Math 235 Practice Midterm 1

Q1. The following matrix is the augmented matrix for a system of linear equations.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 0 & 5 & 5 \end{bmatrix}$$

(a) Write down the linear system of equations whose augmented matrix is A.

(b) Find the reduced echelon form of A.

(c) In the reduced echelon form of A, mark the pivot positions.

(d) Does the system have no solutions, exactly one solution or infinitely many solutions? Justify your answer.

Q2. Determine if the vector $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$. If yes, indicate at least one possible value for the weights. If not, explain why.

$$\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Q3. Convert the given linear system to an augmented matrix and then find all solutions. Write the solutions in parametric form.

$$2x_1 + 6x_2 - 9x_3 - 4x_4 = 0$$

$$-3x_1 - 11x_2 + 9x_3 - x_4 = 0$$

$$x_1 + 4x_2 - 2x_3 + x_4 = 0$$

Q4. (1) Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 5 & -5 \\ -2 & 4 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

(a) Show that $\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{0}$.

(b) Show that the columns of A are linearly dependent by using part (a) to write down an explicit linear dependence relation. That is, find c_1, c_2, c_3 not all zero, such that

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ -5 \\ -8 \end{bmatrix} = \mathbf{0}$$

(2) Suppose that the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ has free variables. Are the columns of A linearly independent? Explain your reasoning.

Q5.

(a) Let T be any linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ and $\mathbf{v}$ be any vector in $\mathbb{R}^2$ such that $T(2\mathbf{v}) = T(3\mathbf{v}) = \mathbf{0}$. Determine whether the following is true or false, and explain why: (i) $\mathbf{v} = \mathbf{0}$, (ii) $T(\mathbf{v}) = \mathbf{0}$.

(b) Find the matrix associated to the geometric transformation on $\mathbb{R}^2$ that first reflects over the y-axis and then contracts in the y direction by a factor of $\frac{1}{3}$ and expands in the x direction by a factor of 2.

Q6. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2).$$

(a) Find the standard matrix for the linear transformation T.

(b) Determine whether the transformation T is onto.

(c) Determine whether the transformation T is one-to-one.

Q7. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ 7 & 2 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 5 \end{bmatrix}$$

(a) Does A commute with B?

(b) A defines a tranformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$. Similarly B defines a tranformation $T_B : \mathbb{R}^3 \to \mathbb{R}^3$. Find the standard matrix associated to the composite mapping $T_B \circ T_A$.

(c) Is the composition $T_A \circ T_B$ equal to the composition $T_B \circ T_A$? Justify your answer.