ALGEBRAIC CYCLES, MODULAR FORMS AND EULER SYSTEMS

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Fix a squarefree integer N and let f be a newform of weight 2 for $\Gamma_0(N)$; we assume that f does not have complex multiplication. It was shown in [14] and [15] that for a set of primes l of density 1 the naive deformation theory of the mod l Galois representation associated to f is unobstructed (in the sense that the universal deformation ring is a power series ring over the Witt vectors). In [31] these methods were modified to obtain results on the deformation problems studied by Taylor-Wiles. In this paper we extend the results of Flach and Mazur to the case of newforms f of weight $\kappa \geq 2$ for $\Gamma_1(N)$.

We now state our results more precisely. Fix $l > \max\{5, \kappa+1\}$, let f be as above and let H be the associated l-adic representation: H is a free module of rank 2 over a certain Hecke algebra A, which itself is a finite, flat, local, Gorenstein $\mathbf{Z}_l$-algebra. Let T be the Tate twist $\mathrm{End}^0_A H(1)$ of the module of trace zero endomorphisms of H. Using techniques of Flach we construct a collection of cohomology classes $\{c^p\}$ in $H^1(\mathbf{Q}, T)$ with tightly controlled ramification. With some mild additional hypotheses, applying the methods of Kolyvagin to these classes yields a certain annihilator $\eta \in A$ of the Selmer group $H^1_f(\mathbf{Q}, T^*)$ of the Cartier dual of T. This Selmer group is dual to the differentials $\Omega_R \otimes_R A$, where R is the universal minimally ramified deformation ring of the residual representation of H. In the case that η is a unit this then implies that both R and A are isomorphic to the ring of Witt vectors over the residue field of A.

In the general case, following Mazur we show that our construction yields a derivation from A to the Selmer group $H^1_f(\mathbf{Q}, T/\eta T)$; it follows by a formal argument that the natural surjection $R \twoheadrightarrow A$ induces an isomorphism $\Omega_R \otimes_R A \cong \Omega_A$. Although not the strongest possible result, this does provide a great deal of information on the structure of the ring R. (It is possible that any such map $R \twoheadrightarrow A$ must be an isomorphism, although as far as I know this question remains open.) We also show that the isomorphism $\Omega_R \otimes_R A \cong \Omega_A$ is characterized by the fact that

$$\Omega_A \cong \Omega_R \otimes_R A \cong \mathrm{Hom}_{\mathbf{Z}_l}\big(H^1_f(\mathbf{Q}, T^*), \mathbf{Q}_l/\mathbf{Z}_l\big)$$

identifies the differential of the trace of a geometric Frobenius at p acting on H with -12 times the dual of c^p under the Bockstein pairing

$$H^1_f(\mathbf{Q}, T/\eta T) \otimes H^1_f(\mathbf{Q}, T^*) \to \mathbf{Q}_l/\mathbf{Z}_l.$$

(The factor of 12 appears as the weight of the modular form Δ used to define the classes c^p.) This non-obvious congruence exhibits a certain naturality of our construction which is not otherwise apparent and answers the question [31], p. 98.

In order to explain the methods of the construction of the classes c^p we consider a more general situation. Let X be a smooth projective variety over a number field

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F. Set $V = H^{2m}_{\text{ét}}(X_{F_{\text{ac}}}, \mathbf{Q}_l/\mathbf{Z}_l(m))$ for some m. The Selmer group $H^1_f(F, V)$ is the subgroup of the Galois cohomology group $H^1(F, V)$ consisting of cohomology classes which are everywhere locally "minimally ramified". This Selmer group often has an interesting arithmetic interpretation (as with the deformation theory case considered above) and is connected to special values of L-functions by the Bloch-Kato conjectures.

The most successful methods to date for understanding Selmer groups involve Kolyvagin's method of Euler systems: one uses certain collections of classes in the Galois cohomology of the Cartier dual of V to produce an annihilator (or even a bound) for $H^1_f(F, V)$. Most of these constructions rely on realizing appropriate elements of motivic cohomology in Galois cohomology. The map we use is an Abel-Jacobi map

$$H^{2m+1}_{\mathcal{M}}\big(X, \mathbf{Q}(m+1)\big) \to H^1\big(F, H^{2m}_{\text{ét}}(X_{F_{\text{ac}}}, \mathbf{Q}_l(m+1))\big).$$

(We actually use a formulation via coniveau spectral sequences so that we can work integrally.) Our key result, Theorem 3.1.1, is a reciprocity law in the sense of Kato describing this Abel-Jacobi map locally as the composition of a certain divisor map and the cycle class map. This generalizes [15], Lemma 3 and is precisely the sort of result needed to check that classes in the image of the Abel-Jacobi map form an Euler system.

The first four sections of this paper can be regarded as the generalization to higher dimensions of [31]. In Section 1 we give our version of standard material on Selmer groups. Very little in this section is new; our treatment is a synthesis of [17], [31] and [37]. Our notion of a "partial geometric Euler system" is weak but well-suited to applications. We also describe Mazur's notion of a cohesive Flach system and give the applications to deformation theory.

The next three sections are concerned with the production of cohesive Flach systems. Section 2 discusses the global behavior of the Abel-Jacobi map. We review its construction and investigate its relations with algebraic correspondences. The main result is the Leibniz relation Proposition 2.3.3; this is used to construct the derivation to Galois cohomology.

Section 3 begins with the statement and proof of the reciprocity law for the Abel-Jacobi map. We then explain how these results can be used, in the presence of appropriate geometric data, to construct partial geometric Euler systems and cohesive Flach systems.

Section 4 is concerned with exhibiting this data for modular curves and Kuga-Sato varieties; this yields our applications to modular forms. The key idea is the production of certain modular units on Hecke correspondences.

In Section 5 we give a proof of our congruence description of the map $\Omega_R \otimes_R A \to \Omega_A$. Our proof is quite computational; it would be useful to have a more conceptual argument. We hope to give applications of this result in a later paper. We include a short appendix on linear algebra over Gorenstein rings and bilateral derivations.

We should note that stronger results on the deformation theory of Galois representations associated to modular forms have been obtained by Diamond, Flach and Guo. Given this, we will not feel a need to maintain the absolute most general hypotheses and we will make additional (not terribly restrictive) hypotheses as needed. We refer to their paper [8] for the applications to the Bloch-Kato conjecture.

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If K is a field we write K_{s} (resp. K_{ac}) for a separable (resp. algebraic) closure of K. (We will tend to write K_{s} even when over a perfect field to emphasize that our perfect hypotheses are often excessive.) We set $G_K = \mathrm{Gal}(K_{\mathrm{s}}/K)$. All cohomology in this paper is either Galois or étale; for Galois cohomology we always use cohomology with continuous cochains as in [37], Appendix B. We write $H^i(L/K, T)$ for $H^i(\mathrm{Gal}(L/K), T)$ and $H^i(K, T)$ for $H^i(\mathrm{Gal}(K_{\mathrm{s}}/K), T)$. By a local (resp. global) field we mean a finite extension of $\mathbf{Q}_p$ or $\mathbf{F}_p((t))$ for some p (resp. $\mathbf{Q}$ or $\mathbf{F}_p(t)$ for some p). If F is a number field and v is a place of F, we write $\mathrm{Fr}(v) \in G_F$ for a geometric Frobenius element at v. If χ is a character of G_F of conductor prime to v, we write $\chi(v)$ for $\chi(\mathrm{Fr}(v))$. If H is a Galois representation, we simply write $\mathrm{End}\, H(1)$ (rather than the cumbersome $(\mathrm{End}\, H)(1)$) for the Tate twist of the endomorphisms of H.

1. Selmer groups and geometric Euler systems

1.1. **Local cohomology groups.**

1.1.1. *l-adic Galois modules.* Fix a prime l and let A be a finite, flat, local $\mathbf{Z}_l$-algebra to be fixed throughout the discussion. Let K be a field. By a *finitely generated (resp. discrete) l-adic G_K-module* we mean an A-module T which is finitely generated over $\mathbf{Z}_l$ (resp. torsion of finite corank), endowed with an A-linear action of G_K which is continuous for the l-adic (resp. discrete) topology on T. The l-adic G_K-modules form a category in the obvious way; morphisms are assumed to be A-linear, continuous and compatible with G_K-actions.

1.1.2. *Local finite/singular structures.* We now restrict to the case of a local field K with residue field k of characteristic p; we allow $p = l$ only if K itself has characteristic zero. Let K_{ur} denote the maximal unramified extension of K and let $\mathcal{I}_K = \mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{ur}})$ denote the inertia group of K. We say that an l-adic G_K-module T is *unramified* if $\mathcal{I}_K$ acts trivially on T. In any case we define the *unramified subgroup* $H^1_{\mathrm{ur}}(K, T)$ of $H^1(K, T)$ by

$$H^1_{\mathrm{ur}}(K, T) = \ker\big(H^1(K, T) \to H^1(K_{\mathrm{ur}}, T)\big).$$

By [37], Lemma 1.3.2, $H^1_{\mathrm{ur}}(K, T)$ identifies with $H^1(k, T^{\mathcal{I}_K})$ via inflation.

Definition 1.1.1. A *local finite/singular structure* $\mathcal{S}$ on T consists of a choice of A-submodule $H^1_{f,\mathcal{S}}(K, T) \subseteq H^1(K, T)$. A *structured G_K-module* $(T, \mathcal{S})$ is a pair of an l-adic G_K-module T and a finite/singular structure $\mathcal{S}$ on T.

We call $H^1_{f,\mathcal{S}}(K, T)$ the *finite subgroup*; we define the *singular quotient* $H^1_{s,\mathcal{S}}(K, T)$ to be $H^1(K, T)/H^1_{f,\mathcal{S}}(K, T)$. We will often omit the finite/singular structure $\mathcal{S}$ from the notation if it is clear from context. We write c_s for the image of $c \in H^1(K, T)$ in $H^1_{s,\mathcal{S}}(K, T)$.

We call the structure with $H^1_{f,\mathcal{S}}(K, T) = H^1_{\mathrm{ur}}(K, T)$ the *unramified structure.* When $p \neq l$ we say that a structured G_K-module $(T, \mathcal{S})$ is *unramified* if T is unramified and $\mathcal{S}$ is the unramified structure. In this case we have

$$H^1_{s,\mathcal{S}}(K,T) \cong T(-1)^{G_k}, \tag{1.1.1}$$

as follows from the fact that the maximal pro-l quotient of $\mathcal{I}_K$ is isomorphic to $\mathbf{Z}_l(1)$ as a G_k-module.

In the case $l \neq p$ the only non-trivial structures we will consider is the minimally ramified structure. For this we assume that T is free over $\mathbf{Z}_l$ (resp. l-divisible). Set $V = T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ (resp. $V = \varprojlim T[l^n] \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$). The *minimally ramified structure* on T is the pullback (resp. pushforward) of $H^1_{\mathrm{ur}}(K,V)$ via the natural map $T \to V$ (resp. $V \to T$). If T is unramified one checks easily that this agrees with the unramified structure.

In the case that $l = p$ we will usually consider only the *crystalline structure* given by the procedure of the preceeding paragraph with

$$H^1_f(K,V) = \ker\bigl(H^1(K,V) \to H^1(K, V \otimes_{\mathbf{Q}_l} B_{\mathrm{cris}})\bigr).$$

1.1.3. *Induced finite/singular structures.* A morphism $f : (T,\mathcal{S}) \to (T',\mathcal{S}')$ of structured G_K-modules is a morphism $f : T \to T'$ of l-adic G_K-modules such that $f_* H^1_{f,\mathcal{S}}(K,T) \subseteq H^1_{f,\mathcal{S}'}(K,T')$. If $(T,\mathcal{S})$ is a structured G_K-module and $i : T' \to T$ (resp. $j : T \to T''$) is a map of l-adic G_K-modules, we defined the *induced finite/singular structure* $i^*\mathcal{S}$ on T' (resp. $j_*\mathcal{S}$ on T'') to be the full inverse image (resp. pushforward) of $H^1_{f,\mathcal{S}}(K,T)$. The map i (resp. j) is then a map of structured modules.

Let $0 \to T' \to T \to T'' \to 0$ be a sequence of structured G_K-modules. We say that this is sequence is *exact* (as a sequence of structured G_K-modules) if it is exact as a sequence of A-modules and if the structures on T' and T'' are induced from the structure on T. A diagram chase and cohomological dimension arguments show that one then obtains exact sequences

$$\begin{aligned} 0 \to H^0(K,T') \to H^0(K,T) \to H^0(K,T'') \to \\ H^1_f(K,T') \to H^1_f(K,T) \to H^1_f(K,T'') \to 0 \end{aligned} \tag{1.1.2}$$

and

$$\begin{aligned} 0 \to H^1_s(K,T') \to H^1_s(K,T) \to H^1_s(K,T'') \to \\ H^2(K,T') \to H^2(K,T) \to H^2(K,T'') \to 0. \end{aligned} \tag{1.1.3}$$

1.1.4. *Cartier dual structures.* The category of l-adic G_K-modules has a natural involution $T \mapsto T^*$ where $T^* = \mathrm{Hom}_{\mathbf{Z}_l}(T, \mu_{l^\infty}(K_{\mathrm{s}}))$ with A-module structure induced from T and adjoint Galois action. Tate local duality yields a perfect pairing

$$\langle \cdot,\cdot \rangle_K : H^1(K,T) \otimes_{\mathbf{Z}_l} H^1(K,T^*) \to \mathbf{Q}_l/\mathbf{Z}_l.$$

If $p \neq l$ and T is unramified, one also has that $H^1_{\mathrm{ur}}(K,T)$ and $H^1_{\mathrm{ur}}(K,T^*)$ are exact orthogonal complements; see [37], Chapter 1, Section 4.

We extend Cartier duality to the category of structured G_K-modules by letting the finite subgroup $H^1_{f,\mathcal{S}^*}(K,T^*)$ on T^* be the exact annihilator of $H^1_{f,\mathcal{S}}(K,T)$ under the Tate pairing. This respects minimally ramified structures for $l \neq p$; by [1], Proposition 3.8 it also respects crystalline structures if $T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ is deRham.

1.1.5. *Archimedean structures.* We briefly consider the archimedean case. Let K denote either $\mathbf{R}$ or $\mathbf{C}$ and let T be an l-adic G_K-module. The cohomology group $H^1(K,T)$ is trivial, so that there is only one choice for the finite/singular structure, unless $K=\mathbf{R}$ and $l=2$. We refer to [37], Remark 1.3.7 for the natural choices in this case.

1.2. **Global cohomology groups.**

1.2.1. *Global finite/singular structures.* Let F be a global field. For every place v of F we fix now and forever embeddings $F_{\mathrm{s}} \hookrightarrow F_{v,\mathrm{s}}$; these induce injections $G_{F_v} \hookrightarrow G_F$. Let k_v denote the residue field of F_v and let $\mathcal{I}_v = \mathrm{Gal}(F_{v,\mathrm{s}}/F_{v,\mathrm{ur}})$ denote the inertia group of F_v.

Let A be a $\mathbf{Z}_l$-algebra as before; we now write $\mathfrak{m}$ for its maximal ideal and k for its residue field, contrary to our earlier notation. We assume that l does not equal the characteristic of F. We modify the definition of an l-adic G_F-module to include only those which are unramified at almost all places of F. Let Σ_l (resp. Σ_∞) denote the set of places of F above l (resp. the set of archimedean places); both are empty if F has positive characteristic.

Definition 1.2.1. A *global finite/singular structure* $\mathcal{S}$ on an l-adic G_F-module T consists of choices of local finite/singular structures $H^1_{f,\mathcal{S}}(F_v,T)$ for every place v of F such that $(T,\mathcal{S})$ is unramified as a stuctured G_{F_v}-module for almost all v.

The structures considered in [1], [17] and [37] are those which are minimally ramified away from Σ_l. The definitions of morphisms, induced structures, exact sequences and Cartier duals extend to structured G_F-modules by considering each place of F individually.

1.2.2. *Selmer groups.* Let $(T,\mathcal{S})$ be a structured G_F-module. For every place v there is a canonical restriction map $\mathrm{res}_v : H^1(F,T) \to H^1(F_v,T)$. We often write $\mathrm{res}_v(c) = c_v$.

Definition 1.2.2. The *Selmer group* $H^1_{f,\mathcal{S}}(F,T)$ of $(T,\mathcal{S})$ is defined by

$$H^1_{f,\mathcal{S}}(F,T) = \ker\bigl(H^1(F,T) \to \prod_v H^1_{s,\mathcal{S}}(F_v,T)\bigr).$$

See [37], Chapter 1, Section 6 for interpretations of Selmer groups in terms of ideal class groups, global units and rational points on abelian varieties.

Let Σ be a set of places which contains Σ_l, Σ_∞ and all places where T is ramified. Let $\mathcal{S}$ be the finite/singular structure which is unramified away from Σ and with $H^1_{f,\mathcal{S}}(F_v,T) = H^1(F_v,T)$ for $v \in \Sigma$. By [43], Proposition 6 inflation induces an isomorphism $H^1_{f,\mathcal{S}}(F,T) \cong H^1(F_\Sigma/F,T)$ where F_Σ is the maximal extension of F unramified outside Σ. It follows from this and [37], Proposition B.2.7 that if $(T,\mathcal{S})$ is a finite (resp. finitely generated, resp. discrete) structured G_F-module, then $H^1_{f,\mathcal{S}}(F,T)$ is finite (resp. finitely generated over $\mathbf{Z}_l$, resp. of finite corank).

Let $0 \to T' \to T \to T'' \to 0$ be an exact sequence of structured G_F-modules. The local exact sequences (1.1.2) yield an exact sequence

$$0 \to H^0(F,T') \to H^0(F,T) \to H^0(F,T'') \to \\ H^1_f(F,T') \to H^1_f(F,T) \to H^1_f(F,T'').$$

As an immediate consequence we have the following useful lemma.

Lemma 1.2.3. *Let $(T,\mathcal{S})$ be a structured G_F-module and let $\alpha \in A$ be such that $(\alpha T)^{G_F} = (T/\alpha T)^{G_F} = 0$. Let $T[\alpha]$ have the induced structure. Then $H^1_f(F,T[\alpha])$ injects into $H^1_f(F,T)$ and under this identification it identifies with $H^1_f(F,T)[\alpha]$.*

1.2.3. *Global pairings.* Let $(T,\mathcal{S})$ be a structured G_F-module. We recall now two global pairings. We begin with the *Kolyvagin pairing*

$$\langle\cdot,\cdot\rangle_F : \left(\bigoplus_v H^1_{s,\mathcal{S}}(F_v,T)\right)\otimes_{\mathbf{Z}_l} H^1_{f,\mathcal{S}^*}(F,T^*) \to \mathbf{Q}_l/\mathbf{Z}_l$$

given by

$$\langle (c^v), d\rangle_F = \sum_v \langle c^v, d_v\rangle_v$$

where $\langle\cdot,\cdot\rangle_v$ is the Tate pairing for F_v. We define the *compactly supported cohomology* $H^1_c(F,T)$ to be the A-submodule of $H^1(F,T)$ of classes which lie in $H^1_f(F_v,T)$ for almost all v. (Note that $H^1_c(F,T) = H^1(F,T)$ if T is discrete.) The basic fact about the Kolyvagin pairing is the following.

Proposition 1.2.4. *Let T be a structured G_F-module. Then the image of $H^1_c(F,T)$ in $\oplus H^1_s(F_v,T)$ is orthogonal to $H^1_f(F,T^*)$ under the Kolyvagin pairing.*

Proof. This follows from global class field theory; see [30], Section 12. □

Now let

$$0 \to T' \xrightarrow{\alpha} T \xrightarrow{\beta} T'' \to 0$$

be an exact sequence of finite structured G_F-modules. There is then a *Bockstein pairing*

$$\{\cdot,\cdot\}_{\alpha,\beta} : H^1_f(F,T'') \otimes H^1_f(F,T'^*) \to \mathbf{Q}_l/\mathbf{Z}_l.$$

We give the definition of $\{x'',y\}_{\alpha,\beta}$ only in the case that x'' is the image of $x \in H^1(F,T)$. Since $x'' \in H^1_f(F,T'')$, it follows from (1.1.3) that for each v the singular restriction $x_{v,s}$ is the image of $x'_v \in H^1_s(F_v,T')$; in fact, we have $x'_v = 0$ for almost all v. We define

$$\{x'',y\}_{\alpha,\beta} = \langle (x'_v), y\rangle_F\,.$$

By Proposition 1.2.4 this is independent of the choice of x. See [17], Chapitre 2, Section 1.4 for the general definition.

1.3. Partial geometric Euler systems.

1.3.1. *Definitions.* Let A and F be as above. Let $(T,\mathcal{S})$ be a structured G_F-module. If C is an A-submodule of $H^1(F,T)$ and v is a places of F, we write $C_{v,s}$ for the image of C in $H^1_s(F_v,T)$.

Definition 1.3.1. Let $\mathcal{L}$ be a (possibly infinite) set of places of F and let η be an ideal of A. A *partial (geometric) Euler system* $\{C^v\}_{v\in\mathcal{L}}$ of depth η for $(T,\mathcal{S})$ is an assignment of A-submodules $C^v \subseteq H^1(F,T)$ for each $v \in \mathcal{L}$ such that

- $C^v_{w,s} = 0$ for all places $w \neq v$;
- $H^1_s(F_v,T)/C^v_{v,s}$ is killed by η.

(That is, C^v is supported in the singular quotients only at v, and here it contains $\eta H^1_s(F_v,T)$.) If in addition C^v vanishes in $H^1_s(F_v,T/\eta T)$ for all $v \in \mathcal{L}$, we say that the partial Euler system has *strict depth* η. In this case the image of each C^v in $H^1(F,T/\eta T)$ lies in $H^1_f(F,T/\eta T)$ and we define the *Euler module* Φ to be the A-submodule of $H^1_f(F,T/\eta T)$ generated by the images of the C^v for all $v \in \mathcal{L}$.

For a set of places $\mathcal{L}$, define

$$H^1_{\mathcal{L}}(F,T) = \ker\left(H^1(F,T) \to \prod_{v\in\mathcal{L}} H^1(F_v,T)\right).$$

The basic utility of a partial Euler system comes from the next lemma.

Lemma 1.3.2. *Let $(T,\mathcal{S})$ be a finitely generated structured G_F-module. Assume that T admits a partial Euler system $\{C^v\}_{v\in\mathcal{L}}$ of depth η. Then*

$$\eta H^1_f(F,T^*) \subseteq H^1_{\mathcal{L}}(F,T^*).$$

Proof. Fix $v \in \mathcal{L}$. By Proposition 1.2.4 the Kolyvagin pairing is trivial on the image of C^v. However, by the definition of C^v the restriction of the Kolyvagin pairing to C^v "coincides" with the restriction

$$\langle\cdot,\cdot\rangle_v : C^v_v \otimes_{\mathbf{Z}_l} \mathrm{im}\big(H^1_f(F,T^*) \to H^1_f(F_v,T^*)\big) \to \mathbf{Q}_l/\mathbf{Z}_l$$

of the Tate local pairing at v. The lemma follows from this and the duality of $H^1_s(F_v,T)$ and $H^1_f(F_v,T^*)$. □

1.3.2. *Annihilation theorems.* Let T be a finite l-adic G_F-module with splitting field F'; set $\Delta = \mathrm{Gal}(F'/F)$. For $\tau \in \Delta$, define $\mathcal{L}_\tau$ to be the set of non-archimedean places of F which are unramified in F' and such that $\mathrm{Fr}(v)$ is conjugate to τ on F'. We say that τ acts on T as a *scalar* if its action factors through the natural map $A^\times \to \mathrm{Aut}_A T$. The next lemma is a variation due to Flach and Mazur of ideas of Kolyvagin.

Lemma 1.3.3. *Let T be a finite l-adic G_F-module with $l \neq 2$. Suppose that $T[\mathfrak{m}]$ is absolutely irreducible as a $k[G_F]$-module and that $\tau \in \Delta$ acts on T as a non-scalar involution. Then for any set of places $\mathcal{L}$ cofinite in $\mathcal{L}_\tau$ we have $H^1_{\mathcal{L}}(F,T) \subseteq H^1(\Delta,T)$.*

Proof. By inflation-restriction it suffices to show that the image of $c \in H^1_{\mathcal{L}}(F,T)$ in $\mathrm{Hom}_\Delta(G^{\mathrm{ab}}_{F'},T)$ is trivial. Let $\psi : G^{\mathrm{ab}}_{F'} \to T$ be this Δ-equivariant homomorphism. Let F'' be the fixed field of $\ker\psi$ and set $\Gamma = \mathrm{Gal}(F''/F')$. Fix a lifting $\tilde{\tau} \in \mathrm{Gal}(F''/\mathbf{Q})$ of τ.

Fix now a $g \in \Gamma$. By the Tchebatorev density theorem there exist infinitely many unramified places v'' of F'' such that $\mathrm{Fr}_{F''/F}(v'') = \tilde{\tau}g$. Let v' (resp. v) denote the restriction of v'' to F' (resp. F). We have $\mathrm{Fr}_{F'/F}(v') = \tau$, so that $v \in \mathcal{L}_\tau$. We pick such a place v'' so that $v \in \mathcal{L}$. Thus by assumption $\psi|_{\mathrm{Gal}(F''_{v''}/F'_{v'})} = 0$. Equivalently, $\psi(\mathrm{Fr}_{F''/F'}(v'')) = 0$.

Since τ has order 2, $F'_{v'}/F_v$ is of degree 2 and $\mathrm{Fr}_{F''/F'}(v'') = (\tilde{\tau}g)^2$. Thus $\psi(\tilde{\tau}g\tilde{\tau}g) = 0$. Since $\tilde{\tau}$ is an involution and Δ acts on $G^{\mathrm{ab}}_{F'}$ by conjugation, this means that $\psi({}^{\tau}g \cdot g) = 0$. By Δ-equivariance we conclude that $\tau\psi(g) = -\psi(g)$.

Thus the A-span Ψ of the image of ψ is a G_F-stable submodule of $T^{\tau=-1}$, It follows that $\Psi[\mathfrak{m}] \subseteq T[\mathfrak{m}]^{\tau=-1}$. As τ is non-scalar, $T[\mathfrak{m}]^{\tau=-1} \neq T[\mathfrak{m}]$; thus $\Psi[\mathfrak{m}] \neq T[\mathfrak{m}]$. Since $T[\mathfrak{m}]$ is absolutely irreducible this implies that $\Psi[\mathfrak{m}] = 0$, and thus that $\Psi = 0$. We conclude that $\psi = 0$, which completes the proof. □

Lemmas 1.3.2 and 1.3.3 yield the following result. Let δ be the A-annihilator of $H^1(\Delta,T^*)$.

Proposition 1.3.4. *Let $(T,\mathcal{S})$ be a finite structured G_F-module. Suppose that:*

- *$T^*[\mathfrak{m}]$ is absolutely irreducible as a G_F-module over k;*

- *There is a non-scalar involution $\tau \in \Delta$;*
- *T admits a partial Euler system $\{C^v\}_{v\in\mathcal{L}}$ of depth η for some set of places $\mathcal{L}$ cofinite in $\mathcal{L}_\tau$.*

Then $\delta\eta$ annihilates the Selmer group $H^1_f(F, T^*)$.

There is a version of Proposition 1.3.4 for finitely generated T, but it involves some additional complications which will not come up in our applications.

1.4. **Flach systems.**

1.4.1. *Review of deformation theory.* We continue with our previous notation. We now require that F is a number field with at least one real embedding; fix a complex conjugation $\tau \in G_F$. We also assume that l is unramified in $F/\mathbf{Q}$. We assume now that $l > 2$, and we require that A is Gorenstein; in particular, $\mathrm{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l)$ is free of rank 1 as an A-module. Fix a Gorenstein trace $\mathrm{tr} : A \to \mathbf{Z}_l$. (See Appendix A.1.) Let $W(k)$ denote the ring of Witt vectors for k.

Definition 1.4.1. Let H be an l-adic G_F-module which is free of rank 2 as an A-module. Let Σ be the set of places at which H is ramified together with Σ_l and Σ_∞. We say that H is *minimally ramified* if the following conditions are satisfied:

- $H \otimes_A k$ is absolutely irreducible;
- For every $v \in \Sigma - \Sigma_l \cup \Sigma_\infty$, the image of the inertia group $\mathcal{I}_v$ in $\mathrm{Aut}_A H \cong \mathrm{GL}_2(A)$ is conjugate to the subgroup $\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) ; a \in A\}$;
- For some $\kappa < l$ and each $v \in \Sigma_l$, H is crystalline of weight κ at v in the following sense: H arises via the Tate module functor from a strongly divisible lattice D in $H^0(F_v, B_{\mathrm{cris}} \otimes_{\mathbf{Z}_l} H)$ such that the filtration $F^i D$ satisfies
$$\mathrm{rank}_{\mathcal{O}_{F_v}} F^i D = \begin{cases} 2 & i \leq 0; \\ 1 & 1 \leq i \leq \kappa - 1; \\ 0 & \kappa \leq i; \end{cases}$$
(see [16] and [1], Section 4);
- The determinant character $\chi : G_F \to A^\times$ of H factors through $W(k)^\times$ and satisfies $\chi(\tau) = -1$;
- The $W(k)$-algebra A is generated by the traces of $\mathrm{Fr}(v)$ acting on the inertia coinvariants $H_{\mathcal{I}_v}$ for all $v \notin \Sigma_l$.

For each $v \notin \Sigma_l$ we define the *Hecke operator* $T_v \in A$ to be the trace of $\mathrm{Fr}(v)$ acting on $H_{\mathcal{I}_v}$; note that this is well-defined since $H_{\mathcal{I}_v}$ is always free for such v by our assumptions. For $v \notin \Sigma$, the characteristic polynomial of $\mathrm{Fr}(v)$ on H is $x^2 - T_v x + \chi(v)$.

We are interested in deformations of $H \otimes_A k$ as a Galois module which satisfy the same local conditions as H. Specifically, let $\mathcal{C}$ denote the category of inverse limits of artinian local rings with residue field k. Let H' be a lifting of $H \otimes_A k$ over a ring B of $\mathcal{C}$; that is, H' is a free B-module of rank 2 with $H' \otimes_B k \cong H \otimes_A k$. Following Diamond, we say that H' is *minimally ramified* if:

- H' is unramified away from Σ;
- For every $v \in \Sigma - \Sigma_l$, the image of $\mathcal{I}_v$ acting on H' is conjugate to a subgroup of $\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) ; b \in B\}$;
- H' is crystalline at every place $v \in \Sigma_l$ (in the sense of Fontaine-Laffaille above) and the filtration on the associated Dieudonné module D satisfies $F^0 D = D$ and $F^\kappa D = 0$;

- H' has determinant χ.

We let $D : \mathcal{C} \to \mathbf{Sets}$ be the corresponding deformation functor, so that $D(B)$ is the set of isomorphism classes of pairs (H', α') of minimally ramified liftings of $H \otimes_A k$ to B and isomorphisms $\alpha' : H' \otimes_B k \cong H \otimes_A k$.

By [32] the functor D is representable; we let R be the universal deformation ring. Let $\pi : R \to A$ be the map corresponding to the minimally ramified deformation H. Let $\hat{T}_v$ be the trace of $\mathrm{Fr}(v)$ acting on the inertia coinvariants of the universal deformation; we have $\pi(\hat{T}_v) = T_v$, from which it follows that π is surjective.

In order to study the ring R via our given deformation H we need to define a second deformation functor. Let $\mathcal{C}'$ be the category of inverse limits of local artinian rings, with residue field k, equipped with a local homomorphism to A (inducing the identity map on k). We define a deformation functor $D_A : \mathcal{C}' \to \mathbf{Sets}$ by letting $D_A(f : B \to A)$ be the inverse image of $(H, \mathrm{id}) \in D(A)$ under the map $f_* : D(B) \to D(A)$. That is, D_A classifies the deformations which are "congruent to H". [32], Section 20, Proposition 4 shows that D_A is represented by $\pi : R \to A$ for the same π and R as above.

1.4.2. *Tangent spaces and Selmer groups.* For any ideal $\mathfrak{a}$ of A let $A_\mathfrak{a}$ be the ring $A[\epsilon]/(\mathfrak{a}\epsilon, \epsilon^2)$. There is a well-known canonical isomorphism

$$D_A(A_\mathfrak{a}) \cong \mathrm{Hom}_A(\Omega_R \otimes_R A, A/\mathfrak{a}) \tag{1.4.1}$$

where Ω_R is the module of differentials of R over $W(k)$ (or equivalently over $\mathbf{Z}_l$); see [32], Section 17.

Set $T = \mathrm{End}^0_A H(1)$. T is a free A-module of rank 3; the composition of the usual trace pairing with the Gorenstein trace tr yields an isomorphism

$$T^* \cong \mathrm{End}^0_A H \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l. \tag{1.4.2}$$

We give T the finite/singular structure $\mathcal{S}$ which is minimally ramified away from Σ_l and crystalline at all places of Σ_l; the Cartier dual structure $\mathcal{S}^*$ on T^* has the same description. For every ideal $\mathfrak{a}$ of A we give $T/\mathfrak{a}T$ and $T^*[\mathfrak{a}]$ the induced structures. We now have a second isomorphism

$$D_A(A_\mathfrak{a}) \cong H^1_f(F, T^*[\mathfrak{a}]); \tag{1.4.3}$$

see [32], Sections 21, 22, 24, 29 and [1], Lemma 4.5 for details.

Combining (1.4.1) and (1.4.3) and passing to the limit yields

$$H^1_f(F, T^*) \cong \mathrm{Hom}_A\big(\Omega_R \otimes_R A, A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l\big). \tag{1.4.4}$$

Postcomposition with the Gorenstein trace yields an isomorphism

$$H^1_f(F, T^*) \cong \mathrm{Hom}_{\mathbf{Z}_l}\big(\Omega_R \otimes_R A, \mathbf{Q}_l/\mathbf{Z}_l\big) \tag{1.4.5}$$

which is fundamental to what follows.

1.4.3. *Singular quotients.* Let F' be the splitting field of $H \otimes_A k$. By assumption τ acts on H as a non-scalar involution; it follows that it acts in the same way on T and T^*. Let $\mathcal{L}$ denote the set of non-archimedean places of F, unramified with Frobenius conjugate to τ on F'.

Note that for $v \in \mathcal{L}$ the characteristic polynomial of $\mathrm{Fr}(v)$ acting on H is congruent to $x^2 - 1$ modulo $\mathfrak{m}$. In particular, $\chi(v) \equiv -1 \pmod{\mathfrak{m}}$ and we have a factorization

$$x^2 - T_v x + \chi(v) \equiv (x-1)(x+1) \pmod{\mathfrak{m}}.$$

By Hensel's lemma this lifts to a factorization

$$x^2 - T_v x + \chi(v) = (x-\alpha)(x-\beta)$$

with $\alpha, \beta \in A$ and $\alpha \equiv -\beta \equiv 1 \pmod{\mathfrak{m}}$. An argument using Nakayama's lemma and a dimension count now yields the following.

Lemma 1.4.2. *Let v be a place of $\mathcal{L}$. There is a direct sum decomposition (depending on v) $H = H_\alpha \oplus H_\beta$, where H_α (resp. H_β) is free of rank 1 over A and $\mathrm{Fr}(v)$ acts on it as the scalar α (resp. β).*

Corollary 1.4.3. *For all places $v \in \mathcal{L}$ and all ideals $\mathfrak{a}$ of A, the singular quotient $H^1_s(F_v, T/\mathfrak{a}T)$ is a free $A/\mathfrak{a}$-module of rank 1.*

Proof. By (1.1.1) we have $H^1_s(F_v,T) \cong T(-1)^{G_{k_v}} \cong (\mathrm{End}^0_A H)^{G_{k_v}}$. G_{k_v} is generated by $\mathrm{Fr}(v)$, and $\mathrm{Fr}(v)$ acts on $\mathrm{End}^0_A H$ (with respect to the basis of Lemma 1.4.2) as conjugation by $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$. This sends a matrix $\left(\begin{smallmatrix}a & b\\ c & -a\end{smallmatrix}\right) \in \mathrm{End}^0_A H$ to

$$\begin{pmatrix} a & \frac{\alpha^2}{\chi(v)} b \\ \frac{\beta^2}{\chi(v)} c & -a \end{pmatrix}.$$

Note that α^2 and β^2 are not equal to $\chi(v)$ (or even congruent to it modulo $\mathfrak{m}$) since $\chi(v) \equiv -1 \pmod{\mathfrak{m}}$. It follows that $(\mathrm{End}^0_A H)^{G_{k_v}}$ is free of rank 1 over A, generated by $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. The result for $\mathfrak{a} \neq 0$ is proven in the same way. □

1.4.4. *Flach systems.* We now introduce a slightly refined version of a partial geometric Euler system. We assume that $T \otimes_A k$ is absolutely irreducible and that $H^1(F(T^*[\mathfrak{a}])/F, T^*[\mathfrak{a}]) = 0$ for all ideals $\mathfrak{a}$ of A. By [44], Proposition III.5.1 this second condition holds if $\#k \neq 5$ and $G_F \to \mathrm{Aut}_A H$ is surjective; see also [14], Lemma 1.2 and [15], Section 4, Remark 1..

Definition 1.4.4. Let η be a non-zero divisor in A. A *Flach system* $\{c^v\}_{v\in\mathcal{L}}$ of depth η for T is a partial Euler system $\{C^v\}_{v\in\mathcal{L}}$ of strict depth ηA such that each C^v is a cyclic A-module, generated by $c^v \in H^1(F,T)$.

Let $\{c^v\}_{v\in\mathcal{L}}$ be a Flach system for T and let Φ denote its Euler module. By Corollary 1.4.3 we see that this Flach system induces (by pushforward) a partial Euler system of depth η for $T/\mathfrak{a}T$ for every ideal $\mathfrak{a}$ of finite index in A. Proposition 1.3.4 and our vanishing assumption above now show that $\eta H^1_f(F, T^*[\mathfrak{a}]) = 0$ for every $\mathfrak{a}$; thus $\eta H^1_f(F,T^*) = 0$. Since $T \otimes_A k$ is absolutely irreducible the hypotheses of Lemma 1.2.3 are satisfied and we conclude that $H^1_f(F,T^*) = H^1_f\big(F, T^*[\eta]\big)$.

Consider now the Bockstein pairing

$$\{\cdot,\cdot\}_\eta : H^1_f(F, T/\eta T) \otimes_{\mathbf{Z}_l} H^1_f(F,T^*) \to \mathbf{Q}_l/\mathbf{Z}_l \tag{1.4.6}$$

associated to the exact sequence

$$0 \to T/\eta T \xrightarrow{\eta} T/\eta^2 T \to T/\eta T \to 0.$$

(One checks easily that each factor of $T/\eta T$ receives the same induced finite/singular structure.)

Lemma 1.4.5. *The restriction of $\{\cdot,\cdot\}_\eta$ to Φ on the left is right non-degenerate. In particular it induces an injection*

$$H^1_f(F,T^*) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}_l}(\Phi, \mathbf{Q}_l/\mathbf{Z}_l). \tag{1.4.7}$$

Proof. Let $c \in \Phi$ be the image of c^v for some $v \in \mathcal{L}$. Fix $y \in H^1_f(F, T^*)$. We compute the Bockstein pairing $\{c, y\}_\eta$. Proceeding as in Section 1.2.3, we first must lift c to $H^1(F, T/\eta^2 T)$. The image of c^v yields such a lift. $c^v_{w,s}$ is 0 for $w \neq v$, and it follows that

$$\{c, y\}_\eta = \left\langle \tfrac{1}{\eta} c^v_{v,s}, y_v \right\rangle_v .$$

Here $\frac{1}{\eta} c^v_{v,s}$ denotes the element of $H^1_s(F_v, T/\eta T)$ which maps to the class $c^v_{v,s} \in H^1_s(F_v, T/\eta^2 T)$ under multiplication by η.

By definition of a Flach system, $c^v_{v,s}$ generates $\eta H^1_s(F_v, T/\eta^2 T)$ as an A-module. Thus $\frac{1}{\eta} c^v_{v,s}$ generates $H^1_s(F_v, T/\eta T)$ as an A-module. In particular, since the Tate pairing is perfect we conclude that $\{c, y\}_\eta = 0$ only when $y_v = 0$ in $H^1_f(F_v, T^*)$.

Thus if y is orthogonal to all of Φ, then we have $y_v = 0$ for all $v \in \mathcal{L}$. By Lemma 1.3.3 and our assumptions we then have $y = 0$. This completes the proof. □

(1.4.7) and (1.4.5) yield a surjection $\Phi \twoheadrightarrow \Omega_R \otimes_R A$. In particular $\Omega_R \otimes_R A$ is η-torsion. If η is a unit this implies that $\Omega_R \otimes_R A = 0$; an easy argument then shows that $W(k) \cong R \cong A$. In order to obtain strong results when η is not a unit we will need to impose more structure on our Flach system.

1.4.5. *Cohesive Flach systems.* We continue with the assumptions of the previous section.

Definition 1.4.6. A *cohesive Flach system of depth* η for T is a collection of classes $c^v \in H^1(F, T)$ for $v \notin \Sigma_l \cup \Sigma_\infty$ such that:

- $\{c^v\}_{v \in \mathcal{L}}$ is a Flach system of depth η;
- c^v vanishes in $H^1_s(F_w, T)$ for all $w \neq v$;
- c^v vanishes in $H^1_s(F_v, T/\eta T)$;
- The map $\Theta : A \to H^1(F, T/\eta T)$ sending each T_v to c^v is a well-defined continuous derivation.

(Note that the second two conditions are redundant for $v \in \mathcal{L}$.)

Note that since the T_v generate A the image of Θ is contained in $H^1_f(F, T/\eta T)$. Note also that Θ is automatically $W(k)$-linear since $W(k)$ is unramified over $\mathbf{Z}_l$.

Θ induces an A-linear map $\Omega_A \to H^1_f(F, T/\eta T)$. This yields a surjection $\Omega_A \twoheadrightarrow \operatorname{im}\Theta$. Of course, there is an injection $\Phi \hookrightarrow \operatorname{im}\Theta$. We also have the surjection $\Phi \twoheadrightarrow \Omega_R \otimes_R A$ of the previous section. Lastly, since $\pi : R \to A$ is surjective we have a surjection $\Omega_R \otimes_R A \twoheadrightarrow \Omega_A$. The existence of these four maps and [28], Theorem 2.4 imply that all four are isomorphisms. We define the *Flach automorphism* $\Xi : \Omega_A \xrightarrow{\simeq} \Omega_A$ to be the appropriate composition

$$\Omega_A \twoheadrightarrow \operatorname{im}\Theta = \Phi \twoheadrightarrow \Omega_R \otimes_R A \twoheadrightarrow \Omega_A.$$

Returning to the isomorphism $\Omega_R \otimes_R A \xrightarrow{\simeq} \Omega_A$, we see that the surjection $\pi : R \to A$ induces an isomorphism on differentials. Such a π is called an *evolution.* If A is a complete intersection, then such a π is necessarily an isomorphism. As far as we know no examples of non-trivial evolutions are known in our setting. See [9] for examples of non-trivial evolutions in positive characteristic.

1.4.6. *Cohesive Flach systems of Eichler-Shimura type.* One can give an explicit description of the map Ξ introduced above for certain cohesive Flach systems. We first fix some notation. Fix a place $v \notin \Sigma$ and let λ_0 be a uniformizer of F_v. Fix a compatible system $\lambda = (\lambda_n)$ of l-power roots of λ_0. Fix also a generator $\zeta = (\zeta_n)$ of $\mathbf{Z}_l(1)$. These choices determine a generator σ of $\mathrm{Gal}(F_{v,\mathrm{ur}}(\lambda)/F_{v,\mathrm{ur}}) \cong \mathbf{Z}_l(1)$ by requiring $\sigma(\lambda_n) = \zeta_n \lambda_n$ for all n. The isomorphism (1.1.1) is given explicitly by

$$H^1_s(F_v, T) \cong \mathrm{Hom}_{G_{k_v}}\big(\mathrm{Gal}(F_{v,\mathrm{ur}}(\lambda)/F_{v,\mathrm{ur}}), T\big).$$

Let $\mathrm{Fr}(v)$ denote the endomorphism of H induced by Frobenius at v. We define the *verschiebung* $\mathrm{Ver}(v)$ to be the endomorphism $\chi(v)\,\mathrm{Fr}(v)^{-1}$.

Definition 1.4.7. A cohesive Flach system $\{c^v\}$ is said to be of *Eichler-Shimura type* of weight $2w$ if for each $v \notin \Sigma$ the class $c^v_{v,s} \in H^1_s(F_v, T)$ is given by

$$\begin{aligned} c^v_{v,s} : \mathrm{Gal}\big(F_{v,\mathrm{ur}}(\lambda)/F_{v,\mathrm{ur}}\big) &\to T \\ \tau^j &\mapsto wj\eta\big(\mathrm{Fr}(v) - \mathrm{Ver}(v)\big) \otimes \zeta. \end{aligned}$$

One checks immediately that this is a well-defined cocycle with values in T and that this definition is independent of the choices of λ and ζ. Our main result on cohesive Flach systems is the following, which is proved in Section 5.

Theorem 1.4.8. *Let $\{c^v\}$ be a cohesive Flach system of Eichler-Shimura type of depth η and weight $2w$ for T. Assume also that $H^1_s(F_v, T) = 0$ for $v \in \Sigma - \Sigma_l$. Then the Flach automorphism $\Xi : \Omega_A \to \Omega_A$ is multiplication by $2w$.*

Theorem 1.4.8 can be restated in the following terms. Consider the pairing on Ω_A induced from the Bockstein pairing, the cohesive Flach system, (1.4.5) and the isomorphism $\Omega_R \otimes_R A \cong \Omega_A$:

$$\begin{array}{ccc} H^1_f(F, T/\eta T) \otimes_{\mathbf{Z}_l} H^1_f(F, T^*) & \longrightarrow & \mathbf{Q}_l/\mathbf{Z}_l \\ \uparrow \qquad\qquad \uparrow & & \\ \Omega_A \otimes_{\mathbf{Z}_l} \mathrm{Hom}_{\mathbf{Z}_l}(\Omega_A, \mathbf{Q}_l/\mathbf{Z}_l) & & \end{array}$$

Theorem 1.4.8 says that this induced pairing is precisely $2w$ times the canonical duality pairing.

1.5. Additional applications to deformation theory.

A simple generalization of [14], Section 3 yields the following application to deformation theory. We will only use it in the situation of the previous section; nevertheless, as the general case is no more difficult we relax our hypotheses. Let F be a global field of characteristic different from l. Let A be a finite, flat, local, Gorenstein $\mathbf{Z}_l$-algebra with residue field k and let H be an l-adic G_F-module which is unramified away from a set of places Σ (containing Σ_l and Σ_∞) and which is free of finite rank as an A-module. We assume that $H \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ is deRham over F_v for each $v \in \Sigma_l$. Assume also that the determinant character of H factors through $W(k)^\times$. Set $T = \mathrm{End}^0_A H(1)$ (resp. $T' = \mathrm{End}^0_A H$) with finite/singular structure $\mathcal{S}$ (resp. $\mathcal{S}'$) minimally ramified away from Σ_l and crystalline at Σ_l. We will also use the finite/singular structure $\mathcal{S}_0$ on T which is unramified away from Σ and with $H^1_{f,\mathcal{S}_0}(F_v, T) = 0$ for $v \in \Sigma$. We have $T' \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l = T^*$. Let $G_{F,\Sigma} = \mathrm{Gal}(F_\Sigma/F)$.

Proposition 1.5.1. *Assume that $H^0(F_v, T \otimes_A k) = 0$ for all $v \in \Sigma - \Sigma_\infty$; that $T' \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ is critical in the sense of Deligne; and that $H^1_{f,\mathcal{S}^*}(F, T^*) = 0$. Then the* *deformation problem (with fixed determinant) associated to $H \otimes_A k$ as a $k[G_{F,\Sigma}]$-module is unobstructed.*

Proof. By [29], Section 1.6 it suffices to show that $H^2(G_{F,\Sigma}, T' \otimes_A k) = 0$. By the Poitou-Tate exact sequence

$$\prod_{v \in \Sigma - \Sigma_\infty} H^0(F_v, T \otimes_A k) \to H^2(G_{F,\Sigma}, T' \otimes_A k)^\vee \to H^1_{f,\mathcal{S}_0}(F, T \otimes_A k)$$

and our assumptions it suffices to show that $H^1_{f,\mathcal{S}_0}(F, T \otimes_A k) = 0$. This group injects into $H^1_{f,\mathcal{S}}(F, T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l)$, and we will show that this Selmer group vanishes.

We begin with the exact sequence (as in [14], Section 1)

$$0 \to \pi_* H^1_{f,\mathcal{S}'}(F, T' \otimes_{\mathbf{Z}_l} \mathbf{Q}_l) \to H^1_{f,\mathcal{S}^*}(F, T^*) \to Ш(F, T^*) \to 0.$$

Here $\pi : T' \otimes_{\mathbf{Z}_l} \mathbf{Q}_l \to T^*$ is the natural map and we are using the obvious notion of the Selmer group of $T' \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$. By assumption we thus have $H^1_{f,\mathcal{S}'}(F, T' \otimes_{\mathbf{Z}_l} \mathbf{Q}_l) = Ш(F, T^*) = 0$. By [13] and [12], Corollary 1.5 (using that $T' \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ is critical), it follows that $H^1_{f,\mathcal{S}}(F, T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l) = Ш(F, T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l) = 0$. Thus $H^1_{f,\mathcal{S}}(F, T \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l) = 0$, as claimed. □

2. The Abel-Jacobi map

2.1. Coniveau spectral sequences.

2.1.1. *Étale cohomology.* Let X be a scheme of finite Krull dimension, let Y be a closed subscheme of X and let $\mathcal{F}$ be a torsion étale sheaf on X. As in [21], Section 10.1 and [3], Section 1 there is a spectral sequence

$$E^{p,q}_{1,Y}(X, \mathcal{F}) = \bigoplus_{x \in X^p \cap Y} H^{p+q}_x(X, \mathcal{F}) \Rightarrow H^{p+q}_Y(X, \mathcal{F}) \tag{2.1.1}$$

coming from filtration by codimension of support and excision. Here X^p denotes the points of X of codimension p and

$$H^i_x(X, \mathcal{F}) \overset{\text{def}}{=} \varinjlim_{Z \subsetneq \overline{\{x\}}} H^i_{\overline{\{x\}} - Z}(X - Z, \mathcal{F}).$$

It is clear from its construction that (2.1.1) is contravariant for flat morphisms, covariant for finite flat morphisms and that these functorialities are compatible with edge maps when they exist. When $Y = X$ we write (2.1.1) simply as $E^{p,q}_1(X, \mathcal{F})$.

There is a localization sequence when $U = X - Y$ is dense in X. Specifically, for each q there is a short exact sequence of complexes

$$0 \to E^{\bullet q}_{1,Y}(X, \mathcal{F}) \to E^{\bullet q}_1(X, \mathcal{F}) \to E^{\bullet q}_1(U, \mathcal{F}) \to 0$$

which yields a long exact sequence

$$\cdots \to E^{p,q}_{2,Y}(X, \mathcal{F}) \to E^{p,q}_2(X, \mathcal{F}) \to E^{p,q}_2(U, \mathcal{F}) \to E^{p+1,q}_{2,Y}(X, \mathcal{F}) \to \cdots.$$

The boundary maps of this exact sequence are compatible with edge maps when they exist; see [15], Proposition 3.

2.1.2. *Purity.* Assume now that X is a regular scheme of finite type over a perfect field (resp. smooth over a discrete valuation ring with perfect residue field). Let $i : Y \hookrightarrow X$ be a regular closed subscheme of X (resp. of the closed fiber of X) of constant codimension c and let $\mathcal{F}$ be a locally constant torsion sheaf on X of order invertible in $\mathcal{O}_X$. In this situation one knows that Grothendieck's purity conjecture holds: there are functorial isomorphisms

$$R^j i^! \mathcal{F} \cong \begin{cases} 0 & j \neq 2c; \\ i^* \mathcal{F}(-c) & j = 2c; \end{cases}$$

where $i^! \mathcal{F}$ is the sheaf of sections of $\mathcal{F}$ supported on Y. In particular, one then has

$$H_Y^{j+2c}(X, \mathcal{F}) \cong H^j\big(Y, i^* \mathcal{F}(-c)\big)$$

for all j. See [24], Exposé 16, Section 3 (resp. [35], Lemma 2.1) for details. (One can extend much of what we will say to local and global fields of positive characteristic with some mild hypotheses on the sheaf $\mathcal{F}$; see [41], Corollary 3.7 for the relevant purity statements. We do not treat this except to say that, assuming that purity holds, the rest of our arguments go through.)

Return now to the closed immersion $i : Y \hookrightarrow X$; note that we also know purity for the inclusion of every regular closed subscheme of Y into X. In this situation one can use purity, the existence of an open locus of regularity [23], Corollary 6.12.6 and the compatibility of étale cohomology with direct limits to obtain an isomorphism

$$H_x^i(X, \mathcal{F}) \cong H^{i-2p}\big(\operatorname{Spec} k(x), \mathcal{F}(-p)\big)$$

for any $x \in X^p \cap Y$. In particular,

$$E_{1,Y}^{p,q}(X, \mathcal{F}) \cong \bigoplus_{x \in X^p \cap Y} H^{q-p}\big(\operatorname{Spec} k(x), \mathcal{F}(-p)\big). \tag{2.1.2}$$

2.1.3. *K-theory.* There is an analogous spectral sequence in K-theory; see [15], Sections 5.1 and 5.2 and [34], Section 7, Theorem 5.4. Here we require that X is a regular noetherian scheme of finite Krull dimension and that Y is a closed subscheme of X; the spectral sequence is

$$E_{1,Y}^{p,q}(X) = \bigoplus_{x \in X^p \cap Y} K_{-p-q} k(x) \Rightarrow K_{-p-q,Y} X. \tag{2.1.3}$$

It is contravariant for flat morphisms, covariant for finite flat morphisms and has a localization sequence as in Section 2.1.1; all of these are compatible with edge maps when they exist.

Note that

$$E_{1,Y}^{p-1,-p}(X) = \bigoplus_{x \in X^{p-1} \cap Y} k(x)^\times; \qquad E_{1,Y}^{p,-p}(X) = \bigoplus_{x \in X^p \cap Y} \mathbf{Z}.$$

By [34], Proposition 5.14 and Remark 5.17 and [20] the spectral sequence differential between these terms is nothing other than the divisor map. In particular $E_2^{p,-p}(X)$ identifies with the codimension p Chow group $\mathrm{CH}^p X$ and $E_2^{p,-p-1}(X)$ is a quotient of the abelian group of formal sums $\sum (Z_i, f_i)$ of pairs of codimension p cycles Z_i and rational functions f_i on Z_i such that $\sum \operatorname{div}_{Z_i} f_i = 0$. Alternately, by [25], Lemma 6.12.4 we may write $E_2^{p,-p}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $E_2^{p,-p-1}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ as the motivic cohomology groups $H_{\mathcal{M}}^{2p}(X, \mathbf{Q}(p))$ and $H_{\mathcal{M}}^{2p+1}(X, \mathbf{Q}(p+1))$.

2.1.4. *Chern characters.* The coniveau spectral sequences in K-theory and étale cohomology are connected by the Chern characters of [19]; see also [27], Chapter 3. Specifically, let X be a regular noetherian scheme of finite Krull dimension n and let Y be a closed subscheme of X. For any i and for any N invertible on X and relatively prime to $p!$ there is a map of spectral sequences

$$E^{p,q}_{r,Y}(X) \to E^{p,q+2i}_{r,Y}\big(X, \mathbf{Z}/N\mathbf{Z}(i)\big)$$

which is compatible with all functorialities and localization sequences. For the case $p = -q$, there is a commutative diagram

$$\begin{array}{ccc} E^{p,-p}_{1,Y}(X) & \longrightarrow & E^{p,p}_{1,Y}\big(X, \mathbf{Z}/N\mathbf{Z}(p)\big) \\ \Vert & & \Vert \\ \bigoplus_{x\in X^p\cap Y} \mathbf{Z} & \longrightarrow & \bigoplus_{x\in X^p\cap Y} \mathbf{Z}/N\mathbf{Z} \end{array} \tag{2.1.4}$$

with the bottom map the obvious one. (At the second stage this becomes the statement that the cycle class map is a special case of the Chern character.)

2.2. **The Abel-Jacobi map.** We are now in a position to define the Abel-Jacobi map to Galois cohomology. Let F be a perfect field and let X be a smooth separated F-scheme of finite type and dimension n. Fix m, $0 \le m \le n$ and an integer N relatively prime to $m!$ and the characteristic of F. Let $\mathcal{F}$ denote the constant sheaf $\mathbf{Z}/N\mathbf{Z}$ on X.

We begin with the Chern character

$$E^{m,-m-1}_2(X) \to E^{m,m+1}_2\big(X, \mathcal{F}(m+1)\big). \tag{2.2.1}$$

Using the expression (2.1.2) we see that there is an edge map

$$E^{m,m+1}_2\big(X, \mathcal{F}(m+1)\big) \to H^{2m+1}\big(X, \mathcal{F}(m+1)\big). \tag{2.2.2}$$

Consider now the Leray spectral sequence for $u : X \to \operatorname{Spec} F$:

$$H^p\big(\operatorname{Spec} F, R^q u_* \mathcal{F}(m+1)\big) \Rightarrow H^{p+q}\big(X, \mathcal{F}(m+1)\big).$$

We define $H^{2m+1}(X, \mathcal{F}(m+1))_0$ to be the kernel of the edge map

$$H^{2m+1}\big(X, \mathcal{F}(m+1)\big) \to H^0\big(\operatorname{Spec} F, R^{2m+1}u_*\mathcal{F}(m+1)\big).$$

(Of course, $H^{2m+1}(X, \mathcal{F}(m+1))_0$ and $H^{2m+1}(X, \mathcal{F}(m+1))$ coincide whenever $H^{2m+1}(X_{F_s}, \mathcal{F}(m+1))$ has no G_F-invariants.) There is an edge map

$$H^{2m+1}\big(X, \mathcal{F}(m+1)\big)_0 \to H^1\big(F, R^{2m}u_*\mathcal{F}(m+1)\big).$$

Let $E^{m,m+1}_2(X)_{0,\mathcal{F}}$ be the inverse image of $H^{2m+1}(X, \mathcal{F}(m+1))_0$ under (2.2.2) and (2.2.1). The *Abel-Jacobi map* is the map

$$\sigma_m : E^{m,-m-1}_2(X)_{0,\mathcal{F}} \to H^1\big(\operatorname{Spec} F, R^{2m}u_*\mathcal{F}(m+1)\big).$$

We will usually identify the range with $H^1(F, H^{2m}(X_{F_s}, \mathcal{F}(m+1)))$.

The Abel-Jacobi map is functorial for change of base field; for flat morphisms of relative dimension 0; and (covariantly) for finite flat morphisms. All of these follow easily from what we have said so far and compatibilities of Leray spectral sequences with edge maps.

It is clear that the Abel-Jacobi map is compatible with change in N. In particular, we can pass to the limit to obtain an l-adic Abel-Jacobi map

$$\sigma_m : E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} \to H^1\big(F, H^{2m}(X_{F_s}, \mathbf{Z}_l(m+1))\big).$$

Note that by the Weil conjectures $H^{2m+1}(X_{F_s}, \mathbf{Z}_l(m+1))$ has no G_F-invariants (so that $E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} = E_2^{m,-m-1}(X)$) if it is torsion-free, F is a local field (and X has good reduction) or a global field and X is projective.

2.3. **The Leibniz relation.**

2.3.1. *Algebraic correspondences.* Let F be a perfect field and let X and Y be smooth proper varieties of dimension n over F. We will be forced to work with a fairly restrictive notion of algebraic correspondences.

Definition 2.3.1. An *irreducible correspondence* from X to Y is an integral closed subscheme $\alpha \hookrightarrow X \times_F Y$ such that the projections $\alpha \to X$ and $\alpha \to Y$ are finite and faithfully flat. A *correspondence* is a formal sum (with integer or $\mathbf{Z}_l$-coefficients as appropriate) of irreducible correspondences.

We use algebraic correspondences to define maps in K-theory and étale cohomology in the usual way: for any irreducible correspondence $\alpha \hookrightarrow X \times_F Y$, we define maps

$$\alpha_* : H^i(X, \mathbf{Z}_l) \to H^i(\alpha, \mathbf{Z}_l) \to H^i(Y, \mathbf{Z}_l)$$
$$\alpha^* : H^i(Y, \mathbf{Z}_l) \to H^i(\alpha, \mathbf{Z}_l) \to H^i(X, \mathbf{Z}_l)$$

as the composition of pullback and trace maps on étale cohomology. In the same way we obtain analogous maps of coniveau spectral sequences in étale cohomology and K-theory. We can also apply the construction to $\alpha_{F_s} \hookrightarrow X_{F_s} \times_{F_s} Y_{F_s}$; in this case $\alpha_{F_s *}$ and $\alpha_{F_s}^*$ commute with the action of G_F since α is defined over F. In particular, one obtains maps on the Galois cohomology of the étale cohomology groups of X_{F_s} and Y_{F_s}. All of these definitions extend immediately to general correspondences by linearity.

The functorialities of the Abel-Jacobi map show that it is compatible with algebraic correspondences in the sense that there is a commutative diagram

$$\begin{array}{ccc}
E_2^{m,-m-1}(X)_{0,\mathbf{Z}_l} & \xrightarrow{\alpha_*} & E_2^{m,-m-1}(Y)_{0,\mathbf{Z}_l} \\
\downarrow \sigma_m & & \downarrow \sigma_m \\
H^1\big(F, H^{2m}(X_{F_s}, \mathbf{Z}_l(m+1))\big) & \xrightarrow{\alpha_{F_s *}} & H^1\big(F, H^{2m}(Y_{F_s}, \mathbf{Z}_l(m+1))\big)
\end{array}$$

and similarly for α^*.

We say that a variety X is *cohomologically torsion-free at l* if all of the groups $H^i(X_{F_s}, \mathbf{Z}_l)$ are torsion-free. Let $\alpha \hookrightarrow X \times Y$ and $\beta \hookrightarrow X' \times Y'$ be irreducible correspondences and assume that X, X', Y, Y' are all cohomologically torsion-free at l. One then has Künneth projections which are compatible with algebraic correspondences:

$$\begin{array}{ccc}
H^{i+j}(X_{F_s} \times X'_{F_s}, \mathbf{Z}_l) & \longrightarrow & H^i(X_{F_s}, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} H^j(X'_{F_s}, \mathbf{Z}_l) \\
\downarrow (\alpha\times\beta)_{F_s *} & & \downarrow \alpha_{F_s *} \otimes \beta_{F_s *} \\
H^{i+j}(Y_{F_s} \times Y'_{F_s}, \mathbf{Z}_l) & \longrightarrow & H^i(Y_{F_s}, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} H^j(Y'_{F_s}, \mathbf{Z}_l)
\end{array}$$

and similarly for contravariant maps.

2.3.2. *Markings.* Assume that for every F-variety X of dimension n we are given a Zariski sheaf $\mathcal{L}_X$ which is invertible on the smooth locus of X; further assume that this assignment is functorial in the sense that for a map $f : X \to Y$ over $\operatorname{Spec} F$ there is an induced map $f^*\mathcal{L}_Y \to \mathcal{L}_X$. (For example, if $n = 1$ one can take $\mathcal{L}_X$ to be any fixed pluricanonical sheaf $\Omega^{\otimes w}_{X/F}$.) We define an *$\mathcal{L}$-marking* ω_X on an F-variety X to be a non-zero rational section of $\mathcal{L}_X$; a *marked variety* is a variety X together with an $\mathcal{L}$-marking ω_X.

Let X and Y be smooth proper marked varieties of dimension n over $\operatorname{Spec} F$. Let $\alpha \hookrightarrow X \times Y$ be an irreducible correspondence. We define a rational function f_α on α as the ratio of the pullbacks of ω_X and ω_Y to $\mathcal{L}_\alpha$. We may view the pair (α, f_α) as an element of the coniveau spectral sequence $E_1^{n,-n-1}(X \times Y)$; if $\alpha = \sum m_i\alpha_i$ by this we mean the element $\sum(\alpha_i, f_{\alpha_i}^{m_i})$. We say that α is *admissible* for the given markings if the Weil divisor of f_α is trivial on α; in this case (α, f_α) defines an element of $E_2^{n,-n-1}(X \times Y)$.

2.3.3. *Composition of algebraic correspondences.* Let X, Y, Z be smooth proper varieties of dimension n over F. Let $\alpha \hookrightarrow X \times Y$ and $\beta \hookrightarrow Y \times Z$ be irreducible correspondences. Under certain circumstances we can define a composition $\beta \circ \alpha$ as a correspondence from X to Z.

Let Γ denote the scheme-theoretic intersection of $\alpha \times Z$ and $X \times \beta$ in $X \times Y \times Z$. An easy argument with tangent spaces shows that each irreducible component Γ_i of Γ is generically reduced. Let $\gamma_i \hookrightarrow X \times Z$ be the scheme-theoretic image of Γ_i under the map $X \times Y \times Z \to X \times Z$. Checking on the level of geometric points and using [22], Proposition 4.4.2 one can show that each γ_i is finite and surjective over X and Z. If in addition each γ_i is flat over X and Z, we define $\beta \circ \alpha$ to be the correspondence $\sum[k(\Gamma_i) : k(\gamma_i)]\gamma_i$. We extend this definition to arbitrary correspondences by distributivity, at least when all subproducts are defined.

Lemma 2.3.2. *Suppose that X, Y, Z are marked varieties and α and β are admissible. If $\beta \circ \alpha$ is defined, then it is admissible as well.*

Proof. This is straightforward from the definitions; one does need to use the fact that if $\pi : X \to Y$ is a finite surjective morphism, then $\pi_*\pi^*$ is injective on cycles. □

2.3.4. *The Leibniz relation.* Let X, Y, Z be smooth proper marked varieties of dimension n over F and let $\alpha \hookrightarrow X \times Y$ and $\beta \hookrightarrow Y \times Z$ be admissible correspondences such that $\gamma = \beta \circ \alpha$ is defined (and thus, by Lemma 2.3.2, admissible). We assume further that X, Y, Z are cohomologically torsion-free at l. It was observed by Mazur and Beilinson in the case $n = 1$ that there is a remarkable relation between the Abel-Jacobi classes

$$\sigma_{X\times Y}(\alpha, f_\alpha) \in H^1\big(F, H^{2n}(X_{F_s} \times Y_{F_s}, \mathbf{Z}_l(n+1))\big);$$
$$\sigma_{Y\times Z}(\beta, f_\beta) \in H^1\big(F, H^{2n}(Y_{F_s} \times Z_{F_s}, \mathbf{Z}_l(n+1))\big);$$
$$\sigma_{X\times Z}(\gamma, f_\gamma) \in H^1\big(F, H^{2n}(X_{F_s} \times Z_{F_s}, \mathbf{Z}_l(n+1))\big).$$

We now prove their Leibniz relation for arbitrary n.

We first need some notation. Let $\Delta_Z \hookrightarrow Z \times Z$ be the diagonal viewed as an algebraic correspondence from Z to Z; both Δ_{Z*} and Δ_Z^* are the identity map on étale cohomology and K-theory. We view $\alpha \times \Delta_Z$ as a correspondence from $X \times Z$ to $Y \times Z$ and $\Delta_X \times \beta$ as a correspondence from $X \times Y$ to $X \times Z$.

Proposition 2.3.3. *With the above hypotheses and notation we have*

$$\sigma_{X\times Z}(\gamma, f_\gamma) = (\alpha\times\Delta_Z)^*\sigma_{Y\times Z}(\beta, f_\beta) + (\Delta_X\times\beta)_*\sigma_{X\times Y}(\alpha, f_\alpha).$$

Proof. By linearity we may assume that α and β are irreducible. Since the Abel-Jacobi map is compatible with algebraic correspondences it suffices to prove that

$$(\gamma, f_\gamma) = (\alpha\times\Delta_Z)^*(\beta, f_\beta) + (\Delta_X\times\beta)_*(\alpha, f_\alpha)$$

in $E_1^{n,-n-1}(X\times Z)$.

Consider first $(\alpha\times\Delta_Z)^*(\beta, f_\beta)$. The "cycle" part of this is obtained as follows: one pulls back and pushes forward $\beta \hookrightarrow Y\times Z$ in the diagram

$$\begin{array}{ccc} & \alpha\times\Delta_Z & \\ \swarrow & & \searrow \\ Y\times Z & & X\times Z \end{array}$$

Let β' be the image of β under the map $\mathrm{id}\times\Delta_Z : Y\times Z \to Y\times Z\times Z$. Pulling back β to $\alpha\times\Delta_Z$ is the same as forming the scheme-theoretic intersection

$$(X\times\beta')\cap(\alpha\times\Delta_Z) \hookrightarrow X\times Y\times Z\times Z. \tag{2.3.1}$$

The projection from here to $X\times Z$ factors through $X\times Y\times Z$; here the image of (2.3.1) is just the intersection of $X\times\beta$ and $\alpha\times Z$. In particular, the image of β in $X\times Z$ is nothing other than $\beta\circ\alpha = \gamma$.

Since f_β "is" ω_Y/ω_Z (we will systematically omit pullback maps from our notation for the remainder of this proof), tracing through the maps we see that the induced rational function on an irreducible component γ_i of γ is

$$\frac{N_{k(\Gamma_i)/k(\gamma_i)}\omega_Y}{\omega_Z^{m_i}}$$

where Γ_i is the irreducible component of $\Gamma = (\alpha\times Z)\cap(X\times\beta)$ surjecting onto γ_i and $m_i = [k(\Gamma_i) : k(\gamma_i)]$. That is, writing $\gamma = \sum m_i\gamma_i$ as a sum of irreducible correspondences, we have

$$(\alpha\times\Delta_Z)^*(\beta, f_\beta) = \sum\left(\gamma_i, \frac{N_{k(\Gamma_i)/k(\gamma_i)}\omega_Y}{\omega_Z^{m_i}}\right). \tag{2.3.2}$$

Similarly, we have

$$(\Delta_X\times\beta)_*(\alpha, f_\alpha) = \sum\left(\gamma_i, \frac{\omega_X^{m_i}}{N_{k(\Gamma_i)/k(\gamma_i)}\omega_Y}\right). \tag{2.3.3}$$

Adding (2.3.2) and (2.3.3) in $E_1^{n,-n-1}(X\times Z)$ yields precisely (γ, f_γ), as claimed. □

2.4. **Derivations to Galois cohomology.**

2.4.1. *Algebras of correspondences.* Let X be a smooth proper marked variety of dimension n over F; we assume that X is cohomologically torsion-free at l and that $E_2^{n,-n-1}(X)_{0,\mathbf{Z}_l} = E_2^{n,-n-1}(X)$. (This second condition is redundant if F is a global field.) By an *algebra of correspondences* $\mathcal{A}_0$ on X we will mean a set of correspondences from X to X which is closed under addition and composition (in particular, we assume that every product is defined) and contains Δ_X; $\mathcal{A}_0$ is naturally a ring with identity Δ_X. We say that $\mathcal{A}_0$ is *admissible* if every element of $\mathcal{A}_0$ is admissible (with respect to the fixed marking on X).

Set $V = H^n(X_{F_s}, \mathbf{Z}_l)$ and $\mathcal{A} = \mathcal{A}_0 \otimes_{\mathbf{Z}} \mathbf{Z}_l$; $\mathcal{A}$ admits two maps to $\mathrm{End}_{\mathbf{Z}_l} V$, one given by $\alpha \mapsto \alpha_*$ and the other by $\alpha \mapsto \alpha^*$. We let B_* and B^* denote the images of these maps.

Assuming that $\mathcal{A}_0$ is admissible, we have a map

$$\sigma : \mathcal{A} \to E_2^{n,-n-1}(X \times X) \otimes_{\mathbf{Z}} \mathbf{Z}_l \to H^1\big(F, V \otimes_{\mathbf{Z}_l} V(n+1)\big)$$

given as the composition of $\alpha \mapsto (\alpha, f_\alpha)$, the Abel-Jacobi map and the Künneth projection. By Proposition 2.3.3 we have

$$\sigma(\beta\alpha) = (\alpha^* \otimes 1)\sigma(\beta) + (1 \otimes \beta_*)\sigma(\alpha).$$

2.4.2. *Bilateral derivations.* We assume now that $\mathcal{A}$ is commutative. We can use the Poincaré pairing $\varphi : V \otimes_{\mathbf{Z}_l} V(n) \to \mathbf{Z}_l$ to identify $V(n)$ with $V^\dagger = \mathrm{Hom}_{\mathbf{Z}_l}(V, \mathbf{Z}_l)$ by $v \mapsto \varphi(\cdot, v)$. Identifying $V \otimes_{\mathbf{Z}_l} V^\dagger$ with $\mathrm{End}_{\mathbf{Z}_l} V$, we can view σ as a map

$$\sigma' : \mathcal{A} \to H^1\big(F, \mathrm{End}_{\mathbf{Z}_l} V(1)\big).$$

However, the Poincaré pairing satisfies $\varphi(v, \alpha_* v') = \varphi(\alpha^* v, v')$ and $\varphi(v, \alpha^* v') = \varphi(\alpha_* v, v')$. Thus the isomorphism $V(n) \cong V^\dagger$ interchanges the actions of B_* and B^*; in other words, σ' satisfies

$$\sigma'(\beta\alpha) = (\alpha^* \otimes 1)\sigma'(\beta) + (1 \otimes \beta^*)\sigma'(\alpha)$$

where we view $\mathrm{End}_{\mathbf{Z}_l} V$ as a $B^* \otimes_{\mathbf{Z}_l} B^*$-module via $(\alpha^* \otimes \beta^*) f(v) = \alpha^* f(\beta^* v)$. Thus σ' is an $\mathcal{A}$-bilateral derivation when V is viewed as $\mathcal{A}$-module via $\alpha \mapsto \alpha^*$.

2.4.3. *Derivations.* Choose now a maximal ideal $\mathfrak{m}$ of B^* and set $A = B^*_{\mathfrak{m}}$, $H = V \otimes_{B^*} A$. We have natural maps $i : H \hookrightarrow V$ and $j : V \twoheadrightarrow H$.

We say that $\mathfrak{m}$ is *dualizing* if A is reduced and Gorenstein and H is free of rank 2 over A. (This last hypotheses will not be relevant until later sections.) Given such an $\mathfrak{m}$, fix a Gorenstein trace $\mathrm{tr} : A \to \mathbf{Z}_l$ with congruence element $\eta \in A$. Define an $\mathcal{A}$-bilateral derivation

$$\mathcal{D} : \mathcal{A} \to H^1\big(F, \mathrm{End}_{\mathbf{Z}_l} H(1)\big)$$

to be the composition of σ' with the map on cohomology induced by the map $\mathrm{End}_{\mathbf{Z}_l} V \to \mathrm{End}_{\mathbf{Z}_l} H$ given by $f \mapsto jfi$. Define an $\mathcal{A}$-derivation

$$\partial : \mathcal{A} \to H^1\big(F, \mathrm{End}_A H(1)\big)$$

as the composition of $\mathcal{D}$ with the map $h_{\mathrm{tr}} : \mathrm{End}_{\mathbf{Z}_l} H \to \mathrm{End}_A H$ of Appendix A.1.2.

Let I denote the kernel of $\mathcal{A} \to A$. By Lemma A.2.1 the maps above restrict to $\mathcal{A}$-module homomorphisms

$$\tilde{\mathcal{D}} : I/I^2 \to H^1\big(F, \mathrm{End}_{\mathbf{Z}_l} H(1)\big)_\delta$$

$$\tilde{\partial} : I/I^2 \to H^1\big(F, \mathrm{End}_A H(1)\big).$$

By Lemma A.2.3 our choice of Gorenstein trace tr yields an A-linear Galois equivariant isomorphism $(\mathrm{End}_{\mathbf{Z}_l} H)_\delta \cong \mathrm{End}_A H$. If we assume that the hypotheses of Lemma A.2.2 are satisfied (as is the case if $H \otimes_A k$ is absolutely irreducible), we can use this to view $\tilde{\mathcal{D}}$ as a map

$$\tilde{\mathcal{D}} : I/I^2 \to H^1\big(F, \mathrm{End}_A H(1)\big)$$

which by Lemma A.2.3 satisfies $\eta\tilde{\mathcal{D}} = \tilde{\partial}$. That is, there is a commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{\tilde{\mathcal{D}}} & H^1\big(F, \mathrm{End}_A H(1)\big) \\ \big\downarrow & & \big\downarrow \eta \\ \mathcal{A} & \xrightarrow{\partial} & H^1\big(F, \mathrm{End}_A H(1)\big) \end{array}$$

This induces a derivation on cokernels

$$\Theta : A \to H^1\big(F, \mathrm{End}_A(H/\eta H)(1)\big).$$

In Section 3.2 we will use this derivation to construct cohesive Flach systems.

It is worth mentioning an argument which is often implicit in the proof that a maximal ideal $\mathfrak{m}$ is dualizing. For this we ignore Galois actions; in particular, we fix an isomorphism $\mathbf{Z}_l \cong \mathbf{Z}_l(1)$ so that we can regard φ as a pairing $\varphi : V \otimes_{\mathbf{Z}_l} V \to \mathbf{Z}_l$. Suppose that we are given a $\mathbf{Z}_l$-linear automorphism w of V such that $w(\alpha_* v) = \alpha^* w(v)$ and $w(\alpha^* v) = \alpha_* w(v)$ for all $\alpha \in \mathcal{A}$ and $v \in V$. Suppose also that we know that $H = V_{\mathfrak{m}}$ is free of rank 2 over $A = B^*_{\mathfrak{m}}$. The pairing $\varphi = \varphi \circ (\mathrm{id} \otimes w)$ on V is now a perfect B^*-pairing. Localizing yields a perfect A-pairing $H \otimes_{\mathbf{Z}_l} H \to \mathbf{Z}_l$. By Section A.1.2 it follows that A is Gorenstein. In the case of modular forms the automorphism w is given by an Atkin-Lehner involution w^*_ζ. In fact, one can keep track of the Galois twist given by w^*_ζ; in this way one can rewrite the above construction to take values in a twist of the symmetric square of H. This point of view is closer to that of Flach and Mazur.

3. Construction of Euler systems

3.1. A reciprocity law for the Abel-Jacobi map. The Abel-Jacobi map produces Galois cohomology classes from appropriate pairs of algebraic cycles and rational functions; in order to show that such classes form a partial Euler system we will need some control over the local behavior of these classes. This is accomplished in the next result which relates this local behavior to divisor maps in positive characteristic; it is an explicit reciprocity law in the sense of Kato and is formally quite similar to [40], Corollary 3.2.4.

Let R be a complete local ring with perfect residue field k of characteristic $p \geq 0$ and perfect fraction field K. (As before our arguments go through for any R satisfying appropriate purity hypotheses.) Let $\mathfrak{X}$ be a smooth proper scheme over $\operatorname{Spec} R$ of relative dimension n; let X denote the generic fiber of $\mathfrak{X}$. Fix an integer m and let $\mathcal{F}$ denote either the constant sheaf $\mathbf{Z}/N\mathbf{Z}$ or $\mathbf{Z}_l$ (for N, l relatively prime to $m!p$). Let V denote the G_K-module $H^{2m}(X_{K_s}, \mathcal{F}(m+1))$; it is unramified by smooth base change and we give it the unramified finite/singular structure. We have an Abel-Jacobi map

$$\sigma_m : E_2^{m,-m-1}(X)_{0,\mathcal{F}} \to H^1(K, V).$$

For $Z \subseteq X$ let $\bar{Z}$ denote the closure of Z in $\mathfrak{X}$. Let

$$\mathrm{div}_k : E_2^{m,-m-1}(X) \to \mathrm{CH}^m \mathfrak{X}_k$$

be the map sending a pair (Z, f) to the divisor of f on $\bar{Z}$; it is supported on $\bar{Z}_k$. (That div_k is well-defined will become apparent in the proof below. Alternately, div_k can be viewed as a localization map in motivic cohomology.)

Theorem 3.1.1. *With the notation above there is a commutative diagram*

$$\begin{array}{ccc} E_2^{m,-m-1}(X)_{0,\mathcal{F}} & \xrightarrow{\mathrm{div}_k} & \mathrm{CH}^m \mathfrak{X}_k \\ \downarrow{\scriptstyle \sigma_m} & & \\ H^1(K,V) & & \downarrow{\scriptstyle s} \\ \downarrow & & \\ H^1_s(K,V) & \xrightarrow{\simeq} & H^{2m}\big(\mathfrak{X}_{k_s}, \mathcal{F}(m)\big)^{G_k} \end{array}$$

Here s is the cycle class map in étale cohomology, and the bottom isomorphism is that of (1.1.1) and smooth base change.

Somewhat more "motivically", one can write this diagram over $\mathbf{Q}$ as

$$\begin{array}{ccc} H^{2m+1}_{\mathcal{M}}\big(X, \mathbf{Q}(m+1)\big) & \longrightarrow & H^{2m}_{\mathcal{M}}\big(\mathfrak{X}_k, \mathbf{Q}(m)\big) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{2m+1}_{\text{ét}}\big(X, \mathbf{Q}_l(m+1)\big) & & H^{2m}_{\text{ét}}\big(\mathfrak{X}_{k_s}, \mathbf{Q}_l(m)\big)^{G_k} \\ \downarrow & & \downarrow{\scriptstyle \simeq} \\ H^1\big(K, H^{2m}_{\text{ét}}(X_{K_s}, \mathbf{Q}_l(m+1))\big) & \longrightarrow & H^1_s\big(K, H^{2m}_{\text{ét}}(X_{K_s}, \mathbf{Q}_l(m+1))\big) \end{array}$$

We use the formulation in terms of coniveau spectral sequences so that we can work integrally.

Proof. Assume first that $\mathcal{F} = \mathbf{Z}/N\mathbf{Z}$. We show that the map div_k relates the Abel-Jacobi map for X to a relative Abel-Jacobi map for the pair $\mathfrak{X}, \mathfrak{X}_k$. This relative Abel-Jacobi map will be identified with the cycle class map via purity.

We claim that there is a commutative diagram

(3.1.1)
$$\begin{array}{ccc} E_2^{m,-m-1}(X)_{0,\mathcal{F}} & \xrightarrow{\delta_1} & E_{2,\mathfrak{X}_k}^{m+1,-m-1}(\mathfrak{X}) \\ \downarrow & & \downarrow{\scriptstyle a_1} \\ E_2^{m,m+1}\big(X, \mathcal{F}(m+1)\big)_0 & \xrightarrow{\delta_2} & E_{2,\mathfrak{X}_k}^{m+1,m+1}\big(\mathfrak{X}, \mathcal{F}(m+1)\big) \\ \downarrow & & \downarrow{\scriptstyle a_2} \\ H^{2m+1}\big(X, \mathcal{F}(m+1)\big)_0 & \xrightarrow{\delta_3} & H^{2m+2}_{\mathfrak{X}_k}\big(\mathfrak{X}, \mathcal{F}(m+1)\big) \\ \downarrow & & \downarrow{\scriptstyle a_3} \\ H^1\big(\mathrm{Spec}\, K, R^{2m} u_{K*}\mathcal{F}(m+1)\big) & \xrightarrow{\delta_4} & H^2_{\mathrm{Spec}\, k}\big(\mathrm{Spec}\, R, R^{2m} u_* \mathcal{F}(m+1)\big) \end{array}$$

where $u : \mathfrak{X} \to \mathrm{Spec}\, R$, $u_K : X \to \mathrm{Spec}\, K$ are the structure maps. Here the left-hand vertical maps are those in the definition of the Abel-Jacobi map.

The map a_1 is a relative Chern character and δ_1 and δ_2 are the boundary maps of the localization sequences for the coniveau spectral sequences as in Section 2.1.1. We already commented on the commutativity of this square in Section 2.1.4. The map a_2 is an edge map which exists by purity for closed subschemes of $\mathfrak{X}_k$ in $\mathfrak{X}$.

The map δ_3 is the boundary map in étale cohomology for the pair $\mathfrak{X}, \mathfrak{X}_k$. This square commutes by Section 2.1.1.

The map a_3 is an edge map of a Leray spectral sequence (with supports); its existence uses purity for the pair $\operatorname{Spec} R, \operatorname{Spec} k$. The map δ_4 is a boundary map in étale cohomology combined with a base change map. The commutativity of this square is a fairly straightforward and general fact of homological algebra; see [44], Proposition A.7.1. This completes the proof of the commutativity of (3.1.1).

We claim that the right-hand side of (3.1.1) identifies via purity with the sequence

$$\text{(3.1.2)}\quad \mathrm{CH}^m \mathfrak{X}_k = E_2^{m,-m}(\mathfrak{X}_k) \xrightarrow{a_1'} E_2^{m,m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \xrightarrow{a_2'} H^{2m}\big(\mathfrak{X}_k, \mathcal{F}(m)\big) \xrightarrow{a_3'} H^0\big(\operatorname{Spec} k, R^{2m} u_{k*}\mathcal{F}(m)\big).$$

Here $u_k : \mathfrak{X}_k \to \operatorname{Spec} k$ is the structure map, a_1' is a Chern character and the next two are the obvious edge maps.

To see this, note first that from the construction of the coniveau spectral sequence in K-theory as in [34], Section 7 there is a canonical isomorphism of spectral sequences $E_r^{p,q}(\mathfrak{X}_k) \xrightarrow{\simeq} E_{r,\mathfrak{X}_k}^{p+1,q-1}(\mathfrak{X})$ which for $r = 1$ coincides with the obvious identification of the corresponding expressions (2.1.3). There is an analogous isomorphism of spectral sequences $E_r^{p,q}(\mathfrak{X}_k, \mathcal{F}(m)) \xrightarrow{\simeq} E_{r,\mathfrak{X}_k}^{p+1,q+1}(\mathfrak{X}, \mathcal{F}(m+1))$ coming from purity and the construction of the coniveau spectral sequence; for $r = 1$ it coincides with the obvious identifications of (2.1.2). The identification of a_1 with a_1' now follows from these identifications and the description of the Chern character in (2.1.4). The identifications of a_2 with a_2' and a_3 with a_3' are straightforward from the definition of the purity isomorphism (see for example [19], pp. 205–207) and easy compatibilities of edge maps. That the composition (3.1.2) is the cycle map follows easily from (2.1.4). (All of this is really just an integral version of standard computations of motivic cohomology.)

It remains to identify the maps

$$\text{(3.1.3)}\qquad E_2^{m,-m-1}(X) \xrightarrow{\delta_1} E_{2,\mathfrak{X}_k}^{m+1,-m-1}(\mathfrak{X}) \xleftarrow{\simeq} E_2^{m,-m}(\mathfrak{X}_k) = \mathrm{CH}^m \mathfrak{X}_k$$

$$\text{(3.1.4)}\quad H^1(K,V) = H^1\big(\operatorname{Spec} K, R^{2m} u_{K*}\mathcal{F}(m+1)\big) \xrightarrow{\delta_4} H^2_{\operatorname{Spec} k}\big(\operatorname{Spec} R, R^{2m} u_*\mathcal{F}(m+1)\big) \xleftarrow{\simeq} H^0\big(\operatorname{Spec} k, R^{2m} u_{k*}\mathcal{F}(m)\big) \cong H^1_s(K,V)$$

with div_k and the singular restriction map, respectively. For (3.1.3) this follows from the definition of δ_1 as a boundary map of the localization sequence and the description of the spectral sequence differential as the divisor map. The desired description of (3.1.4) is straightforward from the definition of purity via cup product with the fundamental class; we omit the details.

For the case of $\mathcal{F} = \mathbf{Z}_l$ it suffices to observe that everything above is obviously compatible with change in N. □

3.2. **Construction of cohesive Flach systems.**

3.2.1. *Divisorial liftings.* Let F be a number field and let S be an open subscheme of the spectrum of the ring of integers of F. Let $\mathfrak{X}$ be a smooth proper scheme of relative dimension n over S with generic fiber X. Fix a closed point $v \in S$ and let Z be a codimension m cycle of the closed fiber $\mathfrak{X}_{k_v}$. We say that a finite set $\{(Z_i, f_i)\}$ of pairs of codimension m cycles Z_i on X and rational functions f_i on Z_i

is a *divisorial lifting* of Z if $\sum \operatorname{div}_{\bar{Z}_i} f_i = Z$; here $\bar{Z}_i$ is the closure of Z_i in $\mathfrak{X}$ and Z is considered as a vertical cycle of codimension $m+1$ on $\mathfrak{X}$.

Note that a divisorial lifting of Z yields an element $\sum(Z_i, f_i)$ of $E_2^{m,-m-1}(X)$ such that

$$\operatorname{div}_{k_w}\left(\sum(Z_i, f_i)\right) = \begin{cases} Z & w = v; \\ 0 & w \in S - \{v\}. \end{cases}$$

By Theorem 3.1.1 such an element will yield a Galois cohomology class which is unramified at $w \in S - \{v\}$ and locally at v looks like the cycle class of Z.

3.2.2. *Partial Euler systems.* Let F, S and $\mathfrak{X}$ be as above. Fix an integer m and a prime $l > m$ such that $\Sigma_l \subseteq S$; set $V = H^{2m}(X_{F_s}, \mathbf{Z}_l(m+1))$. We assume that V is torsion-free.

Fix a $\mathbf{Z}_l$-algebra A of scalars and let T be a torsion-free l-adic G_F-module admitting a map $V \to T$ with finite cokernel. (V itself need not have any structure of A-module.) We will consider the Abel-Jacobi map

$$\sigma : E_2^{m,-m-1}(X) \to H^1(F, V) \to H^1(F, T).$$

V is unramified at all places of $S - \Sigma_l$ by smooth base change; thus T is unramified at all such places as well. We let $\mathcal{S}$ denote the finite/singular structure on T with

$$H^1_{f,\mathcal{S}}(F_v, T) = \begin{cases} H^1_{\mathrm{ur}}(F_v, T) & v \in S - \Sigma_l; \\ H^1_{f,\mathrm{cris}}(F_v, T) & v \in \Sigma_l; \\ H^1(F_v, T) & v \notin S. \end{cases}$$

Definition 3.2.1. Let v be a closed point of $S - \Sigma_l$ and let η be an element of A. We say that a collection of codimension m cycles $Z_1, \ldots, Z_r$ on $\mathfrak{X}_{k_v}$ *generate* T with depth η if the A-submodule of $T(-1)^{G_{k_v}}$ generated by the cycle classes of the Z_i contains $\eta T(-1)^{G_{k_v}}$.

Lemma 3.2.2. *Let $Z_1, \ldots, Z_r$ be cycles on $\mathfrak{X}_{k_v}$ which generate T with depth η. Assume that each of the Z_i admits a divisorial lifting to X. Then there is an A-submodule C of $H^1(F,T)$ such that $C_{w,s} = 0$ for $w \in S - \{v\}$ and such that $C_{v,s}$ has depth η in $H^1_s(F_v, T)$.*

Proof. Let $\{(Z_{ij}, f_{ij})\}$ be a divisorial lifting of Z_i and define C to be the A-submodule of $H^1(F,T)$ generated by the $\sigma(\sum(Z_{ij}, f_{ij}))$. That C has the desired properties at $w \in S - \Sigma_l$ follows immediately from the definition of a divisorial lifting and Theorem 3.1.1. The fact that C has trivial singular restriction at $w \in \Sigma_l$ follows from [33], Theorem 3.1. □

Proposition 3.2.3. *Let $\mathcal{L}$ be a set of closed points of $S - \Sigma_l$. Assume that for each $v \in \mathcal{L}$ there is a set of codimension m cycles on $\mathfrak{X}_{k_v}$ which generate T with depth η and which admit divisorial liftings. Then there is a partial Euler system $\{C^v\}_{v \in \mathcal{L}}$ of depth η for $(T, \mathcal{S})$.*

Proof. This is immediate from Lemma 3.2.2. □

Of course, if one can choose $\mathcal{L}$ appropriately above, then Proposition 1.3.4 yields an annihilator of the Selmer group of T.

In the special case of a product variety $X \times X$ one can often use powers of the graph of Frobenius as generators. This is essentially the point of view we will take in our construction of cohesive Flach systems.

3.2.3. *Construction of cohesive Flach systems: preparation.* Assume now that F is as in Section 1.4 and let S and $\mathfrak{X}$ be as above. Fix a prime $l > n$ with $\Sigma_l \subset S$ and such that X is cohomologically torsion-free at l. Set $V = H^n(X_{F_s}, \mathbf{Z}_l)$.

Let $\mathcal{A} = \mathcal{A}_0 \otimes \mathbf{Z}_l$ be a commutative l-adic algebra of correspondences on X. Let B_* and B^* denote the images of $\mathcal{A} \to \mathrm{End}_{\mathbf{Z}_l} V$. Assume also that we have a dualizing maximal ideal $\mathfrak{m}$ of B^*; set $A = B^*_{\mathfrak{m}}$ and $H = V \otimes_{B^*} A$. We assume that H is minimally ramified in the sense of Section 1.4.1. (Note that H is automatically unramified away from the set Σ consisting of Σ_l and the places not lying in S. H is also crystalline at every place v of Σ_l by [10].) A is Gorenstein by assumption; fix a Gorenstein trace tr with congruence element η.

Let T denote $\mathrm{End}^0_A H(1)$ endowed with the finite/singular structure as in Section 1.4.2. Consider the Galois equivariant map

$$V \otimes_{\mathbf{Z}_l} V(n) \cong V \otimes_{\mathbf{Z}_l} V^{\dagger} \cong \mathrm{End}_{\mathbf{Z}_l} V \to \mathrm{End}_{\mathbf{Z}_l} H \to \mathrm{End}_A H \twoheadrightarrow \mathrm{End}^0_A H$$

as in Section 2.4.3. Since by the Künneth theorem $V \otimes_{\mathbf{Z}_l} V(n)$ is a quotient of $H^{2n}(X_{F_s} \times X_{F_s}, \mathbf{Z}_l(n))$, we see that in this way we obtain a map

$$H^{2n}\big(X_{F_s} \times X_{F_s}, \mathbf{Z}_l(n+1)\big) \to T. \tag{3.2.1}$$

3.2.4. *Construction of cohesive Flach systems: generators.* For a place v of $S - \Sigma_l$, by a *diamond operator* for v we mean an automorphism $\langle v \rangle$ of X such that $j\, \langle v \rangle^*\, i = \chi(v)\varepsilon(v)^n$ as an automorphism of H; here χ is the determinant character of H, ε is the cyclotomic character and i, j are as in Section 2.4.3. Assuming that such diamond operators exist, we let Γ_v be the graph of the Frobenius morphism $\mathrm{Fr}(v)$ on $\mathfrak{X}_{k_v}$ and let Γ'_v be the image of $\mathrm{Fr}(v) \times \langle v \rangle : \mathfrak{X}_{k_v} \to \mathfrak{X}_{k_v} \times \mathfrak{X}_{k_v}$. Let $\mathcal{L}$ denote the set of places of F with Frobenius conjugate to complex conjugation on $H \otimes_A k$.

Lemma 3.2.4. *Fix a place $v \in \mathcal{L}$ and integers a, b such that l does not divide $a - b$. Then $a\Gamma_v + b\Gamma'_v$ generates T with depth η via (3.2.1).*

Proof. By Lemma 1.4.3 we know that $T(-1)^{G_{k_v}} \cong (\mathrm{End}^0_A H)^{G_{k_v}}$ is a free A-module of rank 1 generated by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (for an appropriate choice of basis of H). We must compute the image of the cycle class of $a\Gamma_v + b\Gamma'_v$ in $T(-1)$. The image of Γ_v in $\mathrm{End}_{\mathbf{Z}_l} V$ is precisely the geometric Frobenius endomorphism of V by [18], pp. 155–156 and Chapter II, Section 4. This maps to the Frobenius endomorphism $\mathrm{Fr}(v)$ of H in $\mathrm{End}_{\mathbf{Z}_l} H$. Since Frobenius is A-linear, by Section A.1.2 this maps to η times the Frobenius morphism in $\mathrm{End}_A H$.

By [18], pp. 155–156, the image of Γ'_v in $\mathrm{End}_{\mathbf{Z}_l} V$ is $\langle v \rangle^* \, \mathrm{Fr}(v)^{\mathrm{adj}}$ where $\mathrm{Fr}(v)^{\mathrm{adj}}$ is the Poincaré adjoint of $\mathrm{Fr}(v)$, characterized by $\varphi(\mathrm{Fr}(v)^{\mathrm{adj}} x, y) = \varphi(x, \mathrm{Fr}(v) y)$. Since φ is Galois equivariant, one computes that $\mathrm{Fr}(v)^{\mathrm{adj}}$ is just $\varepsilon(v)^{-n} \, \mathrm{Fr}(v)^{-1}$. Thus $\langle v \rangle^* \, \mathrm{Fr}(v)^{\mathrm{adj}} = \chi(v) \, \mathrm{Fr}(v)^{-1}$. We therefore conclude as above that Γ'_v maps to $\eta \chi(v) \, \mathrm{Fr}(v)^{-1}$ in $\mathrm{End}_A H$.

Choose a basis of H with respect to which $\mathrm{Fr}(v)$ is given by a matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$. Since H has determinant χ, $\chi(v) \, \mathrm{Fr}(v)^{-1}$ is given by $\left(\begin{smallmatrix} \beta & 0 \\ 0 & \alpha \end{smallmatrix}\right)$. It follows that $a\Gamma_v + b\Gamma'_v$ maps to

$$\tfrac{1}{2}(a-b)(\alpha-\beta)\eta \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

in $\mathrm{End}^0_A H$. The lemma follows. □

3.2.5. *Construction of cohesive Flach systems: conclusion.* We are now in a position to prove our main result on the construction of cohesive Flach systems. Recall that if ω is a marking on X and $\alpha \in \mathcal{A}_0$, we write f_α for the induced rational function on α. We continue to assume that H is minimally ramified.

Proposition 3.2.5. *Let ω be an admissible marking for the algebra of correspondences $\mathcal{A}$ on X. Assume that for all $v \in S - \Sigma_l$ there is a correspondence $\mathcal{T}_v \in \mathcal{A}_0$ and integers a_v, b_v such that:*

- *$(\mathcal{T}_v, f_{\mathcal{T}_v})$ is a divisorial lifting of $a_v\Gamma_v + b_v\Gamma'_v$;*
- *l does not divide $a_v - b_v$;*
- *$\mathcal{T}_v^*$ agrees with the Hecke operator $T_v = \operatorname{trace}\operatorname{Fr}(v)$ on H;*

and that for $v \notin S$ there is a correspondence $\mathcal{T}_v \in \mathcal{A}_0$ with $\mathcal{T}_v^* = T_v$. Assume further that:

- *$T \otimes_A k$ is absolutely irreducible;*
- *$H^1(F(T^*[\mathfrak{a}])/F, T^*[\mathfrak{a}]) = 0$ for every ideal $\mathfrak{a}$ of finite index in A;*
- *$H^1_s(F_v, T) = 0$ for every $v \notin S$.*

Then T admits a cohesive Flach system of depth η. If the differences $a_v - b_v$ are a constant c independent of v, then this cohesive Flach system is of Eichler-Shimura type of weight c.

Proof. The classes c^v are defined to be $\sigma(\mathcal{T}_v, f_{\mathcal{T}_v})$, with σ the map

$$\mathcal{A} \to E_2^{n,-n-1}(X\times X)\otimes_{\mathbf{Z}}\mathbf{Z}_l \to H^1\big(F, H^{2n}(X_{F_s}\times X_{F_s}, \mathbf{Z}_l(n+1))\big) \to H^1(F,T).$$

That $\{c^v\}_{v\in\mathcal{L}}$ is a Flach system of depth η and that each c^v vanishes in $H^1_s(F_w, T)$ for $w \neq v$ is immediate from the assumptions and Lemmas 3.2.2 and 3.2.4. That each c^v vanishes in $H^1_s(F_v, T/\eta T)$ is proved in the same way as Lemma 3.2.4. The fact that the map $\Theta : A \to H^1(F, T/\eta T)$ sending $\mathcal{T}_v$ to c^v is a derivation was proved in Section 2.4.3. The last statement is clear from the proof of Lemma 3.2.4. □

4. Applications to modular forms

4.1. The modular curve $X_1(N)$.

4.1.1. *Definitions.* It is a fairly straightforward exercise to apply Proposition 3.2.5 in the case of classical modular forms of squarefree level. (The squarefree assumption simplifies a few points; it is probably not essential.) Fix a squarefree integer $N \geq 5$ and let $\mathfrak{X}_1(N) \to \operatorname{Spec}\mathbf{Z}[\frac{1}{N}]$ be the modular curve of level N as in [7]; it is a fine moduli scheme for generalized elliptic curves with sections of exact order N on fibers (which are required to meet all irreducible components of fibers which are Néron polygons). $\mathfrak{X}_1(N) \to \operatorname{Spec}\mathbf{Z}[\frac{1}{N}]$ is proper, smooth, geometrically connected and of relative dimension 1. The maps $\langle d\rangle : (E, P) \mapsto (E, dP)$ on moduli realize $(\mathbf{Z}/N\mathbf{Z})^\times$ as a group of automorphisms of $\mathfrak{X}_1(N)$.

For any prime number p we let $\mathfrak{X}_1(N;p)$ be the modular curve classifying triples (E, P, C) of elliptic curves, points of exact order N and cyclic subgroups of order p; one requires that C contains no non-trivial multiple of P in the case that p divides N. $\mathfrak{X}_1(N;p)$ is a proper $\mathbf{Z}[\frac{1}{N}]$-scheme of relative dimension 1 and is smooth over $\mathbf{Z}[\frac{1}{Np}]$. It admits two natural degeneracy maps $j_p, j'_p : \mathfrak{X}_1(N;p) \rightrightarrows \mathfrak{X}_1(N)$; on moduli the first sends a triple (E, P, C) to (E, P) and the second sends it to $(E/C, P)$.

4.1.2. *Hecke correspondences.* We define the p^{th} *Hecke correspondence* $\mathfrak{T}_p$ to be the scheme-theoretic image of the map

$$j_p \times j'_p : \mathfrak{X}_1(N;p) \to \mathfrak{X}_1(N) \times_{\operatorname{Spec} \mathbf{Z}[\frac{1}{N}]} \mathfrak{X}_1(N).$$

For p not dividing N the fiber $\mathfrak{T}_{p,\mathbf{F}_p}$ is reduced and has two irreducible components: the graph of Frobenius (which we write as Γ_p) and the image of $\mathrm{Fr} \times \langle p \rangle$ (which we write as Γ'_p).

The modular curve $X_1(N)$ is the generic fiber of $\mathfrak{X}_1(N)$; it is a smooth projective curve over $\mathbf{Q}$. Write $\mathcal{T}_p$ for the generic fiber of the Hecke correspondence $\mathfrak{T}_p$. One can more generally define Hecke correspondences $\mathcal{T}_n$ for any n; see [36], Chapter 2, Sections 1–3, for example. Each $\mathcal{T}_n$ is a divisor on $X_1(N) \times X_1(N)$, and thus is Cohen-Macaulay. The projections $\mathcal{T}_n \rightrightarrows X_1(N)$ are visibly quasi-finite, proper and surjective, so that by [23], Proposition 15.4.2 and [22], Proposition 4.4.2 they are finite and faithfully flat. In particular the $\mathcal{T}_n$ are correspondences in our sense. Regarding the diamond operators as correspondences via their graphs, it follows from the relations [26], Chapter 7, Theorem 2.1 (which already hold on the level of moduli) that the Hecke correspondences $\mathcal{T}_p$ and the diamond operators generate a commutative algebra of correspondences $\mathbf{T}_1(N)$.

4.1.3. *Galois representations.* We fix a prime $l \geq 5$ not dividing N and let $V = H^1(X_1(N)_{\mathbf{Q}^{\mathrm{ac}}}, \mathbf{Z}_l)$. Let B_* and B^* denote the images of $\mathbf{T}_1(N)$ in $\operatorname{End}_{\mathbf{Z}_l} V$ as usual. By [6], Lemma 4.1 we can omit $\mathcal{T}_l$ from the generating set of B_* and B^*.

Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of B^* associated to a newform of level N; see [6], Chapter 4 and [42], Theorem 3.4 for definitions. Assume also that $\mathfrak{m}$ does not contain l. Then by the above references $\mathfrak{m}$ is dualizing and $H \otimes_A k$ is absolutely irreducible. In particular $A = B^*_{\mathfrak{m}}$ is Gorenstein. Fix a Gorenstein trace $\mathrm{tr} : A \to \mathbf{Z}_l$ with congruence element η. Set $H = V \otimes_{B^*} A$ and $T = \operatorname{End}^0_A H(1)$.

4.1.4. *The modular unit* Δ. For any curve X over $\mathbf{Q}$ we set $\mathcal{L}_X = \Omega^{\otimes 6}_{X/\mathbf{Q}}$. Following Flach and Mazur, we use the modular form Δ as a marking on the curve $X_1(N)$. In particular, Δ is a global section of $\mathcal{L}_{X_1(N)}$.

Lemma 4.1.1. $\mathbf{T}_1(N)$ *is an admissible algebra of correspondences for* $X_1(N)$ *with the* $\mathcal{L}$*-marking* Δ*.*

Proof. It is clear that the diamond operators are admissible for Δ, so that by Lemma 2.3.2 we need only check that the divisor of $f_p = j_p^*\Delta / j_p^{*\prime}\Delta$ vanishes on $\mathcal{T}_p$ for each p. It suffices to work on the level of geometric points. We first compute the divisor of Δ on $X_1(N)$. For this we recall the combinatorics of the cusps of $X_1(N)$: for each d dividing N there are $\phi(N)/2$ distinct $\Gamma_1(N)$-structures on Néron d-gons, so that the cusps of $X_1(N)$ are indexed as $c_{d,i}(N)$ with d dividing N and $i \in (\mathbf{Z}/N\mathbf{Z})^\times / \pm 1$. A cusp $c_{d,i}(N)$ has ramification degree d over the unique cusp c of $X(1)$; since the divisor of Δ on $X(1)$ is c, it follows that the divisor of Δ on $X_1(N)$ is $\sum d c_{d,i}(N)$.

Assume now that p does not divide N. There are two cusps $c_{d,i}(N;p)$ and $c_{dp,i}(N;p)$ of $X_1(N;p)$ lying over $c_{d,i}(N)$ under j_p and j'_p; under j_p (resp. j'_p) the first is unramified (resp. ramified of degree p) while the second is ramified of degree p (resp. unramified). Thus the divisor of f_p on $X_1(N;p)$ is $\sum d(1-p)(c_{d,i}(N;p) - c_{dp,i}(N;p))$. Since both $c_{d,i}(N;p)$ and $c_{dp,i}(N;p)$ map to $c_{d,i}(N) \times c_{d,i}(N)$ under $j_p \times j'_p$, it follows that the divisor of f_p on $\mathcal{T}_p$ is trivial.

The argument is similar for p dividing N. Here the behavior of the cusps is more complicated: there are three cusps $c_{d,p,i}(N;p)$, $c_{dp,1,i}(N;p)$, $c_{dp,p,i}(N;p)$ lying over the pair of cusps $c_{d,i}(N)$ and $c_{dp,i}(N)$. Under j_p (resp. j'_p) the cusp $c_{d,p,i}(N;p)$ (resp. $c_{dp,1,i}(N;p)$) is ramified of degree p over $c_{d,i}(N)$, while $c_{dp,1,i}(N;p)$ (resp. $c_{d,p,i}(N;p)$) is unramified over $c_{dp,i}(N)$. In both cases $c_{dp,p,i}(N;p)$ is ramified of degree $p-1$ over $c_{dp,i}(N)$. It follows easily that f_p has trivial divisor already on $X_1(N;p)$, and thus on $\mathcal{T}_p$ as well. This completes the proof. □

If one follows through the calculations in the second case above, it becomes clear that the only cusp forms for which $\mathbf{T}_1(N)$ is admissible are multiples of powers of Δ.

4.1.5. *Divisors in positive characteristic.* We now prove the following key lemma.

Lemma 4.1.2. *For all p not dividing N the pair $(\mathcal{T}_p, f_p)$ is a divisorial lifting of $6(\Gamma'_p - \Gamma_p)$ on $\mathfrak{X}_1(N)\times\mathfrak{X}_1(N) \to \operatorname{Spec}\mathbf{Z}[\frac{1}{N}]$.*

Proof. By Lemma 4.1.1 we know that the divisor of f_p on $\mathfrak{T}_p$ has no horizontal component. In characteristics not dividing Np the analysis is identical to that in characteristic 0, so that the divisor of f_p on $\mathfrak{T}_p$ is supported in characteristic p.

We compute on $\mathfrak{T}_{p,\mathbf{F}_p} = \Gamma_p + \Gamma'_p$. Since the divisor of f_p on $\mathfrak{T}_{p,\mathbf{F}_p}$ is of codimenson zero, it suffices to compute it generically on each irreducible component. Recall that Γ_p is the scheme-theoretic image of the map

$$\mathrm{id}\times\mathrm{Fr} : \mathfrak{X}_1(N)_{\mathbf{F}_p} \to \mathfrak{X}_1(N)_{\mathbf{F}_p}\times\mathfrak{X}_1(N)_{\mathbf{F}_p};$$

on this irreducible component f_p is simply $\pi_1^*\Delta/\pi_2^*\Delta$ where π_1, π_2 are the two projections. In particular, $\pi_1 = \mathrm{id}$ is an isomorphism so that the divisor of $\pi_1^*\Delta$ has no generic support on Γ_p. However, $\pi_2 = \mathrm{Fr}$ is purely inseparable; since Δ is in the sixth power of the canonical sheaf, it follows that $\pi_2^*\Delta$ vanishes to order 6 on Γ_p. Thus the divisor of f_p on Γ_p is $-6\Gamma_p$. In the same way one computes that the divisor of f_p on Γ'_p is $6\Gamma'_p$ (using that $\langle p\rangle$ is an isomorphism). This completes the proof. □

4.1.6. *The cohesive Flach system.* We are now in a position to prove our main theorem. Recall that $N \geq 5$ is squarefree and $l \geq 5$ is a prime not dividing N. A is the localization of the "contravariant" image of $\mathbf{T}_1(N)$ in $\operatorname{End}_{\mathbf{Z}_l} H^1(X_1(N)_{\mathbf{Q}_{\mathrm{ac}}}, \mathbf{Z}_l)$ at a non-Eisenstein maximal ideal $\mathfrak{m}$ of B^* corresponding to a newform and $H = H^1(X_1(N)_{\mathbf{Q}_{\mathrm{ac}}}, \mathbf{Z}_l)_{\mathfrak{m}}$. A is Gorenstein and we have fixed a Gorenstein trace tr with congruence element η. H is minimally ramified of weight 2 by [6], Lemma 3.27 and [38]. Set $T = \operatorname{End}^0_A H(1)$ endowed with the usual finite/singular structure.

Theorem 4.1.3. *Let H be a modular Galois representation of weight 2 as above. Assume that $T\otimes_A k$ is absolutely irreducible and that $H^1(\mathbf{Q}(T^*[\mathfrak{a}])/\mathbf{Q}, T^*[\mathfrak{a}])$ vanishes for all ideals $\mathfrak{a}$ of finite index in A (this holds in particular if $G_F \to \operatorname{Aut}_A H$ is surjective). Then T admits a cohesive Flach system of Eichler-Shimura type of depth η and weight -12.*

Proof. We need to check the hypotheses of Proposition 3.2.5 for the algebra of correspondences $\mathbf{T}_1(N)$ with the marking Δ. That the $\langle p\rangle$ are diamond operators in our sense follows from [6], Theorem 3.1. The first two conditions on $(\mathcal{T}_p, f_p)$ follow from Lemma 4.1.2. That $\mathcal{T}_p^*$ agrees with the trace of $\mathrm{Fr}(p)$ on H is also [6], Theorem 3.1. It remains to check that $H^1_s(\mathbf{Q}_p, T) = 0$ for every p dividing N. By [2], Théorème A, as a $G_{\mathbf{Q}_p}$-module, H is either ordinary or a direct sum of an unramified character and a tamely ramified character. The fact that $H^1_s(\mathbf{Q}_p, T) = 0$ is straightforward in the second case; for the first case see [44], Lemma I.5.2 or [14], Lemma 2.10. This completes the proof. □

We should note that it is not necessary to use [33] (as in Lemmaa 3.2.2) to check the crystalline condition in this theorem; one can easily modify the original methods of [14] to deal with it.

4.2. **Kuga-Sato varieties.**

4.2.1. *Universal elliptic curves.* Fix a squarefree integer N and let $\mathfrak{Y}_1(N) \to \mathbf{Z}[\frac{1}{N}]$ be the open complement of the cusps in $\mathfrak{X}_1(N) \to \mathbf{Z}[\frac{1}{N}]$. Let $f : \mathcal{E}_1(N) \to \mathfrak{Y}_1(N)$ be the universal elliptic curve and let $\bar{f} : \bar{\mathcal{E}}_1(N) \to \mathfrak{X}_1(N)$ be the universal generalized elliptic curve; the first is a smooth open subscheme of the second, which is proper but not smooth over $\mathbf{Z}[\frac{1}{N}]$. For $\kappa > 0$ we let $\mathcal{E}_1^\kappa(N)$ (resp. $\bar{\mathcal{E}}_1^\kappa(N)$) denote the κ-fold fiber product of $\mathcal{E}_1(N)$ over $\mathfrak{Y}_1(N)$ (resp. $\bar{\mathcal{E}}_1(N)$ over $\mathfrak{X}_1(N)$). $\mathcal{E}_1^\kappa(N) \to \mathbf{Z}[\frac{1}{N}]$ is smooth but not proper, while $\bar{\mathcal{E}}_1^\kappa(N) \to \mathbf{Z}[\frac{1}{N}]$ is proper but not smooth. $\bar{\mathcal{E}}_1^\kappa(N)$ has a proper smooth model $\tilde{\mathcal{E}}_1^\kappa(N)$ which contains $\mathcal{E}_1^\kappa(N)$ as a dense open subscheme; see [39], Section 2 and [4], Section 4.3. We use the obvious notation for the generic fibers of each of these $\mathbf{Z}[\frac{1}{N}]$-schemes.

The Galois representation associated to a modular form of weight $\kappa + 2 \geq 3$ is initially a localization of

$$\begin{aligned}\tilde{H}^1\big(Y_1(N)_{\mathbf{Q}_{\mathrm{ac}}}, \operatorname{Sym}^\kappa R^1 f_{\mathbf{Q}_{\mathrm{ac}}*}\mathbf{Z}_l\big) &\stackrel{\mathrm{def}}{=} \\ \operatorname{im}\big(H^1_c(Y_1(N)_{\mathbf{Q}_{\mathrm{ac}}}, \operatorname{Sym}^\kappa R^1 f_{\mathbf{Q}_{\mathrm{ac}}*}\mathbf{Z}_l) &\to H^1(Y_1(N)_{\mathbf{Q}_{\mathrm{ac}}}, \operatorname{Sym}^\kappa R^1 f_{\mathbf{Q}_{\mathrm{ac}}*}\mathbf{Z}_l)\big).\end{aligned}$$

It can also be realized in the cohomology group $\tilde{H}^{\kappa+1}(E_1^\kappa(N)_{\mathbf{Q}_{\mathrm{ac}}}, \mathbf{Z}_l)$, which is itself a quotient of $H^{\kappa+1}(\tilde{E}_1^\kappa(N)_{\mathbf{Q}_{\mathrm{ac}}}, \mathbf{Z}_l)$. This is the group in which we will work.

4.2.2. *Hecke correspondences.* Let $\mathfrak{Y}_1(N;p) \to \mathbf{Z}[\frac{1}{N}]$ be the open complement of the cusps in $\mathfrak{X}_1(N;p) \to \mathbf{Z}[\frac{1}{N}]$ and let $\mathcal{E}_1(N;p) \to \mathfrak{Y}_1(N;p)$ be the universal elliptic curve. There are two natural degeneracy maps $j_p, j_p' : \mathcal{E}_1(N;p) \rightrightarrows \mathcal{E}_1(N)$ and we define the Hecke correspondence $\mathfrak{T}_p^\kappa$ to be the scheme-theoretic image of

$$j_p^\kappa \times j_p'^{\,\kappa} : \mathcal{E}_1^\kappa(N;p) \to \mathcal{E}_1^\kappa(N) \times_{\mathbf{Z}[\frac{1}{N}]} \mathcal{E}_1^\kappa(N).$$

We let $\bar{\mathfrak{T}}_p^\kappa$ (resp. $\tilde{\mathfrak{T}}_p^\kappa$) be the closure of $\mathfrak{T}_p^\kappa$ in $\bar{\mathcal{E}}_1^\kappa(N) \times \bar{\mathcal{E}}_1^\kappa(N)$ (resp. $\tilde{\mathcal{E}}_1^\kappa(N) \times \tilde{\mathcal{E}}_1^\kappa(N)$). $\tilde{\mathfrak{T}}_p^\kappa$ is also the strict transform of $\bar{\mathfrak{T}}_p^\kappa$ under the blow-up $\tilde{\mathcal{E}}_1^\kappa(N) \times \tilde{\mathcal{E}}_1^\kappa(N) \to \bar{\mathcal{E}}_1^\kappa(N) \times \bar{\mathcal{E}}_1^\kappa(N)$. We write $\mathcal{T}_p^\kappa$, $\bar{\mathcal{T}}_p^\kappa$ and $\tilde{\mathcal{T}}_p^\kappa$ for the generic fibers of these correspondences; that they are correspondences in our sense is shown in [39], Section 4. There are also diamond operators $\langle d \rangle^\kappa$ on $\mathcal{E}_1^\kappa(N)$ and $\tilde{\mathcal{E}}_1^\kappa(N)$. The diamond operators and the $\mathcal{T}_p^\kappa$ generate an algebra $\mathbf{T}_1(N)^\kappa$ of correspondences on $\mathcal{E}_1^\kappa(N)$ which is isomorphic to the algebra of correspondences on $\tilde{\mathcal{E}}_1^\kappa(N)$ generated by the diamond operators and the $\tilde{\mathcal{T}}_p^\kappa$.

For p not dividng N, the characteristic p fiber $\mathfrak{T}^\kappa_{p,\mathbf{F}_p}$ is reduced and has two irreducible components Γ_p^κ, $\Gamma_p^{\kappa\prime}$; the first is the image of $\mathrm{id} \times \mathrm{Fr}$ and the second is the image of $\mathrm{Fr} \times \langle p \rangle^\kappa$. See [4], Theorem 5.3.3.1.

4.2.3. *Galois representations.* We follow the presentation of [4] via the integral theory of [11]. Fix a prime $l > \max\{5, \kappa+1\}$ not dividing N. Let V be the Galois module $\tilde{H}^1(Y_1(N)_{\mathbf{Q}_{\mathrm{ac}}}, \mathrm{Sym}^\kappa R^1 f_{\mathbf{Q}_{\mathrm{ac}}*}\mathbf{Z}_l)$. By [4], Section 5.4 and [39], Proposition 4.1.1, V is a subquotient of $\tilde{H}^{\kappa+1}(E_1^\kappa(N)_{\mathbf{Q}_{\mathrm{ac}}}, \mathbf{Z}_l)$; thus it is a subquotient of $V_0 = H^{\kappa+1}(\tilde{E}_1^\kappa(N)_{\mathbf{Q}_{\mathrm{ac}}}, \mathbf{Z}_l)$ as well.

We must assume that $\tilde{E}_1^\kappa(N)$ is cohomologically torsion-free at l and that V is a direct summand of V_0 as a $\mathbf{Z}_l$-module; both of these conditions hold for almost all l. We will apply the procedure of Proposition 3.2.5 directly to V rather than to V_0; it is straightforward to check that this is valid as V is stable under all relevant operations.

Let B_* and B^* denote the images of $\mathbf{T}_1(N)^\kappa$ in $\mathrm{End}_{\mathbf{Z}_l} V$. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of B^* associated to a newform of weight $\kappa+2$ as in [11], Theorem 2.1; such an $\mathfrak{m}$ is dualizing. In particular, $H = V \otimes_{B^*} A$ is free of rank 2 over the Gorenstein ring $A = B^*_{\mathfrak{m}}$ and by [11], Theorem 3.38 $H \otimes_A k$ is absolutely irreducible. Fix a Gorenstein trace $\mathrm{tr} : A \to \mathbf{Z}_l$ with congruence element η. Set $T = \mathrm{End}_A^0 H(1)$ with the usual finite/singular structure.

4.2.4. *Admissible markings.* We have maps $\tilde{E}_1^\kappa(N) \to X_1(N)$ and $\tilde{\mathcal{T}}_p^\kappa \to \mathcal{T}_p$; we assign sheaves $\mathcal{L}_\bullet$ to $\tilde{E}_1^\kappa(N)$ and $\tilde{\mathcal{T}}_p^\kappa$ as the pullback of the sixth power of the canonical sheaf on $X_1(N)$ and $\mathcal{T}_p$ via these maps. In particular, Δ is a global section of $\mathcal{L}_{\tilde{E}_1^\kappa(N)}$.

Lemma 4.2.1. $\mathbf{T}_1(N)^\kappa$ *is an admissible algebra of correspondences for* $\tilde{E}_1^\kappa(N)$ *with the* $\mathcal{L}$*-marking* Δ.

Proof. This is clear for the diamond operators. By Lemma 4.1.1 we know that $f_p = f_{\mathcal{T}_p}$ has trivial divisor on each $\mathcal{T}_p$. It follows that $f_p^\kappa = f_{\bar{\mathcal{T}}_p^\kappa}$ has trivial divisor on $\bar{\mathcal{T}}_p^\kappa$. We need to check that this divisor does not become non-trivial as $\bar{E}_1^\kappa(N) \times \bar{E}_1^\kappa(N)$ is blown-up to $\tilde{E}_1^\kappa(N) \times \tilde{E}_1^\kappa(N)$. This is straightforward from the explicit description of the resolution process as in [39], Section 2; one finds that the singularities of $\bar{\mathcal{T}}_p^\kappa$ coming from the singularities of $E_1^\kappa(N;p)$ are resolved into a single κ-cycle. (The singularity of $\bar{\mathcal{T}}_p^\kappa$ coming from its embedding into $\bar{E}_1^\kappa(N) \times \bar{E}_1^\kappa(N)$ is not resolved.) Since the divisor of f_p^κ on $\tilde{\mathcal{T}}_p^\kappa$ must be supported on this κ-cycle, it follows that it is still trivial. □

Lemma 4.2.2. *For all* p *not dividing* N *the pair* $(\tilde{\mathcal{T}}_p^\kappa, f_p^\kappa)$ *is a divisorial lifting of the cycle* $6(\Gamma_p^{\kappa\prime} - \Gamma_p^\kappa)$ *on* $\tilde{\mathcal{E}}_1^\kappa(N) \times \tilde{\mathcal{E}}_1^\kappa(N) \to \mathbf{Z}[\frac{1}{N}]$.

Proof. The proof of this is identical to that of Lemma 4.1.2; one also uses the description of $\tilde{\mathcal{T}}_p^\kappa$ given in the proof of Lemma 4.2.1. □

4.2.5. *The cohesive Flach system.* Recall that we have assumed that N is square-free; that $l > \max\{5, \kappa+1\}$; that $\tilde{E}_1^\kappa(N)$ is cohomologically torsion-free at l; and that V is a direct summand of V_0. H is minimally ramified by [2] and [38].

Theorem 4.2.3. *Let* H *be a modular Galois representation of weight* $\kappa+2$ *as above and set* $T = \mathrm{End}_A^0 H(1)$. *Assume that* $T \otimes_A k$ *is absolutely irreducible and that the cohomology group* $H^1(\mathbf{Q}(T^*[\mathfrak{a}])/\mathbf{Q}, T^*[\mathfrak{a}])$ *vanishes for all ideals* $\mathfrak{a}$ *of finite index in* A. *Then* T *admits a cohesive Flach system of depth* η *and weight* -12.

Proof. The proof is essentially the same as that of Theorem 4.1.3, using our extra hypotheses to pass all constructions to V. □

Of course, assuming the hypotheses above we now obtain immediate applications to deformation theory as in Section 1.4.5 and Section 1.4.6. If η is a unit, then one can also apply Proposition 1.5.1.

5. Flach systems of Eichler-Shimura type

In this section we give the proof of Theorem 1.4.8. Since A is generated by the T_v with $v \notin \Sigma_l$ and $H^1_s(F_v, T) = 0$ for $v \in \Sigma - \Sigma_l$ it suffices to show that $\Xi(\partial T_v) = 2w\partial T_v$ for $v \notin \Sigma$.

5.1. The map on differentials.

We begin by recalling the details of the construction of the map Ξ. Fix a power l^n of l such that η divides l^n in A; such a power exists since η is a non zero-divisor. We have

$$H^1_f(F, T^*) = H^1_f\big(F, T^*[\eta]\big) = H^1_f\big(F, T^*[l^n]\big).$$

We can and will work at finite levels since everything is l^n-torsion. For any $\mathbf{Z}/l^n\mathbf{Z}$-module M we write $M^\vee$ for its Pontrjagin dual $\mathrm{Hom}_{\mathbf{Z}_l}(M, \mathbf{Z}/l^n\mathbf{Z})$.

The map $\Xi : \Omega_A \to \Omega_A$ is defined to be the composition

$$\Omega_A \xrightarrow{\xi_1} H^1_f(F, T/\eta T) \xrightarrow{\xi_2} H^1_f(F, T^*[l^n])^\vee \xrightarrow{\xi_3} H^1_f\big(F, \mathrm{End}^0_A(H/l^nH)\big)^\vee \xrightarrow{\xi_4}$$

$$\mathrm{Hom}_A(\Omega_R \otimes_R A, A/l^nA)^\vee \xrightarrow{\xi_5} \mathrm{Hom}_{\mathbf{Z}_l}(\Omega_R \otimes_R A, \mathbf{Z}/l^n\mathbf{Z})^\vee \xrightarrow{\xi_6} \Omega_R \otimes_R A \xrightarrow{\xi_7} \Omega_A$$

ξ_1 is the A-linear map induced by Θ; we have $\xi_1(\partial T_v) = c^v$. ξ_2 is induced by the Bockstein pairing (1.4.6) and we computed in the proof of Lemma 1.4.5 that $\xi_2(c^v)(\kappa) = \left\langle \frac{1}{\eta} c^v_{v,s}, \kappa_v \right\rangle_v$. ξ_3 comes from (1.4.2); ξ_4 is the dual of the isomorphism of l^n-torsion in (1.4.4); ξ_5 is induced by the Gorenstein trace tr; ξ_6 is the double duality isomorphism; and ξ_7 is induced by $\pi : R \twoheadrightarrow A$.

Note that the cohesive Flach system enters only into the very first map ξ_1; the remaining maps are at most dependent on the choice of Gorenstein trace tr, although the composite does not depend on that choice.

5.2. The Tate pairing.

In order to explicitly compute the map Ξ we will need to work with the Tate pairing. Let M be a finite G_{F_v}-module of exponent m and let $M^* = \mathrm{Hom}_{\mathbf{Z}}(M, \mu_m)$ be its Cartier dual. Recall that the Tate pairing is the map

$$H^1(F_v, M) \otimes H^1(F_v, M^*) \to \mathbf{Q}/\mathbf{Z}$$

defined as the composition of

$$H^1(F_v, M) \otimes H^1(F_v, M^*) \xrightarrow{\text{cup}} H^2(F_v, M \otimes_{\mathbf{Z}} M^*) \xrightarrow{\text{Cartier}} H^2(F_v, \mu_m) \xrightarrow{\simeq}$$

$$H^2(L/F_v, L^\times) \xrightarrow{\text{val}} H^2(L/F_v, \mathbf{Z}) \xleftarrow{\delta} H^1(L/F_v, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\text{eval}} \mathbf{Q}/\mathbf{Z}.$$

Here L is the unique unramified extension of F_v of degree m. The unlabeled isomorphism comes from local class field theory. The map δ is the boundary map for the exact sequence $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$; it is an isomorphism.

In our computation we will compute as far as $H^2(L/F_v, L^\times)$; we now give the inverse image in here of $\frac{e}{m} \in \mathbf{Q}/\mathbf{Z}$ so that we have something to compare with. Let $\{\cdot\} : \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}$ be the map sending $x \in \mathbf{Q}/\mathbf{Z}$ to the unique $\tilde{x} \in \mathbf{Q}$ such that

$0 \le \tilde{x} < 1$ and $x \equiv \tilde{x} \pmod{\mathbf{Z}}$. Using the definition of δ one finds that the cocycle $C_e \in H^2(L/F_v, L^\times)$ given by

$$C_e(\mathrm{Fr}(v)^i, \mathrm{Fr}(v)^j) = \lambda_0^{\{\frac{e(i+j)}{m}\}-\{\frac{ei}{m}\}-\{\frac{ej}{m}\}} \tag{5.2.1}$$

maps to $\frac{e}{m}$ under $\mathrm{eval} \circ \delta^{-1} \circ \mathrm{val}$.

5.3. **A special case.**

5.3.1. *Additional hypotheses.* In this section we compute $\Xi(\partial T_v)$ with some additional simplifying hypotheses; this computation will still contain most of the content of the general case, but it is significantly simpler algebraically.

We make two assumptions. First, assume that the action of $\mathrm{Fr}(v)$ on H is diagonal with respect to a fixed basis x, y; that is, $\mathrm{Fr}(v)$ acts on H by a matrix $\left(\begin{smallmatrix} s & 0 \\ 0 & t \end{smallmatrix}\right)$ with $s, t \in A$. In particular, $st = \chi(v)$ and $s + t = T_v$. It follows from the definition of a cohesive Flach system of Eichler-Shimura type that in this case the cocycle $c^v_{v,s}$ is given by

$$\begin{aligned} c^v_{v,s} : \mathrm{Gal}\big(F_v^{\mathrm{ur}}(\lambda)/F_v^{\mathrm{ur}}\big) &\to T \\ \tau^j &\mapsto wj\eta(s-t)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \otimes \zeta_n \end{aligned} \tag{5.3.1}$$

with the notation of Section 1.4.6.

The second simplifying assumption is that the map $\pi : R \to A$ is an isomorphism; that is, A is the universal minimally ramified deformation ring of $H \otimes_A k$ and H is the universal deformation. Of course, we will identify R with A via π.

5.3.2. *Preliminaries.* To compute $\Xi(\partial T_v)$, we begin by computing the image of ∂T_v in $\mathrm{Hom}_A(\Omega_A, A/l^n A)^\vee$. So let $\omega : \Omega_A \to A/l^n A$ be a fixed map; we will compute its image in $\mathbf{Z}/l^n\mathbf{Z}$ under $\xi_4 \circ \xi_3 \circ \xi_2 \circ \xi_1(\partial T_v)$.

5.3.3. Making explicit the definition of ξ_4 (see [44], Section V.1) one computes that this is the same as the image under $\xi_3 \circ \xi_2 \circ \xi_1(\partial T_v)$ of the cohomology class represented by the cocycle $\kappa' : G_F \to \mathrm{End}^0_A(H/l^n H)$ given by

$$\kappa'(\sigma) = \tfrac{1}{ad-bc}\begin{pmatrix} d\omega(\partial a) - b\omega(\partial c) & d\omega(\partial b) - b\omega(\partial d) \\ a\omega(\partial c) - c\omega(\partial a) & a\omega(\partial d) - c\omega(\partial b) \end{pmatrix}.$$

Here $\rho_A(\sigma) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A)$ with $\rho_A : G_F \to \mathrm{GL}_2(A)$ the Galois representation on H with respect to the basis x, y

5.3.4. Using the definition of ξ_3, this is just the image under $\xi_2 \circ \xi_1(\partial T_v) \in H^1_f(F, T^*[l^n])^\vee$ of the cohomology class represented by the cocycle

$$\begin{aligned} \kappa : G_F \to T^*[l^n] &= \mathrm{Hom}_{\mathbf{Z}_l}\big(\mathrm{End}^0_A(H/l^n H)(1), \mu_{l^n}\big) \\ &= \mathrm{Hom}_{\mathbf{Z}_l}\big(\mathrm{End}^0_A(H/l^n H), \mathbf{Z}/l^n\mathbf{Z}\big) \end{aligned}$$

explicitly given as

$$\begin{aligned} \kappa(\sigma)\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \mathrm{tr}\Big(\tfrac{1}{ad-bc}\big(&\alpha d\omega(\partial a) - \alpha b\omega(\partial c) + \gamma d\omega(\partial b) - \gamma b\omega(\partial d) + \\ &\beta a\omega(\partial c) - \beta c\omega(\partial a) - \alpha a\omega(\partial d) + \alpha c\omega(\partial b)\big)\Big). \end{aligned} \tag{5.3.2}$$

5.3.5. Using the definition of ξ_2 and its explicit expression for $c^v = \xi_1(\partial T_v)$, we find that the desired element of $\mathbf{Z}/l^n\mathbf{Z}$ is the value of the Tate pairing $\left\langle \frac{1}{\eta} c^v_{v,s}, \kappa_v \right\rangle_v$; here we are identifying the image $\frac{1}{l^n}\mathbf{Z}/\mathbf{Z}$ with $\mathbf{Z}/l^n\mathbf{Z}$. It remains to compute this.

5.3.6. *The Tate pairing : preliminaries.* To begin with, note that κ_v factors through $\mathrm{Gal}(F_v^{\mathrm{ur}}/F_v)$, as it is unramified at v; thus we need only concern ourselves with κ evaluated at powers of $\mathrm{Fr}(v)$. Using the fact that $\rho_A(\mathrm{Fr}(v)^i) = \left(\begin{smallmatrix} s^i & 0 \\ 0 & t^i \end{smallmatrix}\right)$ which has determinant $\chi(v)^i$, we find that (5.3.2) simplifies to

$$\kappa(\mathrm{Fr}(v)^i)\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) = \mathrm{tr}\left(\chi(v)^{-i}\alpha(t^i\omega(\partial s^i) - s^i\omega(\partial t^i))\right). \tag{5.3.3}$$

Next, note that

$$s^i\partial t^i = s^i t^{i-1} i\partial t = \chi(v)^{i-1} s i\partial t; \qquad t^i\partial s^i = \chi(v)^{i-1} t i\partial s.$$

We therefore can write (5.3.3) as

$$\kappa(\mathrm{Fr}(v)^i)\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) = \mathrm{tr}\left(i\chi(v)^{-1}\alpha\left(t\omega(\partial s) - s\omega(\partial t)\right)\right).$$

Setting $K = F_v(H/l^nH)$ (so that K/F_v is unramified), we see that κ_v factors through $\mathrm{Gal}(K/F_v)$.

For c^v, we computed in (5.3.1) that

$$\begin{aligned} c^v_{v,s} : \mathrm{Gal}(F_v^{\mathrm{ur}}(\lambda)/F_v^{\mathrm{ur}}) &\to T/l^nT \\ \tau^j &\mapsto wj\eta(s-t)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\otimes\zeta_n \end{aligned}$$

Since τ^{l^n} goes to 0 under this map, c^v factors through $\mathrm{Gal}(F_v^{\mathrm{ur}}(\lambda_n)/F_v^{\mathrm{ur}})$.

In order to compute the Tate pairing of κ and c^v we first must descend c^v to a cocycle over F_v. We can do this over the field $K(\lambda_n)$ as follows: let $G = \mathrm{Gal}(K(\lambda_n)/F_v)$. Denote by φ the element of G which acts as Frobenius on K and fixes λ_n, and denote by τ the element of G which is the identity on K and sends λ_n to $\zeta_n\lambda_n$. Then φ and τ generate G with the relations

$$\varphi^{[K:F_v]} = \tau^{l^n} = 1, \qquad \tau\varphi = \varphi\tau^{\varepsilon(v)}.$$

One checks that $\frac{1}{\eta}c^v_{v,s}$ is represented by the cocycle

$$\begin{aligned} \theta^v : G &\to T/l^nT \\ \varphi^i\tau^j &\mapsto w\varepsilon(v)^i j(s-t)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\otimes\zeta_n. \end{aligned}$$

Via inflation we can represent κ_v by the cocycle $G \to T^*[l^n]$ given by

$$\kappa_v(\varphi^i\tau^j)\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) = \mathrm{tr}\left(i\chi(v)^{-1}\alpha(t\omega(\partial s) - s\omega(\partial t))\right). \tag{5.3.4}$$

5.3.7. *The Tate pairing : cup product.* We now compute the Tate pairing. The first step is to form the cup product $\theta^v \cup \kappa_v$. By the formula for the cup product we see that $\theta^v \cup \kappa_v$ sends the pair $(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) \in G \times G$ to

$$\theta^v(\varphi^i\tau^j) \otimes \kappa_v^{\varphi^i\tau^j}(\varphi^{i'}\tau^{j'}) \in T/l^nT \otimes_{\mathbf{Z}_l} T^*[l^n].$$

Under Cartier duality this maps to the cocycle $C \in H^2(G, \mu_{l^n})$ given by

$$C(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) = \kappa_v^{\varphi^i\tau^j}(\varphi^{i'}\tau^{j'})\left(w\varepsilon(v)^i j(s-t)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)\zeta_n.$$

Recall that $\kappa_v(\varphi^{i'}\tau^{j'})$ is a map $\mathrm{End}^0_A(H/l^nH) \to \mathbf{Z}/l^n\mathbf{Z}$. In particular, $\varphi^i\tau^j$ acts trivially on both the domain and the range. Thus by the definition of the adjoint Galois action we find that

$$C(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) = \mathrm{tr}\left(i'\chi(v)^{-1}w\varepsilon(v)^i j(s-t)(t\omega(\partial s) - s\omega(\partial t))\right)\zeta_n.$$

If we let $C' : G \times G \to \mu_{l^n}$ be the cocycle $C'(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) = \varepsilon(v)^i i' j\zeta_n$, then we conclude that ω maps to

$$\mathrm{tr}\left(\omega(w\chi(v)^{-1}(t\partial s - s\partial t))\right) I \tag{5.3.5}$$

where I is the image of C' under the invariant map

$$H^2(K(\lambda_n)/F_v, \mu_{l^n}) \to \mathbf{Z}/l^n\mathbf{Z}.$$

5.3.8. At this point, thankfully, we get the maps ξ_5 and ξ_6 for free. Specifically, suppose that we began with $\omega_0 : \Omega_A \to \mathbf{Z}/l^n\mathbf{Z}$ and wished to compute its image in $\mathbf{Z}/l^n\mathbf{Z}$ under $\xi_5 \cdots \xi_1(\partial T_v)$. This is the same as the image under $\xi_4 \cdots \xi_1(\partial T_v)$ of the unique $\omega : \Omega_A \to A/l^nA$ such that $\mathrm{tr} \circ \omega = \omega_0$. By (5.3.5) this is visibly just

$$\omega_0\big(w\chi(v)^{-1}(s-t)(t\partial s - s\partial t)\big)I. \tag{5.3.6}$$

Similarly, in $\mathrm{Hom}_{\mathbf{Z}_l}(\Omega_A, \mathbf{Z}/l^n\mathbf{Z})^\vee$ the element (5.3.6) is just the evaluation at

$$w\chi(v)^{-1}(s-t)(t\partial s - s\partial t)I \tag{5.3.7}$$

map, so that (5.3.7) is the final image of ∂T_v in Ω_A. It remains, then, to compute I and to simplify our expression.

5.3.9. *Computation of the invariant.* We begin by computing I; it is the image of C' under the maps

$$H^2(K(\lambda_n)/F_v, \mu_{l^n}) \to H^2(L/F_v, L^\times) \to \mathbf{Z}/l^n\mathbf{Z},$$

where L is the unique unramified extension of F_v of degree l^n. We first need to modify C' by a coboundary to get it to factor through $\mathrm{Gal}(L/F_v)$ and to take values in $L^\times$. We can do this using the cochain $f : G \to K(\lambda_n)^\times$ given by $f(\varphi^i\tau^j) = \lambda_n^{\langle i\rangle}$ where $\langle i\rangle$ is the unique integer in $\{0, 1, \ldots, l^n - 1\}$ which is congruent to i modulo l^n. The coboundary formula states that

$$\begin{aligned} C'\partial f(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) &= \frac{C'(\varphi^i\tau^j, \varphi^{i'}\tau^{j'})f(\varphi^i\tau^j\varphi^{i'}\tau^{j'})}{{}^{\varphi^i\tau^j}f(\varphi^{i'}\tau^{j'})f(\varphi^i\tau^j)} \\ &= \lambda_n^{\langle i+i'\rangle - \langle i\rangle - \langle i'\rangle}. \end{aligned}$$

This is simply the inflation to $H^2(K(\lambda_n)/F_v, K(\lambda_n)^\times)$ of the cocycle C_1 of (5.2.1). Since C_1 was defined to map to 1 under the invariant map, we see that C' does as well. Thus $I = 1$.

5.3.10. *Differentials and Hecke operators.* We conclude by (5.3.7) that

$$\Xi(\partial T_v) = w\chi(v)^{-1}(s-t)(t\partial s - s\partial t) \in \Omega_A.$$

It remains to simplify this expression. Using that $st = \chi(v)$ we find that

$$(s-t)(t\partial s - s\partial t) = 2\chi(v)\left(\partial s + \partial t\right).$$

Thus we conclude that $\Xi(\partial T_v) = 2w\partial(s+t) = 2w\partial T_v$. This completes the proof of Theorem 1.4.8 in this case.

5.4. **A matrix computation.** The key to removing both of the assumptions of the previous computation is the following matrix lemma; the proof is an unenlightening induction and we omit it.

Lemma 5.4.1. *Let R be a ring, S an R-algebra and M an S-module. Let $\partial : S \to M$ be an R-linear derivation. Let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(S)$ be a matrix with determinant $\delta = ad - bc$ and trace $t = a + d$. Assume that δ lies in the image of R in S. Let e be a positive integer, and write $T^e = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GL}_2(S)$. Then*

$$2e\delta^e \partial t = (-2bC + aD - dD)\partial A + (2cD + aC - dC)\partial B + (2bA - aB + dB)\partial C + (-2cB - aA + dA)\partial D$$

5.5. **Computation of Ξ in the non-diagonal case.** We now explain how to compute $\Xi(\partial T_v)$ when $\mathrm{Fr}(v)$ is not necessarily diagonal. We continue to assume that π is an isomorphism. This computation is fundamentally the same as the previous special case, just a bit messier and with the simple computation of Section 5.3.10 replaced by the more elaborate computation of Lemma 5.4.1.

The complication is that $\mathrm{Fr}(v)$ no longer acts diagonally. Write $\rho_A(\mathrm{Fr}(v)) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\rho_A(\mathrm{Fr}(v)^i) = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ for some fixed basis x, y of H. Note that $a + d = T_v$, $ad - bc = \chi(v)$ and $a_i d_i - b_i c_i = \chi(v)^i$. The formula for the cocycle κ is exactly as computed in (5.3.2), replacing a, b, c, d with a_i, b_i, c_i, d_i for $\sigma = \mathrm{Fr}(v)^i$. The Flach class is now

$$\begin{aligned} c^v_{v,s} : \mathrm{Gal}(F_v^{\mathrm{ur}}(\lambda)/F_v^{\mathrm{ur}}) &\to T \\ \tau^j &\mapsto wj\eta \left(\begin{smallmatrix} a-d & 2b \\ 2c & d-a \end{smallmatrix}\right) \otimes \zeta_n. \end{aligned}$$

In order to compute the Tate pairing of $\frac{1}{\eta} c^v_{v,s}$ and κ_v we first must lift $\frac{1}{\eta} c^v_{v,s}$ to $H^1(F_v, T/l^n T)$. In fact, the lifting

$$\begin{aligned} \theta^v : \mathrm{Gal}(K(\lambda_n)/F_v) &\to T/l^n T \\ \varphi^i \tau^j &\mapsto w\varepsilon(v)^i j \left(\begin{smallmatrix} a-d & 2b \\ 2c & d-a \end{smallmatrix}\right) \otimes \zeta_n \end{aligned}$$

still works.

We now compute the cup product of κ_v and θ^v as cohomology classes for G. Writing $C = \theta^v \cup \kappa_v$, one finds that

$$\begin{aligned} &C(\varphi^i \tau^j, \varphi^{i'} \tau^{j'}) = \\ &\mathrm{tr}\Big(w\varepsilon(v)^i j \chi(v)^{-i'} \Big((-2bc_{i'} + ad_{i'} - dd_{i'})\omega(\partial a_{i'}) + (2cd_{i'} + ac_{i'} - dc_{i'})\omega(\partial b_{i'}) + \\ &\qquad (2ba_{i'} - ab_{i'} + db_{i'})\omega(\partial c_{i'}) + (-2cb_{i'} - aa_{i'} + da_{i'})\omega(\partial d_{i'})\Big)\Big)\zeta_n. \end{aligned}$$

Applying Lemma 5.4.1, we conclude that

$$C(\varphi^i \tau^j, \varphi^{i'} \tau^{j'}) = 2w\varepsilon(v)^i i' j \,\mathrm{tr}\big(\omega(\partial T_v)\big)\zeta_n$$

and from here the computation is identical to the earlier case; we conclude that $\Xi(\partial T_v) = 2w\partial T_v$.

5.6. **Computation of Ξ in the general case.** We now remove the assumption that π is an isomorphism. The computation in this case is essentially the same as in the previous case. First, recall that universality of R means that, fixing a universal deformation $\rho_R : G_F \to \mathrm{GL}_2(R)$, there is some basis of H with respect to which $\rho_A = \pi\rho_R$. We can conjugate in $\mathrm{GL}_2(A)$ from this basis to our fixed basis

x, y; since π is surjective (and R is local) we can lift this conjugation to $\mathrm{GL}_2(R)$. That is, we can conjugate ρ_R so as to assume that $\rho_A = \pi\rho_R$ where ρ_A is now the representation on our fixed basis x, y of H.

To compute Ξ this time, we begin with $\omega : \Omega_R \otimes_R A \to A/l^n A$ and compute its image in $\mathbf{Z}/l^n\mathbf{Z}$. Proceeding as before, we find that this is the image under the Tate pairing of two cocycles $\kappa_v : G \to T^*[l^n]$ and $\theta^v : G \to T/l^n T$. Writing $\rho_R(\mathrm{Fr}(v)^i) = \left(\begin{smallmatrix} \hat{a}_i & \hat{b}_i \\ \hat{c}_i & \hat{d}_i \end{smallmatrix}\right)$, $\rho_A(\mathrm{Fr}(v)^i) = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ we find that

$$\kappa_v(\varphi^i\tau^j)\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) \mathrm{tr}\Big(\chi(v)^{-i}\big(\alpha d_i\omega(\partial\hat{a}_i) - \alpha b_i\omega(\partial\hat{c}_i) + \gamma d_i\omega(\partial\hat{b}_i) - \gamma b_i\omega(\partial\hat{d}_i) + \beta a_i\omega(\partial\hat{c}_i) - \beta c_i\omega(\partial\hat{c}_i) - \alpha a_i\omega(\partial\hat{d}_i) + \alpha c_i\omega(\partial\hat{b}_i)\big)\Big).$$

The cocycle θ^v is given by

$$\theta^v(\varphi^i\tau^j) = w\varepsilon(v)^i j \left(\begin{smallmatrix} a-d & 2b \\ 2c & d-a \end{smallmatrix}\right) \otimes \zeta_n$$

where $\rho_A(\mathrm{Fr}(v)) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ as before.

From these expressions the computation works out exactly as in the previous case, with ∂a_i, ∂b_i, ∂c_i, ∂d_i replaced by $\partial\hat{a}_i$, $\partial\hat{b}_i$, $\partial\hat{c}_i$, $\partial\hat{d}_i$ respectively. Lemma 5.4.1 applies to show that

$$C(\varphi^i\tau^j, \varphi^{i'}\tau^{j'}) = 2w\varepsilon(v)^i i' j\, \mathrm{tr}\big(\omega(\partial\hat{a} + \partial\hat{d})\big)\zeta_n,$$

where $\rho_R(\mathrm{Fr}(v)) = \left(\begin{smallmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{smallmatrix}\right)$. From here the computation is as before, with the fact that $\xi_7(\partial\hat{a}) = \partial a$ and $\xi_7(\partial\hat{d}) = \partial d$ showing that Ξ is still multiplication by $2w$. This completes the proof of Theorem 1.4.8.

Appendix A. Linear algebra

In this appendix we give a quick summary of the basic constructions we will need with Gorenstein rings. We give no proofs; all of the results we state are straightforward.

A.1. **Gorenstein rings.**

A.1.1. *Definitions.* Let A be a finite, flat, local $\mathbf{Z}_l$-algebra with maximal ideal $\mathfrak{m}$ and residue field k. A is *Gorenstein* if the following equivalent conditions hold:

- $\mathrm{Ext}^1_A(k, A) \cong k$;
- $\dim_k(A/lA)[\mathfrak{m}] = 1$;
- $\mathrm{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l)$ is free of rank 1 as an A-module.

See [42], Section 1. Note that local complete intersections are Gorenstein; in particular, this includes rings of the form $\mathbf{Z}_l[x]/f(x)$ for $f(x)$ monic.

By a *Gorenstein trace* we mean an A-basis of $\mathrm{Hom}_{\mathbf{Z}_l}(A, \mathbf{Z}_l)$. Fix a Gorenstein trace tr and consider the ring $A \otimes_{\mathbf{Z}_l} A$ which we regard as an A-algebra via multiplication on the right factor. $\mathrm{tr} \otimes 1$ is an $A \otimes_{\mathbf{Z}_l} A$-generator of the free rank 1 module $\mathrm{Hom}_A(A \otimes_{\mathbf{Z}_l} A, A)$. Thus we can write the diagonal map $\Delta : A \otimes_{\mathbf{Z}_l} A \to A$ as $\iota(\mathrm{tr} \otimes 1)$ for a unique $\iota \in A \otimes_{\mathbf{Z}_l} A$; we define the *congruence element* $\eta_{\mathrm{tr}} \in A$ of tr to be $\Delta(\iota)$. For $u \in A^\times$ we have $\eta_{u\,\mathrm{tr}} = u^{-1}\eta_{\mathrm{tr}}$ so that the *congruence ideal* $\eta_{\mathrm{tr}} A$ is independent of the choice of tr. One also checks that η_{tr} is a non-zero divisor if and only if A is reduced.

A.1.2. *Duality.* Let M be a finitely generated free A-module. It follows immediately from the definition of a Gorenstein trace that the map $f \mapsto \mathrm{tr} \circ f$ sets up an isomorphism $\mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_{\mathbf{Z}_l}(M, \mathbf{Z}_l)$. If M is any finitely generated A-module, on taking a resolution by free A-modules one finds that $f \mapsto \mathrm{tr} \circ f$ yields an isomorphism

$$\mathrm{Hom}_A(M, A \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l) \cong \mathrm{Hom}_{\mathbf{Z}_l}(M, \mathbf{Q}_l/\mathbf{Z}_l).$$

We return now to the case that M is free. We define a map $h_{\mathrm{tr}} : \mathrm{End}_{\mathbf{Z}_l} M \to \mathrm{End}_A M$ as the composition

$$\mathrm{End}_{\mathbf{Z}_l} M \cong \mathrm{Hom}_{\mathbf{Z}_l}(M, \mathbf{Z}_l) \otimes_{\mathbf{Z}_l} M \cong \mathrm{Hom}_A(M, A) \otimes_{\mathbf{Z}_l} M \twoheadrightarrow \mathrm{Hom}_A(M, A) \otimes_A M \cong \mathrm{End}_A M.$$

Here the second isomorphism is induced by tr as above. The composition of the inclusion $\mathrm{End}_A M \hookrightarrow \mathrm{End}_{\mathbf{Z}_l} M$ with h_{tr} is multiplication by η_{tr}.

Consider now a free A-module H of rank 2. It is straightforward to check that the condition that A is Gorenstein is equivalent to the existence of a perfect A-pairing $\psi : H \otimes_{\mathbf{Z}_l} H \to \mathbf{Z}_l$ (one such pairing is given by $\psi(ax + by, cx + dy) = \mathrm{tr}(ad - bc)$ for an A-basis x, y of H).

A.2. **Bilateral derivations.** We review the (very) basic theory of bilateral derivations developed in [31].

A.2.1. *Definitions.* Let $\mathcal{A}$ be a commutative $\mathbf{Z}_l$-algebra and let M be an $\mathcal{A} \otimes_{\mathbf{Z}_l} \mathcal{A}$-module. A *bilateral derivation* from $\mathcal{A}$ to M is a $\mathbf{Z}_l$-linear map $\mathcal{D} : \mathcal{A} \to M$ such that

$$\mathcal{D}(\beta\alpha) = (\alpha \otimes 1)\mathcal{D}(\beta) + (1 \otimes \beta)\mathcal{D}(\alpha)$$

for all $\alpha, \beta \in \mathcal{A}$. The fundamental example is the map $\delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathbf{Z}_l} \mathcal{A}$ given by $\delta(\alpha) = \alpha \otimes 1 - 1 \otimes \alpha$; note that the image lies in the kernel I of the diagonal map Δ. One can show that $\delta : \mathcal{A} \to I$ is the universal bilateral derivation, although we will not use this.

Define M_δ to be the set of $m \in M$ such that $\delta(\alpha)m = 0$ for all $\alpha \in \mathcal{A}$; it is canonically an $\mathcal{A}$-module via Δ. The next two lemmas are straightforward.

Lemma A.2.1. *Let $\mathcal{D} : \mathcal{A} \to M$ be a bilateral derivation and let $\mathfrak{a}$ be an ideal of $\mathcal{A}$ such that $1 \otimes \mathfrak{a}$ and $\mathfrak{a} \otimes 1$ annihilate M. Then the restriction of $\mathcal{D}$ to $\mathfrak{a}$ yields an $\mathcal{A}$-module homomorphism $\tilde{\mathcal{D}} : \mathfrak{a}/\mathfrak{a}^2 \to M_\delta$.*

If M is an $\mathcal{A}$-module, we give $\mathrm{End}_{\mathbf{Z}_l} M$ the structure of $\mathcal{A} \otimes_{\mathbf{Z}_l} \mathcal{A}$-module by $(\alpha \otimes \beta) f(m) = \alpha f(\beta m)$ for $f \in \mathrm{End}_{\mathbf{Z}_l} M$.

Lemma A.2.2. *Let M be a free $\mathcal{A}$-module of finite rank with a continuous $\mathcal{A}$-linear action of some group G. Assume also that every Jordan-Holder constituent of $\mathrm{End}_{\mathbf{Z}_l} M$ has trivial G-invariants. Then there is a canonical isomorphism*

$$H^1(G, \mathrm{End}_{\mathbf{Z}_l} M)_\delta \cong H^1\big(G, (\mathrm{End}_{\mathbf{Z}_l} M)_\delta\big).$$

A.2.2. *Bilateral derivations from Gorenstein rings.* Let A be a finite, flat, local, reduced, Gorenstein $\mathbf{Z}_l$-algebra. Fix a Gorenstein trace tr and let η be the associated congruence element; η is a non-zero divisor since A is reduced. Let M and N be free A-modules of finite rank; $\mathrm{Hom}_{\mathbf{Z}_l}(M, N)$ is an $A \otimes_{\mathbf{Z}_l} A$-module in the obvious way.

Lemma A.2.3. *There is a unique A-module isomorphism $\nu : \mathrm{Hom}_{\mathbf{Z}_l}(M,N)_\delta \xrightarrow{\simeq} \mathrm{Hom}_A(M,N)$ fitting into a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Z}_l}(M,N)_\delta & \hookrightarrow & \mathrm{Hom}_{\mathbf{Z}_l}(M,N) \\ \nu\downarrow & & \downarrow\!\!\downarrow \\ \mathrm{Hom}_A(M,N) & \overset{\eta}{\hookrightarrow} & \mathrm{Hom}_A(M,N) \end{array}$$

Proof. The uniqueness of such a ν is clear. To define it it suffices to consider the case $M = N = A$; we leave this to the reader. □

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