios of the eigenvalues of $\rho(g)$. In particular, every $g \in G_{T,\infty}$ has $\dim_K T_K^{g=1} \geq r-1$, with equality precisely when g has distinct eigenvalues. The lemma follows. □

Note that there may not exist any γ such that $\rho(\gamma)$ has distinct eigenvalues; in that case the minimal $\mathrm{ad}^0\rho$-isotropic elements include those with the smallest number of trivial eigenvalues on H.

For any place $v \in S$ we let Γ_v^i denote the graph of the i^{th} power of the Frobenius morphism on the fiber X_v; we can regard Γ_v^i as a codimension d cycle on $X_v \times X_v$ or a codimension $d+1$ cycle on $\mathfrak{X} \times_S \mathfrak{X}$. We let $A_F^d(X_v \times X_v)$ denote the subgroup of $A^d(X_v \times X_v)$ generated by $\Gamma_v^1, \ldots, \Gamma_v^{r-1}$.

Proposition 5.3. *Let $\gamma \in G_{H,\infty}$ be such that $\rho(\gamma)$ has distinct eigenvalues. Then there is an m such that the map*

$$A_F^d(X_v \times X_v) \otimes_{\mathbf{Z}} \mathcal{O} \xrightarrow{c_v} H^{2d}_{\acute{e}t}\big(\bar{X}_v \times \bar{X}_v, \mathcal{O}(d)\big)^{\mathrm{Fr}(v)=1} \xrightarrow{h_T} T^{\mathrm{Fr}(v)=1} \tag{5.1}$$

has cokernel bounded independent of $v \in \mathcal{P}_{\nu(\gamma)}(G_T, m)$. In particular, to prove Conjecture 4.2 for $\mathfrak{X} \times_S \mathfrak{X} \to S$, T, and $\nu(\gamma)$, it suffices to show that there is a non-zero integer e such that $e\Gamma_v^1, \ldots, e\Gamma_v^{r-1}$ are trivial in $A^{d+1}(\mathfrak{X} \times_S \mathfrak{X})$ for all $v \in \mathcal{P}_{\nu(\gamma)}(G_T, m)$.

Proof. By assumption $\rho(\gamma)$ has distinct eigenvalues; thus we can choose m large enough so that $\rho(\mathrm{Fr}(v))$ has distinct eigenvalues for all places $v \in P_{\nu(\gamma)}(G_T, m)$. For any such v, it follows from basic linear algebra that the endomorphisms

$$\rho(\mathrm{Fr}(v)), \ldots, \rho(\mathrm{Fr}(v))^{r-1}$$

generate $(\mathrm{End}_K^0 H_K)^{\mathrm{Fr}(v)=1}$ over K. One then sees easily that the order of

$$T^{\mathrm{Fr}(v)=1}/\big(\mathcal{O}\rho(\mathrm{Fr}(v)) + \cdots + \mathcal{O}\rho(\mathrm{Fr}(v))^{r-1}\big)$$

is bounded independent of $v \in \mathcal{P}_{\nu(\gamma)}(G_T, m)$. By standard compatibilities in étale cohomology we have $h_T \circ c(\Gamma_v^i) = \rho(\mathrm{Fr}(v))^i$, so that this proves the first statement.

For the second statement, fix v and i. Since we are assuming that $e\Gamma_v^i$ is trivial in $A^{d+1}(\mathfrak{X} \times_S \mathfrak{X})$, we can write

$$e\Gamma_v^i = \sum \mathrm{div}_{Z_j} f_j \tag{5.2}$$

where the Z_j are irreducible codimension d cycles on $\mathfrak{X} \times_S \mathfrak{X}$ and f_j is a rational function on Z_j. In particular, $\sum \mathrm{div}_{Z_j} f_j$ has no support on the generic fiber $X \times X$, so that we can regard $\mathfrak{z} = \sum (Z_j, f_j)$ as an element of $H^{2d+1}_{\mathcal{M}}\big(X \times X, \mathbf{Z}(d+1)\big)$. (It may happen that some Z_j have no support on the generic fiber, but this causes no problems in the argument.) By (5.2), we have

$$\operatorname{div}_w \mathfrak{z} = \begin{cases} 0 & w \notin P_0 \cup \{v\}; \\ e\Gamma_v^i & w = v. \end{cases}$$

Thus $\mathfrak{z} \in \mathcal{S}_{\mathcal{M}}^{P_0 \cup \{v\}}(H^{2d+1}(X), \mathbf{Z}(d+1))$ satisfies $\operatorname{div}_v \mathfrak{z} = e\Gamma_v^i$. The proposition follows from this and the first statement. ☐

5.2. **Classical modular forms.** We now specialize to the case $F = \mathbf{Q}$. In [24, Theorem 4.2.3] optimal annihilators are obtained for the adjoint Selmer group attached to a sufficiently well-behaved classical modular form of squarefree level. Using the results above it is straightforward to extend these methods to prove the finiteness of the adjoint Selmer group for a more general class of classical modular forms of arbitrary level.

Let f be a newform of weight $k+2$ (with $k \geq 0$), level N, and arbitrary character. For sufficiently large K one can associate to f a two-dimensional K-representation $H_{f,K}$ of $\operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. This representation can be realized as a direct summand of $H_{\text{ét}}^{k+1}(\bar{E}_k, K)$ where E_k is a certain canonical resolution of the k-fold product of the universal generalized elliptic curve over the modular curve $X_1(N)$. E_k has a proper, smooth model $\mathcal{E}_k \to \operatorname{Spec} \mathbf{Z}[\frac{1}{N}]$. See [24, Section 4.2.1] for details and references.

Fix a Galois stable $\mathcal{O}$-lattice H_f in $H_{f,K}$; it is pre-motivic for $(E_k, k+1)$. Let $\rho_f : G_{H,f} \to \operatorname{GL}(H_f)$ be the associated representation. Set $T_f = \operatorname{End}_{\mathcal{O}}^0 H_f$ with representation $\operatorname{ad}^0\rho_f : G_{T,f} \to \operatorname{GL}(T_f)$. Let $\nu : G_{H,f} \to G_{T,f}$ be the natural map. T_f is pre-motivic for $(E_k \times E_k, 2k+2)$ by Lemma 5.1.

Proposition 5.4. *Let T_f be as above and let $\gamma \in G_{H,f,\infty}$ be such that $\rho_f(\gamma)$ has distinct eigenvalues. Then Conjecture 4.2 is true for $\mathcal{E}_k \times \mathcal{E}_k \to \operatorname{Spec} \mathbf{Z}[\frac{1}{N}]$, T_f, and $\nu(\gamma)$.*

Proof. Fix p not dividing N and let Γ_p denote the graph of Frobenius in $E_{k,p} \times E_{k,p}$. We will show that $12\Gamma_p$ is trivial in $A^{k+2}(\mathcal{E}_k \times \mathcal{E}_k)$. The proposition then follows from Proposition 5.3.

Let T_p be the p^{th} Hecke correspondence; it is a codimension $k+1$ cycle on $\mathcal{E}_k \times \mathcal{E}_k$. Let Δ be the unique normalized cusp form of weight 12 and level 1; we regard Δ as a pluricanonical form of degree 6 on $X_1(N)$. The two projections $T_{p,\mathbf{Q}} \to E_k$ give rise to two maps $T_{p,\mathbf{Q}} \to X_1(N)$. We let f_p be the rational function on T_p which is the ratio of the pullbacks of Δ under these two maps.

As observed in the weight 2 case by Flach (see [3]), it is a consequence of the Eichler-Shimura congruence relation that $\operatorname{div}_{T_p} f_p = 6\Gamma_p^{\vee} - 6\Gamma_p$ with $\Gamma_p^{\vee}$ the Verschiebung; see [24, Lemma 4.1.1 and Lemma 4.2.1] for the higher weight case. (The essential idea is explained in a different context in the proof of Proposition 5.6 below. Note that it is assumed in [24] that N is squarefree; however, the same proof works in general since we assume here that p does not divide N.) Since $E_{k,p} \times E_{k,p} = \Gamma_p + \Gamma_p^{\vee}$, it follows that $\operatorname{div}_{T_p} p^{-6} f_p = -12\Gamma_p$. This proves the proposition. ☐

Theorem 5.5. *Assume that f is not of CM-type; that f is special or supercuspidal at all p dividing N; and that ℓ does not divide N. Then $\mathcal{S}(\mathbf{Q}, T_{f,\infty})$ is finite.*

Proof. Since f is not of CM-type, there exist $\gamma \in G_{H,f,\infty}$ such that $\rho_f(\gamma)$ has distinct eigenvalues. Thus by Proposition 5.4 and Corollary 4.4 we need only check that $H^1_s(\mathbf{Q}_p, T^*_f)$ is finite for p dividing N. This follows from our hypotheses and [19, 2.3.13]. □

We omit the case that f is principal series at some p dividing N as then $H^1_s(\mathbf{Q}_p, T^*_f)$ is infinite. However, one should be able to deal with this case by a more careful analysis of the geometry of $E_k \times E_k$ at places of bad reduction.

5.3. **Drinfeld modular forms.** Let $\mathbf{F}$ be a finite field and let $F = \mathbf{F}(t)$. In this section we adapt the ideas of Flach as in the previous section to study adjoint representations of certain Drinfeld modular representations over F. We restrict ourselves to the case in which the geometry of the associated Drinfeld modular curve is sufficiently well understood.

Let $\mathfrak{n}$ be a squarefree ideal of $\mathbf{F}[t]$ and let S be the complement of $\mathfrak{n}$ in $\operatorname{Spec} \mathbf{F}[t]$. Let $M_0(\mathfrak{n})$ be the Drinfeld modular curve of level $\mathfrak{n}$ studied in [7]; it admits a proper, smooth model $\mathcal{M}_0(\mathfrak{n}) \to S$. Let $\mathbf{A}$ denote the adeles of F and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ of weight 2, level $\mathfrak{n}$, and trivial character as in [20, Section 3]. For an appropriate choice of $\mathcal{O}$ one can associate to π a pre-motivic Galois representation H_π for $(M_0(\mathfrak{n}), 1)$; H_π is free of rank 2 over $\mathcal{O}$ and there is an equality

$$L(s - \tfrac{1}{2}, \pi_v) = L(s, H_{\pi,v})$$

of local L-factors for almost all places v of F. One also knows that H_π is special for $v \notin S$. Set $T_\pi = \operatorname{End}^0_{\mathcal{O}} H_\pi$; it is pre-motivic for $(M_0(\mathfrak{n}) \times M_0(\mathfrak{n}), 2)$ by Lemma 5.1. Let $\rho_\pi : G_{H,\pi} \to \mathrm{GL}(H_\pi)$, $\mathrm{ad}^0\rho_\pi : G_{T,\pi} \to \mathrm{GL}(T_\pi)$ be the associated representations and let $\nu : G_{H,\pi} \to G_{T,\pi}$ be the natural map.

Proposition 5.6. *Let T_π be as above and let $\gamma \in G_{H,\pi,\infty}$ be such that $\rho_\pi(\gamma)$ has distinct eigenvalues. Then Conjecture 4.2 is true for $\mathcal{M}_0(\mathfrak{n}) \times \mathcal{M}_0(\mathfrak{n}) \to S$, T_π, and $\nu(\gamma)$.*

Proof. The proof is nearly identical to the proof of Proposition 5.4 once we assemble the corresponding geometric data: we will show that $2(1-q^2)\Gamma_v$ is trivial in $A^2(\mathcal{M}_0(\mathfrak{n}) \times \mathcal{M}_0(\mathfrak{n}))$ for any $\mathfrak{p} \in S$, where q is the order of $\mathbf{F}$. The proposition then follows from Proposition 5.3.

Fix $\mathfrak{p} \in S$ and let $T_\mathfrak{p}$ be the Hecke correspondence at $\mathfrak{p}$ regarded as a codimension 1 cycle in $\mathcal{M}_0(\mathfrak{n}) \times \mathcal{M}_0(\mathfrak{n})$. We let Δ be the Drinfeld cusp form of weight q^2-1 and level 1 defined in [7, Section 2]; we regard Δ^2 as a pluricanonical form of degree q^2-1 on $M_0(\mathfrak{n})$ as in [6, Section 5]. We then define $f_\mathfrak{p}$ as the ratio of the pullbacks of Δ^2 under the two projections $T_{\mathfrak{p},F} \to M_0(\mathfrak{n})$.

Set $S' = S - \{\mathfrak{p}\}$; $T_\mathfrak{p} \times_S S'$ is birationally isomorphic to $\mathcal{M}_0(\mathfrak{n}\mathfrak{p})$. By [7, Corollary 3.4 and Section 4], the divisor of $f_\mathfrak{p}$ on $\mathcal{M}_0(\mathfrak{n}\mathfrak{p})$ is a linear combination of differences $(0_i) - (\infty_i)$ of cuspidal divisors. Here 0_i and ∞_i are a pair of cusps lying over a single cusp of $M_0(\mathfrak{n})$. They thus coincide on $T_\mathfrak{p} \subseteq \mathcal{M}_0(\mathfrak{n}) \times \mathcal{M}_0(\mathfrak{n})$ as well, and it follows that the divisor of $f_\mathfrak{p}$ on $T_\mathfrak{p} \times_S S'$ is trivial.

We compute the divisor of $f_\mathfrak{p}$ on the fiber of $T_\mathfrak{p}$ over $\mathfrak{p}$ via the Eichler-Shimura relation of [7, Section 5]. As a cycle on $M_0(\mathfrak{n})_\mathfrak{p} \times M_0(\mathfrak{n})_\mathfrak{p}$, the fiber of $T_\mathfrak{p}$ is the sum of $\Gamma_\mathfrak{p}$ and its transpose $\Gamma_\mathfrak{p}^\vee$. We compute the divisor of $f_\mathfrak{p}$ separately on each component. For $\Gamma_\mathfrak{p}$, the first projection to $M_0(\mathfrak{n})_\mathfrak{p}$ is an isomorphism while the second is totally inseparable. Since Δ^2 is a pluricanonical form of degree q^2-1, it follows that $f_\mathfrak{p}$ has a pole of order q^2-1 on $\Gamma_\mathfrak{p}$. By a similar computation we see that $f_\mathfrak{p}$ has a zero of order q^2-1 on $\Gamma_\mathfrak{p}^\vee$.

We conclude that the divisor of $f_\mathfrak{p}$ on $T_\mathfrak{p}$ is $(1-q^2)(\Gamma_\mathfrak{p}-\Gamma_\mathfrak{p}^\vee)$. If $\varpi_\mathfrak{p}$ is a uniformizer at $\mathfrak{p}$, the divisor of $\varpi_\mathfrak{p}^{1-q^2} f_\mathfrak{p}$ on $T_\mathfrak{p}$ is thus $2(1-q^2)\Gamma_\mathfrak{p}$. This completes the proof. □

As in the classical case, we immediately obtain the following result.

Theorem 5.7. *Assume that π is not of CM-type. Then $\mathcal{S}(\mathbf{F}(t), T_{\pi,\infty})$ is finite.*

We make explicit the bound we have obtained on $\mathcal{S}(\mathbf{F}(t), T_{\pi,\infty})$ in the case that ρ_π is surjective and $l \geq 7$. Since $l \geq 7$ and $M_0(\mathfrak{n})$ is a curve, one can define the regulator map with $r=0$. Using both assumptions, one can show that $\mathcal{S}_{P_n}(F, T_n)$ in Proposition 2.1 is trivial for all n; see [23, Proposition III.5.1]. If γ is chosen so that its eigenvalues have distinct residues in $\mathcal{O}/\lambda$, the cokernels of Lemma 2.2 and (5.1) are trivial as well. Finally, the groups $H^1_s(F_w, T_\pi^*)$ vanish for $w \notin S$ by [23, Lemma I.5.2]. We conclude that $2(q^2-1)\eta$ annihilates $\mathcal{S}(F, T_{\pi,\infty})$ where η is a constant depending on the cokernel of the map $H^1_{\text{ét}}(\bar{M}_0(\mathfrak{n}), \mathcal{O}) \to H_\pi$. One expects that η should be related to congruences between π and other automorphic representations.

It would certainly be preferable to construct a cohesive Flach system for T_π as in [24]. Unfortunately it appears that not enough is yet known about the structure of the Hecke algebra to complete this construction; see [20, pp. 241–242].

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