Finite symmetries in dimension four: 10 open questions – in memory of Slawomir Kwasik

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Abstract

We shall discuss 10 open questions about finite group actions on 4-manifolds. The answers to many of these questions are beyond the reach by the current techniques, so hopefully these questions may stimulate inventions of new ideas or methods. On the other hand, we choose these questions in order to explore the differences between the locally-linear, smooth, symplectic, and holomorphic categories in the study of finite group actions in dimension 4, and to understand the subtleties of some of the issues in group actions when being considered under the different categories.

Locally linear versus smooth: the 4-sphere

The 4-sphere S^4 is the lowest dimensional sphere which admits non-linear, smooth, finite group actions.

(i) Nonlinear actions with co-dimension 1 fixed-point set
Theorem(Poenaru 1960, Mazur 1961,de Rham 1962) There exist smooth involutions on S^4 whose fixed-point set is a Z-homology 3-sphere $\Sigma \neq S^3$.

(ii)Co-dim 2 fixed-point set- counterexample of Smith Conjecture
Theorem(Giffen 1966) There exist smooth Z_p-actions, for odd p, on S^4 whose fixed-point set is a knotted 2-sphere.

(iii)Free actions
Theorem(Fintushel-Stern, 1981) There exists a smooth free involution on S^4 which is not equivalent to the antipodal map.

Locally linear versus smooth: the 4-sphere

Thus, the "linearity" question in dimension 4 necessarily takes a weaker form: if a finite group G acts on S^4 locally linearly (resp. smoothly), is G necessarily isomorphic to a subgroup of $O(5)$? (First studied by Mecchia-Zimmermann, 2006, 2011)

Theorem(C-Kwasik-Schultz, 2015)

If a finite group G acts locally linearly and orientation preservingly on a homology 4-sphere, then G must be isomorphic to a subgroup of $SO(5)$.

Remarks: There exists a finite group G acting topologically and orientation preservingly on S^4 (semifreely with two fixed points) such that G is not isomorphic to a subgroup of $SO(5)$.

Thus, we have a complete understanding for the case of orientation preserving actions.

Locally linear versus smooth: the 4-sphere

Question 1

Does there exist a locally linear, orientation reversing action on S^4 by a finite group G, where G is not isomorphic to a subgroup of $O(5)$?

Remarks:

(1) There exist orientation reversing, topological actions by finite groups G on S^4, where G is not isomorphic to a subgroup of $O(5)$. However, the locally-linear case is still open.

(2) A strategy for constructing orientation reversing, locally-linear actions on S^4 by some of the Milnor groups was proposed by C-Kwasik–Schultz, 2015 (using topological surgery theory). It is possible that Question 1 has an affirmative answer.

Question 2

If an orientation reversing action of a finite group G on S^4 is smooth, must G be isomorphic to a subgroup of $O(5)$?

Remarks:

(1) Constructing an example of a smooth action on S^4 by a finite group G which is not isomorphic to a subgroup of $O(5)$ seems to be quite difficult.

(2) On the other hand, how to do gauge theory in a non-orientable setting is unclear.

(3) Is it necessary that the smooth structure on S^4 be standard?

Locally linear versus smooth: the 4-sphere

An Observation:

For all the results obtained so far concerning smooth finite group actions on S^4, an alternative proof can be found which works in the locally-linear category, so the same result holds true for locally linear actions on a homology 4-sphere. In other words, the smoothness assumption becomes unnecessary.

This said, if the answers to both Question 1 and Question 2 are affirmative, then this would be the first example to indicate that the locally-linear and smooth categories are different for finite group actions on S^4.

Rigidity and smooth structure: the $K3$ surface

The study of smooth actions is largely modeled on holomorphic actions (i.e. finite automorphism groups of $K3$ surfaces).

Homological Rigidity: Any nontrivial automorphism of a $K3$ surface induces a nontrivial isometry of the $K3$ lattice.

Classification: Finite automorphism groups of $K3$ surfaces have been classified, which are called $K3$ groups. Those which fixes a nowhere vanishing holomorphic 2-form are called symplectic $K3$ groups. There are 11 maximal groups, and there is a large body of literature.

Rigidity and smooth structure: the $K3$ surface

Question 3

Does there exist a nontrivial periodic diffeomorphism of the standard $K3$ surface which is homologically trivial?

Remarks: This question was asked by Alan Edmonds (See Kirby's Problem List, Problem 4.124(B)). There exist nontrivial, locally linear, periodic homeomorphisms of the $K3$ surface of any odd prime order which are homologically trivial.

Since this question was proposed, there has been a lot of new developments concerning exotic smooth structures on $K3$ surface.

Perhaps, if we insist that the smooth structure be standard, would the answer be negative? With the current techniques, this seems to be a long shot.

Rigidity and smooth structure: the $K3$ surface

Symplectic symmetries of the standard $K3$ surface

Theorem(C-Kwasik, 2007, 2011)

(1) A nontrivial symplectic finite group action on the standard $K3$ surface must be homologically nontrivial.
(2) If a finite group G acts on the standard $K3$ surface via symplectic symmetries such that the induced action on $H^{2,+}$ is trivial, then G must be isomorphic to a symplectic $K3$ group. Moreover, the fixed-point set structure of the symplectic G-action must coincide with that of a holomorphic G-action.

Remarks: The condition "standard smooth structure" can be replaced by the condition "$c_1 = 0$".

Theorem(C-Kwasik, 2011, C, 2020) Let $G = T_{48} \times Z_2$. If a symplectic homotopy $K3$ surface admits a symplectic G-action, then it must be the standard $K3$ surface.

Question 4

Does there exist a smooth action on a homotopy $K3$ surface by a finite group which is not a $K3$ group?

Remarks:

Since we do not require the smooth structure be standard, perhaps this is achievable with existing techniques.

Note that $G = Z_p$, where p is prime, is a $K3$ group if and only if $p \leq 19$. On the other hand, a smooth Z_p-action, for $p > 19$, on a homotopy $K3$ surface must be homologically trivial.

Symplectic symmetries on homotopy $K3$ surfaces

Question 5

Must a symplectic symmetry of a homotopy $K3$ surface induce a faithful representation on the $K3$ lattice?

Question 6

Does there exist a symplectic exotic $K3$ surface which admits no symplectic Z_p-actions for some prime number $p \leq 19$?

Remarks:

One can arrange to have exotic smooth structures on $K3$ surface such that some of the $K3$ groups (e.g. T_{24}) can no longer act smoothly (C-Kwasik, 2008). The arguments require the $K3$ groups to have a somewhat "complicated" group structure, even if the group actions are assumed to be symplectic.

Bounding the order: 4-manifolds of "general type"

Theorem(Hurwitz, 1893) Let Σ be a compact Riemann surface of genus $g > 1$. Then the automorphism group of Σ is finite, and its order obeys

$$|Aut(\Sigma)| \leq 84(g-1) = 42 deg K_\Sigma.$$

Theorem(G. Xiao, 1994) Let X be a minimal surface of general type. Then

$$|Aut(X)| \leq (42)^2 c_1(K_X)^2.$$

Remarks: Xiao's theorem has been extended to higher dimensional minimal, non-singular varieties of general type by C.D. Hacon, J. Mckernan and C. Xu.

Bounding the order: 4-manifolds of "general type"

The locally linear topological category

Theorem(Kwasik, 1986) The fake CP^2 admits a locally linear, topological Z_p-action for any odd prime p, but admits no locally linear, topological Z_2-actions.

Remarks: Note that the fake CP^2 admits no locally linear, topological S^1-actions.

Theorem(Edmonds, 1987) Let a closed, simply-connected 4-manifold M be given. For any prime number $p > 3$, there exists a locally linear, pseudo-free, topological Z_p-action on M, which is homologically trivial.

Remarks: There exist no locally linear, topological S^1-actions on a simply-connected 4-manifold with even intersection form and nonzero signature (Fintushel, 1977).

Beyond the locally linear topological category

Theorem(C, 2011) Let M be a compact complex surface with $\kappa(M) \geq 0$. Suppose M does not admit any holomorphic S^1-actions. Then there exists a universal constant $c > 0$ such that for any prime order, holomorphic Z_p-action on M, one has

$$p \leq c(1 + b_1 + b_2 + |TorH_2|).$$

Theorem(C, 2011) For any prime number $p \geq 5$, there is a symplectic 4-manifold M_p, where M_p is homeomorphic to $CP^2\#9\overline{CP^2}$ and $\kappa^s(M_p) = 1$, such that M_p admits a symplectic Z_p-action, but admits no smooth S^1-actions.

Remarks: $c_1(K_{M_p}) = (2p-1)[T]$ for some $[T] \in H_2(CP^2\#9\overline{CP^2})$.

Bounding the order: 4-manifolds of "general type"

Theorem(C, 2014) There exist smooth 4-manifolds X_n with $b_2^+ > 1$, where $n \geq 2$, such that the manifolds X_n have the same integral homology, intersection form, and Seiberg-Witten invariant, and each X_n supports no smooth S^1-actions but admits a smooth Z_n-action. Moreover, the Seiberg-Witten invariant of X_n is nonzero.

Remarks: (1) The fundamental group of X_n is increasingly more and more "complicated" as $n \to \infty$.
(2) In higher dimensions, there are examples of manifolds which supports no smooth S^1-actions but admits smooth Z_p-actions for infinitely many primes p.
(3) In dimension 3, for any closed 3-manifold M, the order of finite subgroups of $Diff(M)$ is bounded unless M admits a S^1-action.

Question 7

Let X be a simply-connected, even, and smoothable topological 4-manifold which has nonzero signature. Does there exist a constant $C > 0$, depending only on X, such that there are no smoothable Z_p-actions on X for any prime $p > C$?

Question 8

Let X be a minimal symplectic 4-manifold of $\kappa^s(X) = 2$. Does there exist a universal constant $c > 0$ such that for any finite subgroup G of symplectomorphisms, the order of G satisfies the following bound?

$$|G| \leq c \cdot c_1^2(TX)$$

Exoticness: symplectic vs. holomorphic, top. vs. smooth

Background: Primary examples of finite group actions on 4-manifolds are provides by automorphism groups of algebraic surfaces (or more generally, holomorphic actions on Kähler surfaces); in the case of S^4, primary examples are given by the restrictions of linear actions on R^5. A natural question asks whether there are "exotic" smooth actions which deviate from these standard ones. A basic method of producing exotic actions has been to construct actions whose fixed-point set structures are non-standard. However, if one requires the exotic actions to resemble the standard actions in some strong way, then the construction becomes considerably much harder.

Exoticness: symplectic vs. holomorphic, top. vs. smooth

Question 9

Does there exist a symplectic finite group action on a Kähler surface which is not smoothly equivalent to a holomorphic action?

Question 10

Are there any orientation-preserving, pseudo-free, smooth finite cyclic actions on a simply-connected 4-manifold, which are topologically equivalent but smoothly non-equivalent?

Exoticness: symplectic vs. holomorphic, top. vs. smooth

Remarks: (1) Concerning Question 9, if a symplectic finite group action has a 2-dimensional fixed-point set, then each component is an embedded symplectic surface. "Knotting" of such surfaces is harder, although it is not entirely impossible. On the other hand, symplectic Z_n-actions on CP^2 or a Hirzebruch surface are smoothly equivalent to a holomorphic action.

(2) Concerning Question 10, since the smooth actions are to be pseudo-free, making the 2-dimensional fixed components "knotted" is no longer an applicable method. On the other hand, the topological classification of pseudo-free actions is practically manageable only for "small" 4-manifolds such as S^4, CP^2, or a Hirzebruch surface. This limitation also makes the question harder.

The End

Thank You !