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A CHARACTERIZATION OF THE STANDARD SMOOTH STRUCTURE OF $K3$ SURFACE

WEIMIN CHEN

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ABSTRACT. We characterize the standard smooth structure of $K3$ among all smooth structures by the existence of a certain symplectic symmetry of order 96.

1. Introduction and the main theorem

In the theory of differentiable transformation groups, a classical problem concerns the correlation between the exoticness of the smooth structure and possible smooth (effective) compact Lie group actions on the manifold. In higher dimensions, this problem has been extensively studied in the case of exotic spheres (cf. [15]). In dimension four, however, the case of homotopy $K3$ surfaces (instead of homotopy 4-spheres) was first studied (cf. [6, 7]). In particular, it was shown in [7] that the existence of symplectic symmetries by certain maximal symplectic $K3$ groups will force the corresponding symplectic homotopy $K3$ surface to be "minimally exotic", meaning that the 4-manifold has the same Seiberg–Witten invariant of the standard $K3$. In this paper, we shall improve the results in [7] from "minimally exotic" to "standard" by employing some new ideas from the recent paper [5].

By way of definition, a $K3$ group is a finite group which can be realized as an automorphism group of a $K3$ surface. Furthermore, the $K3$ group (as well as the corresponding automorphism group of $K3$ surface) is called symplectic if its induced action on the holomorphic canonical line bundle is trivial. Symplectic $K3$ groups are completely classified, and there are 11 groups which are maximal (see Mukai [14]).

Recall that a symplectic homotopy $K3$ surface is a symplectic 4-manifold which is homotopy equivalent to a $K3$ surface. By a theorem of Freedman [11], a symplectic homotopy $K3$ surface is homeomorphic to a $K3$ surface. On the other hand, all complex $K3$ surfaces are diffeomorphic as smooth 4-manifolds [2]; the underlying smooth structure is referred to as the standard smooth structure of $K3$ surface.

Main Theorem. *There is a finite group G of order 96 which gives the following characterization of the standard smooth structure of $K3$ surface: Let M be a symplectic homotopy $K3$ surface. Then M is diffeomorphic to a $K3$ surface if and only if M admits a symplectic G-action.*

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Remarks. (1) The Main Theorem above is the first example (in any dimension) where the standard smooth structure of a manifold is characterized by the existence of a finite symmetry group of certain geometric structure. Earlier results of such a nature (e.g., cf. [15]) are all characterized by smooth Lie group actions of positive dimension. However, we should mention that, strictly speaking, our result in the Main Theorem is not characterized by smooth actions of a finite group, as there are homotopy $K3$ surfaces which admit no symplectic structures; or even though the homotopy $K3$ surface admits a symplectic structure, it is not known whether for any smooth finite group action it always possesses an invariant symplectic structure.

On the other hand, we point out that in the case of $K3$ surface, there do exist infinitely many distinct symplectic homotopy $K3$ surfaces with an exotic smooth structure, e.g., those constructed using the Fintushel-Stern knot surgery [10]. However, none of these examples are minimally exotic; it is an open question as to whether there is a minimally exotic but non-standard smooth structure on $K3$ surface (cf. [8,12]).

(2) In dimension four, the smooth structures which support a smooth $\mathbb{S}^1$-action are known to be quite restricted. For example, a simply connected smooth 4-manifold admitting a smooth $\mathbb{S}^1$-action must be a connected sum of $\mathbb{S}^4$, $\pm\mathbb{CP}^2$, or $\mathbb{S}^2\times\mathbb{S}^2$. This was proved in the late 1970s (cf. [9,19]), modulo the 3-dimensional Poincaré Conjecture which is now resolved, cf. [13]. On the other hand, for fixed-point free $\mathbb{S}^1$-actions, it was shown in [4] that for any given finitely presented group G with infinite center, there are only finitely many distinct smooth orientable 4-manifolds admitting a smooth fixed-point free $\mathbb{S}^1$-action, whose fundamental group is isomorphic to G, with the number of such 4-manifolds bounded by a constant depending only on G. With this said, we point out that homotopy $K3$ surfaces do not admit any smooth $\mathbb{S}^1$-actions by a theorem of Atiyah and Hirzebruch [1].

In the Main Theorem, the group $G = T_{48}\times\mathbb{Z}_2$, where T_{48} is one of the 11 maximal symplectic $K3$ groups which has order 48. In fact, G is a $K3$ group, i.e., it can be realized as an (non-symplectic) automorphism group of a $K3$ surface. By choosing a G-invariant Kähler form which is naturally a symplectic structure on the $K3$ surface, this fact immediately implies half of the Main Theorem, i.e., the "only if" part. We shall give a brief description below (see Mukai [14], p. 193, for further details).

The $K3$ surface with G as an automorphism group can be constructed as follows. Recall that T_{48} is a subgroup of $GL(2,\mathbb{C})$ with the binary tetrahedral group $T_{24}\subset SL(2,\mathbb{C})$ as an index-2 normal subgroup. Furthermore, the linear representation of T_{48} on $\mathbb{C}^2$ leaves the degree 6 polynomial $xy(x^4+y^4)$ invariant. With this understood, consider the hypersurface in $\mathbb{C}^4$,

$$Z := \{(x,y,z,w)\in\mathbb{C}^4 | w^2 = xy(x^4+y^4)+z^6\}.$$

We consider the following action of G on Z: for any $g\in T_{48}\subset GL(2,\mathbb{C})$, the action of g on Z is induced by

$$g\cdot(x,y,z,w) = (g(x,y),(\det g)z,w),$$

and the action of the nontrivial element $\tau\in\mathbb{Z}_2$ is given by

$$\tau\cdot(x,y,z,w) = (x,y,z,-w).$$

The G-action commutes with the following $\mathbb{C}^*$-action on Z,

$$\lambda\cdot(x,y,z,w) = (\lambda x,\lambda y,\lambda z,\lambda^3 w),\ \forall\lambda\in\mathbb{C}^*.$$

With this understood, let $S := (Z \setminus \{(0,0,0,0)\})/\mathbb{C}^*$. Then $\pi : S \rightarrow \mathbb{CP}^2$ induced by the projection $(x,y,z,w) \mapsto (x,y,z)$ defines S as a double branched covering of $\mathbb{CP}^2$ with branch loci given by the degree 6 curve $xy(x^4+y^4)+z^6=0$. Hence S is a $K3$ surface. The G-action on Z naturally descends to a G-action on S, where the $\mathbb{Z}_2$-factor is the deck transformation group of the double-branched covering $\pi : S \rightarrow \mathbb{CP}^2$, and T_{48} acts as a symplectic automorphism group of the $K3$ surface S.

Note that the 4-manifold M in the Main Theorem is a symplectic homotopy $K3$ surface equipped with a symplectic T_{48}-action, as T_{48} is contained in $G = T_{48} \times \mathbb{Z}_2$ as a subgroup. Hence by Theorem 1.1 of [7], M must have trivial canonical class, i.e., $c_1(K_M) = 0$. With this understood, the main technical result of this paper, proving the "if" part of the Main Theorem, is the following theorem.

Theorem 1.1. *Let M be a symplectic homotopy $K3$ surface with $c_1(K_M) = 0$. Suppose M admits a symplectic G-action, where $G = T_{48} \times \mathbb{Z}_2$. Then M is diffeomorphic to the standard $K3$ surface.*

2. Proof of Theorem 1.1

We begin by recalling some relevant results from [7], which will be used in the proof of Theorem 1.1. To this end, let M be a symplectic homotopy $K3$ surface with $c_1(K_M) = 0$, equipped with a symplectic G-action (where G is finite). Denote by ω the symplectic structure on M, which is invariant under the G-action. Then

- (†) There is a natural homomorphism $\rho : G \rightarrow \mathbb{S}^1$, defined as follows. We fix a G-invariant metric such that the symplectic form ω is self-dual. This allows us to identify $H^{2,+}(M;\mathbb{R})$ with the space of self-dual harmonic 2-forms on M, under which $[\omega] \in H^{2,+}(M;\mathbb{R})$ is identified to ω. Consequently, we obtain an induced action of G on $H^{2,+}(M;\mathbb{R})$ through the G-action on the space of self-dual harmonic 2-forms, and it is easily seen that the choice of the metric is non-essential here. Now observe that $[\omega] \in H^{2,+}(M;\mathbb{R})$ is fixed by the G-action, so that there is an induced G-action on the orthogonal complement $[\omega]^\perp \subset H^{2,+}(M;\mathbb{R})$ (with respect to the cup-product). Since $b_2^+(M) = 3$, $[\omega]^\perp \cong \mathbb{R}^2$. With this understood, it was shown in [7], Lemma 2.1, that the G-action on $[\omega]^\perp \cong \mathbb{R}^2$ is orientation-preserving, which gives rise to the homomorphism $\rho : G \rightarrow \mathbb{S}^1$.
- (‡) For any subgroup H of G such that $\rho(H) = \{1\}$, it was shown in Theorem 1.2 of [7] that H is a symplectic $K3$ group, and moreover, there is a $K3$ surface with H as a symplectic automorphism group such that the fixed-point set structure of the holomorphic H-action on the $K3$ surface is the same as the fixed-point set structure of the symplectic H-action on M.

With the preceding understood, we now let M and G be as in Theorem 1.1. We denote by G_0 the subgroup of G which is the kernel of the homomorphism $\rho : G \rightarrow \mathbb{S}^1$. In other words, G_0 is the maximal normal subgroup of G such that $b_2^+(M/G_0) = 3$. The following lemma identifies the subgroup G_0.

Lemma 2.1. *The $\mathbb{Z}_2$-factor in $G = T_{48} \times \mathbb{Z}_2$ has a non-trivial image under $\rho : G \rightarrow \mathbb{S}^1$. Consequently, G is isomorphic to $G_0 \times \mathbb{Z}_2$, where G_0 is isomorphic to T_{48}.*

Proof. We recall that T_{48} is a nontrivial semi-direct product $T_{24} \times_\phi \mathbb{Z}_2$ and the commutator $[T_{48}, T_{48}] = T_{24}$. It follows easily that the subgroup $T_{24} \subset G$ is contained in G_0. Assume, to the contrary, that the $\mathbb{Z}_2$-factor in $G = T_{48} \times \mathbb{Z}_2$ has a trivial image under $\rho : G \to \mathbb{S}^1$. Then the subgroup $H := T_{24} \times \mathbb{Z}_2$ of G is contained in G_0. Now as we mentioned in (‡) above, H must be a symplectic $K3$ group. On the other hand, examining Table 2 in [18], we find that H is not listed there, as H has order 48 and $[H, H] = [T_{24}, T_{24}] = Q_8$. The lemma follows easily from this contradiction. □

In what follows, we identify G with $G_0 \times \mathbb{Z}_2$, and denote by τ the generator of the $\mathbb{Z}_2$-factor. A key fact we need to establish is that the action of τ is not free.

To this end, recall that by Theorem 1.2 of [7] (as we mentioned in (‡) above), the action of $G_0 = T_{48}$ on M has the same fixed-point set structure of a symplectic T_{48}-automorphism of a $K3$ surface. With this understood, and by examining Table 2 of [18], we find that the quotient space M/G_0 contains 7 isolated singular points, with the corresponding resolution graphs being E_6, A_7, A_2, and $4A_1$. We shall particularly look at the singular point whose resolution graph is A_7. The preimage of this singular point in M consists of a set of six points $\{p_1, p_2, \ldots, p_6\}$, where each p_i has an isotropy subgroup isomorphic to $\mathbb{Z}_8$ whose action on the tangent space $T_{p_i}M$ has weights $(k, -k)$. On the other hand, since τ commutes with G_0, there is an induced action of τ on the quotient space M/G_0. The action must fix the isolated singular point whose resolution graph is A_7 because such a singular point is unique in M/G_0. Consequently, the set $\{p_1, p_2, \ldots, p_6\}$ must be invariant under the action of τ.

Lemma 2.2. *The set $\{p_1, p_2, \ldots, p_6\}$ is contained in the fixed-point set of τ; in particular, the action of τ is not free.*

Proof. Suppose to the contrary that τ acts nontrivially on $\{p_1, p_2, \ldots, p_6\}$. Then without loss of generality, we assume $\tau \cdot p_1 = p_2$. We denote by $\Gamma \subset G_0$ the isotropy subgroup of p_1. Since τ commutes with G_0, it follows easily that the isotropy subgroup of p_2 is also Γ. On the other hand, note that G_0 acts transitively on the set $\{p_1, p_2, \ldots, p_6\}$. Hence, there is an element $h \in G_0$ such that $h \cdot p_2 = p_1$. Since p_1, p_2 have the same isotropy subgroup Γ, it follows easily that $h^{-1}\Gamma h = \Gamma$. Finally, note that $(h\tau) \cdot p_1 = p_1$, which implies easily that $h^2 \in \Gamma$. In particular, h has an even order. We set $x := h\tau$.

To proceed further, we shall give a description of Γ next. Let γ be a generator of Γ. Then consider the image of γ under the homomorphism $G_0 = Q_8 \times_\phi S_3 \to S_3$ (here S_n is the symmetry group of n letters). Since γ has order 8, the image of γ in S_3 must be an element of order 2, to be denoted by δ. It follows easily that $\gamma\delta$ must be an element of order 4 in Q_8. Without loss of generality, we assume $\gamma\delta = i \in Q_8$, or equivalently, $\gamma = i\delta$. With this understood, note that $i\delta i\delta = \gamma^2 \in Q_8$ is an element of order 4. We may assume $i\delta i\delta = k$, or $\delta i\delta = j$ without loss of generality. Then $\delta j\delta = i$ and $\delta k\delta = -k$ follows easily. Note that $k \in \Gamma$.

Next we analyze the normalizer of Γ; in particular, the possible values for the element h. First, we let g_0 be an element of order 3 in S_3. Then since $g_0^{-1}Q_8 g_0 = Q_8$, it follows easily that the image of $g_0^{-1}\gamma g_0$ under the homomorphism $G_0 = Q_8 \times_\phi S_3 \to S_3$ equals $g_0^{-1}\delta g_0$, which is not equal to δ. This shows that g_0 is not in the normalizer of Γ. On the other hand, one can easily check that both Q_8 and δ are contained in the normalizer of Γ. So the normalizer of Γ is the semi-direct product of Q_8 with δ. In particular, the possible values of h are $\delta, \pm i, \pm j$. Since Γ is of index 2 in its normalizer, it is easy to see that the action of h on Γ by conjugation is independent of the possible choice of h. In particular, we obtain

$$h^{-1}\gamma h = \delta^{-1}\gamma\delta = \delta(i\delta)\delta = -i(i\delta i\delta)\delta = -i\gamma^2\delta = \gamma^2 i\delta = \gamma^3.$$

It follows then that $x^{-1}\gamma x = \gamma^3$, where $x = h\tau$, which fixes the point p_1.

Now we are ready to derive a contradiction by looking at the action of x on the tangent space $T_{p_1}M$. Recall that the action of Γ on $T_{p_1}M$ has weights $(k,-k)$. This means that the eigenvalues of γ in $T_{p_1}M$ are λ, λ^{-1} for some 8-th root of unity λ. Let $v \in T_{p_1}M$ be an eigenvector of γ with eigenvalue λ. Then

$$(x^{-1}\gamma x)\cdot v = \gamma^3 \cdot v = \lambda^3 v,$$

which implies that $\gamma\cdot(x\cdot v) = \lambda^3(x\cdot v)$, i.e., $x\cdot v$ is an eigenvector of γ with eigenvalue $\lambda^3 \neq \lambda, \lambda^{-1}$. This contradiction completes the proof of the lemma. □

Next we analyze the fixed-point set of τ, denoted by M^τ, by exploiting the natural action of G_0 on it, noting that since τ and G_0 commute, M^τ is invariant under G_0.

First of all, we observe that each of $p_1, \cdots, p_6 \in M^\tau$ in Lemma 2.2 must be contained in a 2-dimensional component of M^τ. This is because from the proof of Lemma 2.2, we see that $-1 \in Q_8$ fixes each p_i and its action on $T_{p_i}M$ is given by $-Id$. If p_i were an isolated fixed point of τ, then action of τ on $T_{p_i}M$ would also have been given by $-Id$. That would imply that the product of τ and $-1 \in Q_8$ acts trivially on $T_{p_i}M$, which is a contradiction, hence the claim. In particular, M^τ must have 2-dimensional components.

At this point, we shall review the new ingredient which is crucial for the proof of Theorem 1.1, by recalling the relevant constructions from [5]. To this end, let (X,ω) be any symplectic 4-orbifold, and let Σ denote its singular set, i.e.,

$$\Sigma = \{p \in X | \text{the isotropy group } \Gamma_p \text{ is nontrivial}\}.$$

If we fix an ω-compatible (orbifold) almost complex structure J, and let g_J be the corresponding Riemannian metric, then at each $p \in \Sigma$, the tangent space T_pX can be identified with $\mathbb{C}^2$, with the action of Γ_p on T_pX given by a subgroup of $U(2)$. Consequently, Σ can be decomposed as a disjoint union $\Sigma^0 \sqcup \Sigma^* \sqcup \Sigma^1$, where

- $\Sigma^0 = \{p \in \Sigma | \text{the action of } \Gamma_p \text{ on } T_pX \setminus \{0\} \text{ is free}\}$.
- $\Sigma^* = \{p \in \Sigma | \Gamma_p \text{ fixes a complex line in } (T_pX, J)\}$.
- $\Sigma^1 = \{p \in \Sigma | \text{the action of } \Gamma_p \text{ on } T_pX \setminus \{0\} \text{ is not free but is fixed-point free}\}$.

Both Σ^0, Σ^1 consist of finitely many points, but Σ^* is a 2-dimensional smooth manifold such that $\omega|_{\Sigma^*}$ is an area form. We can compactify each connected component of Σ^* in X by adding points from Σ^1. Let $\{\Sigma_i\}$ be the set of compactified connected components of Σ^*. Then each Σ_i is a symplectic orbifold surface in (X,ω) (possibly immersed), with the points of self-intersection of each Σ_i and the points of intersection of distinct Σ_i, Σ_j contained in Σ^1. We denote by $|X|$ the underlying space of X, which is naturally a smooth 4-orbifold with at most isolated singularities; in particular, Σ^* lies in the smooth locus of the orbifold $|X|$ (see [5] for more details).

With this understood, the main result of [5] constructs a canonical symplectic resolution for the symplectic 4-orbifold (X,ω). The key step is to first de-singularize the symplectic form ω along the 2-dimensional singular strata Σ^*, making $|X|$ naturally a symplectic 4-orbifold with at most isolated singularities. This construction can even be done equivariantly when (X, ω) admits a finite symplectic G-action.

Theorem 2.3 (Theorem 1.1 of [5])**.** *Let (X, ω) be a symplectic 4-orbifold, and let G be a finite group acting smoothly on the 4-orbifold X, preserving the symplectic structure ω. There are G-invariant neighborhoods U of Σ^1 in $|X|$, which can be taken arbitrarily small, such that for any choice of U, there is a G-invariant symplectic structure ω' on the orbifold $|X|$, such that $\omega' = \omega$ on $|X| \setminus (\Sigma^* \cup U)$ (as symplectic forms) and $\omega' = \omega$ on $\Sigma^* \setminus U$ as area forms. Each Σ_i is a symplectic orbifold surface in $(|X|, \omega')$, which may be singular with respect to the smooth structure of the orbifold $|X|$. The self-intersections and singular points of each Σ_i occur only at points in Σ^1, and there is a G-invariant, ω'-compatible, integrable almost complex structure on U with respect to which each $\Sigma_i \cap U$ is a (genuine) holomorphic curve.*

The symplectic resolution of the symplectic 4-orbifold (X, ω), to be denoted by $(\widetilde{X}, \tilde{\omega})$, is simply taken to be a minimal symplectic resolution of the symplectic 4-orbifold $(|X|, \omega')$ (which has isolated singularities). Note that this can be done equivariantly, so there is a natural symplectic G-action on $(\widetilde{X}, \tilde{\omega})$ (cf. [5], Theorem 1.5).

With the preceding understood, we go back to the proof of Theorem 1.1. We shall apply the symplectic resolution construction to the symplectic 4-orbifold $X := M/\langle\tau\rangle$, and denote the corresponding symplectic resolution $(\widetilde{X}, \tilde{\omega})$ by M_τ. We call M_τ the *resolution* of the τ-action on M. It follows easily from the construction that each 2-dimensional component of the fixed-point set M^τ descends to a symplectic surface in M_τ, and each isolated fixed point in M^τ gives rise to a symplectic (-2)-sphere in M_τ. (Note that in the present situation, the subset Σ^1 of the singular set Σ is empty.)

Now the fact that the fixed-point set M^τ contains a 2-dimensional component has the following important consequence (cf. Lemma 4.1 in [5]):

The resolution M_τ of the τ-action on M is a symplectic rational 4-manifold.

With the resolution M_τ at hand, we shall further analyze the fixed-point set M^τ. To this end, we first collect some standard facts about M^τ in terms of the integral $\mathbb{Z}_2$-representation on $H^2(M)$ induced by the action of τ. Recall that the integral $\mathbb{Z}_2$-representation on $H^2(M)$ splits into a direct sum of 3 types of $\mathbb{Z}_2$-representations: the regular type of rank $p = 2$, the trivial representation of rank 1, and the representation of cyclotomic type of rank $p - 1 = 1$. If we let r, t, s be the number of summands of the above 3 types of $\mathbb{Z}_2$-representations in $H^2(M)$. Then they obey the following constraints: (a) $2r + t + s = b_2(M) = 22$, (b) $\chi(M^\tau) = 2 + t - s$ by the Lefschetz fixed point theorem as the trace of τ on $H^2(M)$ equals $t - s$ (the regular $\mathbb{Z}_2$-representations contribute trivially to the trace of τ), and (c) $b_1(M^\tau) = s$ (see e.g. Proposition 1.1 in [3], for a more detailed review as well as references).

Lemma 2.4. *Let x, y be the number of spheres and tori in M^τ respectively, and let ϵ be the number of 2-dimensional components of M^τ which have genus > 1. Let z be the number of isolated points in M^τ. Then (1) $\epsilon = 0$ or 1, (2) $z = 0$, and (3) $x + 2y + \epsilon \leq 12$.*

Proof. Let $\{\Sigma_i | i \in I\}$ be the set of 2-dimensional components of M^τ, and let g_i be the genus of Σ_i. We first show that $\epsilon = 0$ or 1. To see this, note that $c_1(K_M) = 0$, so that the adjunction formula gives $\Sigma_i^2 = 2g_i - 2$. If $g_i > 1$, then $\Sigma_i^2 > 0$. This implies easily the following upper bound $\epsilon \leq b_2^+(M/\langle\tau\rangle)$. Then $\epsilon = 0$ or 1 follows immediately from the fact that $b_2^+(M/\langle\tau\rangle) = 1$ (cf. Lemma 2.1).

To see $z = 0$, we note that by Proposition 3.2 in [5], $c_1(K_{M_\tau}) = -\frac{1}{2}\sum_{i\in I} PD(B_i)$, where B_i is the descendent of Σ_i in M_τ and $PD(B_i)$ denotes the Poincaré dual of B_i. Observing that $PD(B_i)^2 = 2 \cdot \Sigma_i^2$, we obtain

$$c_1(K_{M_\tau})^2 = \frac{1}{4} \cdot \sum_{i\in I} 2 \cdot \Sigma_i^2 = \sum_{i\in I} (g_i - 1) = s/2 - (x + y + \epsilon),$$

because $\Sigma_i^2 = 2g_i - 2$ and $s = b_1(M^\tau) = \sum_{i\in I} 2g_i$. On the other hand, as a symplectic rational 4-manifold, $c_1(K_{M_\tau})^2 = 10 - b_2(M_\tau)$, so that

$$b_2(M_\tau) = 10 + x + y + \epsilon - s/2.$$

Computing $b_2(M_\tau)$ differently, we have

$$b_2(M_\tau) = b_2(M/\langle\tau\rangle) + z = \frac{1}{2}(b_2(M) + tr(\tau|_{H^2(M)})) + z = 11 + \frac{1}{2}(t - s) + z.$$

It follows easily that $x + y + \epsilon = 1 + t/2 + z$. Finally, $\chi(M^\tau) = t - s + 2$ means $\sum_{i\in I}(2 - 2g_i) + z = t - s + 2$, which gives

$$2(x + y + \epsilon) - s + z = t - s + 2.$$

With $x+y+\epsilon = 1+t/2+z$, it follows easily that $z = 0$. Finally, note that $y \leq s/2$, which implies

$$x + 2y + \epsilon \leq 1 + (t + s)/2 \leq 1 + b_2(M)/2 = 12.$$

□

Lemma 2.5. $x = y = 0$.

Proof. We first show that there are no spheres in M^τ, i.e., $x = 0$. Suppose to the contrary that there is a sphere in M^τ, to be denoted by Σ_0. We consider the image C_0 of Σ_0 in the quotient space M/G_0. Since Σ_0 is a 2-sphere, C_0 is a spherical 2-orbifold, whose singular points are among the singular points of M/G_0. With this understood, recall that M/G_0 has 7 isolated singular points, with the corresponding resolution graphs being E_6, A_7, A_2 and $4A_1$. The singular point whose resolution graph is E_6 can not occur in C_0 because its isotropy group is isomorphic to T_{24} which is non-abelian. The singular points corresponding to A_7, A_2, or A_1 are all possibly contained in C_0; their isotropy groups are cyclic of order 8, 3, or 2.

With the preceding understood, note that as a spherical 2-orbifold C_0 is either a football (two singular points of order n), or a turnover which has three singular points of orders $(n, 2, 2)$, $(3, 3, 2)$, $(4, 3, 3)$, or $(5, 3, 2)$. With the description of the singular points of M/G_0 given in the previous paragraph, it is clear that only the case of football with $n = 2$ or a turnover $(n, 2, 2)$ with $n = 3$ or 8 can occur. The case of football can be ruled out immediately, as the subgroup of G_0 which leaves Σ_0 invariant has order 2, so that there must be at least $48/2 = 24$ spheres in M^τ (i.e., $x \geq 24$). But this is a contradiction as $x + 2y + \epsilon \leq 12$, in particular, $x \leq 12$. The case of turnover with $n = 8$ is also not possible because in this case the subgroup of G_0 which leaves Σ_0 invariant is a dihedral group of order 16, and there is no such subgroup in G_0 (see the proof of Lemma 2.2).

To rule out the possibility that C_0 is a turnover with orders $(3, 2, 2)$, we shall consider the 2-dimensional components of M^τ which contain the points $p_1, \ldots, p_6$ in Lemma 2.2. In the present situation, it is clear that $p_1, \ldots, p_6$ are not contained in spheres in M^τ. Furthermore, since a 2-torus does not admit any $\mathbb{Z}_8$-action with a fixed point, the points are not contained in tori in M^τ either. Hence there must be a genus > 1 component of M^τ which contains $p_1, \ldots, p_6$. We denote this component by Σ, and let g_Σ be its genus. Denote by C the 2-orbifold which is the image of Σ in M/G_0.

To derive a contradiction, we first note that the orbifold Euler number of C is

$$\chi(C) = \frac{2 - 2g_\Sigma}{|G_0|} = \frac{1 - g_\Sigma}{24}.$$

On the other hand, C contains a singular point of order 8, and may contain at most two singular points of order 2. Let κ be the number of singular points of order 2 in C, and denote by $|C|$ the underlying 2-manifold of C. Then

$$\chi(C) = \chi(|C|) - (1 - \frac{1}{8}) - \kappa \cdot (1 - \frac{1}{2}),$$

which implies easily that $g_\Sigma = 22 + 12\kappa - 24 \cdot \chi(|C|)$. Finally, note that

$$g_\Sigma \leq s/2 \leq b_2(M)/2 = 11.$$

This implies that in $g_\Sigma = 22 + 12\kappa - 24 \cdot \chi(|C|)$, we must have $\chi(|C|) = 2$. Then $g_\Sigma = 12\kappa - 26 < 0$ as $\kappa \leq 2$, a contradiction. Hence $x = 0$.

By a similar argument, we can show $y = 0$. More concretely, suppose to the contrary that there is a torus in M^τ, denoted by Σ_0. The image C_0 of Σ_0 in M/G_0 is either a nonsingular torus, or a 2-orbifold with four singular points of order 2 (called a pillowcase), or a turnover with orders $(6, 3, 2)$, $(4, 4, 2)$, or $(3, 3, 3)$. Clearly, only the case of a nonsingular torus or a pillowcase can occur for C_0. In the former case, the subgroup of G_0 which leaves Σ_0 invariant must be abelian as its action on Σ_0 is free. This implies a bound $y \geq 6$ because abelian subgroups of G_0 are of order at most 8. With the constraint $x + 2y + \epsilon \leq 12$, we see that $y = 6$ and $x = \epsilon = 0$. But this is a contradiction because there must be a non-torus component in M^τ which contains the points $p_1, \ldots, p_6$ in Lemma 2.2. To rule out the latter case, we consider the genus > 1 component of M^τ which contains the points in Lemma 2.2. We denote this component by Σ, and let g_Σ be its genus. Denote by C the 2-orbifold which is the image of Σ in M/G_0. Then C contains a singular point of order 8, and possibly a singular point of order 3 and no other singular points. Computing the orbifold Euler number, we have

$$\frac{1 - g_\Sigma}{24} = \chi(|C|) - (1 - \frac{1}{8}) - \kappa \cdot (1 - \frac{1}{3}),$$

where $\kappa = 0$ or 1 is the number of singular points of order 3 in C. By the same argument, $g_\Sigma \leq 11$ implies that $\chi(|C|) = 2$ must be true. But with $\kappa \leq 1$, we then obtain $g_\Sigma < 0$, which is a contradiction. This proves $y = 0$. □

With Lemmas 2.4 and 2.5, we now know that M^τ consists of a single component Σ with genus $g_\Sigma > 1$. We shall extend the arguments in Lemma 2.5 to find out the possible values for g_Σ. We continue to denote by $C = \Sigma/G_0$ the 2-orbifold in M/G_0. Let κ be the number of singular points of order 2 in C.

Case (i). Suppose C does not contain a singular point of order 3. In this case, we have

$$\frac{1-g_\Sigma}{24} = \chi(|C|) - (1-\frac{1}{8}) - \kappa\cdot(1-\frac{1}{2}).$$

With the constraint $g_\Sigma \leq 11$, the only possibility in this case is $g_\Sigma = 10$ with $\kappa = 3$.

Case (ii). Suppose C contains a singular point of order 3. In this case, we have

$$\frac{1-g_\Sigma}{24} = \chi(|C|) - (1-\frac{1}{8}) - (1-\frac{1}{3}) - \kappa\cdot(1-\frac{1}{2}).$$

Then the constraint $g_\Sigma \leq 11$ implies that $\kappa = 1$, $g_\Sigma = 2$ is the only possibility. We shall rule out this possibility by showing that $G_0 = T_{48}$ can not act effectively on a genus-2 surface Σ. To see this, note that as a semi-direct product, $T_{48} = Q_8 \times_\phi S_3$. The element $-1 \in Q_8$ generates the center of T_{48}. The action of $-1 \in Q_8$ on the genus-2 surface Σ is a hyperelliptic involution, with quotient a 2-sphere. There is an induced action by the quotient group $T_{48}/\langle -1\rangle$ of order 24 on the 2-sphere. Hence $T_{48}/\langle -1\rangle$ must be isomorphic to S_4, which is not true. (To see this, note that $T_{48}/\langle -1\rangle$ is a semi-direct product of $(\mathbb{Z}_2)^2$ with S_3.)

Now Theorem 1.1 follows readily: with $g_\Sigma = 10$, we see $c_1(K_{M_\tau})^2 = g_\Sigma - 1 = 9$, implying that $M_\tau = \mathbb{CP}^2$. Let B denote the descendant of Σ in $M_\tau = \mathbb{CP}^2$. Then B is an embedded symplectic surface in M_τ (cf. [5]), which is easily seen of degree 6 by the genus formula. Note that M is a double cover of $M_\tau = \mathbb{CP}^2$ branched over B. With this understood, we note that B is isotopic to a degree 6 algebraic curve by a theorem of Shevchishin [16] (see also Siebert and Tian [17]). Thus M is diffeomorphic to a double branched cover of $\mathbb{CP}^2$ along a degree 6 curve, which is a $K3$ surface.

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Department of Mathematics, University of Massachusetts, Amherst, Amherst, Massachusetts 01003-9305

Email address: wchen@math.umass.edu