LECTURE 2: SYMPLECTIC VECTOR BUNDLES

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1. Symplectic Vector Spaces

Definition 1.1. A **symplectic vector space** is a pair (V, ω) where V is a finite dimensional vector space (over $\mathbb{R}$) and ω is a bilinear form which satisfies

- Skew-symmetry: for any $u, v \in V$,
$$\omega(u, v) = -\omega(v, u).$$
- Nondegeneracy: for any $u \in V$,
$$\omega(u, v) = 0 \ \ \forall v \in V \text{ implies } u = 0.$$

A **linear symplectomorphism** of a symplectic vector space (V, ω) is a vector space isomorphism $\psi : V \to V$ such that
$$\omega(\psi u, \psi v) = \omega(u, v) \quad \forall u, v \in V.$$
The group of linear symplectomorphisms of (V, ω) is denoted by $\mathrm{Sp}(V, \omega)$.

Example 1.2. The Euclidean space $\mathbb{R}^{2n}$ carries a standard skew-symmetric, nondegenerate bilinear form ω_0 defined as follows. For $u = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T$, $u' = (x'_1, x'_2, \cdots, x'_n, y'_1, y'_2, \cdots, y'_n)^T$,
$$\omega_0(u, u') = \sum_{i=1}^{n} (x_i y'_i - x'_i y_i) = -u^T J_0 u',$$
where $J_0 = \begin{pmatrix} 0, -I \\ I, 0 \end{pmatrix}$. (Here I is the $n \times n$ identity matrix.)

The group of linear symplectomorphisms of $(\mathbb{R}^{2n}, \omega_0)$, which is denoted by $\mathrm{Sp}(2n)$, can be identified with the group of $2n \times 2n$ symplectic matrices. Recall a symplectic matrix Ψ is one which satisfies $\Psi^T J_0 \Psi = J_0$. For the case of $n = 1$, a symplectic matrix is simply a matrix Ψ with $\det \Psi = 1$.

Let (V, ω) be any symplectic vector space, and let $W \subset V$ be any linear subspace. The **symplectic complement** of W in V is defined and denoted by
$$W^\omega = \{v \in V | \omega(v, w) = 0, \ \ \forall w \in W\}.$$

Lemma 1.3. *(1)* $\dim W + \dim W^\omega = \dim V$*, (2)* $(W^\omega)^\omega = W$*.*

Proof. Define $\iota_\omega : V \to V^*$ by $\iota_\omega(v) : w \mapsto \omega(v, w)$, $\forall v, w \in V$, where V^* is the dual space of V. Since ω is nondegenerate, ι_ω is an isomorphism. Now observe that $\iota_\omega(W^\omega) = W^\perp$ where $W^\perp \subset V^*$ is the annihilator of W, i.e.,
$$W^\perp \equiv \{v^* \in V^* | v^*(w) = 0 \ \ \forall w \in W\}.$$

Part (1) follows immediately from the fact that

$$\dim W + \dim W^{\perp} = \dim V.$$

Part (2) follows easily from $W \subset (W^{\omega})^{\omega}$ and the equations

$$\dim W = \dim V - \dim W^{\omega} = \dim (W^{\omega})^{\omega}$$

which are derived from (1). □

Theorem 1.4. *For any symplectic vector space (V, ω), there exists a basis of V $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ such that*

$$\omega(u_j, u_k) = \omega(v_j, v_k) = 0, \;\; \omega(u_j, v_k) = \delta_{jk}.$$

(Such a basis is called a **symplectic basis***.) In particular,* $\dim V = 2n$ *is even.*

Proof. We prove by induction on $\dim V$. Note that $\dim V \geq 2$.

When $\dim V = 2$, the nondegeneracy condition of ω implies that there exist $u, v \in V$ such that $\omega(u, v) \neq 0$. Clearly u, v are linearly independent so that they form a basis of V since $\dim V = 2$. We can replace v by an appropriate nonzero multiple so that the condition $\omega(u, v) = 1$ is satisfied. Hence the theorem is true for the case of $\dim V = 2$.

Now suppose the theorem is true when $\dim V \leq m - 1$. We shall prove that it is also true when $\dim V = m$. Again the nondegeneracy condition of ω implies that there exist $u_1, v_1 \in V$ such that u_1, v_1 are linearly independent and $\omega(u_1, v_1) = 1$. Set $W \equiv \text{span}(u_1, v_1)$. Then we claim that $(W^{\omega}, \omega|_{W^{\omega}})$ is a symplectic vector space. It suffices to show that $\omega|_{W^{\omega}}$ is nondegenerate. To see this, suppose $w \in W^{\omega}$ such that $\omega(w, z) = 0$ for all $z \in W^{\omega}$. We need to show that w must be zero. To this end, note that $W \cap W^{\omega} = \{0\}$, so that $V = W \oplus W^{\omega}$ by (1) of the previous lemma. Now for any $z \in V$, write $z = z_1 + z_2$ where $z_1 \in W$ and $z_2 \in W^{\omega}$. Then $\omega(w, z_1) = 0$ because $w \in W^{\omega}$ and $\omega(w, z_2) = 0$ by $z_2 \in W^{\omega}$ and the assumption on w. Hence $\omega(w, z) = 0$, and therefore $w = 0$ by the nondegeneracy condition of ω on V.

Note that $\dim W^{\omega} = \dim V - 2 \leq m - 1$, so that by the induction hypothesis, there is a symplectic basis $u_2, \cdots, u_n, v_2, \cdots, v_n$ of $(W^{\omega}, \omega|_{W^{\omega}})$. It is clear that $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ is a symplectic basis of (V, ω), and the theorem is proved. □

Corollary 1.5. *Let ω be any skew-symmetric bilinear form on V. Then ω is nondegenerate if and only if* $\dim V = 2n$ *is even and*

$$\omega^n = \omega \wedge \cdots \wedge \omega \neq 0.$$

Proof. Suppose ω is nondegenerate, then by the previous theorem, $\dim V = 2n$ is even, and there exists a symplectic basis $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$. Clearly

$$\omega^n(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n) \neq 0,$$

hence $\omega^n \neq 0$.

Suppose $\dim V = 2n$ is even and $\omega^n \neq 0$. Then ω must be nondegenerate, because if otherwise, there exists a $u \in V$ such that $\omega(u, v) = 0$ for all $v \in V$. We complete u into a basis $u, v_1, v_2, \cdots, v_{2n-1}$ of V. One can easily check that

$$\omega^n(u, v_1, v_2, \cdots, v_{2n-1}) = 0,$$

which contradicts the assumption that $\omega^n \neq 0$.

$\square$

Corollary 1.6. *Let (V,ω) be any symplectic vector space. Then there exists an $n>0$ and a vector space isomorphism $\phi : \mathbb{R}^{2n} \to V$ such that*

$$\omega_0(z,z') = \omega(\phi z, \phi z'), \ \ \forall z, z' \in \mathbb{R}^{2n}.$$

Consequently, $Sp(V,\omega)$ is isomorphic to $Sp(2n)$.

Proof. Let $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ be a symplectic basis of (V,ω). The corollary follows by defining

$$\phi : (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \mapsto \sum_{i=1}^{n} (x_i u_i + y_i v_i).$$

$\square$

Definition 1.7. A **complex structure** on a real vector space V is an automorphism $J : V \to V$ such that $J^2 = -\mathrm{id}$. A **Hermitian structure** on (V,J) is an inner product g on V which is J-invariant, i.e., $g(Jv,Jw) = g(v,w)$, for all $v,w \in V$.

Let J be a complex structure on V. Then V becomes a complex vector space by defining the complex multiplication by

$$\mathbb{C} \times V \to V : (x+iy, v) \mapsto xv + yJv.$$

If g is a Hermitian structure on (V,J), then there is an associated Hermitian inner product

$$h(v,w) \equiv g(v,w) + ig(v,Jw), \ \ \forall v,w \in V,$$

i.e., $h : V \times V \to \mathbb{C}$ satisfies (1) h is complex linear in the first V and anti-complex linear in the second V, and (2) $h(v,v) > 0$ for any $0 \neq v \in V$.

Note that there always exists a Hermitian structure on (V,J), by simply taking the average $\frac{1}{2}(g(v,w) + g(Jv,Jw))$ of any inner product g on V.

Definition 1.8. Let (V,ω) be a symplectic vector space. A complex structure J on V is called **ω-compatible** if

- $\omega(Jv,Jw) = \omega(v,w)$ for all $v,w \in V$,
- $\omega(v,Jv) > 0$ for any $0 \neq v \in V$.

We denote the set of ω-compatible complex structures by $\mathcal{J}(V,\omega)$. Note that $\mathcal{J}(V,\omega)$ is nonempty: let $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ be a symplectic basis of (V,ω), then $J : V \to V$ defined by $Ju_i = v_i, Jv_i = -u_i$ is a ω-compatible complex structure. Finally, for any $J \in \mathcal{J}(V,\omega)$, $g_J(v,w) \equiv \omega(v,Jw)$, $\forall v,w \in V$, is a (canonically associated) Hermitian structure on (V,J).

Example 1.9. J_0 is a complex structure on $\mathbb{R}^{2n}$ which is ω_0-compatible. The associated Hermitian structure $g_0(\cdot,\cdot) \equiv \omega_0(\cdot, J_0 \cdot)$ is the usual inner product on $\mathbb{R}^{2n}$. J_0 makes $\mathbb{R}^{2n}$ into a complex vector space by

$$\mathbb{C} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n} : (x+iy, v) \mapsto xv + yJ_0 v,$$

which coincides with the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T \mapsto (z_1, z_2, \cdots, z_n)^T$$

where $z_j = x_j + iy_j$ with $i = \sqrt{-1}$. With this understood, the associated Hermitian inner product $h_0 \equiv g_0 - i\omega_0$ on $(\mathbb{R}^{2n}, J_0)$ is the usual one on $\mathbb{C}^n$: $h_0(z, w) = \bar{w}^T z$.

We would like to understand the set $\mathcal{J}(V, \omega)$.

Lemma 1.10. *Suppose* $\dim V = 2n$. *Then for any* $J \in \mathcal{J}(V, \omega)$, *there is a vector space isomorphism* $\phi_J : \mathbb{R}^{2n} \to V$ *such that*

$$\phi_J^* \omega = \omega_0, \quad \phi_J^* J \equiv \phi_J^{-1} \circ J \circ \phi_J = J_0.$$

Moreover, $\phi_J^* : J' \mapsto \phi_J^{-1} \circ J' \circ \phi_J$ *identifies* $\mathcal{J}(V, \omega)$ *with* $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$.

Proof. Let $u_1, u_2, \cdots, u_n$ be a unitary basis of (V, h_J), where h_J is the Hermitian inner product on V associated to the canonical Hermitian structure $g_J(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$. Then one can easily check that $u_1, u_2, \cdots, u_n, Ju_1, Ju_2, \cdots, Ju_n$ form a symplectic basis of (V, ω). If we define

$$\phi_J : (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \mapsto \sum_{i=1}^{n} (x_i u_i + y_i J u_i),$$

then it is obvious that $\phi_J^* \omega = \omega_0$ and $\phi_J^* J = J_0$. Moreover, the verification that $\phi_J^* : J' \mapsto \phi_J^{-1} \circ J' \circ \phi_J$ identifies $\mathcal{J}(V, \omega)$ with $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ is straightforward.

□

We remark that in the proof the definition of ϕ_J may depend on the choice of the unitary basis $u_1, u_2, \cdots, u_n$, but the identification $\phi_J^* : \mathcal{J}(V, \omega) \to \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ does not, it is completely determined by J.

In order to understand $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$, we need to go over some basic facts about the Lie group $\mathrm{Sp}(2n)$. To this end, recall that $\mathbb{R}^{2n}$ is identified with $\mathbb{C}^n$ by

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T \mapsto (z_1, z_2, \cdots, z_n)^T, \text{ where } z_j = x_j + iy_j.$$

Under this identification, $GL(n, \mathbb{C})$ is regarded as a subgroup of $GL(2n, \mathbb{R})$ via

$$Z = X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

Note that $\psi \in GL(2n, \mathbb{R})$ belongs to $GL(n, \mathbb{C})$ iff $\psi J_0 = J_0 \psi$.

Lemma 1.11.

$$U(n) = Sp(2n) \cap O(2n) = Sp(2n) \cap GL(n, \mathbb{C}) = O(2n) \cap GL(n, \mathbb{C}).$$

Proof. Let $\psi \in GL(2n, \mathbb{R})$, then

- $\psi \in GL(n, \mathbb{C})$ iff $\psi J_0 = J_0 \psi$,
- $\psi \in \mathrm{Sp}(2n)$ iff $\psi^T J_0 \psi = J_0$,
- $\psi \in O(2n)$ iff $\psi^T \psi = I$.

The last two identities in the lemma follows immediately.

It remains to show that a $\psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in GL(n, \mathbb{C})$ lies in $\mathrm{Sp}(2n)$ iff $\psi = X + iY$ lies in $U(n)$. But one can check easily that both conditions are equivalent to the following set of equations

$$X^T Y = Y^T X, \quad X^T X + Y^T Y = I.$$

The lemma follows immediately.

□

The previous lemma in particular says that $U(n)$ is a subgroup of $\mathrm{Sp}(2n)$. The space of right orbits

$$\mathrm{Sp}(2n)/U(n) \equiv \{\psi \cdot U(n) | \psi \in \mathrm{Sp}(2n)\}$$

is known to be naturally a smooth manifold. We denote by $*$ the orbit of identity $I \in \mathrm{Sp}(2n)$ in $\mathrm{Sp}(2n)/U(n)$.

Theorem 1.12. *There exists a canonically defined smooth map*

$$H : Sp(2n)/U(n) \times [0,1] \to Sp(2n)/U(n)$$

such that $H(\cdot, 0) = id$, $H(\cdot, 1)(x) = \{*\}$ for any $x \in Sp(2n)/U(n)$, and $H(*, t) = *$ for any $t \in [0,1]$. In particular, $Sp(2n)/U(n)$ is contractible.

Proof. First of all, for any $\psi \in \mathrm{Sp}(2n)$, ψ^T is also in $\mathrm{Sp}(2n)$, so that $\psi\psi^T$ is a symmetric, positive definite symplectic matrix. We will show that $(\psi\psi^T)^\alpha$ is also a symplectic matrix for any real number α.

To this end, we decompose $\mathbb{R}^{2n} = \oplus_\lambda V_\lambda$ where V_λ is the λ-eigenspace of $\psi\psi^T$, and $\lambda > 0$. Then note that for any $z \in V_\lambda$, $z' \in V_{\lambda'}$, $\omega_0(z, z') = 0$ if $\lambda\lambda' \neq 1$. Our claim that $(\psi\psi^T)^\alpha$ is a symplectic matrix for any real number α follows easily from this observation.

Now for any $\psi \in \mathrm{Sp}(2n)$, we decompose $\psi = PQ$ where $P = (\psi\psi^T)^{1/2}$ is symmetric and $Q \in O(2n)$. Note that $Q = \psi P^{-1} \in \mathrm{Sp}(2n) \cap O(2n) = U(n)$, which shows that ψ and $P = (\psi\psi^T)^{1/2}$ are in the same orbit in $\mathrm{Sp}(2n)/U(n)$. With this understood, we define

$$H : (\psi \cdot U(n), t) \mapsto (\psi\psi^T)^{(1-t)/2} \cdot U(n), \ \ \psi \in \mathrm{Sp}(2n), \ t \in [0,1].$$

□

Lemma 1.13. *$\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ is canonically identified with $Sp(2n)/U(n)$, under which J_0 is sent to $*$.*

Proof. By Lemma 1.11, for any $J \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$, there exists a $\phi_J \in \mathrm{Sp}(2n)$ such that $\phi_J^* J \equiv \phi_J^{-1} \cdot J \cdot \phi_J = J_0$, or equivalently, $J = \phi_J \cdot J_0 \cdot \phi_J^{-1}$. The correspondence $J \mapsto \phi_J$ induces a map from $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ to $\mathrm{Sp}(2n)/U(n)$, which is clearly one to one and onto. Note that under the correspondence, J_0 is sent to $*$.

□

The set of ω-compatible complex structures $\mathcal{J}(V, \omega)$ can be given a natural topology so that it becomes a smooth manifold. Lemma 1.10, Theorem 1.12 and Lemma 1.13 give rise to the following

Corollary 1.14. *Given any $J \in \mathcal{J}(V,\omega)$, there exists a canonically defined smooth map*
$$H_J : \mathcal{J}(V,\omega) \times [0,1] \to \mathcal{J}(V,\omega)$$
depending smoothly on J, such that $H_J(\cdot,0) = id$, $H_J(\cdot,1)(J') = \{J\}$ for any $J' \in \mathcal{J}(V,\omega)$, and $H(J,t) = J$ for any $t \in [0,1]$.

Recall that for any $J \in \mathcal{J}(V,\omega)$, there is a canonically associated Hermitian structure (i.e. a J-invariant inner product) $g_J(\cdot,\cdot) \equiv \omega(\cdot, J\cdot)$. The next theorem shows that one can construct ω-compatible complex structures from inner products on V. Let $\mathrm{Met}(V)$ denote the space of inner products on V.

Theorem 1.15. *There exists a canonically defined map $r : Met(V) \to \mathcal{J}(V,\omega)$ such that*
$$r(g_J) = J, \;\; r(\psi^* g) = \psi^* r(g)$$
for all $J \in \mathcal{J}(V,\omega)$, $g \in Met(V)$, and $\psi \in Sp(V,\omega)$.

Proof. For any given $g \in \mathrm{Met}(V)$, we define $A : V \to V$ by
$$\omega(v,w) = g(Av,w), \quad \forall v,w \in V.$$
Then the skew-symmetry of ω implies that A is g-skew-adjoint. It follows that $P \equiv -A^2$ is g-self-adjoint and g-positive definite. Set $Q \equiv P^{1/2}$, which is also g-self-adjoint and g-positive definite.

We define the map r by $g \mapsto J_g \equiv Q^{-1}A$. Then $J_g^2 = Q^{-1}AQ^{-1}A = Q^{-2}A^2 = -I$ is a complex structure. To check that J_g is ω-compatible, note that
$$\omega(Q^{-1}Av, Q^{-1}Aw) = g(AQ^{-1}Av, Q^{-1}Aw) = -g(v, Aw) = \omega(v,w), \;\; \forall v,w \in V,$$
$$\omega(v, Q^{-1}Av) = g(Av, Q^{-1}Av) > 0 \;\; \forall 0 \neq v \in V$$
because Q^{-1} is g-self-adjoint and g-positive definite.

Finally, for any $\psi \in \mathrm{Sp}(V,\omega)$, replacing g with $\psi^* g$ changes A to $\psi^{-1}A\psi$, and therefore changes Q to $\psi^{-1}Q\psi$. This implies $r(\psi^* g) = \psi^* r(g)$. If $g = g_J$, then $A = J$ and $Q = I$, so that $r(g_J) = J$.

□

2. Symplectic Vector Bundles

Definition 2.1. A **symplectic vector bundle** over a smooth manifold M is a pair (E,ω), where $E \to M$ is a real vector bundle and ω is a smooth section of $E^* \wedge E^*$ such that for each $p \in M$, (E_p, ω_p) is a symplectic vector space. (Here E^* is the dual of E.) The section ω is called a **symplectic bilinear form** on E. Two symplectic vector bundles (E_1,ω_1), (E_2,ω_2) are said to be **isomorphic** if there exists an isomorphism $\phi : E_1 \to E_2$ (which is identity over M) such that $\phi^*\omega_2 = \omega_1$.

The standard constructions in bundle theory carry over to the case of symplectic vector bundles. For example, for any smooth map $f : N \to M$ and symplectic vector bundle (E,ω) over M, the pull-back $(f^*E, f^*\omega)$ is a symplectic vector bundle over N. In particular, for any submanifold $Q \subset M$, the restriction $(E|_Q, \omega|_Q)$ is a symplectic vector bundle over Q. Let F be a sub-bundle of E such that for each

$p \in M$, $(F_p, \omega_p|_{F_p})$ is a symplectic vector space. Then $(F, \omega|_F)$ is naturally a symplectic vector bundle. We call F (or $(F, \omega|_F)$) a **symplectic sub-bundle** of (E, ω). The **symplectic complement** of F is the sub-bundle

$$F^\omega \equiv \cup_{p\in M} F_p^{\omega_p} = \cup_{p\in M}\{v \in E_p | \omega_p(v, w) = 0,\ \forall w \in F_p\},$$

which is naturally a symplectic sub-bundle of (E, ω). Note that as a real vector bundle, F^ω is isomorphic to the quotient bundle E/F.

Given any symplectic vector bundles (E_1, ω_1), (E_2, ω_2), the symplectic direct sum $(E_1 \oplus E_2, \omega_1 \oplus \omega_2)$ is naturally a symplectic vector bundle. With this understood, note that for any symplectic sub-bundle F of (E, ω), one has

$$(E, \omega) = (F, \omega|_F) \oplus (F^\omega, \omega|_{F^\omega}).$$

Example 2.2. Let (M, ω) be a symplectic manifold. Note that ω as a 2-form on M is a smooth section of $\Omega^2(M) \equiv T^*M \wedge T^*M$. The nondegeneracy condition on ω implies that (TM, ω) is a symplectic vector bundle. Note that the closedness of ω is irrelevant here.

Suppose Q is a symplectic submanifold of (M, ω). Then TQ is a symplectic sub-bundle of $(TM|_Q, \omega|_Q)$. The normal bundle $\nu_Q \equiv TM|_Q/TQ$ of Q in M is also naturally a symplectic sub-bundle of $(TM|_Q, \omega|_Q)$ by identifying ν_Q with the symplectic complement TQ^ω of TQ. Notice the symplectic direct sum

$$TM|_Q = TQ \oplus \nu_Q.$$

Definition 2.3. Let (E, ω) be a symplectic vector bundle over M. A **complex structure** J of E, i.e., a smooth section J of $\mathrm{Aut}(E) \to M$ such that $J^2 = -I$, is said to be **ω-compatible** if for each $p \in M$, J_p is ω_p-compatible, i.e., $J_p \in \mathcal{J}(E_p, \omega_p)$. The space of all ω-compatible complex structures of E is denoted by $\mathcal{J}(E, \omega)$.

Example 2.4. Let (M, ω) be a symplectic manifold. Then a complex structure of TM is simply what we call an almost complex structure on M. An almost complex structure J on M is said to be **ω-compatible** if $J \in \mathcal{J}(TM, \omega)$. In this context, we denote $\mathcal{J}(TM, \omega)$, the set of ω-compatible almost complex structures on M, by $\mathcal{J}(M, \omega)$. Notice that the closedness of ω is irrelevant here.

In what follows, we will address the issue of classification of symplectic vector bundles up to isomorphisms, and determine the topology of the space $\mathcal{J}(E, \omega)$.

Lemma 2.5. *Let (E, ω) be a symplectic vector bundle over M of rank $2n$.*

(1) There exists an open cover $\{U_i\}$ of M such that for each i, there is a symplectic trivialization $\phi_i : (E|_{U_i}, \omega|_{U_i}) \to (U_i \times \mathbb{R}^{2n}, \omega_0)$. In particular, the transition functions $\phi_{ji}(p) \equiv \phi_j \circ \phi_i^{-1}(p) \in Sp(2n)$ for each $p \in U_i \cap U_j$, and E becomes a $Sp(2n)$-vector bundle. Conversely, any $Sp(2n)$-vector bundle is a symplectic vector bundle, and their classification up to isomorphisms is identical.

(2) (E, ω) as a $Sp(2n)$-vector bundle admits a lifting to a $U(n)$-vector bundle if and only if there exists a $J \in \mathcal{J}(E, \omega)$.

Proof. For any $p \in M$, one can prove by induction as in Theorem 1.4 (with a parametric version) that there exists a small neighborhood U_p of p and smooth sections

$u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ of E over U_p such that for each $q \in U_p$,

$$u_1(q), u_2(q), \cdots, u_n(q), v_1(q), v_2(q), \cdots, v_n(q)$$

form a symplectic basis of (E_q, ω_q). Part (1) follows immediately from this by defining $\phi_p : (E|_{U_p}, \omega|_{U_p}) \to (U_p \times \mathbb{R}^{2n}, \omega_0)$ to be the inverse of

$$(q, (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T) \mapsto \sum_{i=1}^{n} (x_i u_i(q) + y_i v_i(q)).$$

For part (2), if (E, ω) as a $\mathrm{Sp}(2n)$-vector bundle admits a lifting to a $U(n)$-vector bundle, then the corresponding complex structure J on E is ω-compatible because the Hermitian structure and J determines ω completely. If there exists a $J \in \mathcal{J}(E, \omega)$, then one can show that there are local smooth sections $u_1, u_2, \cdots, u_n$ which form a unitary basis at each point for (E, J, h_J). This makes E into a $U(n)$-vector bundle, which is a lifting of the $\mathrm{Sp}(2n)$-vector bundle because $u_1, u_2, \cdots, u_n, Ju_1, Ju_2, \cdots, Ju_n$ are local smooth sections which form a symplectic basis at each point for (E, ω).

□

Lemma 2.6. *Any $Sp(2n)$-vector bundle over a smooth manifold admits a lifting to a $U(n)$-vector bundle, which is unique up to isomorphisms (as $U(n)$-vector bundles). Consequently, for any $J_1, J_2 \in \mathcal{J}(E, \omega)$, the complex vector bundles (E, J_1), (E, J_2) are isomorphic. (In other words, every symplectic vector bundle has a underlying complex vector bundle structure unique up to isomorphisms.)*

Proof. By Theorem 1.12, $\mathrm{Sp}(2n)/U(n)$ is contractible. This implies that the classifying spaces $B\mathrm{Sp}(2n)$ and $BU(n)$ are homotopy equivalent via $i_* : BU(n) \to B\mathrm{Sp}(2n)$ induced by $i : U(n) \subset \mathrm{Sp}(2n)$. In particular, any $\mathrm{Sp}(2n)$-vector bundle over a smooth manifold M, which is classified by a map from M into $B\mathrm{Sp}(2n)$ unique up to homotopy, can be lifted to a $U(n)$-vector bundle by lifting the classifying map to a map from M into $BU(n)$, and such a lifting is unique up to isomorphisms.

□

Theorem 2.7. *Let (E_1, ω_1), (E_2, ω_2) be two symplectic vector bundles. Then they are isomorphic as symplectic vector bundles iff they are isomorphic as complex vector bundles.*

Proof. Pick $J_1 \in \mathcal{J}(E_1, \omega_1)$, $J_2 \in \mathcal{J}(E_2, \omega_2)$. Then by the previous lemma (E_1, ω_1), (E_2, ω_2) are isomorphic as symplectic vector bundles iff (E_1, J_1, ω_1), (E_2, J_2, ω_2) are isomorphic as $U(n)$-vector bundles. But the classification of $U(n)$-vector bundles up to isomorphisms is the same as classification of the underlying complex vector bundles because $GL(n, \mathbb{C})/U(n)$ is contractible. The theorem follows immediately.

□

Theorem 2.8. *For any symplectic vector bundle (E, ω), the space of ω-compatible complex structures $\mathcal{J}(E, \omega)$ is nonempty and contractible.*

Proof. There are actually two proofs of this important fact.

Proof 1: The nonemptiness of $\mathcal{J}(E, \omega)$ follows from Lemmas 2.5 and 2.6. On the other hand, for any $J \in \mathcal{J}(E, \omega)$, a parametric version of Corollary 1.14 gives rise to a deformation retraction of $\mathcal{J}(E, \omega)$ to $\{J\}$, which shows that $\mathcal{J}(E, \omega)$ is contractible.

Proof 2: A parametric version of Theorem 1.15 gives rise to a similar map $r : \mathrm{Met}(E) \to \mathcal{J}(E,\omega)$. Contractibility of $\mathcal{J}(E,\omega)$ follows from convexity of $\mathrm{Met}(E)$. $\square$

Corollary 2.9. *For any symplectic manifold (M,ω), the space of ω-compatible almost complex structures on M is nonempty and contractible.*

Proof 2 of Theorem 2.8 is less conceptual than proof 1 but more useful in various concrete constructions. As an example of illustration, we prove the following

Proposition 2.10. *Let Q be a symplectic submanifold of (M,ω). Then for any $J \in \mathcal{J}(Q,\omega|_Q)$, there exists a $\hat{J} \in \mathcal{J}(M,\omega)$ such that $\hat{J}|_{TQ} = J$. In particular, every symplectic submanifold of (M,ω) is a pseudo-holomorphic submanifold for some ω-compatible almost complex structure on M.*

Proof. Recall the symplectic direct sum decomposition

$$(TM|_Q, \omega|_Q) = (TQ, \omega|_{TQ}) \oplus (\nu_Q, \omega|_{\nu_Q}),$$

where ν_Q is the normal bundle of Q in M. For any $J \in \mathcal{J}(Q,\omega|_Q)$, we can extend it to $J' = (J, J^\nu)$ by choosing a $J^\nu \in \mathcal{J}(\nu_Q, \omega|_{\nu_Q})$. We then extend the corresponding metric $\omega(\cdot, J'\cdot)$ on $TM|_Q$ over the whole M to a metric g on TM. Let $r : \mathrm{Met}(M) \to \mathcal{J}(M,\omega)$ be the parametric version of the map in Theorem 1.15. Then $\hat{J} \equiv r(g)$ satisfies $\hat{J}|_{TQ} = J$, and in particular, Q is a pseudo-holomorphic submanifold with respect to the ω-compatible almost complex structure $\hat{J}$ on M. $\square$

We end this section with a brief discussion about integrability of almost complex structures. Recall that an almost complex structure J on a manifold M is said to be **integrable**, if M is the underlying real manifold of a complex manifold and J comes from the complex structure.

Let (M, J) be an almost complex manifold with almost complex structure J. The **Nijenhuis tensor** of J is defined by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X,Y]$$

for two vector fields $X, Y : M \to TM$. N_J is a bilinear map $T_pM \times T_pM \to T_pM$ for each $p \in M$, and has properties $N_J(X,X) = 0$ and $N_J(X, JX) = 0$ for any vector field X. In particular, $N_J = 0$ if M is 2-dimensional. Moreover, one can also check easily that $N_J = 0$ if J is integrable. The converse is given by the following highly nontrivial theorem of A. Newlander and L. Nirenberg.

Theorem 2.11. *(Newlander-Nirenberg). An almost complex structure is integrable if and only if the Nijenhuis tensor vanishes.*

In particular, every symplectic 2-dimensional manifold is Kähler.

References

[1] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, 2nd edition, Oxford Univ. Press, 1998.