LECTURE 6: J-HOLOMORPHIC CURVES AND APPLICATIONS

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1. Basic elements of J-holomorphic curve theory

Let (M, ω) be a symplectic manifold of dimension $2n$, and let $J \in \mathcal{J}(M, \omega)$ be an ω-compatible almost complex structure. Let $g_J(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$ be the corresponding hermitian metric (i.e. J-invariant Riemannian metric) on M.

Let (Σ, j) be a Riemann surface (not necessarily compact) with complex structure j. A smooth map $u : \Sigma \to M$ is called a (J, j)**-holomorphic map** (or simply a J**-holomorphic map**) if $du \circ j = J \circ du$, or equivalently,

$$\bar{\partial}_J(u) \equiv \frac{1}{2}(du + J \circ du \circ j) = 0.$$

The equation $\bar{\partial}_J(u) = 0$ is a first order, non-linear equation of Cauchy-Riemann type. We give a description of it in a local coordinate system. Let $z_0 \in \Sigma$ be any point and let $p = u(z_0) \in M$ be its image in M under u. Suppose $s + it$ is a local holomorphic coordinate centered at z_0 and $\phi : U \to \mathbb{R}^{2n}$ is a local chart centered at $p \in M$. Set $\phi \circ u = (u^1, \cdots, u^{2n})^T$. Then

$$\bar{\partial}_J(u) = \frac{1}{2}((\partial_s u^j) + J(u^1, \cdots, u^{2n})(\partial_t u^j))ds + \frac{1}{2}((\partial_t u^j) - J(u^1, \cdots, u^{2n})(\partial_s u^j))dt,$$

and $\bar{\partial}_J(u) = 0$ is equivalent to

$$(\partial_s u^j) + J(u^1, \cdots, u^{2n})(\partial_t u^j) = 0.$$

If J is integrable and $(u^1, \cdots, u^{2n})$ is coming from a local holomorphic coordinate system $(z^1, \cdots, z^n)$ with $z^j = u^j + iu^{j+n}$, $j = 1, \cdots, n$, then $J(u^1, \cdots, u^{2n})$ is constant in $u^1, \cdots, u^{2n}$ and equals the matrix

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

where I denotes the $n \times n$ identity matrix. In this case, $\bar{\partial}_J(u) = 0$ becomes the Cauchy-Riemann equations

$$\partial_s u^j - \partial_t u^{j+n} = 0, \;\; \partial_s u^{j+n} + \partial_s u^j = 0, \;\; j = 1, \cdots, n.$$

Hence when J is integrable, J-holomorphic maps are simply the usual holomorphic maps. On the other hand, it is easy to see that for a general J, the linearization of the non-linear equation $\bar{\partial}_J(u) = 0$ is a zero-th order perturbation of the Cauchy-Riemann equations.

Local properties. We shall next list several relevant local analytical properties of J-holomorphic maps.

Let $u, v : \Sigma \to M$ be two smooth maps and let $z_0 \in \Sigma$ be a point. We say that u, v agree to the infinite order at z_0 if $u(z_0) = v(z_0) = p_0$, and there is a local chart centered at p_0, $\phi : U \to \mathbb{R}^{2n}$, such that all partial derivatives of the $\mathbb{R}^{2n}$-valued function $\phi \circ u - \phi \circ v$ vanish at z_0.

Proposition 1.1. *(Unique continuation). If $u, v : \Sigma \to M$ are two J-holomorphic maps which agree to the infinite order at a point $z_0 \in \Sigma$, then $u \equiv v$ in the connetced component of Σ which contains z_0.*

Let $u : \Sigma \to M$ be a J-holomorphic map. A point $z \in \Sigma$ is called a **critical point** if $du(z) = 0$. Correspondingly the image $u(z) \in M$ is called a **critical value**. We remark that u is locally an embedding at any point which is not a critical point. To see this, we suppose $du(z) \neq 0$ for some $z \in \Sigma$. Let $u(z) = p$ and let $s + it$ be a local holomorphic coordinate centered at z. Then $du(z) \neq 0$ means that either $\partial_s u(z) \in T_pM$ or $\partial_t u(z) \in T_pM$ is non-zero. But u is J-holomorphic so that $\partial_s u + J(u)\partial_t u = 0$, which implies that both $\partial_s u(z), \partial_t u(z) \in T_pM$ are non-zero. Hence u is locally an embedding near z.

Lemma 1.2. *A critical point of a non-constant J-holomorphic map is isolated. In particular, a non-constant J-holomorphic map from a compact Riemann surface has only finitely many critical points.*

Lemma 1.3. *Let $\Omega \subset \mathbb{C}$ be an open neighborhood of $0 \in \mathbb{C}$ and let $u, v : \Omega \to M$ be J-holomorphic maps such that*

$$u(0) = v(0), \quad du(0) \neq 0.$$

Moreover, assume that there exist sequences $z_n, w_n \in \Omega$ such that

$$u(z_n) = v(w_n), \quad \lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = 0, \quad w_n \neq 0.$$

Then there exists a holomorphic function $\phi : B_\epsilon(0) \to \Omega$ defined in some neighborhood of $0 \in \mathbb{C}$ such that $\phi(0) = 0$ and

$$v = u \circ \phi.$$

Lemmas 1.2 and 1.3 have the following consequence.

Corollary 1.4. *Let $u : \Sigma \to M$ be a non-constant J-holomorphic map from a compact Riemann surface. Then there exists a compact Riemann surface Σ' and a non-constant J-holomorphic map $v : \Sigma' \to M$ such that in the complement of finitely many points, v is an embedding onto its image. Moreover, there exists a biholomorphism or branched covering map $\phi : \Sigma \to \Sigma'$ such that*

$$u = v \circ \phi.$$

The map v in the above corollary is called **simple** and the map u is called **multiply covered** if $\deg(\phi) > 1$. The image $C \equiv \mathrm{Im}\ v$ is called a **J-holomorphic curve** in M, and the map $v : \Sigma' \to M$ is called a **parametrization** of C. We call C a **rational** J-holomorphic curve if $\Sigma' = \mathbb{S}^2$.

Let $u : \Sigma \to M$ be a smooth map, where Σ is given a complex structure j, M is given a $J \in \mathcal{J}(M, \omega)$. We denote by g_J the associated hermitian metric on M. In order to define the **energy** of the map u, we fix a Kähler metric h on Σ, and with h and g_J, the norm $|du|$ is well-defined. We define the energy of u to be

$$E(u) \equiv \int_\Sigma |du|^2 dvol_\Sigma.$$

An important fact about $E(u)$ is that even though the energy density $|du|^2$ may depend on the choice of the Kähler metric h on Σ, the energy $E(u)$ depends only on the complex structure j, i.e., $E(u)$ is invariant under comformal transformations on the domain of u.

The following energy identity can be easily derived

$$E(u) = \int_\Sigma |\bar{\partial}_J(u)|^2 dvol_\Sigma + \int_\Sigma u^*\omega,$$

which has the following important consequence. (This is where the closedness of ω plays a real role.)

Proposition 1.5. *J-holomorphic maps are the absolute minima of the energy functional $E(u)$ amongst the smooth maps u which carry a fixed homology class in M. In particular, J-holomorphic maps are harmonic maps, and the energy of a J-holomorphic map depends only on the homology class it carries, and a J-holomorphic map must be constant if it carries a trivial homology class.*

Finally, we give the following important local analytical property of J-holomorphic maps.

Theorem 1.6. *(Removal of singularities) Let $D \subset \mathbb{C}$ be the unit disc containing 0 and let $u : D \setminus \{0\} \to M$ be a J-holomorphic map such that $E(u) < \infty$. Then u may be extended to a J-holomorphic map $\hat{u} : D \to M$ with $\hat{u}|_{D\setminus\{0\}} = u$.*

Next we consider the moduli space of J-holomorphic maps. For simplicity, we shall assume $\Sigma = \mathbb{S}^2$. In this case, the complex structure j is unique, and the group of biholomorphisms of Σ is the group of Möbius trnsformations $G = \mathrm{PSL}(2,\mathbb{C})$:

$$z \mapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad - bc = 1.$$

Fix a non-zero homology class $0 \neq A \in H_2(M;\mathbb{Z})$. We consider the space of J-holomorphic maps

$$\mathcal{M}(A,J) = \{u : \mathbb{S}^2 \to M | u \text{ is } J\text{-holomorphic and } u_*[\mathbb{S}^2] = A\},$$

and the subspace of $\mathcal{M}(A,J)$ consisting of simple J-holomorphic maps

$$\mathcal{M}^*(A,J) = \{u : \mathbb{S}^2 \to M | u \text{ is } J\text{-holomorphic and simple, and } u_*[\mathbb{S}^2] = A\}.$$

Note that the group $G = \mathrm{PSL}(2,\mathbb{C})$ acts on $\mathcal{M}(A,J)$ via reparametrization

$$\phi \cdot u = u \circ \phi^{-1}, \quad \forall \phi \in G, u \in \mathcal{M}(A,J),$$

which is free when restricted on the subspace $\mathcal{M}^*(A,J)$. We denote the quotient space by $\widetilde{\mathcal{M}}(A,J)$ and $\widetilde{\mathcal{M}}^*(A,J)$ respectively. Note that $\widetilde{\mathcal{M}}^*(A,J)$ is exactly the space of J-holomorphic curves C such that the homology class of C is A. We remark that when A is a primitive class, i.e., A is not an integral multiple of another integral class, $\mathcal{M}(A,J) = \mathcal{M}^*(A,J)$.

Compactness. One of the fundamental issues concerning the moduli spaces is compactness. Note that the group $G = \mathrm{PSL}(2,\mathbb{C})$ acts freely on $\mathcal{M}^*(A,J)$ and G is not a compact group. Hence the moduli space of J-holomorphic maps $\mathcal{M}(A,J)$ and $\mathcal{M}^*(A,J)$ can not be compact, and one could best hope that the quotient spaces $\widetilde{\mathcal{M}}(A,J)$ and $\widetilde{\mathcal{M}}^*(A,J)$ are compact. However, this is also not true in general, as illustrated in the following example.

Example 1.7. Consider a family of holomorphic curves of degree 2 in $\mathbb{CP}^2$ parametrized by $0 \neq \lambda \in \mathbb{C}$

$$C_\lambda = \{[z_0,z_1,z_2] | \lambda z_0^2 = z_1 z_2\} \in \widetilde{\mathcal{M}}^*(2[\mathbb{CP}^1], J_0).$$

Here $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2;\mathbb{Z})$ is the class of a line, and J_0 is the complex structure of $\mathbb{CP}^2$. As $\lambda \to 0$, C_λ converges to a union of two lines

$$C_0 = \{[z_0,z_1,z_2] | z_1 z_2 = 0\} = \{[z_0,0,z_2]\} \cup \{[z_0,z_1,0]\}$$

which intersect transversely at $[1,0,0]$. It is known that C_0 can not be the image of a holomorphic map $u : \mathbb{S}^2 \to \mathbb{CP}^2$, hence C_0 does not lie in $\widetilde{\mathcal{M}}(2[\mathbb{CP}^1], J_0)$. This shows that both $\widetilde{\mathcal{M}}(2[\mathbb{CP}^1], J_0)$ and $\widetilde{\mathcal{M}}^*(2[\mathbb{CP}^1], J_0)$ are non-compact.

The phenomenon illustrated in the above example is called **bubbling**, i.e., during the limiting process as $\lambda \to 0$, the holomorphic curves C_λ split off a (non-constant) J-holomorphic 2-sphere which carries strictly less energy than the original curves. The bubbling phenomenon is the primary cause of non-compactness of moduli space of J-holomorphic curves, and when $\Sigma = \mathbb{S}^2$ as what we currently consider, it is the only cause. In other words, if there is no bubbling, the space $\widetilde{\mathcal{M}}(A,J)$ is compact.

Next we give a simple criterion which ensures compactness. Recall that a homology class $B \in H_2(M;\mathbb{Z})$ is called **spherical** if it may be represented by a map from $\mathbb{S}^2$ into M. Suppose the symplectic manifold (M,ω) contains no spherical classes B such that

$$0 < \omega(B) < \omega(A).$$

Such a condition has two consequences: (1) every element $u \in \mathcal{M}(A,J)$ is simple because otherwise the image of u represents a spherical class B satisfying $0 < \omega(B) < \omega(A)$, this gives $\mathcal{M}^*(A,J) = \mathcal{M}(A,J)$, (2) there is no bubbling for elements in $\widetilde{\mathcal{M}}(A,J)$ because a split-off J-holomorphic 2-sphere would represent a spherical class B satisfying $0 < \omega(B) < \omega(A)$. This gives rise to the following simple version of the Gromov Compactness Theorem.

Theorem 1.8. *(Gromov). Suppose there are no spherical classes B such that*

$$0 < \omega(B) < \omega(A).$$

Then for any compact subset $W \in \mathcal{J}(M,\omega)$ (given with C^∞-topology), $\cup_{J\in W}\widetilde{\mathcal{M}}(A,J)$ is compact with respect to the C^∞-topology.

The full version of the Gromov Compactness Theorem states that the moduli space of J-holomorphic curves carrying a fixed homology class can be suitably compactified. This is where the closedness of ω plays a real role, cf. Proposition 1.5.

Fredholm theory. Finally, we discuss the Fredholm theory of J-holomorphic maps, which allows us to analyze the topological structure of the moduli spaces.

Fix a sufficiently large integer $l > 0$, we consider the Banach manifold

$$B \equiv \{u : \mathbb{S}^2 \to M | u \text{ is a } C^l\text{-map and } u_*[\mathbb{S}^2] = A\}$$

and the Banach bundle $E \to B$, where the fiber over $u \in B$ is

$$E_u \equiv \{v | v \text{ is a } C^{l-1}\text{-section of } Hom(T\mathbb{S}^2, u^*TM) \to \mathbb{S}^2 \text{ such that } v \circ j = -J \circ v\}.$$

The Banach bundle $E \to B$ has a natural smooth section $s : B \to E$ defined by

$$s : u \mapsto (u, \bar{\partial}_J(u)).$$

By the elliptic regularity of the equation $\bar{\partial}_J(u) = 0$, any C^l-solution is automatically a smooth solution, so that the moduli space of J-holomorphic maps $\mathcal{M}(A, J)$ is simply the zero loci of s, i.e.,

$$s^{-1}(\text{zero-section}) = \mathcal{M}(A, J).$$

A crucial fact is that $s : B \to E$ is a Fredholm section, which means that the linearization of $\bar{\partial}_J(u)$ for each $u \in B$, $D_u : T_u B \to E_u$, is a Fredholm operator between the Banach spaces. This has the following implication on the topological structure of the moduli space $\mathcal{M}(A, J)$.

- For any open subset $U \subset \mathcal{M}(A, J)$, if $D_u : T_u B \to E_u$ is onto for any $u \in U$, then U is a canonically oriented, finite dimensional smooth manifold whose dimension is given by the index of D_u, which can be computed via the Atiyah-Singer index theorem in the following formula

$$\text{Index } D_u = 2n(1 - g_{\mathbb{S}^2}) + 2c_1(TM) \cdot A.$$

  Here $2n = \dim M$ and $g_{\mathbb{S}^2} = 0$ is the genus of $\mathbb{S}^2$. Such a J is called **regular** (with respect to U). (We remark that the same holds true if one allows J to vary in an oriented finite dimensional space.)

When J is integrable, the operator $D_u : T_u B \to E_u$ is simply the $\bar{\partial}$-operator $\bar{\partial} : \Omega^0(\mathbb{CP}^1, V) \to \Omega^{0,1}(\mathbb{CP}^1, V)$ where $V = u^*TM$ is a holomorphic vector bundle over $\mathbb{CP}^1$. The cokernel of D_u is simply the Dolbeault cohomology group $H^{0,1}_{\bar{\partial}}(\mathbb{CP}^1, V)$, which by Kodaira-Serre duality is isomorphic to the space of holomorphic sections of $V^* \otimes K$. Here V^* is the dual of V and K is the canonical bundle of $\mathbb{CP}^1$. The following lemma follows immediately from vanishing theorems of holomorphic vector bundles.

Lemma 1.9. *Suppose J is integrable and $V = u^*TM \to \mathbb{CP}^1$ is a holomorphic vector bundle of non-negative curvature tensor. Then D_u is onto.*

In general, using the Sard-Smale theorem one has

Theorem 1.10. *There exists an open, dense subset $\mathcal{J}_{reg}(A) \subset \mathcal{J}(M, \omega)$ of second Bair category such that for any $J \in \mathcal{J}_{reg}(A)$, J is regular with respect to $\mathcal{M}^*(A, J)$, so that $\mathcal{M}^*(A, J)$ is a smooth manifold of dimension*

$$\dim M + 2c_1(TM) \cdot A.$$

Moreover, for any $J_1, J_2 \in \mathcal{J}_{reg}(A)$, there exists a path $J_t \in \mathcal{J}(M,\omega)$ connecting J_1, J_2 such that

$$\cup_t \mathcal{M}^*(A, J_t)$$

is an oriented smooth manifold with boundary which is the disjoint union of $\mathcal{M}^*(A, J_1)$ and $\mathcal{M}^*(A, J_2)$.

2. The non-squeezing theorem and Gromov invariant

As one of the first applications of J-holomorphic curve theory, we describe the proof of the following non-squeezing theorem, where $B^{2n}(R)$ denotes the closed ball of radius R in $\mathbb{R}^{2n}$ which is equipped with the standard symplectic structure ω_0.

Theorem 2.1. *(Gromov, 1985). There exist no symplectic embeddings $B^{2n}(1) \rightarrow B^2(r) \times \mathbb{R}^{2n-2}$ if $r < 1$.*

Proof. Suppose to the contrary, there exists an symplectic embedding $\psi : B^{2n}(1) \rightarrow B^2(r) \times \mathbb{R}^{2n-2}$ for some $r < 1$. Fix any $\epsilon > 0$, we consider $B^2(r)$ as a subset of $\mathbb{S}^2$ which is given a symplectic form σ with total area $\pi r^2 + \epsilon$. On the other hand, since $\psi(B^{2n}(1))$ is compact, its projection into the $\mathbb{R}^{2n-2}$ factor is contained in an open ball of radius λ centered at the origin. Let T^{2n-2} be the torus which is $\mathbb{R}^{2n-2}$ modulo the lattice $\{(x_1, \cdots, x_{2n}) \cdot \lambda | x_j \in \mathbb{Z}\}$, which inherits a natural symplectic form ω_0. We set $M = \mathbb{S}^2 \times T^{2n-2}$, which is given with the product symplectic structure $\omega = \sigma \oplus \omega_0$. With this understood, note that there is a symplectic embedding $\psi : (B^{2n}(1), \omega_0) \rightarrow (M, \omega)$. We set $p_0 = \psi(0)$ where $0 \in B^{2n}(1)$ is the origin.

Lemma 2.2. *For any $J \in \mathcal{J}(M,\omega)$, there exists a ratinal J-holomorphic curve C which contains p_0 and carries the homology class $[\mathbb{S}^2 \times \{pt\}]$.*

Assuming Lemma 2.2 momentarily, the proof of Theorem 2.1 goes as follows. Note that there is a $J \in \mathcal{J}(M,\omega)$ such that the pull-back almost complex structure $\psi^* J$ is the standard complex structure J_0 on $B^{2n}(1)$. Let C be the rational J-holomorphic curve which contains p_0 and carries the homology class $[\mathbb{S}^2 \times \{pt\}]$. We set $C' \equiv \psi^{-1}(C) \subset B^{2n}(1)$. Then C' is a holomorphic curve in $B^{2n}(1)$ containing the origin. Particularly, C' is a minimal surface, and by the theory of minimal surfaces, the area of C' is at least the area of the flat plane contained in $B^{2n}(1)$, which equals π. This gives rise to the following inequalities

$$\pi \leq \text{Area}(C') = \int_{C'} \omega_0 = \int_{\psi(C')} \omega \leq \int_C \omega = \int_{\mathbb{S}^2} \sigma = \pi r^2 + \epsilon.$$

Let $\epsilon \rightarrow 0$, we obtain $\pi \leq \pi r^2$, which contradicts the assumption $r < 1$. This proves the non-squeezing theorem.

□

The basic idea behind the proof of Lemma 2.2 is the so-called **Gromov invariant**, which is the "number" of rational J-holomorphic curves (counted with signs) for a given J, that carries a given homology class and satisfies a certain topological constraint. (Such a count of J-holomorphic curves is supposed to be independent of the choice of J.) Lemma 2.2 basically says that the Gromov invariant which counts the number of rational curves carrying a homology class $[\mathbb{S}^2 \times \{pt\}]$ and passing through a given point in M is non-zero.

We shall next explain how to define such a Gromov invarint in the current context, and explain why the Gromov invariant is non-zero.

To this end, we set $A = [\mathbb{S}^2 \times \{pt\}] \in H_2(M; \mathbb{Z})$. Since $\omega = \sigma \oplus \omega_0$ is a product symplectic structure, $c_1(TM) = c_1(T\mathbb{S}^2) + c_1(T^{2n-2})$, so that

$$c_1(TM) \cdot A = c_1(T\mathbb{S}^2) \cdot A = 2.$$

By Theorem 1.10, there is an open, dense subset of second Bair category $\mathcal{J}_{reg}(A) \subset \mathcal{J}(M, \omega)$, such that for any $J \in \mathcal{J}_{reg}(A)$, the space $\mathcal{M}^*(A, J)$ is an oriented smooth manifold of dimension

$$\dim M + 2c_1(TM) \cdot A = 2n + 4.$$

In the present case, since A is a generator of $H_2(M, \mathbb{Z}) = \mathbb{Z}$, there are no spherical classes B such that $0 < \omega(B) < \omega(A)$, so that by Theorem 1.8, $\mathcal{M}(A, J) = \mathcal{M}^*(A, J)$, and the quotient space $\widetilde{\mathcal{M}}(A, J)$ is compact, which is an oriented smooth manifold of dimension

$$\dim \widetilde{\mathcal{M}}(A, J) = \dim \mathcal{M}^*(A, J) - \dim \mathrm{PSL}(2, \mathbb{C}) = 2n + 4 - 6 = 2n - 2.$$

Denote $\mathrm{PSL}(2, \mathbb{C})$ by G, and set $\mathcal{M}(A, J) \times_G \mathbb{S}^2 \equiv (\mathcal{M}(A, J) \times \mathbb{S}^2)/G$ where G acts on $\mathcal{M}(A, J) \times \mathbb{S}^2$ via $\phi \cdot (u, z) = (u \circ \phi^{-1}, \phi(z))$. Then $\mathcal{M}(A, J) \times_G \mathbb{S}^2$ is a compact, oriented smooth manifold of dimension $2n$, which is a $\mathbb{S}^2$-bundle over $\widetilde{\mathcal{M}}(A, J)$. The evaluation map

$$ev : \mathcal{M}(A, J) \times_G \mathbb{S}^2 \to M, \quad [(u, z)] \mapsto u(z)$$

is a smooth map between two compact, oriented smooth manifolds of the same dimension. The degree of ev, which is the image of the fundamental class of $\mathcal{M}(A, J) \times_G \mathbb{S}^2$ under $ev_* : H_{2n}(\mathcal{M}(A, J) \times_G \mathbb{S}^2; \mathbb{Z}) \to H_{2n}(M; \mathbb{Z}) = \mathbb{Z}$, can be geometrically interpreted as a count with signs of the points in the pre-image $ev^{-1}(p)$ for any generic point $p \in M$. On the other hand, $\mathcal{M}(A, J) \times_G \mathbb{S}^2$ as a $\mathbb{S}^2$-bundle over $\widetilde{\mathcal{M}}(A, J)$ may be regarded as the space of rational J-holomorphic curves $C \in \widetilde{\mathcal{M}}(A, J)$ with a marked point $z \in \mathbb{S}^2$ in the de-singularization of C. Thus the degree of ev is a count with signs of the number of rational J-holomorphic curves with a marked point, which carry the homology class A and pass through a given generic point $p \in M$ at the marked point. The Gromov invariant involved in the current problem is defined to be the degree of the evaluation map $ev : \mathcal{M}(A, J) \times_G \mathbb{S}^2 \to M$. Note that the Gromov invariant is independent of the choice of $J \in \mathcal{J}_{reg}(A)$. This is because by Theorem 1.10, for any $J_1, J_2 \in \mathcal{J}_{reg}(A)$, there exists a path $J_t \in \mathcal{J}(M, \omega)$ connecting J_1, J_2 such that $\cup_t \mathcal{M}(A, J_t)$ is an oriented smooth manifold with boundary which is the disjoint union of $\mathcal{M}(A, J_1)$ and $\mathcal{M}(A, J_2)$. It follows that $\cup_t \mathcal{M}(A, J_t) \times_G \mathbb{S}^2$ is a cobordism between $\mathcal{M}(A, J_1) \times_G \mathbb{S}^2$ and $\mathcal{M}(A, J_2) \times_G \mathbb{S}^2$, hence the degree of ev is the same for J_1, J_2. This shows that the Gromov invariant is independent of the choice of $J \in \mathcal{J}_{reg}(A)$.

In order to show that the Gromov invariant is non-zero, we consider a special $J \in \mathcal{J}_{reg}(A)$. Let j, J_0 be the complex structure on $\mathbb{S}^2$ and T^{2n-2} respectively, and let $J = j \times J_0$ be the product which lies in $\mathcal{J}(M, \omega)$.

For any $u \in \mathcal{M}(A, J)$, since $J = j \times J_0$, the map $pr \circ u : \mathbb{S}^2 \to T^{2n-2}$, where $pr : M \to T^{2n-2}$ is the projection, is J_0-holomorphic. But $pr \circ u$ carries a trivial homology class, hence by Proposition 1.5, $pr \circ u$ is a constant map. This shows that any $u \in \mathcal{M}(A, J)$ has the form $u : z \mapsto (\phi(z), x)$ for some $\phi \in G = \mathrm{PSL}(2, \mathbb{C})$ and $x \in T^{2n-2}$.

There are two consequences of this fact: (1) For any $u \in \mathcal{M}(A, J)$, u^*TM is isomorphic as a holomorphic vector bundle to $T\mathbb{S}^2 \oplus E$ where E is a trivial bundle of rank $n-1$. By Lemma 1.9, D_u is onto for any $u \in \mathcal{M}(A, J)$, so that $J \in \mathcal{J}_{reg}(A)$. (2) The correspondence $u \mapsto (\phi, x)$ gives an identification of $\mathcal{M}(A, J)$ with $G \times T^{2n-2}$, and hence $\widetilde{\mathcal{M}}(A, J)$ with T^{2n-2} and $\mathcal{M}(A, J) \times_G \mathbb{S}^2$ with $\mathbb{S}^2 \times T^{2n-2} = M$. It follows that the evaluation map $ev : \mathcal{M}(A, J) \times_G \mathbb{S}^2 \to M$ is a diffeomorphism, and the degree of ev is ± 1. This proves that the Gromov invariant is non-zero.

Proof of Lemma 2.2.

Note that the non-vanishing of Gromov invariant only implies immediately that for any $J \in \mathcal{J}_{reg}(A)$, and for any generic point $p \in M$, there exists a J-holomorphic curve $C \in \widetilde{\mathcal{M}}(A, J)$ such that $p \in C$. This is different from the claim in Lemma 2.2 that in fact such a J-holomorphic curve exists for any $J \in \mathcal{J}(M, \omega)$ and any point $p \in M$ (in particular, $p_0 \in M$).

To get around of this, we use the Gromov Compactness Theorem, Theorem 1.8. We pick a sequence of $J_n \in \mathcal{J}_{reg}(A)$, since $\mathcal{J}_{reg}(A)$ is dense in $\mathcal{J}(M, \omega)$, which converges to $J \in \mathcal{J}(M, \omega)$ in C^∞-topology, and we pick a sequence of generic points p_n converging to $p_0 \in M$, such that for each n, there exists a J_n-holomorphic curve C_n such that $p_n \in C_n$. By Theorem 1.8, a subsequence of $\{C_n\}$ converges to a $C \in \widetilde{\mathcal{M}}(A, J)$ such that $p_0 = \lim_{n\to\infty} p_n \in C$. This proves Lemma 2.2.

□

3. J-holomorphic curves in dimension 4

The J-holomorphic curve theory in dimension 4 is particularly more powerful because there are additional tools which allow one to analyse the singularies of a J-holomorphic curve. On the other hand, the existence of certain types of J-holomorphic curves actually can be derived from the underlying differential topology of the symplectic 4-manifold, due to the deep analytical work of Cliff Taubes.

Let (M, J) be an almost complex 4-manifold, and let $C \subset M$ be a J-holomorphic curve parametrized by a simple J-holomorphic map $u : \Sigma \to M$. The following theorem gives a criterion, amongst other things, for the embeddedness of C.

Theorem 3.1. *(Adjunction Inequality). Let g_Σ be the genus of Σ. Then the inequality*

$$\frac{1}{2}(C^2 - c_1(TM) \cdot C) + 1 \geq g_\Sigma$$

holds with equality if and only if C is embedded.

In particular, a rational J-holomorphic curve must be embedded if it is homologous to an embedded rational J-holomorphic curve. This explains why the singular curve C_0 in Example 1.7 can not be the image of a holomorphic map $u : \mathbb{S}^2 \to \mathbb{CP}^2$.

Example 3.2. (Algebraic curves in $\mathbb{CP}^2$). For notations we denote by $[\mathbb{CP}^1]$ the generator of $H_2(\mathbb{CP}^2;\mathbb{Z})$ which is the class of a line. Using Poincaré duality, we identify $H^2(\mathbb{CP}^2;\mathbb{Z})$ with $H_2(\mathbb{CP}^2;\mathbb{Z})$.

We first consider non-singular algebraic curves in $\mathbb{CP}^2$. Let C be a line in $\mathbb{CP}^2$. Then $C^2 = 1$, and

$$\frac{1}{2}(C^2 - c_1(T\mathbb{CP}^2)\cdot C) + 1 = 0,$$

which implies that $c_1(T\mathbb{CP}^2) = 3\cdot[\mathbb{CP}^1]$.

Now let C_d be any non-singular algebraic curve of degree d. Then $C_d^2 = d^2$ and $c_1(T\mathbb{CP}^2)\cdot C_d = 3d$. This gives rise to the following genus formula for C_d:

$$\text{genus }(C_d) = \frac{1}{2}(C_d^2 - c_1(T\mathbb{CP}^2)\cdot C_d) + 1 = \frac{1}{2}(d-1)(d-2).$$

Next we consider a singular algebraic curve, the cusp curve

$$C_0 = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 | z_1^3 = z_0 z_2^2\}.$$

C_0 is of degree 3 and has a cusp singularity at $[1,0,0]$. The left-hand side of the adjunction inequality for C_0 is

$$\frac{1}{2}(C_0^2 - c_1(T\mathbb{CP}^2)\cdot C_0) + 1 = \frac{1}{2}(3^2 - 3\cdot 3) + 1 = 1.$$

Since C_0 is singular, C_0 can only be parametrized by a holomorphic map from a genus zero Riemann surface, i.e., $\mathbb{S}^2$, so C_0 belongs to $\widetilde{\mathcal{M}}^*(3[\mathbb{CP}^1], J_0)$. On the other hand, C_0 is the limit of a family of non-singular cubic curves ($\lambda \neq 0$)

$$C_\lambda = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 | z_1^3 = z_0 z_2^2 + \lambda z_0^3\}$$

as $\lambda \to 0$. We remark that this also represents a certain kind of non-compactness phenomenon in the Gromov Compactness Theorem.

As an application of Theorem 3.1, we prove the following non-existence result.

Proposition 3.3. *For a generic almost complex structure J, there exist no rational J-holomorphic curves C such that $C^2 \leq -2$, and there exist at most embedded rational J-holomorphic curves C with $C^2 = -1$.*

Proof. Suppose C is a rational J-holomorphic curve such that $C^2 \leq -2$. Then the adjunction inequality implies that

$$c_1(TM)\cdot C \leq C^2 + 2 \leq -2 + 2 = 0.$$

On the other hand, for a generic almost complex structure J (cf. Theorem 1.10), the space $\mathcal{M}^*([C], J)$ is a smooth manifold of dimension $4 + 2c_1(TM)\cdot C \leq 4$. Since $G = \mathrm{PSL}(2,\mathbb{C})$ is 6-dimensional and acts on $\mathcal{M}^*([C], J)$ freely if it is non-empty, we see that $\mathcal{M}^*([C], J)$ must be at least 6-dimensional. This proves that for a generic almost complex structure, there exist no rational curves C with $C^2 \leq -2$. The proof for the case of $C^2 = -1$ is similar and we leave the details to the reader.

□

The above proposition shows that in a symplectic 4-manifold, the only interesting rational J-holomorphic curves are those with non-negative self-intersection. Because if J is taken generic, the only rational J-holomorphic curves with negative self-intersection are the embedded ones with self-intersection -1. By the symplectic neighborhood theorem, the symplectic 4-manifold can be symplectically blown down along these (-1)-curves, and the resulting symplectic 4-manifold does not have any rational J-holomorphic curves with negative self-intersection for a generic J. A symplectic 4-manifold is called **minimal** if it contains no embedded symplectic 2-spheres with self-intersection -1 (i.e., it can not be symplectically blown down).

The following theorem is useful in analysing the intersection of two distinct J-holomorphic curves.

Theorem 3.4. *(Positivity of Intersection) Let C, C' be two distinct J-holomorphic curves in a compact almost complex 4-manifold. Then the intersection of C, C' consists of at most finitely many points. Moreover, the intersection product*

$$C \cdot C' = \sum_{p \in C \cap C'} k_p$$

where $k_p \in \mathbb{Z}^+$, and $k_p = 1$ if and only if both C, C' are embedded near p and the intersection at p is transverse.

In particular, $C \cdot C' \geq 0$, and if $C \cdot C' = 0$, then C, C' are disjoint. If $C \cdot C' = 1$, then C, C' intersect at exactly one point and the intersection must be transverse.

Suppose C is an embedded rational J-holomorphic curve with $C^2 = 0$, and suppose C' is another rational J-holomorphic curve which is homologous to C. Then on the one hand, the adjunction inequality implies C' must also be embedded, and on the other hand, the positivity of intersection implies that C, C' must be disjoint. Thus if the moduli space of such rational curves has a positive dimension, they may be used to fill up the whole manifold. In order to do this, we need the following regularity criterion for J.

Lemma 3.5. *Suppose C is an immersed rational J-holomorphic curve in an almost complex 4-manifold (M, J) such that $c_1(TM) \cdot C > 0$. Then for any simple J-holomorphic map $u : \mathbb{S}^2 \to M$ parametrizing C, the linearization D_u of $\bar{\partial}_J(u) = 0$ is onto.*

We combine these tools to give a proof of the following structural theorem of symplectic 4-manifolds which contain an embedded rational curve of self-intersection 0.

Theorem 3.6. *Let (M, ω) be a symplectic 4-manifold which contains an embedded symplectic 2-sphere C with $C^2 = 0$. Suppose that M contains no spherical classes B such that $0 < \omega(B) < \omega(C)$. Then M must be diffeomorphic to a $\mathbb{S}^2$-bundle over a surface.*

Proof. Pick a $J \in \mathcal{J}(M, \omega)$ such that C is J-holomorphic. Then by the adjunction inequality every element in $\widetilde{\mathcal{M}}^*([C], J)$ is embedded, and since

$$c_1(TM) \cdot C = C^2 + 2 = 2 > 0,$$

by Lemma 3.5, the space $\mathcal{M}^*([C], J)$ is an oriented smooth manifold of dimension

$$\dim M + 2c_1(TM) \cdot C = 4 + 2 \cdot 2 = 8.$$

Consequently, $\widetilde{\mathcal{M}}^*([C], J)$ is an oriented surface, which is compact by Theorem 1.8. With this understood, M is diffeomorphic to the $\mathbb{S}^2$-bundle over $\widetilde{\mathcal{M}}^*([C], J)$ via

$$ev : \mathcal{M}^*([C], J) \times_G \mathbb{S}^2 \to M, \ \ [(u, z)] \mapsto u(z),$$

where $G = \mathrm{PSL}(2, \mathbb{C})$.

$\square$

Suppose in the above theorem, M contains another embedded symplectic 2-sphere C' such that $C' \cdot C' = 0$, which intersects with C transversely and positively at a single point. Then one can arrange $J \in \mathcal{J}(M, \omega)$ such that both C, C' are J-holomorphic. Suppose furthermore that there are also no spherical classes B such that $0 < \omega(B) < \omega(C')$, then $\widetilde{\mathcal{M}}^*([C'], J)$ is precisely the space of J-holomorphic sections of $\mathcal{M}^*([C], J) \times_G \mathbb{S}^2$ under ev. It is easily seen that in this case there is a diffeomorphism $\psi : \mathbb{S}^2 \times \mathbb{S}^2 \to M$ such that the 2-spheres $\psi(\mathbb{S}^2 \times \{pt\})$ and $\psi(\{pt\} \times \mathbb{S}^2)$ are symplectic in M. This fact was exploited in the proof of the following theorem.

Theorem 3.7. *(Gromov-Taubes). For any symplectic structure ω on $\mathbb{CP}^2$, there is a diffeomorphism $\psi : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $\psi^*\omega$ is a multiple of the standard Fubini-Study form ω_0.*

The proof of Theorem 3.7 consists of two steps. Step 1, which is due to Taubes and uses his deep work on Seiberg-Witten theory of symplectic 4-manifolds, asserts that there exists an embedded symplectic 2-sphere C with $C^2 = 1$. The complement $\mathbb{CP}^2 \setminus \nu(C)$ of a regular neighborhood of C is a homotopy 4-ball W with $\partial W = \mathbb{S}^3$ (Van-Kampen plus Mayer-Vietoris). By the symplectic neighborhood theorem, the symplectic form ω equals the standard symplectic form on $\mathbb{R}^4$ in a regular neighborhood of ∂W, which is identified with $\{x \in \mathbb{R}^4 | \delta_0 - \epsilon < ||x||^2 \leq \delta_0\}$ for some δ_0 and $\epsilon > 0$.

Step 2: (Gromov). There exists a symplectomorphism $B^4(\delta_0) \to W$ which is identity near the boundaries.

The proof of Step 2 goes as follows. Pick a large enough polydisc $D^2 \times D^2 \subset \mathbb{R}^4$ which contains $B^4(\delta_0)$, and embedded $D^2 \times D^2$ into $\mathbb{S}^2 \times \mathbb{S}^2$ via some embedding $D^2 \subset \mathbb{S}^2$. Then one removes $B^4(\delta_0)$ from $\mathbb{S}^2 \times \mathbb{S}^2$ and then glues back W. Call the resulting symplectic 4-manifold M. Apply the remarks following Theorem 3.6 to M (for details see [1], pages 314-319).

References

[1] D. McDuff and D. Salamon, *J-Holomorphic Curves and Symplectic Topology*, Colloquium Publications **52**, AMS, 2004.