AN INTRODUCTION TO COBORDISM THEORY

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1. Introduction

The notion of cobordism is simple; two manifolds M and N are said to be cobordant if their disjoint union is the boundary of some other manifold. Given the extreme difficulty of the classification of manifolds it would seem very unlikely that much progress could be made in classifying manifolds up to cobordism. However, René Thom, in his remarkable, if unreadable, 1954 paper *Quelques propriétés globales des variétés differentiables* [22], gave the full solution to this problem for unoriented manifolds, as well as many powerful insights into the methods for solving it in the cases of manifolds with additional structure. It was largely for this work that Thom was awarded the Fields medal in 1958. The key step was the reduction of the cobordism problem to a homotopy problem, although the homotopy problem is still far from trivial. This was later generalized by Lev Pontrjagin, and this result is now known as the Thom-Pontrjagin theorem.

The first part of this paper will work towards the proof of the generalized Thom-Pontrjagin theorem. We will begin by abstracting the usual notion of cobordism on manifolds. Personally distasteful as this may be, it will be useful to have an abstract definition phrased in the language of category theory. We will then return to the specific situation of manifolds. The notion of a (B, f) structure on a manifold will be introduced, primarily as a means of unifying the many different special structures (orientations, complex structures, spin structures, etc.) which can be put on a manifold. The class of (B, f) manifolds can be made in a natural way into a cobordism category, and in this way we will define the cobordism groups $\Omega(B, f)$. Once these ideas are all established, we will turn towards the statement and proof of the Thom-Pontrjagin theorem. The proof is fairly involved and makes heavy use of differential topology. We have made no attempt to develop these ideas here; for the necessary material, consult [6], [7] or [21, Appendix 2].

The second part of the paper will focus on the solution of the cobordism problem in the case of unoriented manifolds; that is, the determination of the unoriented cobordism groups. The Thom-Pontrjagin theorem reduces this to a homotopy problem, but the solution of that is still difficult. We will have to review several topics before we can give the solution. We will first recall some standard material on Hopf algebras, partitions and symmetric polynomials. Next, we will review the axiomatic definition of the Steenrod algebra, as well as a few of its other properties. Using this we will define the Stiefel-Whitney classes of a vector bundle, and then the Stiefel-Whitney numbers and s-numbers associated with them. We will then compute these numbers for certain submanifolds of projective spaces. With all of this in hand, we will finally turn towards the solution of the unoriented cobordism problem. This will first involve some additional cohomology computations, before we prove Thom's structure theorems.

The third part of the paper will work towards a partial solution of the cobordism problem in the case of oriented manifolds. This will require the introduction of several additional characteristic cohomology classes of vector bundles. We will begin by defining the Euler class of an oriented real vector bundle. Using the Euler class, we can then define the Chern classes of a complex vector bundle. These in turn lead quite naturally to the Pontrjagin classes of a real vector bundle. Using these Pontrjagin classes, we will determine the cohomology of the Thom space $TBSO_r$. Combining this with some results of Serre, we will obtain an approximation for the oriented cobordism groups, as well as explicit generators for the rational oriented cobordism ring. As an application, we will then prove the Hirzebruch signature theorem, which is a special case of the Atiyah-Singer index theorem.

Part 1. The Thom-Pontrjagin Theorem

2. Cobordism Categories

Our motivation in providing the general definition of a cobordism category is the classical situation of compact differentiable manifolds. Let $\mathcal{D}$ be the category with compact smooth manifolds (not necessarily connected, but with locally constant dimension) with boundary as objects and smooth maps preserving boundaries as maps. (Unless otherwise stated all manifolds in this paper will be manifolds with boundary.) $\mathcal{D}$ has finite sums given by disjoint union, and a boundary operator ∂ operating on manifolds by returning their boundary and on maps by restriction to the boundary. We now formalize this situation.

Definition 1. A *cobordism category* $(\mathcal{C}, \partial, \imath)$ is a triple satisfying:

(1) $\mathcal{C}$ is a category having finite sums and an initial object $\emptyset$.
(2) $\partial : \mathcal{C} \to \mathcal{C}$ is an additive functor with $\partial\partial M = \emptyset$ for any object M of $\mathcal{C}$, and $\partial\emptyset = \emptyset$.
(3) $\imath : \partial \to \mathrm{id}$ is a natural transformation of additive functors, with id the identity functor.
(4) $\mathcal{C}$ has a small subcategory $\mathcal{C}_0$ (that is, $\mathcal{C}_0$ is actually a set) such that each element of $\mathcal{C}$ is isomorphic to an element of $\mathcal{C}_0$.

In the case of compact differentiable manifolds, we take $\emptyset$ to be the empty manifold and $\imath$ to be given by the inclusion of ∂M in M. The existence of a small subcategory $\mathcal{D}_0$ follows from the Whitney embedding theorem, stating that every manifold is isomorphic to a submanifold of $\mathbb{R}^\infty$. ([7, Chapter 1, Theorem 3.5])

Of course, there are many other examples of cobordism problems. For a list of 27 distinct instances of cobordism problems, see [21, Chapter 4].

The fundamental definition of cobordism is the following equivalence relation.

Definition 2. In a cobordism category $(\mathcal{C}, \partial, \imath)$, two objects M and N are said to be *cobordant*, $M \equiv N$, if there exist objects U, V of $\mathcal{C}$ for which

$$M + \partial U \cong N + \partial V.$$

We now establish the basic properties of the relation of cobordism.

Proposition 2.1. *Let M, N, M', N' be objects of $\mathcal{C}$.*

(1) *$\equiv$ is an equivalence relation on $\mathcal{C}$, and the equivalence classes form a set.*
(2) *If $M \equiv N$, then $\partial M \cong \partial N$.*
(3) *For all M, $\partial M \equiv \emptyset$.*
(4) *If $M \equiv M'$, $N \equiv N'$, then $M + N \equiv M' + N'$.*

Proof. (1) Reflexivity and symmetry follow from the corresponding properties of isomorphism. For transitivity, suppose that $M \equiv N$ and $N \equiv P$. Then there exist objects U, V, W, X of $\mathcal{C}$ with $M + \partial U \cong N + \partial V$ and $N + \partial W \cong P + \partial X$. Then

$$\begin{aligned} M + \partial(U + W) &\cong M + \partial U + \partial W \\ &\cong N + \partial V + \partial W \\ &\cong P + \partial V + \partial X \\ &\cong P + \partial(V + X) \end{aligned}$$

so $M \equiv P$. The fact that the equivalence classes form a set follows from the existence of the small subcategory $\mathcal{C}_0$. (In fact, this is precisely the reason for that assumption.)

(2) If $M \equiv N$, then there exist objects U, V of $\mathcal{C}$ with $M + \partial U \cong N + \partial V$. Then

$$\begin{aligned} \partial M &\cong \partial M + \emptyset \\ &\cong \partial M + \partial\partial U \\ &\cong \partial(M + \partial U) \\ &\cong \partial(N + \partial V) \\ &\cong \partial N + \partial\partial V \\ &\cong \partial N + \emptyset \\ &\cong \partial N. \end{aligned}$$

(3) $\partial M + \partial\emptyset \cong \emptyset + \partial M$ since $\partial\emptyset = \emptyset$, so $\partial M \equiv \emptyset$.

(4) We have $M + \partial U \cong M' + \partial U'$, $N + \partial V \cong N' + \partial V'$ for some objects U, U', V, V' of $\mathcal{C}$. Then we have $M + N + \partial(U + V) \cong M' + N' + \partial(U' + V')$, so $M + N \equiv M' + N'$.

□

It should be noted that this is not the same as Thom's original definition of cobordism, which was that two manifolds M and N without boundary are cobordant if there exists a manifold T (with boundary) such that $M + N \cong \partial T$. (See [22].) However, it is not hard to see that the two definitions are equivalent.

Proposition 2.2. *Definition 2 agrees with Thom's definition for manifolds without boundary.*

Proof. Suppose M and N are cobordant in our sense. Then there exist manifolds U and V with $M + \partial U \cong N + \partial V$. Let $T_1 = M \times I + U$, $T_2 = N \times I + V$. Then $\partial T_1 = M + M + \partial U$ and $\partial T_2 = N + N + \partial V$. Since $M + \partial U \cong N + \partial V$ we can glue T_1 and T_2 along that common boundary to form a manifold T with $\partial T \cong M + N$.

Now suppose we have T with $\partial T \cong M + N$. Then

$$\begin{aligned} M + \partial T &\cong M + M + N \\ &\cong N + \partial(M \times I) \end{aligned}$$

so $M \equiv N$. □

We now proceed towards the definition of the cobordism semigroup.

Definition 3. We say an object M of $\mathcal{C}$ is *closed* if $\partial M \cong \emptyset$. We say M *bounds* if $M \equiv \emptyset$.

The similarity of the above definitions to the standard definitions in homology are intentional. We will now verify that these definitions satisfy the familiar properties of chains.

Proposition 2.3. *Let M and N be objects of $\mathcal{C}$.*

(1) *Suppose $M \equiv N$. Then M is closed if and only if N is closed, and M bounds if and only if N bounds.*
(2) *If M and N are both closed, then $M + N$ is closed. If M and N both bound, then $M + N$ bounds.*
(3) *If M bounds, then M is closed.*

Proof. (1) The statement about closed objects is immediate from the second part of Proposition 2.1, and the statement about bounding objects follows from the fact that $\equiv$ is an equivalence relation.

(2) If M and N are both closed, then $\partial M \cong \emptyset$ and $\partial N \cong \emptyset$. Thus

$$\begin{aligned}\partial(M+N) &\cong \partial M + \partial N\\ &\cong \emptyset + \emptyset\\ &\cong \emptyset\end{aligned}$$

so $M+N$ is closed. Similarly, if M and N bound, then $M \equiv \emptyset$ and $N \equiv \emptyset$, so $M+N \equiv \emptyset$.

(3) If M bounds then $M \equiv \emptyset$. Thus, by part 2 of Proposition 2.1, $\partial M \cong \partial\emptyset \cong \emptyset$. So M is closed.

$\square$

Combining Propositions 2.1 and 2.3 we obtain the following.

Proposition 2.4. *The set of equivalence classes of $\mathcal{C}$ under $\equiv$ has a commutative, associative operation induced by the addition in $\mathcal{C}$. The class of $\emptyset$ provides an identity element for this operation.*

This allows us to make the following definition.

Definition 4. The *cobordism semigroup* $\Omega(\mathcal{C}, \partial, \imath)$ is the set of equivalence classes of closed objects of $\mathcal{C}$ with the operation induced by addition in $\mathcal{C}$.

Notice that $\Omega(\mathcal{C}, \partial, \imath)$ is simply the quotient of the closed objects by the bounding objects, exactly as in homology theory.

The unoriented cobordism semigroup $\Omega(\mathcal{D}, \partial, \imath)$ is usually written $\mathfrak{N}$.

The fundamental problem of cobordism theory, then, is the determination of this semigroup for specific values of $(\mathcal{C}, \partial, \imath)$. Of course, it is not at all clear how to carry out this calculation. We will focus on that question for the remainder of this paper.

3. (B, f) Manifolds

In order to solve the cobordism problem we will have to endow our manifolds with additional structure, following [10]. First we set notation. Let $G_r(\mathbb{R}^{n+r})$ be the Grassmann manifold of r-planes in $\mathbb{R}^{n+r}$. Let $\gamma^r(\mathbb{R}^{n+r})$ be the canonical r-plane bundle over $G_r(\mathbb{R}^{n+r})$, consisting of pairs of r-planes and points in that r-plane. Using the standard inclusion $\mathbb{R}^{n+r} \hookrightarrow \mathbb{R}^{n+r+1}$, we obtain inclusions $G_r(\mathbb{R}^{n+r}) \hookrightarrow G_r(\mathbb{R}^{n+r+1})$ and $\gamma^r(\mathbb{R}^{n+r}) \hookrightarrow \gamma^r(\mathbb{R}^{n+r+1})$. Using these maps, we define the infinite Grassmannian of r-planes

$$BO_r = \lim_{n\to\infty} G_r(\mathbb{R}^{n+r})$$

and its canonical r-plane bundle

$$\gamma^r = \lim_{n\to\infty} \gamma^r(\mathbb{R}^{n+r}).$$

Recall that BO_r is the universal classifying space for r-plane bundles over paracompact spaces. (See [14, Section 5].) Throughout this paper we will identify vector bundles with their classifying maps. We will denote the total space of a vector bundle ξ by $E(\xi)$, and sometimes will use this to stand for the vector bundle as well. We will also denote the base space by $B(\xi)$, and the projection by $\pi : E(\xi) \to B(\xi)$. Finally, $E(\xi)_0$ will be used for the subset of $E(\xi)$ of non-zero vectors.

Definition 5. Let $f_r : B_r \to BO_r$ be a fibration. Let $\xi : M \to BO_r$ be an r-plane bundle over M. We define a (B_r, f_r) *structure* on ξ to be an equivalence class of liftings $\widetilde{\xi} : M \to B_r$. (That is, $\xi = f_r \circ \widetilde{\xi}$.) We say that two such lifts $\widetilde{\xi_1}$ and $\widetilde{\xi_2}$ are equivalent if they are homotopic; that is, if there is a map $H : M \times I \to B_r$ with $H|_{M\times 0} = \widetilde{\xi_1}$, $H|_{M\times 1} = \widetilde{\xi_2}$ and $f_r H(m,t) = \xi(m)$ for all m and t.

We will pass this definition to manifolds by means of the normal bundle. Let us recall its construction. Suppose we have an embedding $i : M^n \hookrightarrow \mathbb{R}^{n+r}$ of an n-dimensional manifold M^n. (More generally we can consider an immersion of M into any manifold.) The normal bundle $\nu(i)$ of this embedding is defined to be the quotient of the pullback $i^*\tau(\mathbb{R}^{n+r})$ of the tangent bundle $\tau(\mathbb{R}^{n+r})$ by the tangent bundle $\tau(M)$. We can identify the fiber of $\nu(i)$ over a point $m \in M$ with the set of vectors in $\mathbb{R}^{n+r}$ orthogonal to M at m. The classifying map for $\nu(i)$ is obtained as follows : We define a bundle map

$$n : E(\nu(i)) \to \gamma^r(\mathbb{R}^{n+r})$$

by sending a pair (m, x) of a point m and a vector x in the fiber $E(\nu(i))_m$ over m to the pair $(E(\nu(i))_m, x)$ in $\gamma^r(\mathbb{R}^{n+r})$. Composing with the inclusion of $\gamma^r(\mathbb{R}^{n+r})$ in γ^r, and then taking the induced map on base spaces, yields the classifying map $\nu(i) : M \to BO_r$.

Of course, this does not yet give us a well-defined notion of a (B_r, f_r) structure on M, since the normal bundle is dependent on the embedding. However, the following lemma will be enough to allow us to make our definition.

Lemma 3.1. *If r is sufficiently large, depending only on M, then there is a bijective correspondence between the (B_r, f_r) structures on the normal bundle of any two embeddings $i_1, i_2 : M^n \hookrightarrow \mathbb{R}^{n+r}$. (That is, there is a bijective correspondence between lifts of the normal bundles $\nu(i_1)$ and $\nu(i_2)$.)*

Proof. For r sufficiently large any two embeddings i_1, i_2 are regularly homotopic, say by a homotopy $H : M \times I \to \mathbb{R}^{n+r}$. Further, if H' is another such homotopy, then there is a homotopy $K : M \times I \times I \to \mathbb{R}^{n+r}$ of H and H' with $K|_{M\times 0\times s} = i_1$ and $K|_{M\times 1\times s} = i_2$ for all s.

The sequence of normal bundles

$$H|^*_{M\times t}\tau(\mathbb{R}^{n+r})/\tau(M)$$

then gives a homotopy of $\nu(i_1)$ and $\nu(i_2)$. Similarly, K induces a homotopy between the two homotopies defined by H and H'. In this way we get a well-defined equivalence of the two normal bundles. Our desired bijection now follows immediately from the homotopy lifting theorem. □

Now, let (B, f) be a sequence of fibrations $f_r : B_r \to BO_r$ together with maps $g_r : B_r \to B_{r+1}$ such that the diagram

$$\begin{tikzcd} B_r \arrow[r, "g_r"] \arrow[d, "f_r"'] & B_{r+1} \arrow[d, "f_{r+1}"] \\ BO_r \arrow[r, "j_r"] & BO_{r+1} \end{tikzcd}$$

commutes, where $j_r : BO_r \to BO_{r+1}$ is the inclusion induced by the standard inclusions $G_r(\mathbb{R}^{n+r}) \hookrightarrow G_{r+1}(\mathbb{R}^{n+r+1})$ extending an r-plane by direct sum with the copy of $\mathbb{R}$ in the final coordinate. Suppose we have a (B_r, f_r) structure $\widetilde{\nu}(i) : M^n \to B_r$ on the normal bundle $\nu(i)$ of an embedding $i : M \to \mathbb{R}^{n+r}$. This induces a (B_{r+1}, f_{r+1}) structure on the normal bundle $\nu(i')$ of the embedding $i' = i \times 0 : M \to \mathbb{R}^{n+r+1}$ by $\widetilde{\nu}(i') = g_r\widetilde{\nu}(i)$, since

$$\begin{aligned} f_{r+1}\widetilde{\nu}(i') &= f_{r+1}g_r\widetilde{\nu}(i) \\ &= j_r f_r\widetilde{\nu}(i) \\ &= j_r\nu(i) \\ &= \nu(i') \end{aligned}$$

by commutativity and the choices of the various inclusions.

Definition 6. A (B, f) *structure* on a manifold M is an equivalence class of compatible (under the above construction) (B_r, f_r) structures on the normal bundles of inclusions of M, where equivalence is given by agreement for sufficiently large r subject to the bijection of Lemma 3.1. (This guarantees that the notion of (B, f) structure is independent of the embedding of M in Euclidean space.)

(B, f) structures are fairly difficult to get a handle on. We will attempt to illuminate them at least a bit by means of two examples, which will in fact be the examples we will consider later.

(1) Take $B_r = BO_r$ and f_r to be the identity map. Then every manifold will have a unique $(BO, 1)$ structure. Thus the class of $(BO, 1)$ manifolds is simply the class of all unoriented manifolds.
(2) Let BSO_r be the classifying space for oriented r-plane bundles. It is a double cover of BO_r, corresponding to the two possible orientations of an r-plane. Take $B_r = BSO_r$ and f_r to be the map which ignores the orientation. Then every oriented manifold has a unique (BSO, f) structure, the choice of lifting being given by the orientation, so that the class of (BSO, f) manifolds is the same as the class of oriented manifolds.

The second example in particular suggests that perhaps the best way to think of (B, f) structures are as generalizations of orientations. In fact, when we are proving facts about (B, f) manifolds we will be dragging the (B, f) structures along exactly as we would drag orientations along, having to pause every few moments to make sure that we have the correct structure.

More generally, any class of manifolds with a classifying space can be defined through (B, f) structures. For example, the class of complex manifolds is simply the class of manifolds with (BU, f) structures, and the class of spin manifolds is simply the class of manifolds with (B, f) structures where B_r is the classifying space for spin manifolds. (It is a a two-connective covering space of BSO_r.) See [21, Chapter 4].

In order to make use of this in cobordism, we must show how to use a (B, f) structure on a manifold to induce a (B, f) structure on its boundary. More generally, let $M^m \hookrightarrow W^n$ be any embedding with trivial normal bundle ν.

Pick an embedding $i : M \hookrightarrow \mathbb{R}^{m+r}$. Since ν is trivial we can extend i to an embedding $j_0 : U \hookrightarrow \mathbb{R}^{n+r} = \mathbb{R}^{m+r} \times \mathbb{R}^{n-m}$ of some neighborhood $U \subseteq W$ of M with U meeting $\mathbb{R}^{m+r}$ orthogonally along M. We can then extend this to an embedding $j : W \hookrightarrow \mathbb{R}^{n+r}$. Since W meets $\mathbb{R}^{m+r}$ orthogonally along M, the normal planes of a point m of M in $\mathbb{R}^{m+r}$ will simply be the restriction to $\mathbb{R}^{m+r}$ of the normal planes at m considered as a point in W. Thus we have $\nu(j)|_M = \nu(i)$, and therefore if $\widetilde{\nu}(j)$ is a (B, f) structure on W, $\widetilde{\nu}(j)|_M$ will be a (B, f) structure on M.

This is particularly useful in three special cases.

(1) If $M \to W$ is an isomorphism, then the normal bundle is zero dimensional and thus trivial. So a (B, f) structure on W induces one on M.
(2) If $M \hookrightarrow M + W$ is an inclusion of a direct summand, then the normal bundle is again zero dimensional and thus trivial. So a (B, f) structure on a manifold induces one on its direct summands.
(3) If $\partial W \hookrightarrow W$ is the inclusion of the boundary, then the normal bundle is trivial by the choice of either an outer or an inner trivialization. We will always choose the inner trivialization, and in this way a (B, f) structure on a manifold induces one on its boundary.

4. (B, f) Cobordism

If we take all manifolds with (B, f) structures, we obtain a special kind of cobordism category which will be very useful in determining cobordism semigroups.

Definition 7. Let $\mathcal{C}$ be the category whose objects are compact (B, f) manifolds (that is, manifolds with a specified (B, f) structure) and whose maps are the smooth, boundary preserving inclusions with trivial normal bundle inducing compatible (B, f) structures. Let $\partial : \mathcal{C} \to \mathcal{C}$ be the boundary functor, inducing (B, f) structures by the inner trivialization. Let $\imath : \partial \to I$ be the inclusion of the boundary with inner trivialization. Then $(\mathcal{C}, \partial, \imath)$ is a cobordism category, called the *cobordism category of* (B, f) *manifolds.* We will denote by $\Omega(B, f)$ the cobordism semigroup $\Omega(\mathcal{C}, \partial, \imath)$. We can then write

$$\Omega(B, f) = \bigoplus_{n=0}^{\infty} \Omega_n(B, f)$$

where $\Omega_n(B, f)$ is the subsemigroup of equivalence classes of n dimensional manifolds.

At a first glance it may appear that the cobordism category of $(BO, 1)$ manifolds is not the same as the cobordism category of unoriented manifolds $(\mathcal{D}, \partial, \imath)$, since we have thrown away many of the maps in the category. However, since the only maps which are relevant in the formation of the cobordism semigroup are isomorphisms, inclusions of direct summands and inclusions of boundaries, we do get a canonical isomorphism

$$\Omega(BO, 1) \cong \mathfrak{N}.$$

In fact, for (B, f) cobordism categories, $\Omega(B, f)$ is not merely a semigroup.

Proposition 4.1. *$\Omega(B, f)$ is an abelian group.*

Proof. Take a (B, f) manifold $M^n \in \Omega(B, f)$ and choose an embedding $i : M \hookrightarrow \mathbb{R}^{n+r}$ with a lifting $\widetilde{\nu}(i) : M \to B_r$ inducing the correct (B, f) structure on M. Now let $j : M \times I \hookrightarrow \mathbb{R}^{n+r+1}$ be the obvious embedding. Then if $\pi : M \times I \to M$ is the projection onto M, we will have $\nu(j) = \nu(i)\pi$. Now, since $f_r\widetilde{\nu}(i)\pi = \nu(i)\pi = \nu(j)$, we get a (B, f) structure $\widetilde{\nu}(j) : M \times I \to B_r$ given by $\widetilde{\nu}(j) = \widetilde{\nu}(i)\pi$. We see then that the induced (B, f) structure on $M \times 0$ is the same as that on M, so that $M \cong M \times 0$ as (B, f) manifolds. So, if we let $M' = M \times 1$ with the inner induced (B, f) structure, we have that $M + M' \cong \partial(M \times I) \equiv \emptyset$, and thus $M + M' \equiv \emptyset$. Thus M has an inverse, namely M', and $\Omega(B, f)$ is an abelian group. $\square$

We now need to prepare for the statement of the generalized Thom-Pontrjagin theorem. We first recall the notion of the Thom space of a vector bundle. Let $\xi : M \to BO_r$ be a vector bundle with a Riemannian metric induced in the usual way from γ^r. We define the *Thom space* of ξ, $T\xi$, to be the total space $E(\xi)$ with all vectors of $E(\xi)$ of length greater than or equal to 1 identified to a point. (Alternatively, we can identify $T\xi$ with the one-point compactification of $E(\xi)$.) We write $t_0(\xi)$ for this point, or just t_0 if the vector bundle is clear in context. We will write TBO_r for the Thom space $T\gamma^r$ and TB_r for the Thom space $Tf_r^*\gamma^r$.

Suppose we have a vector bundle η on a space N and a continuous map $g : M \to N$. If we let $\xi = g^*\eta$, then we see that g induces a map $Tg : (T\xi, t_0(\xi)) \to (T\eta, t_0(\eta))$.

Let us now apply this construction to the (B, f) commutative diagram. Recall that we have a map $j_r : BO_r \hookrightarrow BO_{r+1}$. By our definition of j_r we see that $j_r^*(\gamma^{r+1}) = \gamma^r \oplus \varepsilon^1$ where ε^1 is the trivial bundle on BO_r. If we consider the Thom space of this bundle, we see that the effect of adding the ε^1 will be to form the reduced suspension $\Sigma T\gamma^r$. In this way we get a map

$$Tj_r : \Sigma T\gamma^r \to T\gamma^{r+1}.$$

Now, we have a map $g_r^* f_{r+1}^* \gamma^{r+1} \to f_{r+1}^* \gamma^{r+1}$ induced by g_r. But by commutativity, $g_r^* f_{r+1}^* \gamma^{r+1} = f_r^* j_r^* \gamma^{r+1}$. Thus, we have a map

$$f_r^* j_r^* \gamma^{r+1} \to f_{r+1}^* \gamma^{r+1}.$$

This yields a map

$$Tg_r : Tf_r^* j_r^* \gamma^{r+1} \to Tf_{r+1}^* \gamma^{r+1}.$$

But, by definition $Tf_{r+1}^* \gamma^{r+1} = TB_{r+1}$, and by our above observations

$$\begin{aligned} Tf_r^* j_r^* \gamma^{r+1} &= Tf_r^*(\gamma^r \oplus \varepsilon^1) \\ &= T(f_r^* \gamma^r \oplus f_r^* \varepsilon^1) \\ &= \Sigma T f_r^* \gamma^r \\ &= \Sigma TB_r. \end{aligned}$$

So, in fact,

$$Tg_r : \Sigma TB_r \to TB_{r+1}.$$

Combining all of these Thom maps, we obtain from our original commutative diagram a new commutative diagram

$$\begin{tikzcd} \Sigma TB_r \arrow[r, "Tg_r"] \arrow[d, "\Sigma Tf_r"'] & TB_{r+1} \arrow[d, "Tf_{r+1}"] \\ \Sigma TBO_r \arrow[r, "Tj_r"] & TBO_{r+1}. \end{tikzcd}$$

Now, since

$$\Sigma_{\#} : \pi_{n+r}(TB_r, t_0) \to \pi_{n+r+1}(\Sigma TB_r, t_0)$$

and

$$Tg_{r\#} : \pi_{n+r+1}(\Sigma TB_r, t_0) \to \pi_{n+r+1}(TB_{r+1}, t_0)$$

we obtain a map

$$Tg_{r\#} \circ \Sigma_{\#} : \pi_{n+r}(TB_r, t_0) \to \pi_{n+r+1}(TB_{r+1}, t_0).$$

This allows us to define the stable homotopy group

$$\lim_{r\to\infty} \pi_{n+r}(TB_r, t_0).$$

Having done all of this preliminary work, we are finally in a position to state the generalized Thom-Pontrjagin Theorem.

Theorem 4.2 (The Thom-Pontrjagin Theorem). *$\Omega_n(B, f)$, the cobordism group of n-dimensional (B, f) manifolds, is isomorphic to the stable homotopy group $\lim_{r\to\infty} \pi_{n+r}(TB_r, t_0)$.*

This truly remarkable theorem transforms the cobordism problem into a homotopy problem. The proof is rather involved and will be the topic of the next section.

5. The Proof of the Thom-Pontrjagin Theorem

Let M^n be a (B, f) manifold. Let $i : M \hookrightarrow \mathbb{R}^{n+r}$ be an embedding, $\nu = \nu(i) : M \to BO_r$ the normal bundle, $N = E(\nu)$ the total space of ν and $\pi : N \to M$ the projection. Choose a lifting $\widetilde{\nu} : M \to B_r$ giving the correct (B, f) structure. We will first construct a map from $\Omega_n(B, f)$ to $\lim_{r\to\infty} \pi_{n+r}(TB_r, t_0)$. The final map is not too complicated, but we will go carefully through its construction to insure that the (B, f) structures are well behaved.

Considering N as a submanifold of $\mathbb{R}^{n+r} \times \mathbb{R}^{n+r}$ (via the inclusion i), we have the exponential map

$$\exp : \mathbb{R}^{n+r} \times \mathbb{R}^{n+r} \to \mathbb{R}^{n+r}$$

given by $\exp(i(m), x) = i(m) + x$. Clearly exp is differentiable and $\exp|_{i(M)\times 0} = i$. Thus, since exp is just the usual exponential map, $\exp|_{N_\varepsilon}$ will be an embedding for some sufficiently small $\varepsilon > 0$, where N_ε is the subset of N of vectors of length less than or equal to ε.

Now, define $c_0 : \mathbb{R}^{n+r} \to N_\varepsilon/\partial N_\varepsilon$ by sending the interior of N_ε to itself, and $\mathbb{R}^{n+r} - \operatorname{int} N_\varepsilon$ to the point ∂N_ε, and extend to

$$c : S^{n+r} \to N_\varepsilon/\partial N_\varepsilon$$

by sending $\infty \in S^{n+r}$ to ∂N_ε as well. Let

$$\varepsilon^{-1} : N_\varepsilon/\partial N_\varepsilon \to TN$$

be multiplication by $1/\varepsilon$; then

$$\varepsilon^{-1} \circ c : S^{n+r} \to TN.$$

Note that $\varepsilon^{-1} \circ c$ sends $\operatorname{int} N_\varepsilon$ diffeomorphically to $TN - t_0$.

Next, let $j_n^r : \gamma^r(\mathbb{R}^{n+r}) \to \gamma^r$ be the standard inclusion and $n : N \to \gamma^r(\mathbb{R}^{n+r})$ the map $(m, x) \mapsto (N_m, x)$ of Section 3. Then we have a map

$$(j_n^r \circ n) \times (\widetilde{\nu} \circ \pi) : N \to \gamma^r \times B_r.$$

Note that this is injective since $n : N \to \gamma^r(\mathbb{R}^{n+r})$ is. Since (here $p : \gamma^r \to BO_r$ is the projection)

$$\begin{aligned} f_r(\widetilde{\nu}\pi(m, x)) &= \nu(m) \\ &= pj_n^r(N_m, x) \\ &= p(j_n^r n(m, x)) \end{aligned}$$

it actually has image inside of $f_r^*\gamma^r$. Thus we have a bundle map

$$(j_n^r \circ n) \times (\widetilde{\nu} \circ \pi) : N \to f_r^*\gamma^r$$

inducing a map

$$T\big((j_n^r \circ n) \times (\widetilde{\nu} \circ \pi)\big) : TN \to Tf_r^*\gamma^r = TB_r.$$

So, finally, define

$$\theta_{i,\widetilde{\nu},\varepsilon}(M) : (S^{n+r}, \infty) \to (TB_r, t_0)$$

to be the composition

$$T\big((j_n^r \circ n) \times (\widetilde{\nu} \circ \pi)\big) \circ \varepsilon^{-1} \circ c.$$

Essentially this map embeds $\operatorname{int} N_\varepsilon$ into $TB_r - t_0$, and sends the rest of S^{n+r} into t_0.

Now, clearly decreasing ε gives a homotopic map θ. Similarly, an equivalent choice of $\widetilde{\nu}$ will give a homotopic map θ, by the definition of equivalence of lifting. In this way we have a well-defined

$$\theta_i(M) \in \pi_{n+r}(TB_r, t_0).$$

Lemma 5.1. *Let $\iota : \mathbb{R}^{n+r} \to \mathbb{R}^{n+r+1}$ be the usual inclusion. Then the inclusion $\iota \circ i : M \hookrightarrow \mathbb{R}^{n+r+1}$ gives rise to the map $Tg_r \circ \Sigma\theta_i$; that is,*

$$\theta_{\iota i} = Tg_r \circ \Sigma\theta_i.$$

Proof. Since all of our extensions were done in the same way, we have the following identities for the data for ιi, where $l_n^r : \gamma^r(\mathbb{R}^{n+r}) \to \gamma^{r+1}(\mathbb{R}^{n+r+1})$, $\eta : N \to E(\nu(\iota i))$ and $s_{n+r} : S^{n+r} \to S^{n+r+1}$ are obtained in the same way : $\widetilde{\nu}(\iota i) = g_r\widetilde{\nu}$, $\varepsilon_{\iota i}^{-1} c_{\iota i} s_{n+r} = T\eta \circ \varepsilon^{-1} c$, $\pi = \pi_{\iota i}\eta$, $n_{\iota i}\eta = l_n^r n$. Thus,

$$\begin{aligned} \theta_{\iota i} s_{n+r} &= T\big((j_n^{r+1} n_{\iota i}) \times (\widetilde{\nu}(\iota i)\pi_{\iota i})\big)\varepsilon_{\iota i}^{-1} c_{\iota i} s_{n+r} \\ &= T\big((j_n^{r+1} n_{\iota i}) \times (\widetilde{\nu}(\iota i)\pi_{\iota i})\big) T\eta \circ \varepsilon^{-1} c \\ &= T\big((j_n^{r+1} n_{\iota i}\eta) \times (\widetilde{\nu}(\iota i)\pi_{\iota i}\eta)\big)\varepsilon^{-1} c \\ &= T\big((j_n^{r+1} l_n^r n) \times (g_r\widetilde{\nu}\pi)\big)\varepsilon^{-1} c \\ &= T\big((j_n^r n) \times (g_r\widetilde{\nu}\pi)\big)\varepsilon^{-1} c \\ &= Tg_r\theta_i. \end{aligned}$$

Therefore we see that $\theta_{\iota i} = Tg_r \circ \Sigma\theta_i$. □

Lemma 5.1 shows that $\theta_i(M)$ can actually be thought of as an element of $\lim_{r\to\infty} \pi_{n+r}(TB_r, t_0)$.

Lemma 5.2. *Let $i' : M + \partial W \hookrightarrow \mathbb{R}^{n+r}$ be an embedding. Then if r is sufficiently large (depending only on M), θ_i and $\theta_{i'}$ are homotopic.*

Proof. The idea here is to get a (B, f) embedding of $M \times I + W$ in $\mathbb{R}^{n+r} \times I$ agreeing with i on $M \times 0$ and with i' on $M \times 1 + \partial W$. We can then use this embedding to construct the desired homotopy.

Let $\widetilde{\nu}(i') : M + \partial W \to B_r$ be a lift of $\nu(i')$ compatible with the (B, f) structure on M. For r sufficiently large we can choose a regular homotopy $H_0 : M \times I \to \mathbb{R}^{n+r}$ of i and $i'|_M$; that is, $H_0|_{M\times 0} = i$ and $H_0|_{M\times 1} = i'|_M$. We can then easily modify H_0 to a regular homotopy

$$H : M \times I \to \mathbb{R}^{n+r}$$

of i and $i'|_M$ with $H|_{M\times t} = i$ for t near 0 and $H|_{M\times t} = i'|_M$ for t near 1.

Now, by the collaring neighborhood theorem we can find a map

$$K : W \to \mathbb{R}^{n+r} \times (0, 1]$$

with $K|_{\partial W} = i'|_{\partial W} \times 1$ (that is, the last coordinate of K is always 1 on ∂W, and here K agrees with i') and which is an embedding on a tubular neighborhood of ∂W.

Consider the map

$$(H \times \pi_I) + K : M \times I + W \to \mathbb{R}^{n+r} \times I$$

where $\pi_I : M \times I \to I$ is the projection. By our choices of H and K this will be an embedding on some neighborhood of the boundary $M \times 0 + M \times 1 + \partial W$. Thus we may find a homotopic embedding

$$F : M \times I + W \hookrightarrow \mathbb{R}^{n+r} \times I$$

agreeing with $(H \times \pi_I) + K$ on that neighborhood. So $F|_{M\times 0} = i$ and $F|_{M\times 1+\partial W} = i'$. We must check that F gives the correct (B, f) structure on $M \times 0$ and $M \times 1 + \partial W$. We have a normal map $\nu(F|_{M\times I}) : M \times I \to BO_r$ for the embedding

$$F|_{M\times I} : M \times I \hookrightarrow \mathbb{R}^{n+r} \times I;$$

choose a lifting $\widetilde{\nu}(F|_{M\times I})$. Since H is constant near 0 and 1, so is $F|_{M\times I}$ and thus so is $\nu(F|_{M\times I})$. Thus we can actually choose $\widetilde{\nu}(F|_{M\times I})$ to agree with $\widetilde{\nu}(i)$ at 0 and $\widetilde{\nu}(i')$ at 1.

Further, since the (B, f) structure on ∂W is induced by that of W, we can choose a lifting of the normal map of $F|_W$ agreeing with $\widetilde{\nu}(i'|_{\partial W})$. Combining these we see that F induces the correct structures and thus is a (B, f) map.

So we have a (B, f) embedding of $M \times I + W$ in $\mathbb{R}^{n+r} \times I$. If we apply our θ construction to this (not to the manifold $M \times I + W$, but rather just running the construction ignoring the factor of I, we obtain a map

$$\bar{\theta} : S^{n+r} \times I \to TB_r$$

which clearly satisfies $\bar{\theta}|_{S^{n+r}\times 0} = \theta_i$ and $\bar{\theta}|_{S^{n+r}\times 1} = \theta_{i'}$. This is the desired homotopy. □

Applying Lemma 5.1 repeatedly to our initial embedding i to get r sufficiently large, and then applying Lemma 5.2 with $W = \emptyset$, we see that $\theta_i(M)$ is independent of the embedding i, at least as an element of $\lim_{r\to\infty} \pi_{n+r}(TB_r, t_0)$.

Further, suppose M and M' are cobordant. Then there exist (B, f) manifolds U and U' with $M + \partial U \cong M' + \partial U'$. Applying Lemma 5.2,

$$\theta(M) \sim \theta(M + \partial U) \sim \theta(M' + \partial U') \sim \theta(M').$$

Thus the homotopy class of θ is constant on cobordism classes. Thus, finally, we have a well-defined map

$$\Theta : \Omega_n(B, f) \to \lim_{r\to\infty} \pi_{n+r}(TB_r, t_0).$$

Roughly, Θ sends a manifold M to a map $S^{n+r} \to TB_r$ embedding its normal bundle around the zero-section in TB_r.

Proposition 5.3. Θ *is a homomorphism.*

Proof. Choose $[M_1], [M_2] \in \Omega_n(B, f)$ and choose embeddings $i_1 : M_1 \to \mathbb{R}^{n+r}$, $i_2 : M_2 \to \mathbb{R}^{n+r}$ which send M_1 and M_2 into different half-planes. Then choose the normal neighborhoods N^i_ε of M_i small enough so that they also lie in these half-planes. We then see that $\Theta(M_1 + M_2)$ is simply given by the composition

$$S^{n+r} \longrightarrow S^{n+r} \vee S^{n+r} \xrightarrow{\Theta(M_1)\vee\Theta(M_2)} TB_r$$

where $\vee$ is the wedge product and the first map is collapsing the equator to a point, yielding the two copies of S^{n+r}. However, this is also the definition of the sum of homotopy classes $\Theta(M_1) + \Theta(M_2)$. So Θ is a homomorphism of abelian groups. □

Proposition 5.4. Θ *is surjective.*

Proof. Choose a representative $\theta : (S^{n+r}, p) \to (TB_r, t_0)$ of a class of $\lim_{r\to\infty} \pi_{n+r}(TB_r, t_0)$. The idea here is simple; we have $Tf_r \circ \theta : (S^{n+r}, p) \to (TBO_r, t_0)$, and we have a Grassmann manifold BO_r embedded as the the zero section of TBO_r. We would like to take $(Tf_r \circ \theta)^{-1}(BO_r)$ for $\Theta^{-1}(\theta)$. Unfortunately, this need not even be a manifold, and a bit of work will be required to remedy this.

So, we have

$$Tf_r \circ \theta : (S^{n+r}, p) \to (TBO_r, t_0).$$

Since TBO_r has $\{T\gamma^r(\mathbb{R}^{r+s})\}$ as an open cover, and since $Tf_r \circ \theta(S^{n+r})$ is compact, we must have $Tf_r \circ \theta(S^{n+r}) \subseteq T\gamma^r(\mathbb{R}^{r+s})$ for some s. We now deform $Tf_r \circ \theta$ to a map h_0 satisfying four conditions.

(1) h_0 is differentiable on the preimage of some neighborhood of $G_r(\mathbb{R}^{r+s})$.
(2) h_0 is transverse regular on $G_r(\mathbb{R}^{r+s})$.
(3) Setting $M = h_0^{-1}(G_r(\mathbb{R}^{r+s}))$ (M is a manifold by (1) and (2)) there is some tubular neighborhood N of M (so N is isomorphic to the normal bundle of M) such that $h_0|_N$ is a bundle map.
(4) There is a closed set Z, containing t_0 in its interior, for which $Tf_r \circ \theta = h_0$ on $h_0^{-1}(Z)$.

(See [21, Appendix 2, p. 24].) Note that (1) and (2) use the fact that $TBO_r - t_0$ is a manifold. In fact, since $h_0|_M$ classifies the normal bundle of M, we can further deform it to a map

$$h : (S^{n+r}, p) \to (TBO_r, t_0)$$

satisfying the above properties and such that

$$h|_M = \nu : M \to G_r(\mathbb{R}^{r+s}) \hookrightarrow BO_r$$

and h is simply the usual translation of vectors in some tubular neighborhood of M.

Now, $Tf_r : TB_r \to TBO_r$ is a fibration away from t_0, and $t_0 \notin Tf_r \circ \theta(S^{n+r} - h^{-1}(\operatorname{int} Z))$, so by the covering homotopy theorem we can find a homotopy

$$H_0 : (S^{n+r} - h^{-1}(\operatorname{int} Z)) \times I \to TB_r$$

of θ on $S^{n+r} - h^{-1}(\operatorname{int} Z)$ covering $Tf_r \circ \theta$. That is, we have $H_0 = \theta$ at 0 and $Tf_r \circ H = h$ for all $t \in I$. By (4) we may take H_0 to be pointwise fixed on the boundary of Z. Thus, we may extend H_0 to a homotopy

$$H : (S^{n+r}, p) \times I \to (TB_r, t_0)$$

by sending all of $h^{-1}(Z)$ to that point p. Set $\theta_1 = H|_{S^{n+r}\times 1}$.

We have $\theta_1^{-1}(B_r) = h^{-1}(BO_r) = h^{-1}(G_r(\mathbb{R}^{r+s})) = M$. In fact, $\theta_1|_M$ gives a lift of the normal map $h|_M$ since $Tf_r \circ H|_{S^{n+r}\times 1} = h$ and we chose h to agree with the normal map of M. This then makes M into a (B, f) manifold.

Now, take $\Theta(M)$ with this (B, f) structure. Since we chose h to be just translation near M, tracing through the definition of $\Theta(M)$ we see than we can find N_ε such that $\theta_1|_{N_\varepsilon} = \Theta(M)|_{N_\varepsilon}$. Since $TB_r - B_r$ can be deformed to t_0, we can further homotope θ_1 all the way to $\Theta(M)$. So $\theta \sim \theta_1 \sim \Theta(M)$, and Θ is surjective. □

Proposition 5.5. Θ *is injective.*

Proof. Let M be a (B, f) manifold with $\Theta(M) = 0$. Then there is an r such that $\Theta(M) : (S^{n+r}, p) \to (TB_r, t_0)$ is homotopic to the constant map $\theta_0 : S^{n+r} \to t_0$ by a homotopy

$$H : S^{n+r} \times I \to TB_r.$$

Choose H so that $H|_{S^{n+r}\times t} = \Theta(M)$ for $t \leq \delta$. As above, by compactness $Tf_r \circ H(S^{n+r} \times I) \subseteq T\gamma^r(\mathbb{R}^{r+s})$ for some s. Again, as above, we can deform $Tf_r \circ H$ to a map

$$K : S^{n+r} \times I \to TBO_r$$

which is smooth near $G_r(\mathbb{R}^{r+s})$, transverse regular on $G_r(\mathbb{R}^{r+s})$ and such that $K = Tf_r \circ H$ on $N_\varepsilon \times [0, \delta]$, for some $\delta > 0$. By transversity, $W = H^{-1}(G_r(\mathbb{R}^{r+s}))$ is a submanifold of $\mathbb{R}^{n+r} \times I$. In fact, since $K|_{S^{n+r}\times 1} = Tf_r \circ H|_{S^{n+r}\times 1}$ is the constant map at t_0 we see that $\partial W \subseteq \mathbb{R}^{n+r} \times 0$. Since $K = Tf_r \circ H$ on $N_\varepsilon \times [0, \delta]$, we see that $\partial W = M$.

Thus, we have only to find a (B, f) structure on W compatible with that on M. To do this, further homotope K to get $K|_W$ to be the normal map. Now, applying the covering homotopy theorem to the homotopy from $Tf_r \circ H$ to K, we obtain a homotopy from H to a map

$$\theta : S^{n+r} \times I \to TB_r$$

with $\theta|_{S^{n+r}\times t} = \Theta(M)$ for small t and $\theta|_{S^{n+r}\times 1} = \theta_0$. Further, $\theta|_W$ is a lifting of the normal map $K|_W$. This gives W a (B, f) structure which does indeed induce the correct (B, f) structure on $M = \partial W$. Thus, $M \equiv 0$ in $\Omega_n(B, f)$, so Θ is injective. $\square$

This completes the proof of the generalized Thom-Pontrjagin theorem.

Part 2. The Unoriented Cobordism Ring

6. Hopf Algebras

Having proven the generalized Thom-Pontrjagin theorem, we will now turn our attention towards solving some of the specific cobordism problems. We will begin with the simplest case, that of unoriented cobordism. The homotopy problem in this case is still quite difficult, however, and we will require a number of tools of algebraic topology in order to solve it. We will spend the next six sections reviewing the necessary material. We begin with some simple results on Hopf algebras. For a much more complete treatment, see [13].

We first recall the definition of a coalgebra.

Definition 8. Let M be a graded module over a field k. We will call M a *k-coalgebra* if it possesses a coproduct

$$\psi : M \to M \otimes M$$

(a map of graded modules) together with a counit

$$\varepsilon : M \to k$$

satisfying two conditions.

(1) ψ is coassociative; that is, the diagram

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\psi"'] & M \otimes M \arrow[d, "\psi\otimes\mathrm{id}"] \\ M \otimes M \arrow[r, "\mathrm{id}\otimes\psi"] & M \otimes M \otimes M \end{tikzcd}$$

is commutative.

(2) The two compositions

$$\begin{tikzcd} & & k \otimes M \arrow[dr] & \\ M \arrow[r, "\psi"] & M \otimes M \arrow[ur, "\varepsilon\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}\otimes\varepsilon"'] & & M \\ & & M \otimes k \arrow[ur] & \end{tikzcd}$$

are both the identity map.

We say that M is *connected* if $\varepsilon_0 : M_0 \to k$ is an isomorphism.

We first prove a simple property of coproducts for connected coalgebras.

Lemma 6.1. *If M is a connected coalgebra, then for all $m \in M$,*

$$\psi(m) = m \otimes 1 + 1 \otimes m + \sum_{\deg m', m'' \neq n} m' \otimes m''$$

where the sum is over elements not contained in $M^0 \otimes M^n$ or $M^n \otimes M^0$.

Proof. Let n be the degree of m. If we trace m through the sequence of maps

$$M \xrightarrow{\psi} M \otimes M \xrightarrow{\varepsilon\otimes 1} k \otimes M \longrightarrow M$$

and use condition (2) in the definition of coalgebras we get

$$\sum \varepsilon(m')m'' = m \tag{1}$$

where

$$\psi(m) = \sum m' \otimes m''.$$

Since M is connected, $M^0 \otimes M^n$ is isomorphic to M^n, so we can combine all of the terms

$$\sum_{\deg m''=n} m' \otimes m''$$

into a single tensor $m_0' \otimes m_0''$. But since M is a graded module, (1) still holds if we restrict to m'' of degree n. Thus $\varepsilon(m_o')m_0'' = m$, so

$$m_0' \otimes m_0'' = 1 \otimes m.$$

Using the other sequence of maps, we find that the element of $M^n \otimes M^0$ in $\psi(m)$ is $m \otimes 1$, which completes the proof. $\square$

In some cases M will be both a k-algebra and a k-coagebra.

Definition 9. Let $\mathcal{A}$ be a graded k-module which is both a k-algebra and a k-coalgebra. We say that $\mathcal{A}$ is a *Hopf algebra* if the coproduct ψ is a homomorphism of graded k-algebras. We say that $\mathcal{A}$ is *connected* if it is connected as a coalgebra.

The main result we need concerns faithful modules over Hopf algebras.

Proposition 6.2. *Let $\mathcal{A}$ be a connected Hopf algebra over k. Let M be a connected coalgebra over k which is a left $\mathcal{A}$-module and such that its coproduct Δ is a map of A-modules. Let $\nu : \mathcal{A} \to M$ be defined by $\nu(a) = a \cdot 1$, where 1 is the image of $1 \in k$ in M_0. Then if ν is injective, M is a free left module over A.*

Proof. Let $\mathcal{A}^+$ be the submodule of $\mathcal{A}$ of elements of positive degree. Define $N = M/\mathcal{A}^+M$. Let $\pi : M \to N$ be the quotient map, and choose any k-splitting $f : N \to M$. (That is, $\pi f = 1_N$, and f is a homomorphism of k-vector spaces.) Define

$$\varphi : \mathcal{A} \otimes N \to M$$

by $\varphi(a \otimes n) = af(n)$. Then φ is an $\mathcal{A}$-module homomorphism. We will prove that it is an isomorphism.

We prove surjectivity by induction on the degree of $m \in M$. For the degree zero case,

$$\varphi : (\mathcal{A} \otimes N)^0 = \mathcal{A}^0 \otimes N^0 \to M^0$$

is simply the identity map on k by connectivity, so it is certainly surjective. Now suppose that we know that φ is surjective for all degrees $< \deg m$. We calculate

$$\begin{aligned} \pi\big(m - \varphi(1 \otimes \pi(m))\big) &= \pi\big(m - f\pi(m)\big) \\ &= \pi(m) - \pi f \pi(m) \\ &= \pi(m) - \pi(m) \\ &= 0. \end{aligned}$$

Therefore we can write

$$m - \varphi\big(1 \otimes \pi(m)\big) = \sum a_i m_i$$

where $a_i \in \mathcal{A}^+$ and $m_i \in M$. Then all the m_i have degree less than $\deg m$. Thus, by the induction hypothesis, we can find $x_i \in \mathcal{A} \otimes N$ with $\varphi(x_i) = m_i$. So then

$$m = \varphi\left(1 \otimes \pi(m) + \sum a_i x_i\right),$$

which proves surjectivity.

Now, consider the sequence of maps

$$\mathcal{A} \otimes N \xrightarrow{1\otimes f} \mathcal{A} \otimes M \longrightarrow M \xrightarrow{\Delta} M \otimes M \xrightarrow{1\otimes\pi} M \otimes N$$

where we give $M \otimes M$ an $\mathcal{A}$-module structure by $a(m_1 \otimes m_2) = \sum a'm_1 \otimes a''m_2$ where $\psi(a) = \sum a' \otimes a''$. It is clear that $1 \otimes f$ and Δ are both $\mathcal{A}$-module maps. In fact, $1 \otimes \pi$ is as well, for

$$\begin{aligned} (1 \otimes \pi)\big(a(m_1 \otimes m_2)\big) &= (1 \otimes \pi)\left(\sum a'm_1 \otimes a''m_2\right) \\ &= \sum a'm_1 \otimes \pi(a''m_2) \\ &= am_1 \otimes \pi(m_2) \end{aligned}$$

using Lemma 6.1 and the fact that π annihilates $\mathcal{A}^+M$. But this is simply $a\big(m_1 \otimes \pi(m_2)\big)$ in $\mathcal{A} \otimes N$, so $1 \otimes \pi$ is an $\mathcal{A}$-module map.

Now, tracing through the maps, $a \otimes n$ maps to $a \cdot 1 \otimes n$ plus elements of different bidegrees. Thus, since ν is injective, the above composition is injective. But φ is the composition of the first two maps in it, and thus it is also injective.

Therefore, $M \cong \mathcal{A} \otimes N$, so M is a free $\mathcal{A}$-module. □

7. Partitions and Symmetric Functions

Definition 10. A *partition* I of a positive integer n is an unordered sequence $(i_1, \ldots, i_k)$ of positive integers with sum n. I is said to be *dyadic* if each i_α has the form $2^{s_\alpha} - 1$ for some integer s_α. I is said to be *non-dyadic* if none of the i_α have that form.

We denote by $p(n)$ the total number of partitions of n, and by $p_d(n)$ and $p_{nd}(n)$ the number of dyadic and non-dyadic partitions, respectively.

If $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ are partitions of n and m respectively, then the juxtaposition $IJ = (i_1, \ldots, i_k, j_1, \ldots, j_l)$ is a partition of $n + m$. A partition I' of n is said to be a *refinement* of I if we can write $I' = I_1 \cdots I_k$ with each I_α a partition of i_α.

Lemma 7.1. *For all positive integers n,*

$$\sum_{i=0}^{n} p_d(i) p_{nd}(n-i) = p(n).$$

Further, if $f(n)$ is any function satisfying

$$\sum_{i=0}^{n} p_d(i) f(n-i) = p(n)$$

then $f = p_{nd}$.

Proof. This is a standard combinatorics argument. If I is a partition of n, then it can be written uniquely in the form $I_d I_{nd}$ where I_d is dyadic and I_{nd} is non-dyadic. Conversely, for all i, $0 \leq i \leq n$, if I_d is a dyadic partition of i, and I_{nd} is a non-dyadic partition of $n - i$, we obtain a partition $I_d I_{nd}$ of n. There are $p_d(i)p_{nd}(n-i)$ different ways of making this construction for each i, since by the first argument each pair of a dyadic partition and a non-dyadic partition gives rise to a different partition of n. Summing over i gives the desired result. The second statement is an easy induction. □

We will now briefly investigate symmetric functions. Recall that the ring $\mathcal{S}_n$ of symmetric functions in n variables is the graded subring of $\mathbb{Z}[t_1, \ldots, t_n]$ of polynomials which are fixed by every permutation of the variables. It is a standard theorem of algebra that the elementary symmetric functions $\sigma_1, \ldots, \sigma_n$ form a polynomial basis for $\mathcal{S}_n$, so that

$$\mathcal{S}_n = \mathbb{Z}[\sigma_1, \ldots, \sigma_n]$$

with σ_i of degree i. (See [9, Section 4.6].) Thus, a free basis of the degree m elements $\mathcal{S}_n^m$ of $\mathcal{S}_n$ is given by

$$\{\sigma_I = \sigma_{i_1} \cdots \sigma_{i_k} \mid I = (i_1, \ldots, i_k) \text{ a partition of } m\}.$$

We will now construct another useful basis for $\mathcal{S}_n^m$. If $I = (i_1, \ldots, i_k)$ is a partition of m of length at most n, then we can form the symmetric function

$$\sum t^I = \sum t_1^{i_1} t_2^{i_2} \cdots t_k^{i_k}$$

where the $\sum$ indicates that we take the smallest symmetric function containing the monomial t^I. That is, we include every monomial that can be formed with exponents exactly forming I. It is not hard to show that the set

$$\left\{ \sum t^I \mid I \text{ a partition of } m \text{ of length at most } n \right\}$$

is then a basis for $\mathcal{S}_n^m$.

Now, let I be any partition of m. Choose $n \geq m$. Then as we have already observed the elementary symmetric functions $\sigma_1, \ldots, \sigma_m$ of $t_1, \ldots, t_n$ are algebraically independent. Thus, there is a unique polynomial $s_I \in \mathbb{Z}[t_1, \ldots, t_n]$ satisfying

$$s_I(\sigma_1, \ldots, \sigma_m) = \sum t^I.$$

It is not hard to see that s_I is independent of the choice of n, and that the identity remains valid even for $n < m$, substituting 0 for σ_i with $i > n$.

Returning to the case where $n \geq m$, we see that the $p(m)$ polynomials s_I are linearly independent, and therefore that they form a basis for $\mathcal{S}_n^m$. We list the first twelve such polynomials.

$$\begin{array}{lcl}
s() & = & 1 \\
s_{(1)}(\sigma_1) & = & \sigma_1 \\
s_{(2)}(\sigma_1, \sigma_2) & = & \sigma_1^2 - 2\sigma_2 \\
s_{(1,1)}(\sigma_1, \sigma_2) & = & \sigma_2 \\
s_{(3)}(\sigma_1, \sigma_2, \sigma_3) & = & \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\
s_{(1,2)}(\sigma_1, \sigma_2, \sigma_3) & = & \sigma_1\sigma_2 - 3\sigma_3 \\
s_{(1,1,1)}(\sigma_1, \sigma_2, \sigma_3) & = & \sigma_3 \\
s_{(4)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) & = & \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4 \\
s_{(1,3)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) & = & \sigma_1^2\sigma_2 - 2\sigma_2^2 - \sigma_1\sigma_3 + 4\sigma_4 \\
s_{(2,2)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) & = & \sigma_2^2 - 2\sigma_1\sigma_3 + 2\sigma_4 \\
s_{(1,1,2)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) & = & \sigma_1\sigma_3 - 4\sigma_4 \\
s_{(1,1,1,1)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) & = & \sigma_4
\end{array}$$

8. The Steenrod Algebra

We will now recall the main properties of the Steenrod algebra $\mathfrak{A}_2$. For a complete treatment, see [20].

$\mathfrak{A}_2$ is a graded $\mathbb{Z}_2$-algebra, generated (but not freely) by elements Sq^i of degree i, $i \geq 0$, with $\operatorname{Sq}^0 = 1$. The only relations satisfied by the generators are the *Adem relations*

$$\operatorname{Sq}^a \operatorname{Sq}^b = \sum_{i=0}^{[a/2]} \binom{b-i-1}{a-2i} \operatorname{Sq}^{a+b-i} \operatorname{Sq}^i$$

for $a < 2b$. It follows from this that $\mathfrak{A}_2$ has a vector space basis of elements of the form

$$\operatorname{Sq}^{i_1} \operatorname{Sq}^{i_2} \cdots \operatorname{Sq}^{i_n}$$

with $i_\alpha \geq 2i_{\alpha+1}$ for all α. In fact, the dimension of the vector space $\mathfrak{A}_2^n$ of homogeneous elements of degree n is a familiar quantity.

Lemma 8.1. *The number of partitions $I = (i_1, \ldots, i_k)$ of n with $i_\alpha \geq 2i_{\alpha+1}$ for all α is $p_d(n)$, the number of dyadic partitions of n.*

Proof. Let $I = (i_1, \ldots, i_k)$ be a partition of n with $i_\alpha \geq 2i_{\alpha+1}$ for all α. Let I' be the partition containing $i_\alpha - 2i_{\alpha+1}$ copies of the integer $2^\alpha - 1$. Then I' is certainly dyadic, and it is a partition of

$$\begin{aligned}
\sum_{\alpha=1}^{k}(i_\alpha - 2i_{\alpha+1})(2^\alpha - 1) &= \sum_{\alpha=1}^{k} 2^\alpha i_\alpha - 2^{\alpha+1} i_{\alpha+1} - i_\alpha + 2i_{\alpha+1} \\
&= \sum_{\alpha=1}^{k} i_\alpha \\
&= n.
\end{aligned}$$

The inverse construction is now clear, giving us a bijective correspondence between the two types of partitions and establishing the lemma. ☐

Thus, $\dim_{\mathbb{Z}_2} \mathfrak{A}_2^n = p_d(n)$.

The Steenrod algebra is closely related to the cohomology of spaces with coefficients in $\mathbb{Z}_2$. Precisely, for any pair (X, A) we get a natural pairing

$$\mathfrak{A}_2 \otimes H^*(X, A; \mathbb{Z}_2) \to H^*(X, A; \mathbb{Z}_2)$$

such that

$$\operatorname{Sq}^i : H^n(X, A; \mathbb{Z}_2) \to H^{n+i}(X, A; \mathbb{Z}_2)$$

is an additive homomorphism for all n and all i. In general the precise action of Sq^i is difficult to compute. However, we do have

$$\begin{array}{ll}
\operatorname{Sq}^0 x = x & \text{for all } x; \\
\operatorname{Sq}^i x = x \cup x & \text{if } x \text{ has degree } i; \\
\operatorname{Sq}^i x = 0 & \text{if } x \text{ has degree } < i.
\end{array}$$

Finally, we have the following formula for Sq^i of a cup product:

$$\mathrm{Sq}^i(x \cup y) = \sum_{i_1+i_2=i} \mathrm{Sq}^{i_1} x \cup \mathrm{Sq}^{i_2} y.$$

We can define a coproduct on the Steenrod algebra

$$\psi : \mathfrak{A}_2 \to \mathfrak{A}_2 \otimes \mathfrak{A}_2$$

by $\psi(\mathrm{Sq}^i) = \sum_{i_1+i_2=i} \mathrm{Sq}^{i_1} \otimes \mathrm{Sq}^{i_2}$. We can also define a counit

$$\varepsilon : \mathfrak{A}_2 \to \mathbb{Z}_2$$

by $\varepsilon(\mathrm{Sq}^0) = 1$ and $\varepsilon(\mathrm{Sq}^i) = 0$ for $i \geq 1$. It is not difficult to show that this gives $\mathfrak{A}_2$ the structure of a connected Hopf algebra over $\mathbb{Z}_2$.

The Steenrod algebra, like all cohomology operations, is closely related to Eilenberg-MacLane spaces. Recall that the *Eilenberg-MacLane space* $K(G,n)$ is a CW-complex with

$$\pi_i(K(G,n)) = \begin{cases} G, & \text{if } i = n; \\ 0, & \text{otherwise.} \end{cases}$$

It can be shown that the Eilenberg-MacLane space $K(G,n)$ exists and is unique up to homotopy for all positive integers n and groups G. (G must be abelian if $n \geq 2$, of course. See [19, Section 8.1].)

We will need to know the cohomology of $K(\mathbb{Z}_2, n)$. It is immediate from the Hurewicz theorem ([19, Section 7.5, Theorem 5]) that

$$H^i(K(\mathbb{Z}_2,n);\mathbb{Z}_2) = H^i(K(\mathbb{Z}_2,n);\mathbb{Z}) = 0$$

for $0 < i < n$. It follows from a theorem of Serre ([24, Theorem 8.2]) that

$$H^{n+i}(K(\mathbb{Z}_2,n);\mathbb{Z}_2) \cong H^{n+i}(K(\mathbb{Z}_2,n);\mathbb{Z}) \cong \mathfrak{A}_2^i$$

for $i < n$.

Eilenberg-MacLane spaces also have an important mapping property: for any space X, any group G and any positive integer n there is a natural bijection between the cohomology group $H^n(X;G)$ and homotopy classes of maps from X to $K(G,n)$. In the case $G = \mathbb{Z}_2$, we can describe this correspondence more precisely as follows: Let $x \in H^n(X;\mathbb{Z}_2)$ correspond to a map $f : X \to K(\mathbb{Z}_2,n)$. Then f induces maps $f^* : H^i(K(\mathbb{Z}_2,n);\mathbb{Z}_2) \to H^i(X;\mathbb{Z}_2)$. For $n \leq i < 2n$, we have $H^i(X;\mathbb{Z}_2);\mathbb{Z}_2) \cong \mathfrak{A}_2^{i-n}$, and in this case f^* is given by evaluation on x; that is, $\mathfrak{a} \in \mathfrak{A}_2^{i-n} \cong H^i(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ maps to $\mathfrak{a}x \in H^i(X;\mathbb{Z}_2)$.

9. Stiefel-Whitney Classes

In this section we will introduce the theory of characteristic cohomology classes with coefficients in $\mathbb{Z}_2$. For an excellent treatment of this topic, see [14, Sections 4-8].

We will construct the Stiefel-Whitney classes of a vector bundle ξ over a base space B by means of the Steenrod algebra. First, we recall the Thom isomorphism theorem.

Theorem 9.1 (Unoriented Thom Isomorphism Theorem)**.** *Let ξ be an n-dimensional vector bundle over a base space B with projection map $\pi : E = E(\xi) \to B$. Let E_0 be the complement of the zero section in E. Then there exists a unique cohomology class $u_\xi \in H^n(E, E_0;\mathbb{Z}_2)$ (called the Thom class) such that*

$$u_\xi|_{(F,F_0)} \neq 0$$

for all fibers $F = \pi^{-1}(b)$. Furthermore, for all i the map $x \mapsto x \cup u_\xi$ defines an isomorphism

$$H^i(E;\mathbb{Z}_2) \to H^{i+n}(E,E_0;\mathbb{Z}_2).$$

See [14, Section 10] for a proof.

Definition 11. The *Thom isomorphism*

$$\varphi : H^i(B;\mathbb{Z}_2) \to H^{i+n}(E,E_0;\mathbb{Z}_2)$$

is defined by $\varphi(x) = \pi^* x \cup u_\xi$. ($\pi^*$ is an isomorphism since the zero section embeds B as a deformation retract of E.)

Using the Steenrod squares and the Thom isomorphism, we can now define the Stiefel-Whitney classes.

Definition 12. Let ξ be an n-plane bundle over B. The i^{th} *Stiefel-Whitney class* $w_i(\xi) \in H^i(B;\mathbb{Z}_2)$ is defined by
$$w_i(\xi) = \varphi^{-1}\operatorname{Sq}^i \varphi(1).$$
That is, $w_i(\xi)$ satisfies
$$\pi^* w_i(\xi) \cup u_\xi = \operatorname{Sq}^i(u_\xi).$$

Using the properties of Steenrod squares, it is not difficult to verify the five defining properties of Stiefel-Whitney classes.

(1) $w_0(\xi) = 1$ and $w_i(\xi) = 0$ for $i > n$.

Proof. Since Sq^0 is the identity, it is clear that $w_0(\xi) = 1$. If $i > n$, then $\operatorname{Sq}^i(u_\xi) = 0$, so $w_i(\xi) = 0$. ☐

(2) (Naturality) If $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then
$$w_i(\xi) = f^* w_i(\eta).$$

Proof. Let $\overline{f} : \xi \to \eta$ be the bundle map. Then $\overline{f}$ maps $E(\xi)_0$ into $E(\eta)_0$, and thus induces a map $g : (E(\xi), E(\xi)_0) \to (E(\eta), E(\eta)_0)$. Also, from the definition of the Thom class it is clear that $f^*(u_\eta) = u_\xi$. Thus, the Thom isomorphisms φ_ξ and φ_η satisfy
$$g^* \circ \varphi_\eta = \varphi_\xi \circ f^*.$$
Combining this with the naturality of Steenrod squares, we see that $f^* w_i(\eta) = w_i(\xi)$. ☐

(3) If ξ and ξ' are vector bundles, then
$$w_i(\xi \times \xi') = \sum_{i_1+i_2=i} w_{i_1}(\xi) \times w_{i_2}(\xi').$$

Proof. Let $\xi'' = \xi \times \xi'$. We have $u_\xi \times u_{\xi'} \in H^n(E(\xi) \times E(\xi'), E(\xi) \times E(\xi')_0 \cup E(\xi)_0 \times E(\xi'))$. But that union is simply $E(\xi'')_0$, so $u_\xi \times u_{\xi'} \in H^n(E(\xi''), E(\xi'')_0)$ and we see easily that it is the Thom class of ξ''. From this we see that the Thom isomorphism of ξ'', φ'', is simply given by $\varphi \times \varphi'$. Now, we compute
$$\begin{aligned}\varphi''\big(w_i(\xi'')\big) &= \operatorname{Sq}^i(u''_\xi) \\ &= \operatorname{Sq}^i(u_\xi \times u_{\xi'}) \\ &= \sum_{i_1+i_2=i} \operatorname{Sq}^{i_1}(u_\xi) \times \operatorname{Sq}^{i_2}(u_{\xi'}) \\ &= \sum_{i_1+i_2=i} \varphi\big(w_{i_1}(\xi)\big) \times \varphi'\big(w_{i_2}(\xi')\big) \\ &= \sum_{i_1+i_2=i} \varphi''\big(w_{i_1}(\xi) \times w_{i_2}(\xi')\big).\end{aligned}$$
Applying $(\varphi'')^{-1}$ to both sides gives the desired equality. ☐

(4) (The Whitney product formula) If ξ and ξ' are vector bundles over the same base space, then
$$w_i(\xi \oplus \xi') = \sum_{i_1+i_2=i} w_{i_1}(\xi) \cup w_{i_2}(\xi').$$

Proof. This follows immediately from the previous formula by pulling back under the diagonal embedding and using naturality. ☐

(5) $w_1\big(\gamma^1(\mathbb{R}^2)\big)$ is non-zero.

Proof. This will follow immediately from Proposition 10.1. ☐

We define the *total Stiefel-Whitney class* $w(\xi) \in H^*(B;\mathbb{Z}_2)$ by
$$w(\xi) = 1 + w_1(\xi) + \ldots + w_n(\xi).$$
The Whitney product formula then says precisely
$$w(\xi \oplus \xi') = w(\xi)w(\xi').$$

Another useful application of Stiefel-Whitney classes is the Gysin sequence.

Proposition 9.2 (The Unoriented Gysin Sequence). *Let ξ be an n-plane bundle with projection $\pi : E \to B$. Let $\pi_0 : E_0 \to B$ be the restriction of π to E_0. Then there is a long exact sequence*

$$\cdots \longrightarrow H^i(B;\mathbb{Z}_2) \xrightarrow{\cup w_n(\xi)} H^{i+n}(B;\mathbb{Z}_2) \xrightarrow{\pi_0^*} H^{i+n}(E_0;\mathbb{Z}_2) \longrightarrow \cdots$$

Proof. We have the long exact cohomology sequence

$$\cdots \longrightarrow H^i(E,E_0;\mathbb{Z}_2) \longrightarrow H^i(E;\mathbb{Z}_2) \longrightarrow H^i(E_0;\mathbb{Z}_2) \longrightarrow \cdots$$

of the pair (E, E_0). Now, cup product with the Thom class u_ξ gives us an isomorphism from $H^{i-n}(E;\mathbb{Z}_2)$ to $H^i(E,E_0;\mathbb{Z}_2)$. This gives us an exact sequence

$$\cdots \longrightarrow H^{i-n}(E;\mathbb{Z}_2) \xrightarrow{g} H^i(E;\mathbb{Z}_2) \longrightarrow H^i(E_0;\mathbb{Z}_2) \longrightarrow \cdots$$

where $g(x) = (x \cup u_\xi)|_E = x \cup u_\xi|_E$. Now, we also have an isomorphism $\pi^* : H^i(B;\mathbb{Z}_2) \to H^i(E;\mathbb{Z}_2)$. Substituting this in as well, we get a long exact sequence

$$\cdots \longrightarrow H^{i-n}(B;\mathbb{Z}_2) \longrightarrow H^i(B;\mathbb{Z}_2) \longrightarrow H^i(E_0;\mathbb{Z}_2) \longrightarrow \cdots$$

where the maps are easily seen to be as asserted, using the fact that $\pi^* w_n(\xi) = u_\xi|_E$. □

We wish to have a way to compare the Stiefel-Whitney classes of different manifolds. Of course, this is not immediately possible since they are elements of different cohomology groups. We can remedy this by 'evaluating' the Stiefel-Whitney classes, as we now explain.

Let M^n be a compact manifold of dimension n. Recall that M has a unique *fundamental homology class* $\mu_M \in H_n(M;\mathbb{Z}_2)$ characterized by $\rho_x(\mu_M) \neq 0$ for all $m \in M$, where

$$\rho_x : H_n(M;\mathbb{Z}_2) \to H_n(M, M - m;\mathbb{Z}_2)$$

is the natural map. (See [14, Appendix A].) Equivalently, μ_M is non-trivial on each connected component. Given any cohomology class $x \in H^n(M;\mathbb{Z}_2)$, we can then form the *Kronecker index*

$$\langle x, \mu_M \rangle \in \mathbb{Z}_2.$$

We now apply this construction to Stiefel-Whitney classes. Let ξ be a vector bundle over M. Let $I = (i_1, \ldots, i_k)$ be a partition of n. Define

$$w_I(\xi) = w_{i_1}(\xi) \cdots w_{i_k}(\xi) \in H^n(M;\mathbb{Z}_2).$$

We now define the *Stiefel-Whitney number* $W_I[\xi]$ by

$$W_I[\xi] = \langle w_I(\xi), \mu_M \rangle \in \mathbb{Z}_2.$$

We will say that two vector bundles over manifolds M and N of dimension n have the same Stiefel-Whitney numbers if they agree for all partitions I of n.

Recall that we had another way of creating elements of degree n. Given any partition I of n, we have the polynomial s_I satisfying

$$s_I(\sigma_1, \ldots, \sigma_n) = \sum t^I.$$

Thus,

$$s_I\big(w(\xi)\big) = s_I\big(w_1(\xi), \ldots, w_n(\xi)\big) \in H^n(M;\mathbb{Z}_2),$$

so we can form the *s-number*

$$S_I[w(\xi)] = \big\langle s_I\big(w_1(\xi), \ldots, w_n(\xi)\big), \mu_M \big\rangle \in \mathbb{Z}_2.$$

If I is not a partition of n, we will define $S_I[w(\xi)] = 0$.

The utility of s-numbers comes from the following proposition.

Proposition 9.3. *Let ξ and ξ' be vector bundles over M. Then for any partition I,*

$$s_I\big(w(\xi \oplus \xi')\big) = \sum_{I_1 I_2 = I} s_{I_1}\big(w(\xi)\big) s_{I_2}\big(w(\xi')\big).$$

Also, if η is a vector bundle over another manifold N, then

$$S_I[w(\xi \times \eta)] = \sum_{I_1 I_2 = I} S_{I_1}[w(\xi)] S_{I_2}[w(\eta)].$$

Proof. Consider the polynomial ring $\mathbb{Z}[t_1, \ldots, t_{2n}]$, where n is the larger of the fiber dimensions of ξ and ξ'. Let σ_i' and σ_i'' be the elementary symmetric functions of $t_1, \ldots, t_n$ and $t_{n+1}, \ldots, t_{2n}$ respectively. Letting σ_i be the elementary symmetric functions of $t_1, \ldots, t_{2n}$, we see that

$$\sigma_i = \sum_{i_1+i_2=i} \sigma_{i_1}' \sigma_{i_2}''.$$

Now, pick a partition $I = (i_1, \ldots, i_k)$ of some integer r. Then

$$s_I(\sigma_1, \ldots, \sigma_r) = \sum t_1^{i_1} \cdots t_k^{i_k}$$

where the sum is over all monomials of that form. Let I_1 be the partition formed by the exponents of $t_1, \ldots, t_n$ and let I_2 be the partition formed by the exponents of $t_{n+1}, \ldots, t_{2n}$. It is clear that the sum of all monomials corresponding to I_1 and I_2 will be

$$s_{I_1}(\sigma_1', \ldots, \sigma_r') s_{I_2}(\sigma_1'', \ldots, \sigma_r'').$$

Summing over all decompositions of I now yields

$$s_I(\sigma_1, \ldots, \sigma_r) = \sum_{I_1 I_2 = I} s_{I_1}(\sigma_1', \ldots, \sigma_r') s_{I_2}(\sigma_1'', \ldots, \sigma_r'').$$

Now, the product formula for Stiefel-Whitney classes completes the proof since $w(\xi \oplus \xi')$, $w(\xi)$ and $w(\xi')$ satisfy the same identity

$$w_i(\xi \oplus \xi') = \sum_{i_1+i_2=i} w_{i_1}(\xi) w_{i_2}(\xi')$$

and the elementary symmetric functions are algebraically independent.

For the second statement, simply use the fact that the fundamental homology class $\mu_{M\times N}$ is just $\mu_M \times \mu_N$ and that $\xi \times \eta = \pi_1^* \xi \oplus \pi_2^* \eta$, where π_1 and π_2 are the projections. □

As an easy corollary, we have:

Corollary 9.4. *Let ξ and ξ' be vector bundles over M. Then for all r*

$$s_{(r)}\big(w(\xi \oplus \xi')\big) = s_{(r)}\big(w(\xi)\big) + s_{(r)}\big(w(\xi')\big).$$

Proof. We simply apply Proposition 9.3 with $I = (r)$. I can then only be expressed as a juxtaposition in the two trivial ways, which immediately yields the result. □

10. The Cohomology of Grassmann Manifolds

As an application of Stiefel-Whitney classes and the Gysin sequence, we will now compute the cohomology rings of various Grassmann manifolds. These computations will also be important in our later results.

Proposition 10.1. *Let $\mathbb{R}P^n = G_1(\mathbb{R}^{n+1})$ be real projective n-space. Then,*

$$H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1})$$

where $x = w_1\big(\gamma^1(\mathbb{R}^{n+1})\big)$. Further,

$$H^*(BO_1; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$$

where $x = w_1(\gamma^1)$.

Proof. Let E be the total space of $\gamma^1(\mathbb{R}^{n+1})$. Then E_0 is the set of all pairs (l, v), where l is a line through the origin in $\mathbb{R}^{n+1}$ and v is a non-zero point in l. So E_0 can be identified with $\mathbb{R}^{n+1} - 0$. Thus, the Gysin sequence

$$\cdots \longrightarrow H^i(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\cup x} H^{i+1}(\mathbb{R}P^n; \mathbb{Z}_2) \longrightarrow H^{i+1}(E_0; \mathbb{Z}_2) \longrightarrow \cdots$$

reduces to

$$0 \longrightarrow H^i(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\cup x} H^{i+1}(\mathbb{R}P^n; \mathbb{Z}_2) \longrightarrow 0$$

for $0 \le i \le n-1$. Since clearly $H^0(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$, we have that $H^i(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ with generator x^i, for $i \le n$. This shows that $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$ is as asserted, since $H^i(\mathbb{R}P^n; \mathbb{Z}_2) = 0$ for $i > n$. The second assertion follows by passing to the limit as n goes to infinity. □

In order to calculate the cohomology of BO_r for larger r, we will first need another isomorphism. Let E_0 be the space of non-zero vectors of γ^r. We define a map $f : E_0 \to BO_{r-1}$ as follows : For a pair $(P, v) \in E_0$ of a plane P and a non-zero vector v in P, let $f(P, v)$ be the orthogonal complement of v in P. This is an $(r-1)$-plane through the origin, so f is well-defined.

Lemma 10.2. *For any ring R, the map*

$$f^* : H^i(BO_{r-1}; R) \to H^i(E_0; R)$$

is an isomorphism for all i.

Proof. Restrict f to a map $f_N : E(\gamma^r(\mathbb{R}^N)) \to G_{r-1}(\mathbb{R}^N)$ for some large N. Then for any $Q \in G_{r-1}(\mathbb{R}^N)$, we see that $f_N^{-1}(Q)$ is the set of all pairs $(Q + v\mathbb{R}, v)$ where v is orthogonal to Q. So we can identify f_N with a projection map

$$E(\xi)_0 \to G_{r-1}(\mathbb{R}^N)$$

where ξ is the vector bundle over $G_{r-1}(\mathbb{R}^N)$ whose fiber over an $(r-1)$-plane Q is the orthogonal $(N-r+1)$-plane. In fact, this bundle can easily be given an orientation (by the orientation of v relative to $Q+v\mathbb{R}$), so that the oriented Gysin sequence shows that f_N induces cohomology isomorphisms in dimensions $\leq 2(N-r)$. (See Section 14 and Proposition 14.2. The oriented Gysin sequence is similar to the Gysin sequence we have already seen, except that it allows R-coefficients.) Taking the direct limit as N goes to infinity now yields the result. □

In fact, in the proof of the next theorem we will only need the $\mathbb{Z}_2$ case of the above result, where we could get away with the unoriented Gysin sequence. However, we will need this isomorphism with $\mathbb{Z}_p$ coefficients in the process of computing the unoriented cobordism groups.

Proposition 10.3. *$H^*(BO_r; \mathbb{Z}_2)$ is the free polynomial algebra on the Stiefel-Whitney classes $w_i = w_i(\gamma^r)$. That is,*

$$H^*(BO_r; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \ldots, w_r].$$

Proof. We first show that the w_i are algebraically independent. Let $\xi = \gamma^1 \times \cdots \times \gamma^1$ be the Cartesian product of r copies of γ^1. Then ξ is an r-plane bundle over $(BO_1)^r$, so there is a classifying map $g : (BO_1)^r \to BO_r$ covered by a bundle map $\xi \to \gamma^r$. Set $w_i' = w_i(\xi)$. Now, suppose we have a polynomial q with $q(w_1, \ldots, w_r) = 0$. Then $q(w_1', \ldots, w_r') = g^*q(w_1, \ldots, w_r) = 0$, so it will suffice to show that the w_i' satisfy no polynomial relation.

Now, by the Künneth theorem,

$$H^*((BO_1)^r; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \ldots, x_r].$$

Further,

$$\begin{aligned} w_i' &= w(\gamma_1) \cdots w(\gamma_1) \\ &= (1 + x_1) \cdots (1 + x_r). \end{aligned}$$

Thus w_i' is the i^{th} elementary symmetric function in $x_1, \ldots, x_r$. But as we have previously observed, the σ_i are algebraically independent, so the w_i' are as well.

Now, returning to BO_r, we have a Gysin sequence

$$\cdots \longrightarrow H^i(BO_r; \mathbb{Z}_2) \xrightarrow{\cup w_r} H^{i+r}(BO_r; \mathbb{Z}_2) \xrightarrow{\pi_0^*} H^{i+r}(E_0; \mathbb{Z}_2) \longrightarrow \cdots$$

But by the lemma, we have a map $f : E_0 \to BO_{r-1}$ inducing an isomorphism in cohomology. Inserting this into the Gysin sequence, we get

$$\cdots \longrightarrow H^i(BO_r; \mathbb{Z}_2) \xrightarrow{\cup w_r} H^{i+r}(BO_r; \mathbb{Z}_2) \xrightarrow{g} H^{i+r}(BO_{r-1}; \mathbb{Z}_2) \longrightarrow \cdots$$

But from the definitions of f and π_0 it is clear that $\pi_0^* w_i = f^* w_i(\gamma^{r-1})$, so that g sends w_i to $w_i(\gamma^{r-1})$.

We now proceed by induction on r. The case $r = 1$ is Proposition 10.1. Now, since $H^*(BO_{r-1}; \mathbb{Z}_2)$ is generated by $w_1(\gamma^{r-1}), \ldots, w_{r-1}(\gamma^{r-1})$, g is surjective, and we obtain a short exact sequence

$$0 \longrightarrow H^i(BO_r; \mathbb{Z}_2) \xrightarrow{\cup w_r} H^{i+r}(BO_r; \mathbb{Z}_2) \xrightarrow{g} H^{i+r}(BO_{r-1}; \mathbb{Z}_2) \longrightarrow 0.$$

Take $x \in H^{i+r}(BO_r; \mathbb{Z}_2)$. Then $g(x) = q(w_1(\gamma^{r-1}), \ldots, w_{r-1}(\gamma^{r-1}))$ for some polynomial q, by the induction hypothesis. Thus $g\big(x - q(w_1, \ldots, w_{r-1})\big) = 0$, so $x - q(w_1, \ldots, w_{r-1}) = w_r y$ for some $y \in H^i(BO_r; \mathbb{Z}_2)$. So now a second induction on i will show that $w_1, \ldots, w_r$ generate $H^*(BO_r; \mathbb{Z}_2)$. Combined with the independence shown earlier, this establishes the proposition. □

11. Computations in Projective Space

In our final calculation of the unoriented cobordism groups we will need some information on the s-numbers of projective spaces. In this section we will carry out those computations.

Now, recall that $\gamma^1(\mathbb{R}^{n+1})$ is a line bundle over $\mathbb{R}P^n$. By Proposition 10.1, we have $w(\gamma^1(\mathbb{R}^{n+1})) = 1 + x$, where x is the generator of $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$.

By its very definition $\gamma^1(\mathbb{R}^{n+1})$ is contained as a subbundle in the trivial bundle ε^{n+1}. Let $\gamma^\perp$ be its orthogonal complement in ε^{n+1}. Then $\gamma^1(\mathbb{R}^{n+1}) \oplus \gamma^\perp = \varepsilon^{n+1}$. Now, since ε^1 is trivial, there is a bundle map from ε^1 to a bundle over a point. Thus all of its higher Stiefel-Whitney classes are zero, and $w(\varepsilon^1) = 1$. Then by the Whitney product formula, $w(\varepsilon^{n+1}) = 1$.

We now relate these bundles to the tangent bundle $\tau(\mathbb{R}P^n)$.

Proposition 11.1. $\tau(\mathbb{R}P^n) \cong \mathrm{Hom}(\gamma^1(\mathbb{R}^{n+1}), \gamma^\perp)$.

Proof. Let $f : S^n \to \mathbb{R}P^n$ be the quotient map, $f(x) = \{\pm x\}$. Note that

$$Df(x, v) = Df(-x, -v)$$

for all $x \in S^n, v \in \tau(S^n)_x$. We can then identify the tangent bundle $\tau(\mathbb{R}P^n)$ with the set of pairs

$$\{(x, v), (-x, -v)\}$$

with $x \cdot x = 1$ and $x \cdot v = 0$. But such pairs are in bijection with linear maps

$$l : L \to L^\perp$$

where L is the line through x and $-x$ and l is defined by $l(x) = v$. Therefore the fiber $\tau(\mathbb{R}P^n)_x$ is canonically isomorphic to $\mathrm{Hom}(L, L^\perp)$. It then follows that

$$\tau(\mathbb{R}P^n) \cong \mathrm{Hom}(\gamma^1(\mathbb{R}^{n+1}), \gamma^\perp),$$

since the fibers of $\gamma^1(\mathbb{R}^{n+1})$ are simply the lines L and the fibers of $\gamma^\perp$ are the n-planes $L^\perp$. $\square$

We are now in a position to compute the Stiefel-Whitney classes $w\big(\tau(\mathbb{R}P^n)\big)$.

Proposition 11.2. *Let $x = w_1\big(\gamma^1(\mathbb{R}^{n+1})\big)$ be the generator of $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1})$. Then*

$$w\big(\tau(\mathbb{R}P^n)\big) = (1 + x)^{n+1}.$$

Proof. It is clear that $\mathrm{Hom}(\gamma^1(\mathbb{R}^{n+1}), \gamma^1(\mathbb{R}^{n+1}))$ is the trivial line bundle, since it has a non-zero section. Therefore,

$$\begin{aligned}
\tau \oplus \varepsilon^1 &\cong \mathrm{Hom}(\gamma^1(\mathbb{R}^{n+1}), \gamma^\perp) \oplus \mathrm{Hom}(\gamma^1(\mathbb{R}^{n+1}), \gamma^1(\mathbb{R}^{n+1})) \\
&\cong \mathrm{Hom}(\gamma^1(\mathbb{R}^{n+1}), \gamma^\perp \oplus \gamma^1(\mathbb{R}^{n+1})) \\
&\cong \mathrm{Hom}(\gamma^1(\mathbb{R}^{n+1}), \varepsilon^{n+1}) \\
&\cong \mathrm{Hom}(\gamma^1(\mathbb{R}^{n+1}), \varepsilon^1) \oplus \cdots \oplus \mathrm{Hom}(\gamma^1(\mathbb{R}^{n+1}), \varepsilon^1) \\
&\cong \gamma^1(\mathbb{R}^{n+1}) \oplus \cdots \oplus \gamma^1(\mathbb{R}^{n+1}).
\end{aligned}$$

Now, by the Whitney product formula,

$$\begin{aligned}
w(\tau) &= w(\tau) w(\varepsilon^1) \\
&= w(\tau \oplus \varepsilon^1) \\
&= w(\gamma^1(\mathbb{R}^{n+1}) \oplus \cdots \oplus \gamma^1(\mathbb{R}^{n+1})) \\
&= w(\gamma^1(\mathbb{R}^{n+1}))^{n+1} \\
&= (1 + x)^{n+1}.
\end{aligned}$$

$\square$

Corollary 11.3. *The Stiefel-Whitney numbers of $\tau(\mathbb{R}P^n)$ are given by*

$$W_I[\tau(\mathbb{R}P^n)] = \binom{n+1}{i_1} \cdots \binom{n+1}{i_k} \in \mathbb{Z}_2$$

where $I = (i_1, \ldots i_k)$ is a partition of n. Also,

$$S_{(n)}[w\big(\tau(\mathbb{R}P^n)\big)] = n + 1 \in \mathbb{Z}_2.$$

Proof. By the definition of the fundamental homology class $\mu_{\mathbb{R}P^n}$ we will have $W_I \neq 0$ if and only if each of the factors w_{i_α} is non-zero. Now, by Proposition 11.2,

$$w_{i_\alpha}\big(\tau(\mathbb{R}P^n)\big) = \binom{n+1}{i_\alpha}.$$

The first statement now follows immediately.

For the second statement, recall that by definition

$$s_{(n)}(\sigma_1, \dots, \sigma_n) = t_1^n + \cdots + t_{n+1}^n.$$

Now, in this case, w_i is simply the i^{th} elementary symmetric function of $n+1$ copies of x. Therefore,

$$\begin{aligned} s_{(n)}(w_1, \dots, w_n) &= x^n + \cdots + x^n \\ &= (n+1)x^n. \end{aligned}$$

Therefore, as before,

$$S_{(n)}[w\big(\tau(\mathbb{R}P^n)\big)] = n+1.$$

□

We will also need to consider one other class of manifolds. Choose positive integers m and n with $m \leq n$. Let $\mathbb{R}P^m$ have homogeneous coordinates $[x_0, \dots, x_m]$ and $\mathbb{R}P^n$ have homogeneous coordinates $[y_0, \dots, y_n]$. Define $H_{m,n}$ to be the subset of $\mathbb{R}P^m \times \mathbb{R}P^n$ of points satisfying the homogeneous equation

$$x_0 y_0 + \cdots + x_m y_m = 0.$$

Then $H_{m,n}$ is a manifold of dimension $m+n-1$.

Proposition 11.4. *The s-number $S_{(m+n-1)}[w\big(\tau(H_{m,n})\big)]$ is given by*

$$S_{(m+n-1)}[w\big(\tau(H_{m,n})\big)] = \binom{m+n}{m} \in \mathbb{Z}_2.$$

Proof. Set $H = H_{m,n}$. Let $\pi_1 : \mathbb{R}P^m \times \mathbb{R}P^n \to \mathbb{R}P^m$ and $\pi_2 : \mathbb{R}P^m \times \mathbb{R}P^n \to \mathbb{R}P^n$ be the projections. Let λ be the line bundle $\pi_1^*\gamma^1(\mathbb{R}^{m+1}) \otimes \pi_2^*\gamma^1(\mathbb{R}^{n+1})$. Then we see that $\lambda|_H$ is the orthogonal complement of $\tau(H)$ in $\tau(\mathbb{R}P^m \times \mathbb{R}P^n)|_H$. That is,

$$\tau(H) \oplus \lambda|_H = \tau(\mathbb{R}P^m \times \mathbb{R}P^n)|_H.$$

(Of course, by restriction to H we mean the pullback under the inclusion $H \hookrightarrow \mathbb{R}P^m \times \mathbb{R}P^n$.)

Now, by the Künneth theorem,

$$\begin{aligned} H^*(\mathbb{R}P^m \times \mathbb{R}P^n; \mathbb{Z}_2) &= H^*(\mathbb{R}P^m; \mathbb{Z}_2) \otimes H^*(\mathbb{R}P^n; \mathbb{Z}_2) \\ &= \mathbb{Z}_2[\alpha_1]/(\alpha_1^{m+1}) \otimes \mathbb{Z}_2[\alpha_2]/(\alpha_2^{n+1}) \\ &= \mathbb{Z}_2[\beta_1, \beta_2]/(\beta_1^{m+1}, \beta_2^{n+1}). \end{aligned}$$

where $\beta_i = \pi_i^*\alpha_i$. Also,

$$\begin{aligned} w\big(\tau(\mathbb{R}P^m \times \mathbb{R}P^n)\big) &= \pi_1^* w\big(\tau(\mathbb{R}P^m)\big)\pi_2^* w\big(\tau(\mathbb{R}P^n)\big) \\ &= (1+\beta_1)^{m+1}(1+\beta_2)^{n+1} \end{aligned}$$

and

$$w(\lambda) = 1 + \beta_1 + \beta_2.$$

So

$$w(\lambda|_H) = 1 + \delta_1 + \delta_2$$

where $\delta_i = \beta_i|_H$.

By Corollary 9.4 we have

$$s_{(m+n-1)}\big(w(\tau(\mathbb{R}P^m \times \mathbb{R}P^n)|_H)\big) = s_{(m+n-1)}\big(w(\tau(H))\big) + s_{(m+n-1)}\big(w(\lambda|_H)\big).$$

But

$$\begin{aligned} s_{(m+n-1)}\big(w(\tau(\mathbb{R}P^m \times \mathbb{R}P^n))\big) &= (m+1)\beta_1^{m+n-1} + (n+1)\beta_2^{m+n-1} \\ &= 0 \end{aligned}$$

since $m, n < m+n-1$. Thus, since we are working in $\mathbb{Z}_2$,
$$s_{(m+n-1)}\big(w(\tau(H))\big) = s_{(m+n-1)}\big(w(\lambda|_H)\big).$$

But we can compute
$$s_{(m+n-1)}(w_1(\lambda|_H), 0, \dots, 0) = s_{(m+n-1)}(\delta_1 + \delta_2, 0, \dots, 0)$$
by noting that $w_i(\lambda|_H)$ are the symmetric functions of $t_1 = \delta_1 + \delta_2, t_i = 0$ for $i > 1$. Thus
$$s_{(m+n-1)}(w_1(\lambda|_H), 0, \dots, 0) = (\delta_1 + \delta_2)^{m+n-1}.$$

So
$$s_{(m+n-1)}\big(w(\tau(H))\big) = (\delta_1 + \delta_2)^{m+n-1}$$

and
$$\begin{aligned} S_{(m+n-1)}[w\big(\tau(H)\big)] &= \big\langle (\delta_1 + \delta_2)^{m+n-1}, \mu_H \big\rangle \\ &= \big\langle (\delta_1 + \delta_2)^{m+n-1}, \mu_M|_H \big\rangle \\ &= \big\langle (\beta_1 + \beta_2)^{m+n-1} w_1(\lambda), \mu_M \big\rangle \\ &= \big\langle (\beta_1 + \beta_2)^{m+n}, \mu_M \big\rangle \\ &= \left\langle \binom{m+n}{m} \beta_1^m \beta_2^n, \mu_M \right\rangle \\ &= \binom{m+n}{m} \end{aligned}$$

since all other monomials in $(\beta_1 + \beta_2)^{m+n}$ vanish. □

12. The Cohomology of TBO_r

By the Thom-Pontrjagin theorem, in order to solve the cobordism problem for unoriented manifolds it will be enough to compute the homotopy groups of TBO_r. In order to do this, we will first need some facts about the cohomology of these spaces.

Lemma 12.1. *For any r-plane bundle ξ over B*
$$H^{r+i}(T\xi, t_0; \mathbb{Z}_2) \cong H^i(B; \mathbb{Z}_2)$$
for all i.

Proof. B is embedded as the zero cross section in ξ, and thus also in $T\xi - t_0$. Let T_0 be the complement of B in $T\xi$. Then T_0 is clearly contractible to t_0. So by the exact sequence of the triple $(T\xi, T_0, t_0)$ we quickly conclude
$$H^i(T\xi, t_0; \mathbb{Z}_2) \cong H^i(T\xi, T_0; \mathbb{Z}_2).$$
Further, the excision axiom in cohomology shows that
$$H^i(T\xi, t_0; \mathbb{Z}_2) \cong H^i(E(\xi), E(\xi)_0; \mathbb{Z}_2).$$
Together with the Thom isomorphism, we obtain our desired isomorphism
$$H^i(T\xi, t_0; \mathbb{Z}_2) \cong H^{i-r}(B; \mathbb{Z}_2).$$
□

In particular, we have
$$H^{r+i}(TBO_r, t_0; \mathbb{Z}_2) \cong H^i(BO_r; \mathbb{Z}_2).$$
Since, by Proposition 10.3
$$H^i(BO_r; \mathbb{Z}_2) \cong H^i(BO_s; \mathbb{Z}_2)$$
for $i \leq \min\{r, s\}$, we obtain an isomorphism
$$H^{r+i}(TBO_r, t_0; \mathbb{Z}_2) \cong H^{s+i}(TBO_s, t_0; \mathbb{Z}_2)$$
for $i \leq \min\{r, s\}$.

Definition 13. We define $H^n(TBO, t_0; \mathbb{Z}_2)$ to be this constant value. That is,
$$H^n(TBO, t_0; \mathbb{Z}_2) \cong H^{n+r}(TBO_r, t_0; \mathbb{Z}_2)$$
for any $r \geq n$. Define
$$H^*(TBO, t_0; \mathbb{Z}_2) = \bigoplus_{n=0}^{\infty} H^n(TBO, t_0; \mathbb{Z}_2).$$

Note that under our isomorphism the element $1 \in H^0(TBO_0, t_0; \mathbb{Z}_2)$ maps to the image of the Thom class in $H^r(TBO_r, t_0; \mathbb{Z}_2)$. We will denote this element of $H^*(TBO, t_0; \mathbb{Z}_2)$ by $\mathcal{U}$.

Now, the Whitney sum of vector bundles gives a map
$$BO_r \times BO_s \to BO_{r+s}.$$
This induces a map
$$TBO_r \wedge TBO_s \to TBO_{r+s}.$$
(We take the smash product here because all of $t_0(TBO_r) \times TBO_s$ and $TBO_r \times t_0(TBO_s)$ have been identified to a single point.) Combining these operations for all r and s we get a map
$$\psi : H^*(TBO, t_0; \mathbb{Z}_2) \to H^*(TBO, t_0; \mathbb{Z}_2) \otimes H^*(TBO, t_0; \mathbb{Z}_2).$$
Since the Whitney sum is associative ψ will be coassociative. If we define a counit $\varepsilon : H^*(TBO, t_0; \mathbb{Z}_2) \to \mathbb{Z}_2$ by sending $\mathcal{U}$ to 1 and everything else to 0, we see that $H^*(TBO, t_0; \mathbb{Z}_2)$ is a connected coalgebra over $\mathbb{Z}_2$.

In addition, we see that the action of the Steenrod squares is compatible with our isomorphism of Definition 13. (As complicated as this isomorphism is, all of the isomorphisms are natural except for the two involving the cup product with the Thom class. These actions will cancel each other out, however, so that the overall action of the Steenrod algebra will be compatible.) Thus $H^*(TBO, t_0; \mathbb{Z}_2)$ is a left $\mathfrak{A}_2$-module; for $a \in \mathfrak{A}_2^k$ we can use the action
$$a : H^{n+r}(TBO_r, t_0; \mathbb{Z}_2) \to H^{n+r+k}(TBO_r, t_0; \mathbb{Z}_2)$$
for any $r \geq n + k$ to give the action on $H^n(TBO, t_0; \mathbb{Z}_2)$. Since ψ is also a homomorphism of $\mathfrak{A}_2$-modules, we are almost in a position to apply Proposition 6.2.

Lemma 12.2. *The map $\nu : \mathfrak{A}_2 \to H^*(TBO, t_0; \mathbb{Z}_2)$ given by $\nu(a) = a(\mathcal{U})$ is injective.*

Proof. Suppose that there is an $a \in \mathfrak{A}_2$ with $\nu(a) = 0$. Then the same is true of all of the homogeneous components of a, so it will be enough to show that $\nu : \mathfrak{A}_2^n \to H^*(TBO, t_0; \mathbb{Z}_2)$ is injective for all n.

Now, the action of $a \in \mathfrak{A}_2^n$ on $H^*(TBO, t_0; \mathbb{Z}_2)$ is given by
$$a : H^r(TBO_r, t_0; \mathbb{Z}_2) \to H^{r+n}(TBO_r, t_0; \mathbb{Z}_2)$$
for any $r \geq n$. Thus ν is given by
$$\nu : \mathfrak{A}_2^n \to H^{r+n}(TBO_r, t_0; \mathbb{Z}_2)$$
by evaluation on the image of the Thom class u.

Now, let $\xi = \gamma^1 \times \cdots \times \gamma^1$ be the Cartesian product of r copies of γ^1. This is an r-plane bundle over $(BO_1)^r$, so we have a classifying map $g : (BO_1)^r \to BO_r$ covered by a bundle map $\xi \to \gamma^r$. Now, suppose that we have $a \in \mathfrak{A}_2^n$ with $a(u) = 0$. Then by the naturality of the $\mathfrak{A}_2$ action, $a(u_\xi) = g^*a(u) = 0$, where u_ξ is the Thom class of ξ. So it will be enough to prove that
$$\nu : \mathfrak{A}_2^n \to H^{r+n}(T\xi, t_0; \mathbb{Z}_2)$$
is injective.

Here, however, u_ξ splits as a product $x_1 \cdots x_r$ of the Thom classes of the r different γ^1's. Now, recall that $\mathfrak{A}_2^n$ has a base consisting of Sq^I with $I = (i_1, \ldots, i_k)$ a partition of n with $i_\alpha \geq 2i_{\alpha+1}$. It can be shown by an easy induction, using the product formula for Steenrod squares, that for such an I we will have
$$\mathrm{Sq}^I(x_1 \cdots x_r) = \sum x_1^{2^{v_1}} \cdots x_r^{2^{v_r}}$$
where there are $i_\alpha - 2_{\alpha+1}$ copies of the integer α in the sequence of v's, and the sum is over all monomials of this form. But it is then clear that the sequence of v's determines I, so that for any I', $\mathrm{Sq}^{I'}(x_1 \cdots x_r)$ will consist of an entirely different set of monomials. Therefore no linear combination of them could vanish, which proves that the map ν is injective. □

Corollary 12.3. *$H^*(TBO, t_0; \mathbb{Z}_2)$ is a free left $\mathfrak{A}_2$-module.*

Proof. Simply apply Proposition 6.2. ☐

13. Determination of the Unoriented Cobordism Ring

We are now finally in a position to determine the unoriented cobordism groups. Recall that these are the groups $\mathfrak{N}_n = \Omega_n(BO, 1)$, so that by the Thom-Pontrjagin theorem

$$\mathfrak{N}_n \cong \lim_{r\to\infty} \pi_{n+r}(TBO_r, t_0).$$

In fact, we can give $\mathfrak{N} = \oplus_{n=0}^{\infty}\mathfrak{N}_n$ the structure of a graded $\mathbb{Z}_2$-algebra.

Proposition 13.1. *$\mathfrak{N}$ is a commutative graded $\mathbb{Z}_2$-algebra with product induced by the Cartesian product of manifolds.*

Proof. For any $[M] \in \mathfrak{N}$ we have $M + M + \partial\emptyset \cong \emptyset + \partial(M \times I)$, so $M + M \equiv \emptyset$ and M is its own additive inverse. Now, if M, N, N' are closed with $N \equiv N'$, then there are manifolds U and U' with $N + \partial U \cong N' + \partial U'$, so that

$$M \times N + \partial(M \times U) \cong M \times N' + \partial(M \times U').$$

Thus $M \times N \equiv M \times N'$, so Cartesian product induces a well-defined product on $\mathfrak{N}$. Commutativity and distributivity follow from the corresponding properties of Cartesian product. ☐

We are now ready to prove the first version of the structure theorem.

Theorem 13.2. *The dimension of $\mathfrak{N}_n$ as a $\mathbb{Z}_2$ vector space is $p_{nd}(n)$, the number of non-dyadic partitions of n. Also, two manifolds are cobordant if and only if their tangent bundles have the same Stiefel-Whitney numbers.*

Proof. Since by Corollary 12.3 $H^n(TBO, t_0; \mathbb{Z}_2)$ is a free $\mathfrak{A}_2$-module, and $H^n(TBO, t_0; \mathbb{Z}_2) \cong H^{n+r}(TBO_r, t_0; \mathbb{Z}_2)$ for $r \geq n$, $H^i(TBO_r, t_0)$ is a free $\mathfrak{A}_2$-module for dimensions between r and $2r$. But $H^i(TBO_r, t_0; \mathbb{Z}_2) = 0$ for $i < r$, so $H^i(TBO_r, t_0; \mathbb{Z}_2)$ is actually a free $\mathfrak{A}_2$-module for all dimensions $\leq 2r$.

Now, choose a basis $x_1, \ldots, x_m$ of this part of $H^*(TBO_r, t_0; \mathbb{Z}_2)$ as an $\mathfrak{A}_2$-module, with $x_i \in H^{n_i}(TBO_r, t_0; \mathbb{Z}_2)$. Note that we have $n_i \geq r$ for all i, since the lower cohomology groups are zero. Each x_i corresponds to a map $\varphi_i : TBO_r \to K(\mathbb{Z}_2, n_i)$, and for $j \leq 2n_i$ the map

$$\varphi_i^* : H^j(K(\mathbb{Z}_2, n_i); \mathbb{Z}_2) \cong \mathfrak{A}_2^{j-n_i} \to H^j(TBO_r, t_0; \mathbb{Z}_2)$$

is just given by evaluation of the Steenrod algebra on x_i. (See Section 8.)

Next, set $K = \prod_{i=1}^m K(\mathbb{Z}_2, n_i)$, and let $\varphi : TBO_r \to K$ be the product of the φ_i. By the Künneth theorem,

$$H^j(K; \mathbb{Z}_2) = \bigoplus_{\sum j_\alpha = n+r} H^{j_1}(K(\mathbb{Z}_2, n_1); \mathbb{Z}_2) \otimes \cdots \otimes H^{j_m}(K(\mathbb{Z}_2, n_m); \mathbb{Z}_2). \tag{2}$$

Since $n_i \geq r$ for all i, $H^j(K(\mathbb{Z}_2, n_i); \mathbb{Z}_2) = 0$ for $0 < j < r$ and all i. Thus, if $j \leq 2r$, the only non-zero terms in (2) will be those with one $j_{\alpha_0} = j$ and all other $j_\alpha = 0$. Thus, for $j \leq 2r$,

$$H^j(K; \mathbb{Z}_2) = \bigoplus_{i=1}^m H^j(K(\mathbb{Z}_2, n_i); \mathbb{Z}_2). \tag{3}$$

Now, consider

$$\varphi^* : H^j(K; \mathbb{Z}_2) \to H^j(TBO_r, t_0; \mathbb{Z}_2).$$

By (3), for $j \leq 2r$,

$$H^j(K; \mathbb{Z}_2) \cong \oplus \mathfrak{A}_2^{j-n_i},$$

and φ^* is given by evaluation on $(x_1, \ldots, x_m)$. So, since the x_i are a basis for $H^*(TBO_r, t_0; \mathbb{Z}_2)$ in dimension $\leq 2r$, φ^* will be an isomorphism in $\mathbb{Z}_2$-cohomology for $j \leq 2r$.

We wish to show that φ actually induces an isomorphism in $\mathbb{Z}$-homology. Let E be the total space of γ^r, and E_0 the complement of BO_r in E. Let p be an odd prime. We then have the $\mathbb{Z}_p$-cohomology long exact sequence of the pair

$$\cdots \to H^i(E, E_0; \mathbb{Z}_p) \to H^i(E; \mathbb{Z}_p) \to H^i(E_0; \mathbb{Z}_p) \to \cdots$$

Now, we have that $H^i(E, E_0; \mathbb{Z}_p)$ is canonically isomorphic to $H^i(TBO_r, t_0; \mathbb{Z}_p)$, and that $H^i(E; \mathbb{Z}_p)$ is canonically isomorphic to $H^i(BO_r; \mathbb{Z}_p)$. Also, by Lemma 10.2, we have that $H^i(E_0; \mathbb{Z}_p)$ is isomorphic to $H^i(BO_{r-1}; \mathbb{Z}_p)$. So we obtain a long exact sequence

$$\cdots \to H^i(TBO_r, t_0; \mathbb{Z}_p) \to H^i(BO_r; \mathbb{Z}_p) \to H^i(BO_{r-1}; \mathbb{Z}_p) \to \cdots$$

By Corollary 17.2, the last two are isomorphic for $i \leq 2r$, which implies that
$$H^i(TBO_r, t_0; \mathbb{Z}_p) = 0$$
for $i \leq 2r$.

Now, by the universal coefficient theorem with field coefficients, we also have $H_i(TBO_r, t_0; \mathbb{Z}_p) = 0$ for $i \leq 2r$. Of course, $H_i(K; \mathbb{Z}_p) = 0$ as well in this range. So φ actually induces an isomorphism in $\mathbb{Z}$-homology in these dimensions. So by the Whitehead theorem, $\varphi_{\# i}$ gives a homotopy equivalence for $i < 2r$. (See [19, Section 7.5,Theorem 9].)

We now compute $\pi_{n+r}(K)$. If we let m_k be the number of n_i equal to k, then this is just m_{n+r} copies of $\mathbb{Z}_2$. To compute this, we use (3):
$$H^{n+r}(K; \mathbb{Z}_2) = \bigoplus_{i=1}^{m} H^{n+r}(K(\mathbb{Z}_2, n_i); \mathbb{Z}_2).$$
Therefore,
$$\begin{aligned}\dim H^{n+r}(K; \mathbb{Z}_2) &= \sum_{i=1}^{m} \dim H^{n+r}(K(\mathbb{Z}_2, n_i); \mathbb{Z}_2) \\ &= \sum_{i=1}^{m} \dim \mathfrak{A}_2^{n+r-n_i} \\ &= \sum_{i=1}^{m} p_d(n+r-n_i) \\ &= \sum_{k=r}^{n+r} m_k p_d(n+r-k).\end{aligned}$$
But
$$H^{n+r}(K; \mathbb{Z}_2) \cong H^{n+r}(TBO_r, t_0; \mathbb{Z}_2) \cong H^n(BO_r; \mathbb{Z}_2)$$
which has rank $p(n)$ since $n \leq r$. So
$$p(n) = \sum_{k=r}^{n+r} m_k p_d(n+r-k)$$
and by Lemma 7.1, $m_k = p_{nd}(k-r)$. Thus,
$$\begin{aligned}\dim \pi_{n+r}(TBO_r, t_0) &= \dim \pi_{n+r}(K) \\ &= m_{n+r} \\ &= p_{nd}(n).\end{aligned}$$
Since by the Thom-Pontrjagin theorem
$$\mathfrak{N}_n \cong \lim_{r \to \infty} \pi_{n+r}(TBO_r, t_0)$$
and we have shown that the groups on the right-hand side all have $\mathbb{Z}_2$-dimension $p_{nd}(n)$ for $r \geq n$, we have shown that $\mathfrak{N}_n$ has dimension $p_{nd}(n)$.

Now, since the Hurewicz homomorphism (from homotopy groups to homology groups) is certainly injective for K, it is also injective for TBO_r in dimensions $< 2r$. This gives us an injection of $\mathfrak{N}_n$ into $H^*(TBO_r, t_0; \mathbb{Z}_2)$. Since the map Θ sends an n-manifold to the image of its normal bundle, which in turn maps to Stiefel-Whitney classes, we see that Stiefel-Whitney numbers of the normal bundle are determined by cobordism. But since the Whitney sum of the normal bundle and the tangent bundle is trivial, the Stiefel-Whitney numbers of the normal bundle are the same as those of the tangent bundle, which completes the proof. □

As an immediate corollary, we obtain the following theorem of Rohlin (see [17]), for which a direct proof has never been found.

Corollary 13.3. *A compact* 3 *dimensional manifold (without boundary) is the boundary of a* 4 *dimensional manifold.*

Proof. We calculate $p_{nd}(3) = 0$, so the oriented cobordism group $\mathfrak{N}_3$ is trivial. Thus all closed 3-manifolds bound, which is the assertion. □

Theorem 13.2 seems to suggest very strongly that the entire algebra $\mathfrak{N}$ is a polynomial ring with one generator for each degree not of the form $2^s - 1$. Using our computations of Section 11, this will not be difficult to prove.

Theorem 13.4. *$\mathfrak{N}$ is the free $\mathbb{Z}_2$-algebra with one generator in each dimension not of the form $2^s - 1$. These generators may be taken to be $\mathbb{R}P^n$ for n even and the hypersurface $H_{2^p,2^{p+1}q} \subseteq \mathbb{R}P^{2^p} \times \mathbb{R}P^{2^{p+1}q}$ for $n = 2^p(2q+1)-1$ odd not of the form $2^s - 1$.*

Proof. Let M_i be the asserted generator of dimension i. It follows easily from Corollary 11.3 and Proposition 11.4 that

$$S_{(i)}[w\big(\tau(M_i)\big)] \neq 0$$

for all i.

Consider the set of non-dyadic partitions of n. These have a partial ordering where $I \leq I'$ if I' refines I. Choose a total ordering of the non-dyadic partitions compatible with this partial ordering.

For a non-dyadic partition $I = (i_1, \ldots, i_k)$ of n, define

$$M_I = M_{i_1} \times \cdots M_{i_k}.$$

Let I' be another non-dyadic partition of n. Now, by Proposition 9.3

$$S_{I'}[w\big(\tau(M_I)\big)] = \sum_{I_1 \cdots I_k = I} S_{I_1}[w\big(\tau(M_{I_1})\big)] \cdots S_{I_k}[w\big(\tau(M_{I_k})\big)].$$

So if I' does not refine I, $S_{I'}[w\big(\tau(M_I)\big)]$ must be zero, since there will be no way to choose partitions $I_1, \ldots, I_k$ with each I_α a partition of i_α. Also, $S_I[w\big(\tau(M_I)\big)] = 1$ since in this case there is exactly one choice of $I_1, \ldots, I_k$ giving a non-zero contribution to the sum.

Now, consider the matrix indexed by the non-dyadic partitions of n, ordered by our total ordering. Then the above calculations show that this matrix is triangular with 1's on the diagonal. Therefore it has non-zero determinant, and the s-numbers of the manifolds M_I are linearly independent over $\mathbb{Z}_2$. Since the polynomials s_I are a basis for the symmetric functions of degree n, this is equivalent to the linear independence of the Stiefel-Whitney numbers of the tangent bundles of the M_I. It now follows from Theorem 13.2 that the manifolds M_I are linearly independent over $\mathbb{Z}_2$ as elements of $\mathfrak{N}_n$. Since there are exactly $p_{nd}(n)$ of them, they form a basis for $\mathfrak{N}_n$. This completes the proof. □

Part 3. The Oriented Cobordism Ring

14. Oriented Bundles and the Euler Class

We now turn our attention towards the oriented cobordism ring Ω^{SO}. We will first need to develop the theory of characteristic classes with coefficients in $\mathbb{Z}$. Most of the material in the next three sections will be directly analogous to that in Sections 9, 10 and 11. For a more complete treatment of these topics, see [14, Sections 9-16]. [4, Sections 20-22] provides a much different approach to this material; [21, Chapter 5] unifies all of the characteristic classes into a single theory, although this requires quite a bit more machinery than we will make use of here.

Recall than an *orientation* on a real vector space V of dimension n is a choice of an equivalence class of ordered bases, where two bases are said to be equivalent if they are related by a linear transformation with positive determinant. Evidently, then, there are two possible orientations on a real vector space.

A choice of an orientation on V is equivalent to a choice of a preferred generator $\mu_V \in H_n(V, V_0; \mathbb{Z}) \cong \mathbb{Z}$, where $V_0 = V - 0$. This in turn gives rise to a preferred generator $u_V \in H^n(V, V_0; \mathbb{Z}) \cong \mathbb{Z}$ by the relation $\langle u_V, \mu_V \rangle = +1$.

An *orientation* on a vector bundle, then, is a choice of orientation on each fiber satisfying the usual local triviality condition. In this case, that says that every point b in the base space has a neighborhood W and a cohomology class $u_W \in H^n(\pi^{-1}(W), \pi^{-1}(W)_0; \mathbb{Z})$ such that $u_W|_{(F,F_0)} = u_F$ for every fiber F over W, where π is the projection. Of course, the fundamental example of an oriented bundle is the tangent bundle of an oriented manifold.

The oriented version of the Thom isomorphism theorem asserts that the above local orientation classes glue to give a global Thom class.

Theorem 14.1 (Oriented Thom Isomorphism Theorem)**.** *Let ξ be an oriented n-dimensional vector bundle over a base space B with projection map $\pi : E = E(\xi) \to B$. Let E_0 be the complement of the zero section in E. Then there exists a unique cohomology class $u_\xi \in H^n(E, E_0; \mathbb{Z})$ (called the Thom class) such that*

$$u_\xi|_{(F,F_0)} = u_F$$

for all fibers $F = \pi^{-1}(b)$. Furthermore, for all i the map $x \mapsto x \cup u_\xi$ defines an isomorphism

$$H^i(E; \mathbb{Z}) \to H^{i+n}(E, E_0; \mathbb{Z}).$$

Proof. See [14, Section 10]. □

Definition 14. The *(oriented) Thom isomorphism*

$$\varphi : H^i(B;\mathbb{Z}) \to H^{i+n}(E, E_0;\mathbb{Z})$$

is defined by $\varphi(x) = \pi^*(x) \cup u_\xi$.

Unfortunately, we can not continue to proceed as with Stiefel-Whitney classes, because we have no analogue of the Steenrod algebra with $\mathbb{Z}$-coefficients. We can, however, define an analogue of the top Stiefel-Whitney class $w_n(\xi)$.

Definition 15. Let ξ be an oriented n-plane bundle over B. The *Euler class* $e(\xi) \in H^n(B;\mathbb{Z})$ is defined by

$$e(\xi) = \varphi^{-1}(u_\xi \cup u_\xi).$$

where u_ξ is the Thom class. That is,

$$\pi^* e(\xi) = u_\xi|_E.$$

The Euler class has many of the same properties as Stiefel-Whitney classes. We list some here. The proofs of the first three are similar to the corresponding proofs for Stiefel-Whitney classes, and the proof of the fourth is trivial. Note that we give Whitney sums and Cartesian products of oriented vector bundles orientations in the obvious way.

(1) If $f : B(\xi) \to B(\eta)$ is covered by an orientation preserving bundle map from ξ to η, then

$$e(\xi) = f^* e(\eta).$$

(2) If ξ and ξ' are oriented vector bundles, then

$$e(\xi \times \xi') = e(\xi) \times e(\xi').$$

(3) If ξ and ξ' are oriented vector bundles over the same base space, then

$$e(\xi \oplus \xi') = e(\xi) \cup e(\xi').$$

(4) If ξ' is isomorphic to ξ as an unoriented bundle, but has the opposite orientation, then

$$e(\xi') = -e(\xi).$$

(5) If ξ is an n-plane bundle with n odd, then $2e(\xi) = 0$.

Proof. If ξ has odd fiber dimension, then it has an orientation reversing automorphism $(b, v) \mapsto (b, -v)$. So, by property 4, $e(\xi) = -e(\xi)$. □

(6) Let ξ be an oriented vector bundle over a base space B. Then the canonical map $H^n(B;\mathbb{Z}) \to H^n(B;\mathbb{Z}_2)$ sends $e(\xi)$ to $w_n(\xi)$.

Proof. We have $e(\xi) = \varphi^{-1}(u_\xi \cup u_\xi)$. Now, clearly u_ξ maps to the unoriented Thom class u'_ξ, and therefore $u_\xi \cup u_\xi$ maps to $\mathrm{Sq}^n(u'_\xi)$. Thus, $\varphi^{-1}(u_\xi \cup u_\xi)$ maps to $\varphi^{-1}\big(\mathrm{Sq}^n(u'_\xi)\big)$, which is just $w_n(\xi)$. □

In analogy with property 6, the Euler class $e(\xi)$ gives rise to a well-defined element $e_R(\xi)$ in $H^n(B;R)$ for any ring R, under the homomorphism of cohomology induced by the ring map $\mathbb{Z} \to R$. This class has all the properties of the usual Euler class, as is easy to see.

One final useful application of the Euler class is the construction of an oriented Gysin sequence. The proof is identical to the proof of Proposition 9.2.

Proposition 14.2 (The Gysin Sequence)**.** *Let ξ be an oriented n-plane bundle with projection $\pi : E \to B$. Let $\pi_0 : E_0 \to B$ be the restriction of π to E_0. Then for any coefficient ring R, there is a long exact sequence*

$$\cdots \longrightarrow H^i(B;R) \xrightarrow{\cup e} H^{i+n}(B;R) \xrightarrow{\pi_0^*} H^{i+n}(E_0;R) \longrightarrow \cdots$$

where $e = e_R(\xi)$ is the Euler class with respect to R.

15. Complex Vector Bundles and Chern Classes

Before we can further develop the theory of characteristic classes of real vector bundles, we must turn to the complex case. Let ω be a complex n-plane bundle; that is, each fiber has the structure of a complex vector space of dimension n. We can endow ω with a Hermitian metric in the usual way. Of course, ω can also be considered as a real $2n$-plane bundle; we will denote this by $\omega_{\mathbb{R}}$. In fact, $\omega_{\mathbb{R}}$ has a canonical preferred orientation: if $v_1, \ldots, v_n$ is a $\mathbb{C}$-basis for a fiber F of ω, we take $v_1, iv_1, \ldots, v_n, iv_n$ to be an $\mathbb{R}$-basis for F as a fiber of $\omega_{\mathbb{R}}$. It is easy to show that the orientation is independent of the choice of complex basis.

There are several important constructions on complex vector bundles. First, there is the conjugate bundle $\overline{\omega}$. This bundle has the same underlying real structure as ω (that is, $\omega_{\mathbb{R}} = \overline{\omega}_{\mathbb{R}}$), but has $\mathbb{C}$-action on fibers where $(x+iy)v$ in $\overline{\omega}$ is given by $(x-iy)v$ in ω. Using the Hermitian metric, it is easy to show that

$$\overline{\omega} = \mathrm{Hom}_{\mathbb{C}}(\omega, \varepsilon^1_{\mathbb{C}})$$

where $\varepsilon^1_{\mathbb{C}}$ is the trivial complex line bundle over $B(\omega)$.

We will need one other construction on complex vector bundles. Let ω be a complex n-plane bundle with total space E. We now define a complex $(n-1)$-plane bundle ω_0 over E_0 as follows : Given a pair (b, v) in E_0 of a point in the base space and a non-zero vector in the fiber F_b, let the fiber over (b, v) in ω_0 be the orthogonal complement of v in F_b. This is then an $(n-1)$-plane, and it is easy to see that this construction makes ω_0 into a complex vector bundle over E_0.

Finally, recall that since $\omega_{\mathbb{R}}$ is an oriented $2n$-plane bundle, we have an exact Gysin sequence

$$\cdots \longrightarrow H^{i-2n}(B;\mathbb{Z}) \xrightarrow{\cup e(\omega_{\mathbb{R}})} H^i(B;\mathbb{Z}) \xrightarrow{\pi_0^*} H^i(E_0;\mathbb{Z}) \longrightarrow \cdots$$

Since for $i < 2n-1$ we have $H^{i-2n}(B;\mathbb{Z}) = H^{i-2n+1}(B;\mathbb{Z}) = 0$, we see that

$$\pi_0^* : H^i(B;\mathbb{Z}) \to H^i(E_0;\mathbb{Z})$$

is an isomorphism in this range.

We can now define the Chern classes of a complex vector bundle.

Definition 16. Let ω be a complex n-plane bundle over B. For $i \leq n$, the i^{th} *Chern class* $c_i(\omega) \in H^{2i}(B;\mathbb{Z})$ is defined inductively as follows : Set $c_n(\omega) = e(\omega_{\mathbb{R}})$. For $i < n$, set

$$c_i(\omega) = \pi_0^{-1} c_i(\omega_0).$$

(Note that by the Gysin sequence above π_0 is an isomorphism in this range.) For $i > n$, we set $c_i(\omega) = 0$. We define the *total Chern class* $c(\omega) \in H^*(B;\mathbb{Z})$ by

$$c(\omega) = 1 + c_1(\omega) + \cdots + c_n(\omega).$$

Chern classes satisfy the usual properties of characteristic classes. We omit the proofs of the first two, although it should be noted that they are actually considerably more difficult than the analogous results about Stiefel-Whitney classes and the Euler class. For the proofs, see [14, Lemma 14.2 and pp. 164-167].

(1) If $f : B(\omega) \to B(\omega')$ is covered by a bundle map from ω to ω', then

$$c(\omega') = f^* c(\omega).$$

(2) If ω and ω' are complex vector bundles over the same base space, then

$$c(\omega \oplus \omega') = c(\omega) c(\omega').$$

(3) If ω is a complex vector bundle, then

$$c_i(\overline{\omega}) = (-1)^i c_i(\omega).$$

Proof. This follows immediately from the fact that the underlying real vector bundles $\omega_{\mathbb{R}}$ and $\overline{\omega}_{\mathbb{R}}$ have the same orientation if ω has even fiber dimension, and opposite orientations if ω has odd fiber dimension. □

As an application of Chern classes and the Gysin sequence, we can determine the cohomology of complex projective space $\mathbb{C}P^n$. Recall that $\mathbb{C}P^n$ is the complex Grassmann manifold of lines in $\mathbb{C}^{n+1}$. It has a canonical line bundle $\gamma^1(\mathbb{C}^{n+1})$, consisting of pairs of lines in $\mathbb{C}^{n+1}$ and points in that line.

Proposition 15.1. *Set* $x = c_1\big(\gamma^1(\mathbb{C}^{n+1})\big) \in H^2(\mathbb{C}P^n;\mathbb{Z})$. *Then*

$$H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}).$$

Proof. The proof is nearly identical to the proof of Proposition 10.1. The even dimensional cohomology groups $H^{2i}(\mathbb{C}P^n;\mathbb{Z})$ are all isomorphic to $H^0(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$ and generated by x^i, and the odd dimensional cohomology groups are all isomorphic to $H^1(\mathbb{C}P^n;\mathbb{Z})$, which can be seen to be 0 from the portion

$$\cdots \longrightarrow H^{-1}(\mathbb{C}P^n;\mathbb{Z}) \longrightarrow H^1(\mathbb{C}P^n;\mathbb{Z}) \longrightarrow H^1(E_0;\mathbb{Z}) \longrightarrow \cdots$$

of the Gysin sequence. □

We can now compute the Chern classes of $\tau(\mathbb{C}P^n)$. The computation is entirely analogous to the one for $\mathbb{R}P^n$ (Propositions 11.1 and 11.2), so we will skip most of the details.

Proposition 15.2.

$$c\big(\tau(\mathbb{C}P^n)\big) = (1-x)^{n+1}$$

where $x = c_1\big(\gamma^1(\mathbb{C}^{n+1})\big)$.

Proof. Set $\gamma = \gamma^1(\mathbb{C}^{n+1})$ and $\tau = \tau(\mathbb{C}P^n)$. Proceeding exactly as in the case of $\mathbb{R}P^n$, we have

$$\tau \cong \mathrm{Hom}(\gamma, \gamma^\perp)$$

where $\gamma \oplus \gamma^\perp = \varepsilon_{\mathbb{C}}^{n+1}$. Thus, we find that

$$\tau \oplus \varepsilon^1_{\mathbb{C}} \cong \overline{\gamma} \oplus \cdots \oplus \overline{\gamma}.$$

So, by the product formula for Chern classes,

$$\begin{aligned} c(\tau) &= \big(c(\overline{\gamma})\big)^{n+1} \\ &= (1-x)^{n+1}. \end{aligned}$$

□

16. Pontrjagin Classes

We now return to the study of real vector bundles. Let ξ be a real n-plane bundle. We define the *complexification* of ξ, $\xi \otimes \mathbb{C}$, to be the complex n-plane bundle with the same base space and typical fiber $F \otimes C$, where F is a fiber of ξ.

Visibly, $(\xi \otimes \mathbb{C})_{\mathbb{R}} \cong \xi \oplus \xi$. In fact, we also have $\xi \otimes \mathbb{C} \cong \overline{\xi \otimes \mathbb{C}}$, since conjugation is an $\mathbb{R}$-linear homeomorphism interchanging the two complex structures, and thus is a bundle isomorphism. Therefore, since $c_i(\overline{\xi \otimes \mathbb{C}}) = (-1)^i c_i(\xi \otimes \mathbb{C})$, we see that the odd Chern classes $c_1, c_3, \ldots$ of the complexification of a real vector bundle are 2-torsion elements.

Definition 17. Let ξ be an n-plane bundle over B. The i^{th} *Pontrjagin class* $p_i(\xi) \in H^{4i}(B;\mathbb{Z})$ is defined by

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C}).$$

We define the *total Pontrjagin class* $p(\xi) \in H^*(B;\mathbb{Z})$ by

$$p(\xi) = 1 + p_1(\xi) + \cdots + p_{[n/2]}(\xi).$$

The properties of Pontrjagin classes follow easily from those of Chern classes, so we omit the proofs. However, ignoring the odd Chern classes does complicate the statement of the product formula somewhat.

(1) If $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then
$$p(\xi) = f^* p(\eta).$$
(2) Let ξ be a vector bundle over a base space B. Let ε^k be the trivial k-plane bundle over B. Then
$$p(\xi \oplus \varepsilon^k) = p(\xi).$$
(3) If ξ and ξ' are vector bundles over the same base space, then
$$2p(\xi \oplus \xi') = 2p(\xi)p(\xi').$$
(We need the factors of 2 to annihilate the 2-torsion elements.)

One other useful property of Pontrjagin classes is a connection with the Euler class.

Proposition 16.1. *If ξ is an oriented $2n$-plane bundle, then $p_n(\xi) = e(\xi)^2$.*

Proof. As we have already observed, $(\xi \otimes \mathbb{C})_{\mathbb{R}} \cong \xi \oplus \xi$ as unoriented bundles. Let F be a typical fiber of ξ, with ordered basis $v_1, \ldots, v_{2n}$. Then the orientation $(F \otimes \mathbb{C})_{\mathbb{R}}$ is given by the ordered basis $v_1, iv_1, \ldots, v_{2n}, iv_{2n}$. On the other hand, the orientation on $\xi \oplus \xi$ is given by $v_1, \ldots, v_{2n}, iv_1, \ldots, iv_{2n}$. So we see that a linear transformation between the two bases has sign $(-1)^{2n(2n-1)/2} = (-1)^n$. Thus,

$$\begin{aligned} e\big((\xi \otimes \mathbb{C})_{\mathbb{R}}\big) &= (-1)^n e(\xi \oplus \xi) \\ c_{2n}(\xi \otimes \mathbb{C}) &= (-1)^n e(\xi)^2 \\ p_n(\xi) &= e(\xi)^2. \end{aligned}$$

□

Now, let M^{4n} be a compact oriented manifold. We then have a *fundamental homology class* $\mu_M \in H^{4n}(M; \mathbb{Z})$. As with Stiefel-Whitney classes, for any vector bundle ξ over M and any partition $I = (i_1, \ldots, i_k)$ of n, we can then define a *Pontrjagin number*

$$P_I[\xi] = \langle p_{i_1}(\xi) \cdots p_{i_k}(\xi), \mu_M \rangle \in \mathbb{Z}$$

and an *s-number*

$$S_I[p(\xi)] = \big\langle s_I\big(p_1(\xi), \ldots, p_n(\xi)\big), \mu_M \big\rangle \in \mathbb{Z}.$$

Just as in the case of Stiefel-Whitney classes, we have the following proposition.

Proposition 16.2. *Let ξ and ξ' be vector bundles over M. Then*

$$2 s_I\big(p(\xi \oplus \xi')\big) = 2 \sum_{I_1 I_2 = I} s_{I_1}\big(p(\xi)\big) s_{I_2}\big(p(\xi')\big).$$

If η is a vector bundle over another manifold N, then

$$S_I[p(\xi \times \eta)] = \sum_{I_1 I_2 = I} S_{I_1}[p(\xi)] S_{I_2}[p(\eta)].$$

Proof. The proof is nearly identical to the proof of Proposition 9.3. The 2's are present in the first formula to eliminate the torsion elements but are not needed in the second formula since the Kronecker index of any torsion element is 0. □

We now wish to compute the Pontrjagin classes and numbers of complex projective spaces. This is not quite as easy as it sounds, since we must compute the Chern classes of $(\tau(\mathbb{C}P^n) \otimes \mathbb{C})_{\mathbb{R}}$. This is accomplished by means of the following lemma.

Lemma 16.3. *Let ω be a complex vector bundle. Then $\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}$.*

Proof. This follows easily from the fact that for a complex vector space V, $V \otimes \mathbb{C} \cong V \oplus \overline{V}$, where $\overline{V}$ is the same real vector space as V, with the conjugate complex structure. □

Proposition 16.4. *Let $\tau = \tau(\mathbb{C}P^n)$. Then*

$$p(\tau_{\mathbb{R}}) = (1 + x^2)^{n+1},$$

where $x = c_1\big(\gamma_1(\mathbb{C}^{n+1})\big)$. Further,

$$P_{(n)}[\tau_{\mathbb{R}}] = n + 1.$$

Proof. By Lemma 16.3 we have

$$\tau_{\mathbb{R}} \otimes \mathbb{C} \cong \tau \oplus \overline{\tau}.$$

Thus

$$c(\tau_{\mathbb{R}} \otimes \mathbb{C}) = c(\tau \oplus \overline{\tau}).$$

But the odd Chern class disappear, since these cohomology groups have no 2-torsion. Therefore,

$$\begin{aligned} 1 + c_2(\tau_{\mathbb{R}} \otimes \mathbb{C}) + \cdots + c_{2n}(\tau_{\mathbb{R}} \otimes \mathbb{C}) &= c(\tau) c(\overline{\tau}) \\ 1 - p_1(\tau_{\mathbb{R}}) + p_2(\tau_{\mathbb{R}}) - \cdots \pm p_n(\tau_{\mathbb{R}}) &= (1 - x)^{n+1}(1 + x)^{n+1} \\ &= (1 - x^2)^{n+1}. \end{aligned}$$

Thus, $p(\tau_{\mathbb{R}}) = (1 + x^2)^{n+1}$. The computation of $P_{(n)}[\tau_{\mathbb{R}}]$ is now immediate. □

17. The Cohomology of BSO_r

In analogy with the unoriented Grassmann manifolds $G_r(\mathbb{R}^{n+r})$, we now define the oriented Grassmann manifolds $\widetilde{G}_r(\mathbb{R}^{n+r})$. This is simply the manifold of all oriented r-planes in $\mathbb{R}^{n+r}$. It is a two-sheeted covering of $G_r(\mathbb{R}^{n+r})$. We let $\widetilde{\gamma}^r(\mathbb{R}^{n+r})$ be the canonical r-plane bundle over $\widetilde{G}_r(\mathbb{R}^{n+r})$. Then, in the same way as in the unoriented case, we obtain the Grassmann manifold BSO_r of oriented r-planes in $\mathbb{R}^\infty$. It is a two-sheeted covering space of BO_r, with covering map $f_r : BSO_r \to BO_r$. We denote by $\widetilde{\gamma}^r$ the pullback $f_r^*\gamma^r$. This makes BSO_r into a universal classifying space for oriented r-plane bundles over paracompact base spaces. (The notations BO_r and BSO_r are related to the reduction of the structural groups of vector bundles to the orthogonal group and the special orthogonal group respectively. See [4, Sections 5 and 6].)

We now compute the cohomology of BSO_r. We will be most interested in coefficients in $\mathbb{Q}$ and $\mathbb{Z}_p$ for p an odd prime, so we will do the computation for any integral domain R in which 2 is invertible.

Proposition 17.1. *Let R be an integral domain containing $\frac{1}{2}$. Let p_i be the image of $p_i(\widetilde{\gamma}^r)$ and e be the image of $e(\widetilde{\gamma}^r)$ under the cohomology map induced by the ring map $\mathbb{Z} \to R$. Then, for odd r,*

$$H^*(BSO_r; R) \cong R[p_1, \dots, p_{\frac{r-1}{2}}],$$

and for even r,

$$H^*(BSO_r; R) \cong R[p_1, \dots, p_{\frac{r}{2}-1}, e].$$

Proof. This will be an extension of the methods of Proposition 10.3. Now, since the space $\widetilde{G}_1(\mathbb{R}^n)$ is homeomorphic to S^{n-1}, letting n go to infinity shows that

$$H^*(BSO_1; R) \cong R.$$

We proceed by induction on n. We have a Gysin sequence

$$\cdots \longrightarrow H^i(BSO_r; R) \xrightarrow{\cup e} H^{i+r}(BSO_r; R) \longrightarrow H^{i+r}(E_0; R) \longrightarrow \cdots$$

Just as in the unoriented case, we can replace E_0 by BSO_{r-1}, obtaining a long exact sequence

$$\cdots \longrightarrow H^i(BSO_r; R) \xrightarrow{\cup e} H^{i+r}(BSO_r; R) \xrightarrow{g} H^{i+r}(BSO_{r-1}; R) \longrightarrow \cdots$$

where g sends p_i to the image p_i' of $p_i(\widetilde{\gamma}^{r-1})$.

Now, suppose r is even. Then by the induction hypothesis g is surjective, so we get a short exact sequence

$$0 \longrightarrow H^i(BSO_r; R) \xrightarrow{\cup e} H^{i+r}(BSO_r; R) \xrightarrow{g} H^{i+r}(BSO_{r-1}; R) \longrightarrow 0$$

By the induction hypothesis, $H^*(BSO_{r-1}; R) \cong R[p_1', \dots, p_{\frac{r}{2}-1}']$, and it easily follows that

$$H^*(BSO_r; R) \cong R[p_1, \dots, p_{\frac{r}{2}-1}, e].$$

Next, suppose that r is odd. Then $e = 0$ since 2 is a unit in R, so the Gysin sequence yields a short exact sequence

$$0 \longrightarrow H^i(BSO_r; R) \xrightarrow{g} H^i(BSO_{r-1}; R) \longrightarrow H^{i-r+1}(BSO_r; R) \longrightarrow 0$$

In this way we can consider $H^*(BSO_r; R)$ as a subring of $H^*(BSO_{r-1}; R)$.

Set $A = R[p_1, \dots, p_{\frac{r-1}{2}}] \subseteq H^*(BSO_{r-1}; R)$. We wish to show that $A = H^*(BSO_r; R)$. Of course, we already have

$$A \subseteq g(H^*(BSO_r; R))$$

and therefore

$$\operatorname{rank} A^i \leq \operatorname{rank} H^i(BSO_r; R)$$

where by the *rank* of an R-module we mean the maximal number of elements which are linearly independent over R.

By the induction hypothesis, we have

$$H^*(BSO_{r-1}; R) = R[p_1', \dots, p_{\frac{r-3}{2}}', e']$$

so every $x \in H^*(BSO_{r-1}; R)$ can be written uniquely in the form $a + e'a'$ with $a \in A^i$ and $a' \in A^{i-r+1}$. That is,

$$H^i(BSO_{r-1}; R) \cong A^i \oplus A^{i-r+1}$$

so

$$\operatorname{rank} H^i(BSO_{r-1}; R) = \operatorname{rank} A^i + \operatorname{rank} A^{i-r+1}.$$

By the short exact sequence above, we also have

$$\operatorname{rank} H^i(BSO_{r-1}; R) = \operatorname{rank} H^i(BSO_r; R) + \operatorname{rank} H^{i-r+1}(BSO_r; R).$$

It follows that

$$\operatorname{rank} H^i(BSO_r;R) = \operatorname{rank} A^i$$

for all i. Now, suppose that $g(H^i(BSO_r;R)) \neq A^i$. Then we can find an element $a + e'a' \in H^i(BSO_r;R)$ with $a' \neq 0$, and this element will be linearly independent of A^i. But then

$$\operatorname{rank} A^i < \operatorname{rank} H^i(BSO_r;R)$$

which is a contradiction. Thus, $A^i = g(H^i(BSO_r;R))$, so

$$H^*(BSO_r;R) \cong A$$

which completes the induction. □

Corollary 17.2. *Let R be as above. Let p_i be the image of $p_i(\gamma^r)$ under the cohomology homomorphism induced by the ring map $\mathbb{Z} \to R$. Then*

$$H^*(BO_r;R) \cong R[p_1, \ldots, p_{[r/2]}].$$

Proof. The covering map f_r induces an injection

$$f_r^* : H^*(BO_r;R) \hookrightarrow H^*(BSO_r;R).$$

An element $x \in H^*(BSO_r;R)$ will lie in $f_r^* H^*(BO_r;R)$ if and only if it is fixed by the map $\rho : BSO_r \to BSO_r$ which interchanges the two sheets of the covering. Now, the Pontrjagin classes are independent of orientation, so they are fixed by ρ and thus lie in $fr^*H^*(BO_r;R)$. The Euler class, however, changes sign with a change in orientation, so it is not fixed by ρ and does not lie in $f_r^*H^*(BO_r;R)$. This completes the proof. □

Corollary 17.3. *$H^i(BSO_r;\mathbb{Z})$ is finite if i is not divisible by 4 and has rank $p(i/4)$ if i is divisible by 4.*

Proof. This follows immediately from the universal coefficient theorem and Proposition 17.1 with $R = \mathbb{Q}$. □

18. Determination of $\Omega^{SO} \otimes \mathbb{Q}$

In this section we will determine the structure of $\Omega^{SO} \otimes \mathbb{Q}$. Tensoring with $\mathbb{Q}$ has the effect of killing all torsion, while preserving the free structure. (Although free generators for $\Omega^{SO} \otimes \mathbb{Q}$ need not generate the free part of Ω^{SO}.) By the Thom-Pontrjagin theorem, we already have

$$\Omega_n^{SO} \cong \lim_{r\to\infty} \pi_{n+r}(TBSO_r, t_0).$$

Unfortunately, computing $\pi_{n+r}(TBSO_r, t_0)$ is much more difficult than computing $\pi_{n+r}(TBO_r, t_0)$. Instead, we will use several results of Serre to approximate these homotopy groups by homology groups. For an exposition of the needed results, see [18] or [19, Sections 9.6 and 9.7].

Recall that a CW-complex is said to be *k-connected* if it is connected and $H_i(X;\mathbb{Z}) = \pi_i(X) = 0$ for $1 \leq i \leq k$.

Proposition 18.1. *Let X be a finite k-connected CW complex. Then*

$$\operatorname{rank} H_i(X;\mathbb{Z}) = \operatorname{rank} \pi_i(X)$$

for $i \leq 2k$.

Proof. We will establish this result by approximating X by spheres. First, consider the sphere S^n, with $n \geq k+1$. Then $H_n(S^n;\mathbb{Z})$ and $\pi_n(S^n)$ both have rank 1, and $H_i(S^n;\mathbb{Z})$ and $\pi_i(S^n)$ are both finite for $1 \leq i \leq 2n-2, i \neq n$. Since $2k \leq 2n-2$, this establishes the proposition in this case.

Now, suppose that the result holds for two finite k-connected CW complexes X and Y. Then by the Künneth theorem,

$$\begin{aligned} H_i(X \times Y;\mathbb{Z}) &\cong \bigoplus_{i_1+i_2=i} H_{i_1}(X;\mathbb{Z}) \otimes H_{i_2}(Y;\mathbb{Z}) \\ &\cong H_i(X;\mathbb{Z}) \oplus H_i(Y;\mathbb{Z}) \end{aligned}$$

since for the other terms at least one of $H_{i_1}(X;\mathbb{Z})$ and $H_{i_2}(Y;\mathbb{Z})$ is 0. Since also

$$\pi_i(X \times Y) \cong \pi_i(X) \oplus \pi_i(Y)$$

the proposition is also true for $X \times Y$.

Next, since

$$H_i(X \vee Y;\mathbb{Z}) \cong H_i(X;\mathbb{Z}) \oplus H_i(Y;\mathbb{Z})$$

and

$$\pi_i(X \vee Y) \cong \pi_i(X) \oplus \pi_i(Y) \oplus \pi_{i+1}(X \times Y, X \vee Y)$$

and $\pi_{i+1}(X \times Y, X \vee Y)$ is finite, we see that the proposition is also true for $X \vee Y$. Thus it is true for a finite bouquet of spheres of dimension $\geq k+1$.

Finally, let X be any finite k-connected CW complex. Since the homotopy groups of X are finitely generated, we can choose a finite basis for the free part of $\pi_i(X)$ for $i \leq 2k+1$. Each of these basis elements is represented by a map

$$\rho_i : S^{r_i} \to X$$

where $r_i \geq k+1$. Combining the ρ_i's, we obtain a map

$$\rho : S^{r_1} \vee \cdots \vee S^{r_m} \to X.$$

Since by construction the homotopy groups of $S^{r_1} \vee \cdots \vee S^{r_m}$ and X have the same rank in dimensions $\leq 2k+1$, by the generalized Whitehead theorem the homology groups have the same rank for dimensions $\leq 2k$. But since we already have that

$$\operatorname{rank} \pi_i(S^{r_1} \vee \cdots \vee S^{r_m}) = \operatorname{rank} H_i(S^{r_1} \vee \cdots \vee S^{r_m}; \mathbb{Z})$$

for $i \leq 2k$, it follows that

$$\operatorname{rank} \pi_i(X) = \operatorname{rank} H_i(X; \mathbb{Z})$$

in this range, as asserted. □

With this proposition, we can now give a partial solution to the oriented cobordism problem.

Theorem 18.2. *Ω_n^{SO} is finite for n not divisible by 4, and has rank $p(n/4)$ for n divisible by 4.*

Proof. By the Thom-Pontrjagin theorem,

$$\Omega_n^{SO} \cong \lim_{r\to\infty} \pi_{n+r}(TBSO_r, t_0).$$

Choose $r > n$. By Proposition 18.1, taking the limit of the finite complexes $T\widetilde{\gamma}^r(\mathbb{R}^{n+r})$ as n goes to infinity, we see that

$$\operatorname{rank} \pi_{n+r}(TBSO_r, t_0) = \operatorname{rank} H_{n+r}(TBSO_r; \mathbb{Z}).$$

But this is the same as the rank of $H_{n+r}(TBSO_r, t_0; \mathbb{Z})$, by the exact sequence of the pair $(TBSO_r, t_0)$. This in turn is the same as the rank of $H^{n+r}(TBSO_r, t_0; \mathbb{Z})$, by the universal coefficient theorem. But by Lemma 12.1,

$$H^{n+r}(TBSO_r, t_0; \mathbb{Z}) \cong H^n(BSO_r; \mathbb{Z}).$$

By Corollary 17.3, this is finite for n not divisible by 4, and has rank $p(n/4)$ for n divisible by 4. This completes the proof. □

As with $\mathfrak{N}$, Ω^{SO} can be given the structure of a commutative graded $\mathbb{Z}$-algebra with product induced by the Cartesian product of manifolds. (See Proposition 13.1.) We can in fact give explicit generators for $\Omega^{SO} \otimes \mathbb{Q}$.

Theorem 18.3. *$\Omega^{SO} \otimes \mathbb{Q}$ is the free $\mathbb{Z}$-algebra generated by $\mathbb{C}P^{2n}$ for $n \geq 1$.*

Proof. By Proposition 16.4,

$$P_{(n)}[\tau(\mathbb{C}P^{2n})_{\mathbb{R}}] = 2n+1.$$

Proceeding as in the proof of Theorem 13.4, we conclude that the $\mathbb{C}P^{2n}$ are algebraically independent as elements of $\Omega^{SO} \otimes \mathbb{Q}$. Since $(\Omega^{SO} \otimes \mathbb{Q})^i$ has no torsion, by Theorem 18.2 it is 0 if i is not divisible by 4, and has rank $p(i/4)$ for i divisible by 4. Thus,

$$\operatorname{rank}(\Omega^{SO} \otimes \mathbb{Q})^i = \operatorname{rank}(\mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \ldots])^i$$

for all i, so they are isomorphic. □

We can also relate oriented cobordism to the vanishing of Pontrjagin numbers, although it turns out that this alone is not enough to show that an oriented manifold is a boundary.

Proposition 18.4. *Let M be a compact $4n$-dimensional oriented manifold. If M is the boundary of an oriented $(4n+1)$-dimensional manifold B, then all of the Pontrjagin numbers of the tangent bundle of M are zero.*

Proof. We have the exact sequences

$$H_{4n+1}(B,M;\mathbb{Z}) \xrightarrow{\partial} H_{4n}(M;\mathbb{Z}) \xrightarrow{i_*} H_{4n}(B;\mathbb{Z})$$

and

$$H^{4n}(B;\mathbb{Z}) \xrightarrow{i^*} H^{4n}(M;\mathbb{Z}) \xrightarrow{\delta} H^{4n+1}(B,M;\mathbb{Z})$$

arising from the long exact sequence of the pair (B,M), where δ is the dual of ∂. Let $\mu_{B,M} \in H_{4n+1}(B,M;\mathbb{Z})$ be the fundamental homology class of the pair (B,M) and $\mu_M \in H_{4n}(M;\mathbb{Z})$ the fundamental homology class of M. Then $\partial\mu_{B,M} = \mu_M$.

Now, note that there is a unique outward pointing normal vector along $M \subseteq B$, so

$$\tau(B)|_M = \tau(M) \oplus \varepsilon^1.$$

Thus,

$$p(\tau(B)|_M) = p\big(\tau(M)\big),$$

and therefore for any partition I of $4n$,

$$\begin{aligned} P_I[\tau(M)] &= P_I[\tau(B)|_M] \\ &= \langle p_I(\tau(B)|_M), \mu_M\rangle \\ &= \big\langle i^* p_I\big(\tau(B)\big), \mu_M\big\rangle \\ &= \big\langle i^* p_I\big(\tau(B)\big), \partial\mu_{B,M}\big\rangle \\ &= \big\langle \delta i^* p_I\big(\tau(B)\big), \mu_{B,M}\big\rangle \\ &= \langle 0, \mu_{B,M}\rangle \\ &= 0 \end{aligned}$$

by exactness. □

The complete determination of Ω^{SO} is a very difficult problem. The above result was obtained by Thom in [22]. Additional results were obtained by Milnor, Averbuh and Novikov (see [12], [3] and [16]), before the complete structure theorem was obtained by C.T.C. Wall in [23]. We state it without proof.

Theorem 18.5. *Two oriented manifolds are cobordant if and only if their tangent bundles have the same Pontrjagin numbers and Stiefel-Whitney numbers. Ω^{SO} is the $\mathbb{Z}$-algebra generated by manifolds X_{4i} of dimension $4i$ (for all $i \geq 1$) and by manifolds $Y_{2i-1,j}$ of dimension $2i-1$, subject only to the relations $2Y_{2i-1,j} = 0$. There is one torsion generator $Y_{2i-1,j}$ for each partition of i into distinct positive integers, none a power of 2.*

19. The Hirzebruch Signature Theorem

As an application of cobordism, in this section we will use our cobordism classification to give a proof of the remarkable Hirzebruch signature theorem. This is a special case of the Atiyah-Singer index theorem; for an explanation of this celebrated result, see [1].

Let M be a compact oriented manifold of dimension n. We define the *signature* of M, $\sigma(M)$, as follows: If n is not divisible by 4, then $\sigma(M) = 0$. If n is divisible by 4, say $n = 4m$, we define $\sigma(M)$ to be the signature of the rational quadratic form Q on $H^{2m}(M;\mathbb{Q})$ given by

$$Q(x) = \langle x \cup x, \mu_M\rangle \in \mathbb{Q}$$

where $\mu_M \in H^n(M;\mathbb{Q})$ is the fundamental rational homology class of M. (Recall that the signature of a rational quadratic form is computed as follows : Choose a basis $x_1, \ldots, x_r$ of $H^{2m}(M;\mathbb{Q})$ for which the symmetric matrix $(\langle x_i \cup x_j, \mu_M\rangle)$ is diagonal. Then the signature of the associated quadratic form is the number of positive diagonal entries minus the number of negative diagonal entries.)

The signature satisfies the following properties.

(1) $\sigma(M + N) = \sigma(M) + \sigma(N)$.
(2) $\sigma(M \times N) = \sigma(M)\sigma(N)$.
(3) If M is an oriented boundary, then $\sigma(M) = 0$.

See [8, Section 9] for a proof. These combine to show that σ induces a well-defined $\mathbb{Q}$-algebra homomorphism

$$\sigma : \Omega^{SO} \otimes \mathbb{Q} \to \mathbb{Q}.$$

We now construct another such homomorphism. Define a graded commutative $\mathbb{Q}$-algebra A by

$$A = \mathbb{Q}[t_1, t_2, \ldots]$$

where t_i has degree i. Define an associated ring $\mathcal{A}$ to be the ring of infinite formal sums

$$a = a_0 + a_1 + a_2 + \cdots$$

where $a_i \in A$ is homogeneous of degree i. We let $\mathcal{A}^*$ be the subgroup of the multiplicative group of $\mathcal{A}$ of elements with leading term 1.

Now, suppose we have a sequence of polynomials

$$K_1(t_1), K_2(t_1, t_2), K_3(t_1, t_2, t_3), \ldots \in A$$

where K_n is homogeneous of degree n. For $a = 1 + a_1 + a_2 + \cdots \in \mathcal{A}^*$, we then define $K(a) \in \mathcal{A}^*$ by

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots$$

We say that the K_n form a *multiplicative sequence* if $K(ab) = K(a)K(b)$ for all $a, b \in \mathcal{A}^*$.

A simple example is provided by the sequence

$$K_n(t_1, \ldots, t_n) = \lambda^n t_n$$

for any $\lambda \in \mathbb{Q}$. We will now construct a more interesting example.

Consider the power series expansion of $\sqrt{t}/\tanh\sqrt{t}$:

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + (-1)^{i-1}\frac{2^{2i}B_i}{(2i)!}t^i + \cdots$$

where B_i is the i^{th} Bernoulli number. (We use the conventions $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, \ldots$) Set

$$\lambda_i = (-1)^{i-1}\frac{2^{2i}B_i}{(2i)!}.$$

Further, for any partition $I = (i_1, \ldots, i_k)$ of n, set

$$\lambda_I = \lambda_{i_1} \cdots \lambda_{i_k}.$$

Now, define polynomials $L_n(t_1, \ldots, t_n) \in A$ by

$$L_n(t_1, \ldots, t_n) = \sum_I \lambda_I s_I(t_1, \ldots, t_n)$$

where the sum if over all partitions of n and s_I is the polynomial of Section 7.

We claim that the L_n form a multiplicative sequence. It is clear from the definition of s_I that L_n is homogeneous of degree n. Now, take $a, b \in \mathcal{A}^*$. Then

$$\begin{aligned} L(ab) &= \sum_I \lambda_I s_I(ab) \\ &= \sum_I \lambda_I \sum_{I_1 I_2 = I} s_{I_1}(a) s_{I_2}(b) \\ &= \sum_{I_1 I_2 = I} \lambda_{I_1} s_{I_1}(a) \lambda_{I_2} s_{I_2}(b) \\ &= L(a)L(b). \end{aligned}$$

Therefore the L_n do indeed form a multiplicative sequence.

The first few L-polynomials are :

$$\begin{aligned} L_1(t_1) &= \tfrac{1}{3}t_1 \\ L_2(t_1, t_2) &= \tfrac{1}{45}(7t_2 - t_1^2) \\ L_3(t_1, t_2, t_3) &= \tfrac{1}{945}(62t_3 - 13t_1t_2 + 2t_1^3) \\ L_4(t_1, t_2, t_3, t_4) &= \tfrac{1}{14175}(381t_4 - 71t_1t_3 - 19t_2^2 + 22t_1^2t_2 - 3t_1^4). \end{aligned}$$

Note that the coefficient of t_1^n is L_n is λ_n, since the only s-polynomial containing that monomial is $s_{(n)}$.

Now, let M be a manifold of dimension n. We define the *L-genus* $L[M]$ as follows : If n is not divisible by 4, then $L[M]=0$. If n is divisible by 4, say $n=4m$, then we define

$$L[M]=\left\langle K_m\big(p_1(\tau(M)),\dots,p_m(\tau(M))\big),\mu_M\right\rangle.$$

This makes sense, since $K_m\big(p_1(\tau(M)),\dots,p_m(\tau(M))\big)\in H^n(M;\mathbb{Z})$.

Lemma 19.1. *The correspondence $M\mapsto L[M]$ defines a $\mathbb{Q}$-algebra homomorphism*

$$L:\Omega^{SO}\otimes\mathbb{Q}\to\mathbb{Q}.$$

Proof. The additivity of the correspondence is clear. It follows immediately from Proposition 18.4 that the L-genus of a boundary is zero, and these two facts together show that L is well-defined. Now, consider a product manifold $M\times N$. Since the total Pontrjagin class $p(M\times N)$ of $M\times N$ is given by $p(M)\times p(N)$, up to elements of order 2, we see that

$$L\big(p(M\times N)\big)=L\big(p(M)\big)\times L\big(p(N)\big).$$

Therefore,

$$\begin{aligned}
L[M\times N]&=\left\langle L\big(p(M\times N)\big),\mu_{M\times N}\right\rangle\\
&=\left\langle L\big(p(M\times N)\big),\mu_M\times\mu_N\right\rangle\\
&=\left\langle L\big(p(M)\big)\times L\big(p(N)\big),\mu_M\times\mu_N\right\rangle\\
&=\left\langle L\big(p(M)\big),\mu_M\right\rangle\left\langle L\big(p(N)\big),\mu_N\right\rangle\\
&=L[M]L[N].
\end{aligned}$$

□

The Hirzebruch signature theorem states that the two homomorphisms we have constructed are in fact the same.

Theorem 19.2. *For any oriented manifold M, $\sigma(M)=L[M]$.*

Proof. Since both σ and L are $\mathbb{Q}$-algebra homomorphisms from $\Omega^{SO}\otimes\mathbb{Q}$ to $\mathbb{Q}$, it will suffice to check the theorem on a set of algebra generators of $\Omega^{SO}\otimes\mathbb{Q}$. By Theorem 18.3 we may use the complex projective spaces $\mathbb{C}P^{2n}$ as generators.

Now, by Proposition 15.1, $H^{2n}(\mathbb{C}P^{2n})$ is generated by a single element x^n satisfying

$$\langle x^n\cup x^n,\mu_{\mathbb{C}P^{2n}}\rangle=1.$$

Thus $\sigma(\mathbb{C}P^{2n})=+1$.

Next, recall from Proposition 16.4 that $p(\tau(\mathbb{C}P^{2n})_{\mathbb{R}})=(1+x^2)^{2n+1}$. Since the coefficient of t_1^i in L_i is λ_i, we see that

$$\begin{aligned}
L(1+x^2)&=1+L_1(x^2)+L_2(x^2,0)+L_3(x^2,0,0)+\cdots\\
&=1+\lambda_1x^2+\lambda_2x^4+\cdots\\
&=\frac{\sqrt{x^2}}{\tanh\sqrt{x^2}}\\
&=\frac{x}{\tanh x}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
L\big(p(\tau(\mathbb{C}P^{2n})_{\mathbb{R}})\big)&=L\big((1+x^2)^{2n+1}\big)\\
&=L(1+x^2)^{2n+1}\\
&=\Big(\frac{x}{\tanh x}\Big)^{2n+1}.
\end{aligned}$$

Thus, $L_n\big(p(\tau(\mathbb{C}P^{2n}))\big)$ is simply the x^{2n} term in this power series, and $L[\mathbb{C}P^{2n}]$ is the coefficient of that term.

We can compute that coefficient by classical methods of complex analysis. Recall that the coefficient of z^{2n} in the Taylor expansion of $(z/\tanh z)^{2n+1}$ is simply given by the integral

$$\frac{1}{2\pi i}\oint\Big(\frac{z}{\tanh z}\Big)^{2n+1}\frac{dz}{z^{2n+1}}=\frac{1}{2\pi i}\oint\frac{dz}{(\tanh z)^{2n+1}}.$$

To compute this, we make the substitution $u = \tanh z$, so

$$dz = \frac{du}{1-u^2} = (1 + u^2 + u^4 + \cdots)du.$$

Thus,

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{dz}{(\tanh z)^{2n+1}} &= \frac{1}{2\pi i} \oint \frac{(1 + u^2 + u^4 + \cdots)du}{u^{2n+1}} \\ &= \frac{1}{2\pi i} \oint \frac{du}{u} \\ &= 1. \end{aligned}$$

Thus, $L[\mathbb{C}P^{2n}] = +1 = \sigma(\mathbb{C}P^{2n})$. This completes the proof. $\square$

The power of the cobordism classification here is quite impressive; we have given a formula for the signature of a manifold in terms of certain polynomials derived from the power series of $\sqrt{t}/\tanh\sqrt{t}$ evaluated on certain characteristic numbers simply by checking it in the nearly trivial case of complex projective space. A direct proof can be given, using the Atiyah-Singer index theorem. (See [2, Section 6].) However, the depth of that result only serves to make the cobordism classification even more impressive.

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