Resolving symplectic orbifolds with applications to finite group actions

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ABSTRACT. We associate to each symplectic 4-orbifold X a canonical smooth symplectic resolution $\pi : \tilde{X} \to X$, which can be done equivariantly if X comes with a symplectic G-action by a finite group. Moreover, we show that the resolutions of the symplectic 4-orbifolds X/G and $\tilde{X}/G$ are in the same symplectic birational equivalence class; in fact, the resolution of $\tilde{X}/G$ can be reduced to that of X/G by successively blowing down symplectic (-1)-spheres.

To any finite symplectic G-action on a 4-manifold M, we associate a pair (M_G, D), where $\pi : M_G \to M/G$ is the canonical resolution of the quotient orbifold and D is the pre-image of the singular set of M/G under π. We propose to study the group action on M by analyzing the smooth or symplectic topology of M_G as well as the embedding of D in M_G. In this paper, an investigation on the symplectic Kodaira dimension κ^s of M_G is initiated. In particular, we conjecture that $\kappa^s(M_G) \leq \kappa^s(M)$. The inequality is verified for several classes of symplectic G-actions, including any actions on a rational surface or a symplectic 4-manifold with $\kappa^s = 0$.

1. Introduction and the main results

The purpose of this paper is twofold. On the one hand, we are concerned with the basic question of resolving orbifold singularities in symplectic geometry. We show that any symplectic 4-orbifold admits a canonical symplectic resolution (a more precise description will follow) without imposing any conditions on the structure of its singular set. The second goal of this paper is to introduce some new ideas to the study of symplectic finite group actions on 4-manifolds, which are based on analyzing the symplectic resolution canonically associated to the symplectic quotient orbifold of the group action. In forthcoming papers we shall develop these ideas in the context of several natural problems in group actions and 4-manifolds.

1.1. Resolving symplectic orbifolds

The problem of resolving symplectic singularities was posed by Gromov (cf. [25]) in the 1980s. The first progress was made by McCarthy and Wolfson in [33], where the authors dealt with the case of isolated orbifold singularities of a 4-dimensional space. They formulated a natural notion of symplectic resolution and gave a construction in

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this case. In a subsequent work the construction was extended to isolated algebraic singularities in a symplectic 4-manifold (cf. [34]). In both works the authors employed certain techniques of gluing symplectic manifolds along a certain type of hypersurface to construct the symplectic resolution. Their construction can be regarded as a symplectic analog of resolution of singularities in algebraic geometry.

Going beyond the case of isolated singularities in dimension 4, Cavalcanti, Fernández, and Muñoz [4] constructed symplectic resolutions of isolated orbifold singularities in all dimensions. Their work made use of the algebraic resolution of orbifold singularities as McCarthy and Wolfson did in [34], but the construction of symplectic resolution was different. For non-isolated orbifold singularities, the construction of symplectic resolution requires new ideas. In [39, 40], Niederkrüger and Pasquotto constructed symplectic resolutions of any symplectic orbifolds which arise in the symplectic reduction of a Hamiltonian torus action. Their idea is to construct an auxiliary circle action in a neighborhood of the singularities with the largest isotropy group, and subsequently use it to perform a symplectic cut of Lerman (cf. [29]). This procedure produces a symplectic orbifold of singularities with smaller isotropy groups, giving rise to a "partial resolution" of the original symplectic orbifold. A symplectic resolution is then obtained by a sequence of such partial resolutions. As an application, the authors gave a different method for constructing symplectic resolution of an isolated cyclic quotient singularity (cf. [39]). It uses neither the algebraic resolution nor the gluing techniques as it was done in the works of [33], [34] or [4], but rather, it amounts to performing a sequence of weighted blow-ups (as described in [23]). Constructing a symplectic resolution for an arbitrary symplectic orbifold remains an open problem (see [36] for some very recent development).

In this paper, we give a construction of a symplectic resolution for an arbitrary symplectic 4-orbifold without imposing any conditions on the structure of its singular set. In order to explain the nature of our construction, we shall look at the analogous situation in the complex analytic category. To this end, we let X be a 2-dimensional complex orbifold (e.g., the quotient orbifold of a holomorphic finite group action on a complex surface). The singular set Σ (i.e., the subset of points whose isotropy group is nontrivial) may contain points of complex co-dimension 1 in X. However, these points are automatically non-singular in the underlying space of X. More precisely, if we denote by $|X|$ the underlying analytic space of the orbifold X, then the points in Σ which have complex co-dimension 1 are regular points of $|X|$. In fact, $|X|$ is a complex orbifold with at most isolated singularities. With this understood, one may take a resolution of the analytic space $|X|$ to serve as a resolution of the complex orbifold X (such a resolution is unique if we require it to be a minimal resolution, cf. [28]). Our construction of a symplectic resolution amounts to carrying out an analogous consideration in the symplectic category.

The following notations (concerning symplectic 4-orbifolds) will be used throughout. Let (X, ω) be a symplectic 4-orbifold, with singular set denoted by Σ, i.e.,

$$\Sigma = \{p \in X \mid \text{the isotropy group } \Gamma_p \text{ is nontrivial}\}.$$

If we fix an ω-compatible (orbifold) almost complex structure J, and let g_J be the corresponding Riemannian metric, then at each $p \in \Sigma$, the tangent space $T_p X$ can be identified with $\mathbb{C}^2$, with the action of Γ_p on $T_p X$ given by a subgroup of $U(2)$. Consequently, Σ can be decomposed as a disjoint union $\Sigma^0 \sqcup \Sigma^* \sqcup \Sigma^1$, where

- $\Sigma^0 = \{p \in \Sigma | \text{the action of } \Gamma_p \text{ on } T_p X \setminus \{0\} \text{ is free}\}$.
- $\Sigma^* = \{p \in \Sigma | \Gamma_p \text{ fixes a complex line in } (T_p X, J)\}$.
- $\Sigma^1 = \{p \in \Sigma | \text{the action of } \Gamma_p \text{ on } T_p X \setminus \{0\} \text{ is not free but is fixed-point free}\}$.

Both Σ^0, Σ^1 consist of finitely many points, but Σ^* is a 2-dimensional smooth manifold such that $\omega|_{\Sigma^*}$ is an area form. We can compactify each connected component of Σ^* in X by adding points from Σ^1. Let $\{\Sigma_i\}$ be the set of compactified connected components of Σ^*. Then each Σ_i is a symplectic orbifold surface in (X, ω) (possibly immersed), with the points of self-intersection of each Σ_i and the points of intersection of distinct Σ_i, Σ_j contained in Σ^1. We denote by $|X|$ the underlying space of X.

In the symplectic case, it is not difficult to see that $|X|$ is a smooth orbifold with at most isolated singularities (we shall explain this in more detail in Section 2). In particular, $|X|$ is non-singular along the 2-dimensional components Σ^*. However, the symplectic form ω is singular on $|X|$ along Σ^*. With this understood, the main step in the construction of the symplectic resolution of (X, ω) is to show that the orbifold $|X|$ supports a natural symplectic structure; in fact we shall de-singularize ω along Σ^*. (We remark that in the complex analytic situation, the complex structure is automatically non-singular along Σ^*, so this step is not necessary.) For the purpose of applications in finite group actions, we shall give an equivariant version of this construction. The following theorem is the main technical result.

Theorem 1.1. *Let (X, ω) be a symplectic 4-orbifold, and let G be a finite group acting smoothly on the 4-orbifold X, preserving the symplectic structure ω. There are G-invariant neighborhoods U of Σ^1 in $|X|$, which can be taken arbitrarily small, such that for any choice of U, there is a G-invariant symplectic structure ω' on the orbifold $|X|$, such that $\omega' = \omega$ on $|X| \setminus (\Sigma^* \cup U)$ (as symplectic forms) and $\omega' = \omega$ on $\Sigma^* \setminus U$ as area forms. Each Σ_i is a symplectic orbifold surface in $(|X|, \omega')$, which may be singular with respect to the smooth structure of the orbifold $|X|$. The self-intersections and singular points of each Σ_i occur only at points in Σ^1, and there is a G-invariant, ω'-compatible, integrable almost complex structure on U with respect to which each $\Sigma_i \cap U$ is a (genuine) holomorphic curve.*

Remark 1.2. (1) It is interesting to compare the symplectic de-singularization in Theorem 1.1 with the usual holomorphic de-singularization. Let $p \in \Sigma^* \setminus U$ and let m be the order of the isotropy group Γ_p. If we let (δ, ϕ), (ρ, ψ) be the natural polar coordinates on X and $|X|$ in the normal direction at p, then the fact that the symplectic forms ω, ω' agree in the complement of $\Sigma^* \setminus U$ forces the polar coordinates to obey the following equations

$$\rho^2 = \frac{1}{m} \cdot \delta^2, \ \psi = m \cdot \phi,$$

as in the normal direction the symplectic forms are given by $\delta d\delta \wedge d\phi$ and $\rho d\rho \wedge d\psi$ respectively (see Remark 2.3 for more details). On the other hand, in the holomorphic de-singularization, if z is the normal holomorphic coordinate on X, then $w = z^m$ is the normal holomorphic coordinate on $|X|$. In terms of normal polar coordinates, (δ, ϕ), (ρ, ψ) are related by the equations $\rho = \delta^m$, $\psi = m \cdot \phi$.

(2) Even though the symplectic structure ω' on $|X|$ depends a priori on fixing a small neighborhood U of Σ^1, the corresponding (orbifold) canonical line bundle $K_{\omega'}$ is uniquely determined up to isomorphism. We shall denote it by $K_{|X|}$.

(3) In principle, the procedure in Theorem 1.1 can be reversed, i.e., a symplectic structure on $|X|$ also determines a symplectic structure on X. We shall state and prove the following corollary to this effect in the context of finite group actions.

Corollary 1.3. *Let M be a smooth 4-manifold and $B \subset M$ be a smoothly embedded surface (maybe disconnected). Suppose a smooth 4-manifold $\hat{M}$ is constructed by taking a branched covering of M along B, with Γ being the group of deck transformations. Let G be a finite group acting smoothly on $\hat{M}$ extending the group of deck transformations Γ and inducing a smooth G/Γ-action on M. Then the G-action on $\hat{M}$ preserves a symplectic structure if and only if M admits a G/Γ-invariant symplectic structure with respect to which B is symplectic.*

The idea to construct non-standard group actions by taking branched coverings along a non-standard embedding of a co-dimension 2 submanifold goes back to the work of Giffen [22] which produced the first counterexamples to the generalized Smith conjecture. A much more elaborated version of this idea was used by Fintushel, Stern and Sunukjian [21] to construct the first examples of topologically equivalent but smoothly non-equivalent finite cyclic actions on irreducible 4-manifolds. In particular, they constructed exotic $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$ actions on the $K3$ surface. Corollary 1.3 implies that none of these exotic actions can be made symplectic. For example, consider the exotic involutions which were constructed by taking a double branched covering of $\mathbb{CP}^2$ along a smoothly embedded surface B, where B is topologically isotopic but not smoothly isotopic to the sextic. If the exotic involutions were symplectic, then by Corollary 1.3 there is a symplectic structure on $\mathbb{CP}^2$ with respect to which B is symplectic. But a theorem of Shevchishin [44] (see also Siebert and Tian [45]) implies that B is smoothly isotopic to the sextic, which is a contradiction. We remark that Corollary 1.3 is relevant to the following question which is currently open.

Question 1.4. Does there exist a smooth finite group action on a Kähler surface which preserves a symplectic structure but not a complex structure (i.e., a symplectic finite group action not smoothly equivalent to a holomorphic action)?

With Theorem 1.1 at hand, a symplectic resolution of the symplectic 4-orbifold (X, ω) is obtained by simply applying the results in [33], [34], [4], or [39] to the symplectic 4-orbifold $(|X|, \omega')$. More specifically, we shall adopt the construction in [4]. The result is summarized in the following theorem.

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Theorem 1.5. *Let (X, ω) be a symplectic 4-orbifold. There exists a symplectic 4-manifold, denoted by $\tilde{X}$ and called the* resolution *of (X, ω), and a continuous map $\pi : \tilde{X} \to X$, with the following significance.*

(1) *The map $\pi : \tilde{X} \setminus \pi^{-1}(\Sigma) \to X \setminus \Sigma$ is a diffeomorphism, $\pi^{-1}(\Sigma^*)$ is a smoothly embedded surface in $\tilde{X}$ such that $\pi : \pi^{-1}(\Sigma^*) \to \Sigma^*$ is a diffeomorphism, and for each $p \in \Sigma^0 \sqcup \Sigma^1$ which is a singular point of $|X|$, $\pi^{-1}(p)$ is a configuration of embedded two-spheres $\{S_i\}$, where S_i, S_j, $i \neq j$, are either disjoint or intersect transversely at a single point, and no three distinct S_i intersect in one point. If p is a smooth point of $|X|$, $\pi^{-1}(p)$ is a single point.*

(2) *There are neighborhoods U of $\Sigma^0 \sqcup \Sigma^1$, which can be taken arbitrarily small, such that for any choice of U, there is a symplectic structure $\tilde{\omega}$ on $\tilde{X}$ such that*

$$\pi : (\tilde{X} \setminus \pi^{-1}(U \cup \Sigma^*), \tilde{\omega}) \to (X \setminus (U \cup \Sigma^*), \omega)$$

is a symplectomorphism, $\pi^{-1}(\Sigma^*)$ is symplectic with $\tilde{\omega} = \pi^*\omega$ on $\pi^{-1}(\Sigma^* \setminus U)$ as area forms, and for each $p \in \Sigma^0 \sqcup \Sigma^1$ which is a singular point in $|X|$, the components of $\pi^{-1}(p)$ are symplectic and intersect positively. Furthermore, there exists a neighborhood U' of Σ^1, where $U' \subset U$, with an $\tilde{\omega}$-compatible, integrable almost complex structure on U' such that $U' \cap \pi^{-1}(\Sigma)$ is given by (genuine) holomorphic curves.

(3) *Suppose a finite group G acts on X smoothly, preserving the symplectic structure ω. Then there is a natural smooth G-action on $\tilde{X}$ such that $\pi : \tilde{X} \to X$ is G-equivariant, and that $\tilde{\omega}$ can be made G-invariant if the neighborhood U is chosen G-invariant. Furthermore, let V, W be the resolutions of the symplectic 4-orbifolds $(X/G, \omega)$ and $(\tilde{X}/G, \tilde{\omega})$ respectively. Then either $W = V$, or W can be reduced to V by successively blowing down symplectic (-1)-spheres.*

Remark 1.6. (1) The diffeomorphism type of $\tilde{X}$ is uniquely determined by the smooth orbifold X; in fact, it is the smooth 4-manifold obtained by replacing a neighborhood of each singular point of the smooth orbifold $|X|$ by the minimal resolution of the neighborhood. In particular, the symplectic Kodaira dimension of $\tilde{X}$ is well-defined and depends only on the smooth structure of X (cf. [30]).

(2) Even though the symplectic structure $\tilde{\omega}$ depends on a choice of a neighborhood of $\Sigma^0 \sqcup \Sigma^1$, the canonical line bundle $K_{\tilde{\omega}}$ is uniquely determined up to isomorphism. We shall denote it by $K_{\tilde{X}}$. This said, the Gromov-Taubes invariant of $(\tilde{X}, \tilde{\omega})$ is independent of the choice of $\tilde{\omega}$ (cf. [46]).

1.2. Finite group actions on 4-manifolds

Smooth finite group actions on 4-manifolds remain poorly understood in general, despite the tremendous advances in smooth 4-manifold theory following the revolutionary work of Donaldson in the early 1980s (cf. [17]). A key ingredient in understanding a finite group action is the so-called fixed-point set structure. To be more precise, consider M which is an oriented smooth 4-manifold equipped with an orientation preserving smooth

action of a finite group G. For any $g \in G$, the fixed-point set $Fix(g)$ of g consists of finitely many isolated points and 2-dimensional components. Crucial to the understanding of the G-action is information about $Fix(g)$ and the induced representation of g on the tangent space T_pM at a fixed point $p \in Fix(g)$, for any given $g \in G$ and $p \in Fix(g)$. This information is constrained by the various G-index theorems (e.g., Lefschetz fixed point theorem, G-signature theorem, etc.), however, for a general smooth 4-manifold, the fixed-point set structure of a smooth finite group action remains poorly understood.

Fixed-point set structure is intimately related to the various rigidity properties of the group action. In this sense, locally linear topological actions are the most flexible ones. Indeed, Edmonds and Ewing [20] showed that for any pseudo-free, locally linear $\mathbb{Z}_p$-action of prime order on a simply connected 4-manifold, the Lefschetz fixed point theorem and the G-signature theorem are almost the only constraints on the fixed-point set structure. (Pseudo-free, in this case, means that the fixed-point set of the action contains no 2-dimensional components.) Therefore, any rigidity phenomenon of smooth actions (in comparison with locally linear actions) reflects existence of additional constraints on the fixed-point set structure of the group actions, and vice versa, any additional constraints on the fixed-point set structure may lead to certain rigidity properties of the group actions.

Primary examples of smooth actions are provided by holomorphic actions on complex Kähler surfaces, which serve as a model and motivate the study of smooth actions. A well-known rigidity phenomenon of holomorphic actions is the so-called "homological rigidity" of automorphisms of a $K3$ surface, i.e., a holomorphic automorphism must be trivial if the induced action on the $K3$ lattice is trivial (cf. [2]). Knowing that this fails to be true for locally linear actions on $K3$ surfaces (cf. [19]), Edmonds asked whether there exist nontrivial smooth actions on $K3$ surfaces which are homologically trivial (see Kirby's Problem List [27], Problem 4.124 (B)). Another example of rigidity properties of holomorphic actions concerns the order of automorphism group of a minimal algebraic surface of general type (cf. [48]).

A natural class of smooth actions which generalize the holomorphic actions on Kähler surfaces is given by the symplectic finite group actions, and a central problem in this regard is to what extent the rigidity properties of holomorphic actions can be extended to this class of smooth actions. When M is symplectic and the G-action preserves a symplectic structure, the pseudo-holomorphic curve theory and Taubes' seminal work on symplectic 4-manifolds [46] can be adapted to the equivariant setting which gave some powerful techniques to study the group action (see the survey articles [6, 9], and the more recent papers [11, 16]). In particular, these techniques revealed additional constraints on the fixed-point set structure of the group action, and allowed one to extend (partially) the aforementioned rigidity properties of holomorphic actions to symplectic finite group actions. For example, in joint work with Kwasik [13], the author showed that there are no symplectic finite group actions on the standard smooth $K3$ surface which act trivially on homology (it is an interesting open question as whether this continues to be true if the smooth structure of the $K3$ surface is exotic). For another example, the author

investigated in [10] the problem of bounding the order of a symplectic finite group action, and partially extended the work of Xiao in [48] to the symplectic category.

With the preceding understood, we introduce in this paper some new constructions to the study of symplectic finite group actions. The idea is to associate to each symplectic finite group action a symplectic 4-manifold which is canonically determined by the group action, i.e., the resolution of the quotient orbifold. (We shall call it the *resolution* of the group action or the *resolution* of the corresponding G-manifold.)

To be more concrete, let M be a symplectic 4-manifold equipped with a finite symplectic G-action. Denote by $\pi : M_G \to M/G$ the resolution of the quotient orbifold M/G from Theorem 1.5 and let $D := \pi^{-1}(\Sigma) \subset M_G$ be the pre-image of the singular set of M/G, which is in general a configuration of symplectic surfaces in M_G. We shall investigate the group action on M through M_G and the embedding of D in M_G. Note that topologically, M may be regarded as a branched covering of M_G along D. To be more precise, let $\hat{M}$ be the 4-manifold which is the regular G-branched covering of M_G along D, then $\hat{M}$ and M are related by a successive blowing-down of (-1)-spheres which is equivariant with respect to the G-actions on $\hat{M}$ and M. Particularly, this gives an unified point of view for the construction of symplectic finite group actions, which has been lagging behind for progress. Finally, we remark that in some sense, M_G can be regarded as a certain "manifold approximation" of the singular quotient orbifold M/G, and the subset $D \subset M_G$, which contains vital information about the fixed-point set structure of the G-action, is a substitute for the singular set of M/G.

The idea of studying a finite automorphism group of a complex surface through the resolution of the quotient space had appeared in the algebraic geometry literature, e.g., in Nikulin [41], Xiao [48, 50], and the unpublished work of Xiao [51]. So our new approach to the study of symplectic finite group actions on 4-manifolds is an attempt to generalize this idea to the symplectic category.

One basic invariant associated to a symplectic 4-manifold is its symplectic Kodaira dimension (cf. [30]). We may wonder (1) how the rigidity of a group action is seen through the symplectic Kodaira dimension of M_G, and (2) how the symplectic Kodaira dimensions of M_G and M are related.

For the second question, it is known that for a finite holomorphic G-action on a complex surface M, the Kodaira dimensions of M_G, M obey the following inequality:

$$\kappa(M_G) \leq \kappa(M).$$

(This follows easily from the definition of Kodaira dimension.) As the symplectic Kodaira dimension coincides with the (complex) Kodaira dimension in the Kähler case (cf. [30]), it is natural to speculate that the corresponding inequality might be true in the symplectic category.

Conjecture 1.7. *Let κ^s stand for the symplectic Kodaira dimension. Then*

$$\kappa^s(M_G) \leq \kappa^s(M).$$

Conjecture 1.7 offers, in some very rough way, a measurement as to how much symplectic finite group actions resemble holomorphic actions.

Remark 1.8. (1) Conjecture 1.7 is known to be true in some special cases. For example, it is true for many symplectic G-actions on $\mathbb{CP}^2$ or Hirzebruch surfaces because the corresponding group actions are smoothly equivalent to a holomorphic action (cf. [5, 7, 9, 11]). On the other hand, if M is a homotopy $K3$ surface with trivial canonical line bundle and the symplectic G-action satisfies $b_2^+(M/G) = 3$, then M_G is also a homotopy $K3$ surface with trivial canonical line bundle, cf. [15]. (In this case, M/G has only isolated singularities so the construction of M_G is straightforward.) In this last example, $\kappa^s(M_G) = \kappa^s(M) = 0$ so Conjecture 1.7 is true.

(2) Theorem 1.5(3) allows an inductive approach toward Conjecture 1.7. More precisely, suppose H is a normal subgroup of G, with $K = G/H$ being the quotient group. Then by Theorem 1.5(3), there is an induced symplectic K-action on the resolution M_H, and furthermore, $\kappa^s((M_H)_K) = \kappa^s(M_G)$ (cf. [30]). Hence, if one can show $\kappa^s(M_H) \leq \kappa^s(M)$ and $\kappa^s((M_H)_K) \leq \kappa^s(M_H)$, then one has $\kappa^s(M_G) \leq \kappa^s(M)$.

Concerning the first question, the most interesting and important example of a connection between rigidity of a group action and the (symplectic) Kodaira dimension of the resolution is given in the work of Xiao [48] on the order of the automorphism group of a minimal algebraic surface of general type. In that paper, Xiao showed that if M is a minimal algebraic surface of general type and G is its automorphism group, then the number $|G|/c_1(M)^2$ has an interesting correlation with the topology, particularly the Kodaira dimension of M_G:

   (i) If M_G is of general type, then $|G|/c_1(M)^2 \leq 1$.
   (ii) If $\kappa(M_G) = 1$, then $|G|/c_1(M)^2 \leq 3$.
   (iii) If M_G is a ruled surface over a curve of genus ≥ 2, then $|G|/c_1(M)^2 \leq 10.5$.
   (iv) If $\kappa(M_G) = 0$, then $|G|/c_1(M)^2 \leq 288$.
   (v) If M_G is a rational surface or a ruled surface over an elliptic curve, $|G|/c_1(M)^2 \leq c$ for some universal constant $c > 0$ (the computation of c is given separately in [49]).

In this paper, we give some further evidence for Conjecture 1.7. In particular, we have a pretty good understanding when $\kappa^s(M) = -\infty$ or 0. (Recall that a symplectic 4-manifold is rational or ruled if and only if $\kappa^s = -\infty$.)

Theorem 1.9. *Let M be a symplectic 4-manifold with a finite symplectic G-action.*

   (1) *If M is a rational surface, so is M_G.*
   (2) *If $\kappa^s(M) = 0$, then either $\kappa^s(M_G) = 0$, or M_G is a rational surface, or a ruled surface over T^2.*

We remark that, for the case of $\kappa^s(M) = 0$, the verification of Conjecture 1.7 (in Theorem 1.9) does not rely on any information a priori about the symplectic G-action on M, while for the case where M is rational, it requires some nontrivial results concerning the equivariant symplectic cone of rational G-surfaces in [16].

The case where $\kappa^s(M) = 1$ is more intricate. We shall only consider in this paper the symplectic G-actions such that $b_2^+(M/G) > 1$. Under this assumption, the equivariant version of the Seiberg-Witten-Taubes theory yields very strong constraints for the fixed-point set structure of the action (cf. [13, 10]). Note that $b_2^+(M_G) = b_2^+(M/G) > 1$, so one always has $\kappa^s(M_G) \geq 0$ under this assumption. The results are summarized in the following theorem.

Theorem 1.10. *Let M be a minimal symplectic 4-manifold with $\kappa^s(M) = 1$, equipped with a finite symplectic G-action, where $G = \mathbb{Z}_p$ is of prime order p. Furthermore, assume $b_2^+(M/G) > 1$.*

(1) *M_G has torsion canonical class if and only if the 2-dimensional fixed components of G consist of tori $\{T_i\}$ with self-intersection zero and the isolated fixed points of G are all of type $(1, -1)$, and $c_1(K_M) = (p-1)\sum_i T_i$.*

(2) *In general, $c_1(K_{M_G})^2 = -\frac{2(p-1)^2}{p} \cdot s + \sum_m K_m^2$, where s is the number of (-2)-spheres fixed by G, and K_m denotes the canonical class of the minimal resolution of the singular point of M/G corresponding to the isolated fixed point m of G.*

(3) *If the G-action is homologically trivial, then $c_1(K_{M_G})^2 = -\sum_m \chi_m$, where χ_m is the number of exceptional divisors in the minimal resolution of the singular point of M/G corresponding to the isolated fixed point m of G.*

(4) *If $c_1(K_{M_G})^2 = 0$, then M_G must be minimal.*

Remark 1.11. (1) When M_G has torsion canonical class, the topology of M_G is severely constrained (cf. [30, 31, 3]). In particular, with $b_2^+(M_G) > 1$, M_G is either a $\mathbb{Q}$-homology T^2-bundle over T^2, or a homology $K3$ surface. In turn, this gives severe constraints on the topology of M and the order p of the group G. To see this, let F be the number of isolated fixed points of G (which in this case equals the Lefschetz number), then

$$b_2^-(M/G) + (p-1)F = b_2^-(M_G).$$

With this understood, observe that $b_2^-(M_G) = 2, 3$ or 19.

(2) The expression of $c_1(K_{M_G})^2$ given in part (2) implies that $c_1(K_{M_G})^2 \leq 0$. This is consistent with Conjecture 1.7, because if $c_1(K_{M_G})^2 > 0$ were true, then one would have $\kappa^s(M_G) = 2$ which violates Conjecture 1.7. In fact, when $c_1(K_{M_G})^2 = 0$, Conjecture 1.7 is true if and only if M_G is minimal, which is confirmed in part (4). When $c_1(K_{M_G})^2 < 0$, M_G must not be minimal by Taubes [46]. In fact, M_G should contain at least $-c_1(K_{M_G})^2$ many (-1)-spheres, and Conjecture 1.7 is true if and only if M_G contains exactly $-c_1(K_{M_G})^2$ many (-1)-spheres.

(3) Observe that the expression of $c_1(K_{M_G})^2$ in part (2) consists of rational numbers, so the identity gives rise to some integrability conditions on these rational numbers. However, these integrability conditions do not yield any new constraints on the fixed-point set structure (see Remark 4.2 for more details).

In general, M_G contains a large number of (-1)-spheres, which must all intersect the subset D if M is assumed to be minimal. It is natural to attempt to extract new

constraints for the fixed-point set structure from the interaction of these (-1)-spheres with D. We look into this matter by examining the homologically trivial symplectic $\mathbb{Z}_3$-actions on a homotopy $K3$ surface. It turns out that in each case, we can see the (-1)-spheres in M_G explicitly, and blowing down these (-1)-spheres we obtain the minimal model of M_G. For all these possible group actions, we have $\kappa^s(M_G) \leq 1$, so Conjecture 1.7 is true. On the other hand, our new approach by examining M_G does not yield any new constraints. See Example 4.3 for more details.

The organization of this paper is as follows. Section 2 is devoted to the proof of the main technical result, Theorem 1.1. After a preliminary lemma on smooth finite group actions on a general smooth orbifold, the section proceeds to a detailed discussion on how to de-singularize the symplectic structure along the 2-dimensional singular strata of the symplectic 4-orbifold in question, which forms the bulk of the technical work of this paper. Section 2 ends with the proof of Corollary 1.3. The construction of symplectic resolution (Proof of Theorem 1.5) is given in Section 3, which also contains two propositions (Propositions 3.2 and 3.3): one concerning the canonical class of the symplectic resolution, and the other concerning equivariant blowing down. The final section, Section 4, is mainly devoted to the proofs of Theorems 1.9 and 1.10 (on the symplectic Kodaira dimension of M_G), however, we also include at the end of the section a discussion of homologically trivial symplectic $\mathbb{Z}_3$-actions on a homotopy $K3$ surface from the point of view of symplectic resolution.

2. Symplectic orbifold structure on the underlying space

This section is devoted to the proof of Theorem 1.1. First of all, we shall describe a natural smooth orbifold structure on $|X|$, which has at most isolated singularities. With respect to this smooth orbifold structure, each smooth point in X, i.e., each point in $X \setminus \Sigma$, is a smooth point in $|X|$, and each point of Σ^0 is a singular point of $|X|$ with the same orbifold local chart. As for each point $p \in \Sigma^*$, there is a smooth local chart of complex coordinates (w_1, w_2) of p in $|X|$ with the following property: there is a smooth local orbifold chart of p in X given by complex coordinates (z_1, z_2), with a smooth $\mathbb{Z}_m$-action generated by $(z_1, z_2) \mapsto (z_1, \exp(2\pi i/m)z_2)$, such that $w_1 = z_1$ and $w_2 = z_2^m$. In particular, each point $p \in \Sigma^*$ is a smooth point in $|X|$ and Σ^* is smoothly embedded in $|X|$. Finally, for each point $p \in \Sigma^1$, a local orbifold chart of p in $|X|$ is obtained as follows. We take an orbifold chart $(\mathbb{R}^4, \Gamma_p)$ of p in X, where Γ_p acts on $\mathbb{R}^4$ linearly. Let $\mathbb{S}^3$ be the unit sphere in $\mathbb{R}^4$. Then $\mathbb{S}^3/\Gamma_p$ is a 3-orbifold whose singular set is a link denoted by L. The underlying space $|\mathbb{S}^3/\Gamma_p|$ is a 3-manifold, and a neighborhood of p in $|X|$ is given by a cone over $|\mathbb{S}^3/\Gamma_p|$. With this understood, let N_p be the normal subgroup of Γ_p generated by the isotropy subgroups of the components of the singular set L of the 3-orbifold $\mathbb{S}^3/\Gamma_p$, and let $\Gamma'_p := \Gamma_p/N_p$. Then the fundamental group of the 3-manifold $|\mathbb{S}^3/\Gamma_p|$ is isomorphic to Γ'_p, which is obviously a finite group. The universal cover of $|\mathbb{S}^3/\Gamma_p|$ is $\mathbb{S}^3$, and $|\mathbb{S}^3/\Gamma_p|$ is diffeomorphic to the quotient of $\mathbb{S}^3$ by a linear action of Γ'_p.

When Γ_p' is nontrivial, the point $p \in |X|$ has a natural orbifold chart given by $(\mathbb{R}^4, \Gamma_p')$. If Γ_p' is trivial, then $p \in |X|$ is a smooth point.

2.1. Finite group actions on symplectic orbifolds

We start off with a preliminary lemma concerning symplectic finite group actions on a symplectic orbifold. We include a detailed proof here for completeness; these types of arguments will also be used in the proof of Lemma 2.2 and we shall be brief there. For a general reference on smooth orbifolds we refer the reader to [8].

Lemma 2.1. *Let X be a smooth n-orbifold which is equipped with a smooth finite group action by G. Then the quotient space X/G is naturally a smooth n-orbifold. Furthermore, if ω is a symplectic structure on X which is preserved under the G-action, then ω descends to a symplectic structure on the orbifold X/G, making it naturally a symplectic orbifold.*

Proof. For any $p \in X$, let G_p be the isotropy subgroup at p, i.e.,

$$G_p := \{g \in G \mid g \cdot p = p\},$$

and denote by $\bar{p}$ the image of p in the quotient space. Furthermore, let $(\mathbb{R}^n, \Gamma_p)$ be a local orbifold chart of X at p. With this understood, if G_p is trivial, then $G \cdot p$ is a free orbit, so that a neighborhood of p is mapped homeomorphically onto a neighborhood of $\bar{p}$ in the quotient space. Hence in this case, there is a natural smooth orbifold chart of the quotient space at $\bar{p}$, i.e., $(\mathbb{R}^n, \Gamma_p)$, which is independent of the choice of p up to equivalence.

Suppose G_p is nontrivial. Then there is a Γ_p-invariant open subset U_p of $\mathbb{R}^n$ containing 0, such that U_p/Γ_p is a G_p-invariant neighborhood of p. Furthermore, for any $g \in G_p$, there is a lifting of g to a diffeomorphism $\tilde{g} : U_p \to U_p$, and any two such liftings $\tilde{g}, \tilde{g}'$ differ by an element of Γ_p, i.e., $\tilde{g}' = h \circ \tilde{g}$ for some $h \in \Gamma_p$. With this understood, for each $g \in G_p$ we fix a lifting $\tilde{g} : U_p \to U_p$. Note that for any $g_1, g_2 \in G_p$, the lifting of $g_1 g_2$ differs from $\tilde{g}_1 \circ \tilde{g}_2$ by an element of Γ_p. Now we let $\tilde{\Gamma}_p$ be the group of self-diffeomorphisms of U_p generated by h and $\tilde{g}$ for all $h \in \Gamma_p$ and $g \in G_p$. Then the above property implies that the map from $\tilde{\Gamma}_p$ to G_p sending each $h \in \Gamma_p$ to $1 \in G_p$ and each $\tilde{g}$ to g is a surjective homomorphism, whose kernel is Γ_p. In particular, $\tilde{\Gamma}_p$ is a finite group, with a natural short exact sequence

$$1 \to \Gamma_p \to \tilde{\Gamma}_p \to G_p \to 1.$$

Observing that $U_p/\tilde{\Gamma}_p = (U_p/\Gamma_p)/G_p$, we can define $(U_p, \tilde{\Gamma}_p)$ to be a smooth orbifold chart of the quotient X/G at $\bar{p}$. It is easy to see that it is independent of the choice of p up to equivalence. This puts a natural smooth orbifold structure on X/G.

Let η be a differential form on X. For any $g \in G$, the pullback $g^* \eta$ is defined as follows. Let $p, q \in X$ such that $g \cdot p = q$. Let (U_p, Γ_p), (U_q, Γ_q) be local orbifold charts at p, q respectively, such that a lifting of g, i.e., a diffeomorphism $\tilde{g} : U_p \to U_q$ covering $g : U_p/\Gamma_p \to U_q/\Gamma_q$, exists. Furthermore, let η_q be the local Γ_q-invariant differential form on U_q representing η. Then the pullback $\tilde{g}^* \eta_q$ is a Γ_p-invariant differential form on U_p, which is independent of the choice of the local lifting $\tilde{g}$ of g. This is because for any lifting $\tilde{g}' : U_p \to U_q$, there is a $h \in \Gamma_q$ such that $\tilde{g}' = h \circ \tilde{g}$, and in particular, for any $k \in \Gamma_p$,

$\tilde{g} \circ k = k' \circ \tilde{g}$ for some $k' \in \Gamma_q$. With this understood, the pullback $g^*\eta$ is the differential form on X determined by the local Γ_p-invariant forms $\tilde{g}^*\eta_q$. The above description of pullback forms immediately implies that if ω is a symplectic form on X preserved by the G-action, then for any $p \in X$ and $g \in G_p$, $\tilde{g}^*\omega_p = \omega_p$, where ω_p is the Γ_p-invariant symplectic form on U_p representing ω. It follows easily that each ω_p is also $\tilde{\Gamma}_p$-invariant, and hence ω descends to a symplectic form on the quotient orbifold X/G. This finishes the proof of the lemma.

$\square$

We remark that in the case of 4-orbifolds, if G acts on X smoothly, then the induced G-action on $|X|$ is a smooth action of orbifolds, and furthermore, $|X/G| = ||X|/G|$ as smooth orbifolds.

2.2. De-singularizing symplectic forms along co-dimension 2 singular strata

The key issue in the proof of Theorem 1.1 is to de-singularize the symplectic structure ω of X along the subset Σ^* of the singular set, as Σ^* lies in the smooth locus of $|X|$. Recall that Σ_i denotes the closure of a connected component of Σ^* in Σ. We shall consider first the special case where Σ_i lies entirely in Σ^*, and then explain how to extend the argument to the general case. To this end, let Σ_i be such a component. To ease the notation, we denote Σ_i by S, which is a compact closed Riemann surface. Let H be the subgroup of G which leaves S invariant.

Lemma 2.2. *The symplectic structure ω can be de-singularized along S, i.e., there exists a symplectic structure ω' of $|X|$ defined in a neighborhood of S, such that (1) ω' is H-invariant, (2) $\omega' = \omega$ in the complement of S, and (3) $\omega'|_S = \omega|_S$ as area forms.*

Proof. We begin by noting that for each $p \in \Sigma_i$, the isotropy group Γ_p is cyclic of order $m > 1$ where m is independent of p. Let $\nu \to S$ be the normal bundle of S in the orbifold X, which is defined to be the quotient bundle of $TX|_S$ by the sub-bundle TS, and as such, it comes with a structure of a $\mathbb{Z}_m$-complex line bundle over S once we fix an ω-compatible (orbifold) almost complex structure J. The $\mathbb{Z}_m$-action on the fibers of ν is given by the complex multiplication by an m-th root of unity, and the $\mathbb{Z}_m$-action on the base S is trivial. Furthermore, the corresponding Riemannian metric g_J gives rise to a metric on ν. Let $D(\nu, r)$ be the disc-bundle of ν of radius r. Then via the exponential map associated to g_J, $(D(\nu, r), \mathbb{Z}_m)$ gives a global orbifold chart of X near S for small $r > 0$; in particular, a neighborhood of S in the underlying space $|X|$ is given by $D(\nu, r)/\mathbb{Z}_m$ homeomorphically.

Claim: *There is a finite group $\tilde{H}$, with a short exact sequence*

$$1 \to \mathbb{Z}_m \to \tilde{H} \to H \to 1,$$

such that the H-action near S lifts to a $\tilde{H}$-action on the global orbifold chart near S. Furthermore, ω is $\tilde{H}$-invariant.

Proof of Claim: By averaging the metric g_J, we may assume it is G-invariant. With this assumption, the neighborhood $D(\nu, r)/\mathbb{Z}_m$ of S is H-invariant; in particular, the boundary $\partial(D(\nu, r)/\mathbb{Z}_m) = \partial D(\nu, r)/\mathbb{Z}_m$ is invariant under the H-action. Now note that $\partial D(\nu, r) \to \partial D(\nu, r)/\mathbb{Z}_m$ is an m-fold regular cyclic covering map. It follows easily that for each $h \in H$, $h : \partial D(\nu, r)/\mathbb{Z}_m \to \partial D(\nu, r)/\mathbb{Z}_m$ can be lifted to a diffeomorphism $\tilde{h} : \partial D(\nu, r) \to \partial D(\nu, r)$. Then a similar argument as in the proof of Lemma 2.1 shows that the elements $\xi \in \mathbb{Z}_m$ and $\tilde{h}$, $h \in H$, generate a finite group $\tilde{H}$, with a natural short exact sequence $1 \to \mathbb{Z}_m \to \tilde{H} \to H \to 1$. Furthermore, there is a smooth action of $\tilde{H}$ on $D(\nu, r)$ inducing the H-action on the neighborhood $D(\nu, r)/\mathbb{Z}_m$ of S, and ω is preserved by the $\tilde{H}$-action. Hence the claim.

We continue with the proof of Lemma 2.2. Since g_J is G-invariant, so is J, so that the normal bundle ν is a complex $\tilde{H}$-line bundle. Let $\pi : Y \to S$ be the principal $\mathbb{S}^1$-bundle associated to the complex line bundle ν, which comes with a natural smooth bundle action by $\tilde{H}$. Note that the $\mathbb{Z}_m$-action on Y induced by $\mathbb{Z}_m \to \tilde{H}$ is simply given by the inclusion of $\mathbb{Z}_m$ into $\mathbb{S}^1$. Let $Y' := Y/\mathbb{Z}_m$ be the quotient, and let $\pi' : Y' \to S$ be the corresponding principal $\mathbb{S}^1$-bundle. Then there is a natural smooth H-action on the principal $\mathbb{S}^1$-bundle Y'. With this understood, note that $D(\nu, r)/\mathbb{Z}_m$ is H-equivariantly homeomorphic to a disc-bundle $D(\nu')$ of the complex line bundle $\nu' \to S$ associated to the principal $\mathbb{S}^1$-bundle Y'. Furthermore, with respect to the (orbifold) smooth structure of $|X|$, a neighborhood of S is given by $D(\nu')$ diffeomorphically.

We shall regard $D(\nu')$ as a smooth chart on $|X|$ near S, and as such, ω may be regarded as a symplectic structure in the complement of the zero section of $D(\nu')$. This said, we shall describe next how to construct a symplectic structure ω' on $D(\nu')$, which is H-invariant and agrees with ω in the complement of the zero section of $D(\nu')$.

To this end, note first that ω defines an $\tilde{H}$-invariant symplectic structure on $D(\nu, r)$ via pull-back. We shall begin by describing a model for ω on $D(\nu, r)$. Recall that $\pi : Y \to S$ is the $\tilde{H}$-equivariant principal $\mathbb{S}^1$-bundle associated to ν. We pick an $\tilde{H}$-invariant connection 1-form α on Y and let κ be the 2-form on S such that $\pi^*\kappa = -d\alpha$. Finally, let η be the area form on S which is the restriction of ω on S. Then for sufficiently small $r > 0$, the following 2-form is symplectic on $Y \times (-r, r)$:

$$\omega_0 := \pi^*(\eta + t\kappa) + \alpha \wedge dt,$$

where t is the coordinate on $(-r, r)$. The symplectic form ω_0 is clearly $\tilde{H}$-invariant, where the $\tilde{H}$-action on $Y \times (-r, r)$ is trivial in the last factor.

To relate the symplectic structures on $Y \times (-r, r)$ and $D(\nu, r)$, we employ a technique called symplectic cutting due to Lerman (cf. [29]). To this end, note that the natural $\mathbb{S}^1$-action on $Y \times (-r, r)$ is Hamiltonian with respect to the symplectic structure ω_0, with a Hamiltonian function given by $h : (y, t) \in Y \times (-r, r) \mapsto t$. In particular, each $Y \times \{t\}$ is a level set of h. To describe the procedure of symplectic cutting, consider the symplectic manifold $(Y \times (-r, r)) \times \mathbb{C}$ with the symplectic structure $\omega_0 \oplus \frac{i}{2} dz \wedge d\bar{z}$ and the Hamiltonian

$\mathbb{S}^1$-action given by

$$\lambda \cdot ((y,t),z) = ((\lambda \cdot y, t), \lambda \cdot z), \ \ \forall \lambda \in \mathbb{S}^1.$$

A Hamiltonian function of the $\mathbb{S}^1$-action is given by $\mathcal{H}((y,t),z) = h(y,t) - \frac{1}{2}|z|^2$. One can easily verify that 0 is a regular value of $\mathcal{H}$. The level set $\mathcal{H}^{-1}(0)$ is easily seen to be the subset

$$\bigsqcup_{0 \le t < r} Y \times \{t\} \times \{|z|^2 = 2t\}.$$

With this understood, the symplectic reduction $\mathcal{H}^{-1}(0)/\mathbb{S}^1$ can be easily identified with $D(\nu, r)$, under which $Y \times (0,r) \subset \mathcal{H}^{-1}(0)/\mathbb{S}^1$ is identified with the complement of the zero section in $D(\nu, r)$. Furthermore, the symplectic structure on $\mathcal{H}^{-1}(0)/\mathbb{S}^1$ agrees with the symplectic structure ω_0 on $Y \times (0,r)$. For simplicity, we will continue to denote the symplectic structure on $D(\nu, r) = \mathcal{H}^{-1}(0)/\mathbb{S}^1$ by ω_0. Clearly, this symplectic form on $D(\nu, r)$ is $\tilde{H}$-invariant.

Note that on the zero section $S \subset D(\nu, r)$, $\omega_0 = \eta = \omega$. Hence by the equivariant version of the Weinstein neighborhood theorem, ω is $\tilde{H}$-equivariantly symplectomorphic to ω_0 after taking $r > 0$ sufficiently small. In other words, ω_0 can serve as a model for ω in a neighborhood of S in the orbifold X.

With the preceding understood, denote by $D^*(\nu, r), D^*(\nu')$ the complement of the zero section in $D(\nu, r), D(\nu')$ respectively. The $\mathbb{Z}_m$-action on $D^*(\nu, r)$ is free, so that under the H-equivariant homeomorphism between $D(\nu, r)/\mathbb{Z}_m$ and $D(\nu')$, $D^*(\nu, r)/\mathbb{Z}_m$ is sent to $D^*(\nu')$ H-equivariantly by a diffeomorphism. This said, the symplectic form ω_0 on $D^*(\nu, r)$, which is $\tilde{H}$-invariant, descends to an H-invariant symplectic form on $D^*(\nu, r)/\mathbb{Z}_m$. Denote the corresponding symplectic form on $D^*(\nu')$ by ω_0'.

In order to extend ω_0' to the disc-bundle $D(\nu')$, we shall again employ the technique of symplectic cutting. To this end, we identify $D^*(\nu')$ with $Y' \times (0,r)$, where $Y' = Y/\mathbb{Z}_m$ is the principal $\mathbb{S}^1$-bundle associated to ν', and consider ω_0' as a symplectic form on $Y' \times (0,r)$. To identify ω_0', note that the connection 1-form α on Y descends to a 1-form on Y'. Furthermore, there is a connection 1-form α' on Y' such that $\alpha = \frac{1}{m}\alpha'$. Let κ' be the 2-form on S such that $(\pi')^*\kappa' = -d\alpha'$, where $\pi' : Y' \to S$. Then $\kappa = \frac{1}{m}\kappa'$. With this understood, the symplectic form ω_0', as the descendent of ω_0 under the free $\mathbb{Z}_m$-action, can be written on $Y' \times (0,r)$ as

$$\omega_0' = (\pi')^*(\eta + \frac{t}{m}\kappa') + \alpha' \wedge d(\frac{t}{m}),$$

where t is the coordinate on $(0,r)$. This said, note that $\omega_0' = (\pi')^*(\eta + \frac{t}{m}\kappa') + \alpha' \wedge d(\frac{t}{m})$ is in fact defined on the entire $Y' \times (-r, r)$ and is a symplectic form on it as long as $r > 0$ is sufficiently small. Furthermore, note that the $\mathbb{S}^1$-action on $Y' \times (-r, r)$ is Hamiltonian with respect to ω_0', with a Hamiltonian function given by $h' : (y', t) \mapsto \frac{t}{m}$. A symplectic cutting procedure as we described earlier, done H-equivariantly at the regular value 0 of h', gives rise to an H-invariant symplectic form ω' on $D(\nu')$, extending the symplectic form ω_0' on $D^*(\nu')$. Note that $\omega' = \eta = \omega$ on the zero section as area forms. Now if we

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identify a neighborhood of S in $|X|$ with $D(\nu')$, we obtain the symplectic structure ω' with desired properties. This finishes the proof.

$\square$

Remark 2.3. It is somewhat a disturbing statement that the symplectic forms ω, ω' agree on an open dense subset. We shall examine this more closely in local coordinates. Let (δ, ϕ), (ρ, ψ) be the polar coordinates in the fiber direction on $D(\nu, r)$ and $D(\nu')$, such that the restrictions of ω_0, ω_0' on the fibers are given by $\delta d\delta \wedge d\phi$ and $\rho d\rho \wedge d\psi$ respectively. Then if we follow through the symplectic cutting procedures in the proof of Lemma 2.2, it is easy to see that the polar coordinates are related by the equations

$$\rho^2 = \frac{1}{m} \cdot \delta^2, \ \psi = m \cdot \phi,$$

where m is the order of the isotropy groups along S. Under the above relations, the forms $\delta d\delta \wedge d\phi$ and $\rho d\rho \wedge d\psi$ agree in the complement of the zero sections.

Proof of Theorem 1.1

If Σ^1 is empty, then Theorem 1.1 follows immediately from Lemma 2.2 with $U = \emptyset$. Suppose $\Sigma^1 \neq \emptyset$, and let Σ_i be a component which contains points in Σ^1. We shall extend the argument of Lemma 2.2 to first de-singularize the symplectic structure ω along $\Sigma_i \setminus U$, where U is a certain G-invariant neighborhood of Σ^1 which can be chosen arbitrarily small. We summarize the result in the following lemma.

Lemma 2.4. *There are G-invariant neighborhoods U of Σ^1 in $|X|$, which can be taken arbitrarily small, such that for any choice of U, there is a G-invariant symplectic structure ω' on $|X| \setminus U$, with $\omega' = \omega$ in the complement of $\Sigma^* \setminus U$ (as symplectic forms) and $\omega' = \omega$ on $\Sigma^* \setminus U$ as area forms.*

Proof. Note that $\Sigma_i \setminus U$ is a compact Riemann surface with boundary. Crucial to extending the argument of Lemma 2.2 to the general situation is to describe an appropriate model for the symplectic structure ω near the boundary components of $\Sigma_i \setminus U$. We shall first explain how to choose the neighborhood U.

We begin by fixing a standard model for the symplectic structure ω near each point $p \in \Sigma^1$. Let G_p be the subgroup of G fixing p, $\tilde{\Gamma}_p$ be the isotropy group of the orbifold X/G at p, and Γ_p be the isotropy group of X at p. Then these three groups fit into a short exact sequence (see the proof of Lemma 2.1)

$$1 \to \Gamma_p \to \tilde{\Gamma}_p \to G_p \to 1.$$

By the equivariant Darboux theorem, there is an action of $\tilde{\Gamma}_p$ on $\mathbb{C}^2$ as a subgroup of $U(2)$, preserving the standard symplectic structure ω_0 on $\mathbb{C}^2$, such that for some $\epsilon_0 > 0$ which is independent of p, the induced action of Γ_p on $(B^4(\epsilon_0), \omega_0)$ provides a model for an orbifold chart of X at p, and the induced action of G_p on the quotient $(B^4(\epsilon_0)/\Gamma_p, \omega_0)$ provides a model for the action of G_p in the neighborhood of p in X (here $B^4(r) \subset \mathbb{C}^2$ is the open ball of radius r with respect to the standard metric).

With the preceding understood, the G-invariant neighborhood U will be taken to be the union $\bigsqcup_{p \in \Sigma^1} U_p$, where U_p is a neighborhood of p modeled by $(B^4(t_0)/\Gamma_p, \omega_0)$ for some choice of $t_0 < \epsilon_0$. Fixing a $p \in \Sigma^1$, we let $\mathbb{S}^3(t) \subset B^4(\epsilon_0)$ be the sphere of radius t. Since the action of Γ_p is by a subgroup of $U(2)$, the Hopf fibration on $\mathbb{S}^3(t)$ is preserved under the action, and the Hopf fibration descends to a Seifert fibration on the 3-orbifold $\mathbb{S}^3(t)/\Gamma_p$. Note that the singular set of $\mathbb{S}^3(t)/\Gamma_p$ is a union of singular fibers of the Seifert fibration. Finally, for any Σ_i which is not entirely contained in Σ^*, if $p \in \Sigma_i$, then Σ_i intersects with $\mathbb{S}^3(t)/\Gamma_p$ at a union of singular components.

Now we fix a t_0 and let γ be a singular component of the 3-orbifold $\mathbb{S}^3(t_0)/\Gamma_p$. We shall describe a model for the symplectic structure ω near γ. Without loss of generality, we may assume that the pre-image of γ in $\mathbb{S}^3(t_0)$ is given by $z_1 = 0$. A neighborhood of $\{z_1 = 0\} \cap \mathbb{S}^3(t_0)$ in $B^4(\epsilon_0)$, denoted by W, can be parametrized by coordinates (r, θ, ϕ, t), where $0 \le r < r_0$, $\theta, \phi \in \mathbb{R}/2\pi\mathbb{Z}$, and t lies in a small interval containing t_0, by the following map

$$(z_1, z_2) = (\frac{rt}{\sqrt{1+r^2}} e^{i(\theta+\phi)}, \frac{t}{\sqrt{1+r^2}} e^{i\phi}).$$

With this understood, note that $\omega_0 = r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2$ in polar coordinates (here $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$), hence in the new coordinates, a simple calculation shows that

$$\omega_0 = \frac{r^2 t}{1+r^2} dt \wedge d\theta + \frac{t^2 r}{(1+r^2)^2} dr \wedge d\theta + t \, dt \wedge d\phi.$$

We point out that (r, θ, ϕ) gives a trivialization of the Hopf fibration near $z_1 = 0$ in $\mathbb{S}^3(t_0)$, with (r, θ) for the base and ϕ for the fiber.

Let Γ be the subgroup of Γ_p which leaves the fiber $\{z_1 = 0\} \cap \mathbb{S}^3(t_0)$ invariant. Then Γ must be a cyclic group which acts on the neighborhood W of $\{z_1 = 0\} \cap \mathbb{S}^3(t_0)$ by translations in the θ and ϕ variables. Furthermore, let Γ' be the subgroup of Γ which leaves each fiber of the Hopf fibration invariant, i.e., Γ' is the subgroup which acts by translation in the ϕ variable only. We take the quotient by Γ' first, and by making a change of variable $\phi \mapsto \frac{1}{|\Gamma'|}\phi$ so that we continue to have $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, we have the following expression for ω_0 on the quotient W/Γ':

$$\omega_0 = \frac{r^2 t}{1+r^2} dt \wedge d\theta + \frac{t^2 r}{(1+r^2)^2} dr \wedge d\theta + \frac{1}{|\Gamma'|} t \, dt \wedge d\phi.$$

With this understood, let a be the order of the quotient group Γ/Γ'. Then there is a generator $g \in \Gamma/\Gamma'$ whose action on W/Γ' is given by

$$g \cdot (r, \theta, \phi, t) = (r, \theta + \frac{2\pi}{a}, \phi + \frac{2\pi b}{a}, t).$$

Note that with this description, $m := \gcd(a, b) > 1$ is the order of the isotropy subgroup of γ in the 3-orbifold $\mathbb{S}^3(t_0)/\Gamma_p$. We set $a' = a/m$, $b' = b/m$, and when $b' \ne 0 \pmod{a'}$, we let $0 < c < a'$ be the integer satisfying $b'c = 1 \pmod{a'}$. In fact the case $b' = 0 \pmod{a'}$

corresponds to the case $a' = 1$, in which case we take $c = 0$. With this understood, we perform the following change of variables in the θ, ϕ coordinates

$$\theta = \theta' + \frac{c}{a}\phi', \phi = \frac{1}{a'}\phi'. \text{ where } \theta', \phi' \in \mathbb{R}/2\pi\mathbb{Z}.$$

Then the coordinates (r, θ', ϕ', t) with a $\mathbb{Z}_m$-action given by

$$(r, \theta', \phi', t) \mapsto (r, \theta' + \frac{2\pi}{m}, \phi', t)$$

gives a global orbifold chart for W/Γ, a neighborhood of γ in $B^4(\epsilon_0)/\Gamma_p$, on which the symplectic form ω_0 takes the following form

$$\omega_0 = \frac{1}{a'|\Gamma'|}tdt \wedge d\phi' + \alpha_0 \wedge dh_0,$$

where $\alpha_0 = d\theta' + \frac{c}{a}d\phi'$ and $h_0 = \frac{t^2 r^2}{2(1+r^2)}$. The $\mathbb{S}^1$-action given by translations in θ' is Hamiltonian, with a Hamiltonian function $h_0(r, \theta', \phi', t) = \frac{t^2 r^2}{2(1+r^2)}$. Let $Y(\lambda) = h_0^{-1}(\lambda)$ be the level sets. Then $Y(\lambda)$, $\lambda > 0$, can be regarded as a principal $\mathbb{S}^1$-bundle over an annulus with coordinates ϕ', t, and with this viewpoint α_0 is a connection 1-form on $Y(\lambda)$. Note that $Y(\lambda)$ comes with a natural trivialization.

With these preparations, let Σ'_i be the part of Σ_i contained in

$$X \setminus (\bigsqcup_{p \in \Sigma^1} B^4(t_0)/\Gamma_p),$$

and let H_i be the subgroup of G which leaves Σ'_i invariant. Then Σ'_i is a compact Riemann surface with boundary where the boundary components are singular fibers in $\bigsqcup_{p \in \Sigma^1} \mathbb{S}^3(t_0)/\Gamma_p$ such as γ. We can modify the arguments in Lemma 2.2 to de-singularize the symplectic structure ω along each Σ'_i to obtain a symplectic structure ω' on

$$|X| \setminus (\bigsqcup_{p \in \Sigma^1} |B^4(t_0)/\Gamma_p|).$$

More concretely, let ν_i be the normal bundle of Σ'_i in X and let $\pi_i : Y_i \to \Sigma'_i$ be the associated principal $\mathbb{S}^1$-bundle. Note that the order m of the isotropy subgroup of γ in the 3-orbifold $\mathbb{S}^3(t_0)/\Gamma_p$ is also the order of the isotropy group at Σ'_i. With this understood, there is a finite group $\tilde{H}_i$ with a short exact sequence $1 \to \mathbb{Z}_m \to \tilde{H}_i \to H_i \to 1$ such that ν_i and Y_i are natural $\tilde{H}_i$-bundles. We choose an $\tilde{H}_i$-invariant connection 1-form α_i on Y_i, with the following condition near the boundary of Σ'_i: in the neighborhood W/Γ of γ, we identify the complement of Σ'_i with $Y_i \times (0, r_i)$ for some $r_i > 0$ such that $Y(\lambda) = Y_i \times \{\lambda\}$, and under this identification, we require

$$\alpha_i = \alpha_0 = d\theta' + \frac{c}{a}d\phi'.$$

We let κ_i be the 2-form on Σ'_i such that $\pi_i^* \kappa_i = -d\alpha_i$ (note that $\kappa_i = 0$ near the boundary), and consider the following $\tilde{H}_i$-invariant symplectic form on $Y_i \times (-r_i, r_i)$ for

$r_i > 0$ sufficiently small:
$$\omega_i := \pi_i^*(\eta_i + \lambda\kappa_i) + \alpha_i \wedge d\lambda,$$
where η_i is the restriction of the symplectic form ω on Σ_i'.

As in the proof of Lemma 2.2, the symplectic cutting procedure yields a symplectic structure on a disc-bundle $D(\nu_i)$ associated to ν_i which equals ω_i under the natural identification of $Y_i \times (0, r_i)$ with the complement of the zero section in $D(\nu_i)$. We continue to denote it by ω_i. Then the requirement $\alpha_i = \alpha_0 = d\theta' + \frac{c}{a}d\phi'$ in the neighborhood W/Γ of γ implies easily that $\omega_i = \omega_0$ in W/Γ, where ω_0 is the original symplectic structure ω, given by the following expression in the (r, θ', ϕ', t) coordinates:
$$\omega_0 = \frac{1}{a'|\Gamma'|}tdt \wedge d\phi' + \alpha_0 \wedge dh_0, \quad \text{where } h_0 = \frac{t^2 r^2}{2(1 + r^2)}.$$
With this understood, the relative version of the equivariant Weinstein neighborhood theorem implies that ω_i gives a model for the symplectic structure ω on X near each Σ_i'. We can then use symplectic cutting as in the proof of Lemma 2.2 to extend the symplectic structure ω across each Σ_i'. We thus obtained a symplectic structure ω' on $|X| \setminus (\bigsqcup_{p \in \Sigma^1} |B^4(t_0)/\Gamma_p|)$, which is clearly G-invariant and has the desired properties. This finishes the proof of Lemma 2.4.

$\square$

2.3. Capping-off by symplectic fillings

It remains to extend ω' to the entire $|X|$. The key observation is that the boundary of $|X| \setminus (\bigsqcup_{p \in \Sigma^1} |B^4(t_0)/\Gamma_p|)$ is a concave contact boundary with respect to ω', so we shall extend ω' to $|X|$ by capping off the contact boundary with an appropriate symplectic filling.

Note that each boundary component $|\mathbb{S}^3(t_0)/\Gamma_p|$, $p \in \Sigma^1$, comes with a natural Seifert fibration, which is inherited from that on the 3-orbifold $\mathbb{S}^3(t_0)/\Gamma_p$. Furthermore, the isotropy subgroup G_p acts smoothly on $|\mathbb{S}^3(t_0)/\Gamma_p|$ preserving the Seifert fibration.

Lemma 2.5. *For each $p \in \Sigma^1$, the boundary component $|\mathbb{S}^3(t_0)/\Gamma_p|$ of the symplectic manifold $(|X| \setminus (\bigsqcup_{p \in \Sigma^1} |B^4(t_0)/\Gamma_p|), \omega')$ admits a G_p-invariant, inward-pointing Liouville vector field, such that the induced contact form α_p on $|\mathbb{S}^3(t_0)/\Gamma_p|$ is a G_p-invariant, constant multiple of a connection 1-form with respect to the Seifert fibration on $|\mathbb{S}^3(t_0)/\Gamma_p|$.*

Proof. First, by following through the symplectic cutting procedure in the proof of Lemma 2.4, one can easily show that near the boundary of Σ_i', ω' is given by the expression
$$\omega_i' = \frac{1}{a'|\Gamma'|}tdt \wedge d\tilde{\phi} + (d\tilde{\theta} + \frac{c}{a'}d\tilde{\phi}) \wedge d(\frac{t^2 r^2}{2m(1 + r^2)})$$
in coordinates $(r, \tilde{\theta}, \tilde{\phi}, t)$, where $\tilde{\theta}, \tilde{\phi} \in \mathbb{R}/2\pi\mathbb{Z}$, and $\tilde{\theta}, \tilde{\phi}$ are related to θ', ϕ' by the equations $\tilde{\theta} = m\theta'$ and $\tilde{\phi} = \phi'$.

With this understood, we consider the vector field $V := \frac{1}{2}t\partial_t$ on $|B^4(\epsilon_0)/\Gamma_p|$ defined near the hypersurface $|\mathbb{S}^3(t_0)/\Gamma_p|$. It is clear that V is a Liouville vector field with respect

to ω ($= \omega_0$) in the complement of the singular components of the 3-orbifold $\mathbb{S}^3(t_0)/\Gamma_p$. Near the singular components, one can easily verify that $L_V\omega' = \omega'$ using the expression of ω'_i given above. Finally, V is clearly G_p-invariant, and is inward-pointing with respect to the orientation on $\left(|X| \setminus (\bigsqcup_{p\in\Sigma^1} |B^4(t_0)/\Gamma_p|), \omega' \right)$.

Let $\alpha_p := i_V\omega'$ be the corresponding contact form on $|\mathbb{S}^3(t_0)/\Gamma_p|$. Then near the boundary of each Σ'_i, the expression for ω'_i implies that α_p is given by the expression

$$\alpha'_i = \frac{1}{2}t_0^2\left(\left(\frac{1}{a'|\Gamma'|} - \frac{cr^2}{a'm(1+r^2)} \right) d\tilde{\phi} - \frac{r^2}{m(1+r^2)} d\tilde{\theta} \right).$$

To see that α_p is a constant multiple of a connection 1-form with respect to the Seifert fibration, we note that the Seifert fibration is induced from the Hopf fibration on $\mathbb{S}^3(t_0)$, which is given by translations $\theta \mapsto \theta, \phi \mapsto \phi + \lambda$. In terms of coordinates $\tilde{\theta}, \tilde{\phi}$, it becomes $\tilde{\theta} \mapsto \tilde{\theta} - c\lambda, \tilde{\phi} \mapsto \tilde{\phi} + a'\lambda$. It follows that the $\mathbb{S}^1$-action is generated by the vector field $a'\partial_{\tilde{\phi}} - c\partial_{\tilde{\theta}}$, and

$$\alpha'_i(a'\partial_{\tilde{\phi}} - c\partial_{\tilde{\theta}}) = \frac{t_0^2}{2|\Gamma'|}.$$

It follows easily that $\frac{2|\Gamma'|}{t_0^2}\alpha_p$ is a connection 1-form on $|\mathbb{S}^3(t_0)/\Gamma_p|$ with respect to the Seifert fibration. Finally, α_p is clearly G_p-invariant. This finishes off the proof.

$\square$

The following lemma should be well-known to experts. For completeness and the reader's convenience, we include a proof here.

Lemma 2.6. *Let $\pi : Y \to B$ be an oriented Seifert 3-manifold, and let α_0, α_1 be connection 1-forms on Y which are also positive contact forms. Then there exists a fiber-preserving self-diffeomorphism ψ such that $\psi^*\alpha_1 = \alpha_0$. Furthermore, if a finite group G acts on Y smoothly, preserving the Seifert fibration, and both of α_0, α_1 are G-invariant, then ψ can be chosen G-equivariant.*

Proof. Let κ_i be the 2-form on B such that $\pi^*\kappa_i = d\alpha_i$, $i = 0, 1$. Then since α_i are positive contact forms, κ_i are symplectic forms on B, defining the same orientation. Furthermore, since α_i are connection 1-forms, $\alpha_1 - \alpha_0 = \pi^*\eta$ for some 1-form η on B. For B being 2-dimensional, $\kappa_t := t\kappa_1 + (1 - t)\kappa_0$ is symplectic for each $t \in [0, 1]$. It follows that $\alpha_t := t\alpha_1 + (1 - t)\alpha_0$ is a smooth family of connection 1-forms which are also positive contact forms, and we note that $\frac{d}{dt}\alpha_t = \alpha_1 - \alpha_0 = \pi^*\eta$.

Let X_t be the time-dependent vector field on Y determined by the following conditions

$$X_t \in \ker\alpha_t, \quad i_{X_t}d\alpha_t = -\pi^*\eta \text{ on } \ker\alpha_t.$$

Let Z be the vector field which generates the $\mathbb{S}^1$-action on Y. Then

$$i_{X_t}d\alpha_t(Z) = d\alpha_t(X_t, Z) = \pi^*\kappa_t(X_t, Z) = \kappa_t(\pi_*(X_t), \pi_*(Z)) = 0.$$

Consequently, $i_{X_t}d\alpha_t = -\pi^*\eta$ as 1-forms on Y. It follows that $\frac{d}{dt}\alpha_t + L_{X_t}\alpha_t = 0$, and the 1-family of self-diffeomorphisms ψ_t generated by X_t obeys $\frac{d}{dt}\psi_t^*\alpha_t = 0$, and $\psi_1^*\alpha_1 = \alpha_0$

in particular. To see that ψ_1 preserves the $\mathbb{S}^1$-action, we note that $L_{X_t}Z = 0$. This is because $\alpha_t(L_{X_t}Z) = -L_{X_t}\alpha_t(Z) = \pi^*\eta(Z) = 0$ so that $L_{X_t}Z \in \ker\alpha_t$, and on the other hand, for any $W \in \ker\alpha_t$, noting that $\pi^*\kappa_t = d\alpha_t$, one has

$$d\alpha_t(L_{X_t}Z, W) = -L_{X_t}(d\alpha_t)(Z, W) - d\alpha_t(Z, L_{X_t}W) = \pi^*d\eta(Z, W) - 0 = 0.$$

Hence $L_{X_t}Z = 0$. Finally, in the present of a fiber-preserving G-action, with α_0, α_1 being G-invariant, everything can be done equivariantly. Hence the lemma.

$\square$

In light of the "uniqueness" result in Lemma 2.6, in order to complete the proof of Theorem 1.1, we will construct an appropriate model for the contact manifolds $(|\mathbb{S}^3(t_0)/\Gamma_p|, \alpha_p)$ which is symplectically filled by the corresponding quotient of a 4-ball with the standard symplectic structure ω_0. To this end, let Γ_p' be the fundamental group of $|\mathbb{S}^3(t_0)/\Gamma_p|$. Then there is a finite subgroup $\hat\Gamma_p$ of $U(2)$ and a short exact sequence $1 \to \Gamma_p' \to \hat\Gamma_p \to G_p \to 1$ such that $|\mathbb{S}^3(t_0)/\Gamma_p|$ is identified via a diffeomorphism to $\mathbb{S}^3/\Gamma_p'$ and the action of G_p on $|\mathbb{S}^3(t_0)/\Gamma_p|$ is induced by the action of $\hat\Gamma_p$ on $\mathbb{S}^3$. Furthermore, under this identification, the $\mathbb{S}^1$-action on $|\mathbb{S}^3(t_0)/\Gamma_p|$ defining the Seifert fibration lifts to a $\mathbb{S}^1$-action on $\mathbb{S}^3$, which in general takes the form

$$\lambda \cdot (z_1, z_2) = (\lambda^m z_1, \lambda^n z_2)$$

for some m, n with $\gcd(m, n) = 1$. The key observation is that all the group actions on $\mathbb{S}^3$ that are involved here are complex linear, hence can be extended to $\mathbb{C}^2$. Furthermore, they preserve the standard symplectic structure ω_0 on $\mathbb{C}^2$. Note that the $\mathbb{S}^1$-action is Hamiltonian with a Hamiltonian function

$$h_p(z_1, z_2) = \frac{1}{2}(mr_1^2 + nr_2^2),$$

where $r_i = |z_i|$ for $i = 1, 2$ (note that m, n depend on the point p). Moreover, the vector field $L := \frac{1}{2}(r_1\partial_{r_1} + r_2\partial_{r_2})$ is a Liouville vector field transverse to the level sets of h_p. We consider the ellipsoids

$$E(p, \delta) := \{(z_1, z_2) \in \mathbb{C}^2 | h_p(z_1, z_2) \le \delta\}$$

with the standard symplectic structure ω_0, and let $\alpha_0 := i_L\omega_0$ be the contact form on the boundary $\partial E(p, \delta)$.

Note that $\partial E(p, \delta)$ is equivariantly diffeomorphic to the unit sphere $\mathbb{S}^3$ with respect to all the group actions involved. Consequently, $\partial E(p, \delta)/\Gamma_p'$ is G_p-equivariantly diffeomorphic to $|\mathbb{S}^3(t_0)/\Gamma_p|$ as Seifert manifolds. Note that α_0 is $\hat\Gamma_p$-invariant and is also invariant under the $\mathbb{S}^1$-action. We let α_p' be the descendant of α_0 to $\partial E(p, \delta)/\Gamma_p'$, which is clearly G_p-invariant and is a constant multiple of a connection 1-form with respect to the Seifert fibration on $\partial E(p, \delta)/\Gamma_p'$. By Lemma 2.6, for some choice of $\delta = \delta_p$, there exists a G_p-equivariant, fiber-preserving diffeomorphic $\psi_p : \partial E(p, \delta_p)/\Gamma_p' \to |\mathbb{S}^3(t_0)/\Gamma_p|$ such that $\alpha_p' = \psi_p^*\alpha_p$.

Note that $E(p, \delta_p)$ is Γ'_p-equivariantly diffeomorphic to the 4-ball. Hence the following 4-orbifold

$$|X| \setminus \left(\bigsqcup_{p \in \Sigma^1} |B^4(t_0)/\Gamma_p| \right) \bigsqcup_{p \in \Sigma^1} E(p, \delta_p)/\Gamma'_p,$$

where the gluing along the boundaries is given by $\bigsqcup_{p \in \Sigma^1} \psi_p$, is diffeomorphic to the smooth orbifold $|X|$. The former has a natural symplectic structure since each ψ_p is a contactomorphism. It is by this identification we obtained the desired symplectic 4-orbifold $(|X|, \omega')$.

To finish up the proof of Theorem 1.1, we observe that since each ψ_p is fiber-preserving and $\Sigma_i \cap |\mathbb{S}^3(t_0)/\Gamma_p|$ is a union of fibers of the Seifert fibration on $|\mathbb{S}^3(t_0)/\Gamma_p|$, the part of Σ_i in U_p, which is a cone over $\Sigma_i \cap |\mathbb{S}^3(t_0)/\Gamma_p|$, can be identified with a cone in $E(p, \delta_p)/\Gamma'_p$ over

$$\psi_p^{-1}(\Sigma_i \cap |\mathbb{S}^3(t_0)/\Gamma_p|) \subset \partial E(p, \delta_p)/\Gamma'_p.$$

We should point out that with respect to the smooth structure of $E(p, \delta_p)/\Gamma'_p$, it can be singular at the origin $0 \in E(p, \delta_p)$ because the fiber of the Seifert fibration on $\partial E(p, \delta_p)$ is in general a (m, n)-torus knot. (Examples in [18] show that this can indeed occur.) However, with respect to the standard complex structure on $E(p, \delta_p)$, the cone is given by a holomorphic curve. This completes the proof of Theorem 1.1.

We end this section with a proof of Corollary 1.3.

Proof of Corollary 1.3

We apply Theorem 1.1 to the symplectic 4-orbifold $\hat{M}/\Gamma$, which is equipped with a symplectic G/Γ-action. The underlying space $|\hat{M}/\Gamma|$ is naturally identified with M and the singular set Σ of $\hat{M}/\Gamma$ is identified with the branch locus B. With this understood, it follows that M admits a G/Γ-invariant symplectic structure with respect to which B is a symplectic surface. Note that in the present case, Σ^1 is empty, and so is U.

For the converse suppose M admits a G/Γ-invariant symplectic structure with respect to which B is a symplectic surface. Then if we let $\hat{B}$ be the pre-image of B in $\hat{M}$, then the G/Γ-invariant symplectic structure on M lifts to a G-invariant symplectic structure on $\hat{M} \setminus \hat{B}$. To see that it extends across $\hat{B}$ to a symplectic structure on $\hat{M}$, we simply observe that in the proof of Lemma 2.2, if we are given a symplectic structure on the disc bundle $D(\nu')$, we can lift it to a symplectic structure on $D^*(\nu, r)$, and then the same symplectic cutting procedure will allow us to extend the symplectic structure on $D^*(\nu, r)$ across the zero section to a symplectic structure on the entire disc bundle $D(\nu, r)$. (Compare also [37], Proposition 7, for a different construction to this effect.) This completes the proof of Corollary 1.3.

3. A symplectic resolution and its various properties

This section is devoted to the construction of symplectic resolution and its various properties. In particular, we present a proof of Theorem 1.5. The section also contains the proofs of two propositions, one concerning the canonical class of the symplectic resolution, the other concerning symplectic equivariant blowing-down.

3.1. Construction of a symplectic resolution

As we mentioned in the introduction, the resolution $\tilde{X}$ in Theorem 1.5 is simply the symplectic resolution of the symplectic 4-orbifold $(|X|, \omega')$. For the specific purpose of applications in finite group actions, we shall adopt the method in [4] for its construction.

Proof of Theorem 1.5

First, we construct the resolution $\tilde{X}$, together with the map $\pi : \tilde{X} \to X$. Since this is a local operation, it suffices to focus at a point $p \in \Sigma^0 \sqcup \Sigma^1$, where when $p \in \Sigma^1$, we assume it is a singular point of $(|X|, \omega')$. For simplicity, we denote by $\tilde{\Sigma}^1$ the subset of Σ^1 which consists of singular points of $(|X|, \omega')$.

By the equivariant version of Darboux theorem, there is a neighborhood U_p of p in the orbifold $(|X|, \omega')$, which is modeled by $(B^4, \omega_0)/\Gamma'_p$, where $B^4 \subset \mathbb{C}^2$ is a 4-ball, ω_0 is the standard symplectic structure, and Γ'_p is the isotropy group at p, acting on $\mathbb{C}^2$ as a subgroup of $U(2)$. If $p \in \tilde{\Sigma}^1$, we shall assume B^4 is contained in the ellipsoid $E(p, \delta_p)$ from the proof of Theorem 1.1. Note that the ω'-compatible complex structure in Theorem 1.1, with respect to which $\Sigma_i \cap U$ is given by a holomorphic curve, is simply the standard complex structure on B^4 in the case of $p \in \tilde{\Sigma}^1$. We identify $U_p = B^4/\Gamma'_p$ as a subset of the affine algebraic variety $Z := \mathbb{C}^2/\Gamma'_p$.

Let $\pi_Z : \tilde{Z} \to Z$ be the minimal algebraic resolution of the isolated singularity of Z. Setting $\tilde{U}_p := \pi_Z^{-1}(U_p)$, where $U_p = B^4/\Gamma'_p \subset Z$, we glue each $\tilde{U}_p$ to $|X| \setminus (\Sigma^0 \sqcup \Sigma^1)$ by identifying $\tilde{U}_p \setminus \pi_Z^{-1}(p)$ with $U_p \setminus \{p\}$ via π_Z, which gives us the desired resolution $\tilde{X}$ together with the map $\pi : \tilde{X} \to X$. The claims in Theorem 1.5(1) follow easily. We remark that since we require that the algebraic resolution $\pi_Z : \tilde{Z} \to Z$ be minimal, which is unique (cf. [28]), the diffeomorphism type of $\tilde{X}$ is canonically determined by the smooth orbifold X.

Secondly, fixing any such a choice of U_p, $p \in \Sigma^0 \sqcup \tilde{\Sigma}^1$, we construct a symplectic structure $\tilde{\omega}$ on $\tilde{X}$ as follows. First, we note that $\tilde{Z}$ is a quasi-projective variety, and hence there is a Kähler form Ω on $\tilde{Z}$. Consider for each p the subset $A := \pi_Z^{-1}((\frac{2}{3}\overline{B}^4 \setminus \frac{1}{3}B^4)/\Gamma'_p)$ in $\tilde{U}_p$. Since it is homotopic to $\mathbb{S}^3/\Gamma'_p$, a rational homology sphere, there is a 1-form $\gamma \in \Omega^1(A)$ such that $\Omega - \pi_Z^*\omega' = d\gamma$ on A. We let ρ be a cut-off function which equals zero in $\tilde{U}_p \setminus \pi_Z^{-1}(\frac{2}{3}B^4/\Gamma'_p)$ and equals one in $\pi_Z^{-1}(\frac{1}{3}B^4/\Gamma'_p)$. Then we define a closed 2-form $\tilde{\omega}$ on $\tilde{U}_p$, where

$$\tilde{\omega} := \pi_Z^*\omega' + \epsilon d(\rho\gamma)$$

for some $\epsilon > 0$ sufficiently small. Note that on $\tilde{U}_p \setminus \pi_Z^{-1}(\frac{2}{3}B^4/\Gamma'_p)$, $\tilde{\omega} = \pi_Z^* \omega'$ which is symplectic, and on $\pi_Z^{-1}(\frac{1}{3}B^4/\Gamma'_p)$, $\tilde{\omega} = (1-\epsilon)\pi_Z^*\omega' + \epsilon\Omega$, which is a Kähler form, and finally on A, $\tilde{\omega}$ is symplectic as long as we choose $\epsilon > 0$ sufficiently small since A is compact. It is clear that $\tilde{\omega}$ extends to the whole $\tilde{X}$, giving the desired symplectic structure. Let U be the G-invariant neighborhood of $\Sigma^0 \sqcup \Sigma^1$, where

$$U := \bigsqcup_{p \in \Sigma^0} U_p \bigsqcup_{p \in \Sigma^1} E(p, \delta_p)/\Gamma_p,$$

it follows easily that $\pi : (\tilde{X} \setminus \pi^{-1}(U \cup \Sigma^*), \tilde{\omega}) \to (X \setminus (U \cup \Sigma^*), \omega)$ is a symplectomorphism. Note that U can be taken arbitrarily small. To see that $\pi^{-1}(\Sigma^*)$ is a symplectic surface, we only need to observe that on A the complex structure is $\tilde{\omega}$-tame and $A \cap \pi^{-1}(\Sigma^*)$ is holomorphic. Furthermore, it is clear that $\tilde{\omega} = \pi^*\omega$ on $\pi^{-1}(\Sigma^* \setminus U)$ as area forms. Finally, we take $U' := \bigsqcup_{p \in \tilde{\Sigma}^1} \pi_Z^{-1}(\frac{1}{3}B^4/\Gamma'_p)$. Then $U' \cap \pi^{-1}(\Sigma)$ is given by holomorphic curves. The rest of the claims in Theorem 1.5(2) are obvious.

Finally, we prove the claims in Theorem 1.5(3). First, it is well-known that the algebraic resolution $\pi_Z : \tilde{Z} \to Z$ can be carried out equivariantly (cf. [47]). It follows easily that if the orbifold (X, ω) admits a symplectic G-action by a finite group G, then the constructions of the resolution $\pi : \tilde{X} \to X$ and the symplectic structure $\tilde{\omega}$ can be done G-equivariantly.

It remains to compare the resolutions of the symplectic orbifolds X/G and $\tilde{X}/G$. To this end, using the same notations from the construction of $\tilde{X}$, we consider the G-invariant decompositions of the underlying spaces of X and $\tilde{X}$: $|X|$ as the union of $|X| \setminus (\bigsqcup_{p \in \Sigma^0 \sqcup \tilde{\Sigma}^1} U_p)$ and $\bigsqcup_{p \in \Sigma^0 \sqcup \tilde{\Sigma}^1} U_p$, and $\tilde{X}$ as the union of $|X| \setminus (\bigsqcup_{p \in \Sigma^0 \sqcup \tilde{\Sigma}^1} U_p)$ and $\bigsqcup_{p \in \Sigma^0 \sqcup \tilde{\Sigma}^1} \tilde{U}_p$. Noting that the underlying space of X/G is simply the underlying space of the quotient of $|X|$ with the induced G-action, this gives rise to the corresponding decompositions of the underlying spaces of X/G and $\tilde{X}/G$:

$$|X/G| = \left(|X| \setminus (\bigsqcup_{p \in \Sigma^0 \sqcup \tilde{\Sigma}^1} U_p)\right)/G \bigsqcup_{p \in (\Sigma^0 \sqcup \tilde{\Sigma}^1)/G} |U_p/G_p|$$

and

$$|\tilde{X}/G| = \left(|X| \setminus (\bigsqcup_{p \in \Sigma^0 \sqcup \tilde{\Sigma}^1} U_p)\right)/G \bigsqcup_{p \in (\Sigma^0 \sqcup \tilde{\Sigma}^1)/G} |\tilde{U}_p/G_p|,$$

where G_p is the subgroup of G fixing p, and $p \in (\Sigma^0 \sqcup \tilde{\Sigma}^1)/G$ means that p is running over a set of representatives of the quotient set $(\Sigma^0 \sqcup \tilde{\Sigma}^1)/G$ in $\Sigma^0 \sqcup \tilde{\Sigma}^1$.

With the preceding understood, let $\pi_V : V \to X/G$ and $\pi_W : W \to \tilde{X}/G$ denote the corresponding resolutions. It follows easily that the difference between W and V occurs at $\pi_V^{-1}(U_p/G_p)$ and $\pi_W^{-1}(\tilde{U}_p/G_p)$, where $p \in (\Sigma^0 \sqcup \tilde{\Sigma}^1)/G$. With this understood, the following claim finishes off the proof of Theorem 1.5(3).

Claim: $\pi_W^{-1}(\tilde{U}_p/G_p)$ *is either diffeomorphic to* $\pi_V^{-1}(U_p/G_p)$*, or can be reduced to a manifold diffeomorphic to it by successively blowing down symplectic* (-1)*-spheres.*

Proof of Claim: We begin by noting that $|U_p/G_p|$ is a complex analytic space with a unique, isolated quotient singularity at p. Let $|\widetilde{U_p/G_p}|$ be its minimal resolution. On the other hand, $|\tilde{U}_p/G_p|$ is a complex analytic space with isolated quotient singularities, all contained in $\pi_Z^{-1}(p)/G_p$. Let $|\widetilde{\tilde{U}_p/G_p}|$ denote the minimal resolution of $|\tilde{U}_p/G_p|$. It is easily seen that $|\widetilde{\tilde{U}_p/G_p}|$ is some resolution of the unique singularity of $|U_p/G_p|$.

With the preceding understood, we note, from the construction of the resolutions V and W, that $\pi_V^{-1}(U_p/G_p) = |\widetilde{U_p/G_p}|$ and $\pi_W^{-1}(\tilde{U}_p/G_p)$ is diffeomorphic to $|\widetilde{\tilde{U}_p/G_p}|$. Moreover, regarding $|\widetilde{\tilde{U}_p/G_p}|$ as a resolution of the unique singularity in $|U_p/G_p|$, the exceptional set in $|\widetilde{\tilde{U}_p/G_p}|$ corresponds to a configuration of embedded symplectic two-spheres in $\pi_W^{-1}(\tilde{U}_p/G_p)$ intersecting transversely. Furthermore, there is an almost complex structure J compatible with the symplectic structure such that each symplectic two-sphere in the configuration is J-holomorphic.

By the work of Artin [1], the exceptional set in $|\widetilde{\tilde{U}_p/G_p}|$ has the following properties: any two distinct components are either disjoint or intersect at a single point, and no three distinct components intersect in one point. Furthermore, the dual graph is a tree. Clearly, the corresponding configuration of symplectic two-spheres in $\pi_W^{-1}(\tilde{U}_p/G_p)$ also has these properties.

Now we recall the following fact: $|\widetilde{\tilde{U}_p/G_p}|$ can be reduced to the minimal resolution $|\widetilde{U_p/G_p}|$ by successively blowing down holomorphic (-1)-spheres (cf. [28]). The following lemma, Lemma 3.1, shows that the corresponding successive blowing-downs for $\pi_W^{-1}(\tilde{U}_p/G_p)$ can be done symplectically, i.e., with the holomorphic (-1)-spheres replaced by symplectic (-1)-spheres in each step. With this understood, it follows easily that either $\pi_W^{-1}(\tilde{U}_p/G_p)$ is diffeomorphic to $\pi_V^{-1}(U_p/G_p)$ or can be reduced to a manifold diffeomorphic to $\pi_V^{-1}(U_p/G_p)$. Hence the claim.

Lemma 3.1. *Let S be a symplectic (-1)-sphere in a symplectic 4-manifold (M, ω), and let $\{C_i\}$ be a finite collection of symplectic surfaces in (M, ω). Suppose there is an ω-compatible almost complex structure J such that S and C_i are J-holomorphic, and furthermore, near S the symplectic surfaces C_i are embedded, disjoint, and each C_i intersects S transversely. Then*

    (1) *there is a neighborhood U of S, such that for each i, there is an embedded symplectic surface $\tilde{C}_i$, which is isotopic to C_i through a symplectic isotopy supported in U,*

    (2) *there is a smaller neighborhood $V \subset U$ such that, if we let (M', ω') be the symplectic 4-manifold obtained by removing V and gluing back a standard symplectic 4-ball, then each symplectic surface $\tilde{C}_i \setminus V$ can be naturally extended across the*

4-ball to a closed symplectic surface C_i', such that the surfaces C_i' intersect transversely at the origin of the 4-ball, and

(3) there is an ω'-compatible almost complex structure J', agreeing with J outside U, such that C_i' is J'-holomorphic.

Proof. First of all, one can isotop each C_i near the intersection points with S, which can be made in an arbitrarily small neighborhood of the intersection points, such that the new symplectic surface is ω-orthogonal to S (cf. [24]). With this understood, let U be a neighborhood of S in which the symplectic form ω is modeled by a standard symplectic structure on the disc bundle of S. This is possible by the Weinstein neighborhood theorem. Now observe that in the standard model the fiber disc is ω-orthogonal to S. By further deforming each C_i inside U, we can arrange so that near the intersection point it coincides with the fiber disc. This final new symplectic surface is our $\tilde{C}_i$. Now we take a sufficiently small neighborhood $V \subset U$ of S such that inside V, each $\tilde{C}_i$ is given by the fiber disc. Then clearly, after removing V and gluing back a standard symplectic 4-ball, each $\tilde{C}_i \setminus V$ can be extended across the 4-ball by gluing a standard complex disc to the boundary circle of $\tilde{C}_i \setminus V$. The resulting closed symplectic surfaces C_i' intersect transversely at the origin of the 4-ball. We define the ω'-compatible almost complex structure J' to be J outside the neighborhood U, to be the standard complex structure inside the 4-ball, and to be some ω'-compatible almost complex structure in $U \setminus V$ such that each $\tilde{C}_i$ is pseudo-holomorphic. The last assertion is possible because inside $U \setminus V$ the symplectic surfaces $\tilde{C}_i$ are embedded and disjoint. It is clear that each C_i' is J'-holomorphic. This finishes off the lemma.

□

The proof of Theorem 1.5 is complete.

3.2. The canonical class of the symplectic resolution

As we pointed out in Remark 1.2, the orbifold canonical line bundle $K_{\omega'}$ in Theorem 1.1 is uniquely determined up to isomorphisms. To see this, we first note that on $|X| \setminus U$, ω' is uniquely determined by the original symplectic structure ω because $\omega = \omega'$ on $|X| \setminus (\Sigma^* \cup U)$ and $|X| \setminus (\Sigma^* \cup U)$ is dense in $|X| \setminus U$. Consequently, the restriction of $K_{\omega'}$ over $|X| \setminus U$ is uniquely determined by (X, ω). On the other hand, we observe that up to isomorphism, there is only one way to extend $K_{\omega'}|_{|X| \setminus U}$ over U. Hence $K_{\omega'}$ is uniquely determined up to isomorphism. We also pointed out in Remark 1.6 that the canonical line bundle $K_{\tilde{\omega}}$ is uniquely determined up to isomorphism. This is because $K_{\tilde{\omega}}$ is completely determined by $K_{\omega'}$ and the singularities of the smooth orbifold $|X|$ (recall that we have used the unique, minimal resolutions of the singularities of $|X|$ in the construction of $\tilde{X}$).

In the following proposition, we give an expression of $c_1(K_{\tilde{X}})$ (as a class in $H^2(\tilde{X}, \mathbb{Q})$) in terms of the orbifold canonical class $c_1(K_X)$ and the singularities of $(|X|, \omega')$.

Proposition 3.2. *Let $\{p_j\}$ be the set of singular points of $(|X|, \omega')$, and for each p_j, let $\{E_{j,k}|k \in I_j\}$ be the exceptional set in the minimal resolution of p_j. Then there are*

$a_{j,k} \in \mathbb{Q}$, $a_{j,k} \leq 0$, *such that*

$$c_1(K_{\tilde{X}}) = \pi^* c_1(K_{|X|}) + \sum_j \sum_{k \in I_j} a_{j,k} E_{j,k}.$$

On the other hand, let $\{\Sigma_i\}$ *be the set of compactified connected components of* Σ^*, *and for each* i, *let* m_i *be the order of the isotropy groups along* Σ_i. *Then* $c_1(K_{|X|})$ *and* $c_1(K_X)$ *are related by the following equation*

$$c_1(K_{|X|}) = c_1(K_X) + \sum_i \frac{1 - m_i}{m_i} PD(\Sigma_i),$$

where $PD(\Sigma_i)$ *stands for the Poincaré dual of the symplectic surface* Σ_i *in* $|X|$.

Proof. For each j, we fix a regular neighborhood V_j of p_j and let $\tilde{V}_j$ be the minimal resolution of V_j at p_j. Then by the Mayer-Vietoris theorem (with $\mathbb{Q}$-coefficients),

$$c_1(K_{\tilde{X}}) = c_1(K_{|X| \setminus \sqcup_j V_j}) + \sum_j c_1(K_{\tilde{V}_j}).$$

On the other hand, note that $c_1(K_{|X| \setminus \sqcup_j V_j}) = \pi^* c_1(K_{|X|})$, $c_1(K_{\tilde{V}_j}) = \sum_{k \in I_j} a_{j,k} E_{j,k}$ for some $a_{j,k} \in \mathbb{Q}$, where $a_{j,k} \leq 0$ as $\tilde{V}_j$ is the minimal resolution. The formula

$$c_1(K_{\tilde{X}}) = \pi^* c_1(K_{|X|}) + \sum_j \sum_{k \in I_j} a_{j,k} E_{j,k}.$$

follows immediately.

The main part of the proof is concerned with the equation relating $c_1(K_{|X|})$ and $c_1(K_X)$. To this end, we introduce the following notations: $Z := X \setminus U$, $|Z| := |X| \setminus U$, and $\Sigma_i' := \Sigma_i \setminus U$, where U is a neighborhood of Σ^1 introduced in Theorem 1.1. With this understood, note that the symplectic structures ω and ω' agree in the complement of $\Sigma^* \setminus U = \bigsqcup_i \Sigma_i'$. This implies that for any fixed regular neighborhood U' of $\Sigma^* \setminus U$ in $|Z|$, we can choose an ω-compatible almost complex structure J on Z and an ω'-compatible almost complex structure J' on $|Z|$ such that $J = J'$ in the complement of U'. This in particular yields an identification of K_Z and $K_{|Z|}$ in the complement of U'. On the other hand, note that there is a continuous orbifold map $\lambda : Z \to |Z|$ which induces the identity map between the underlying spaces (cf. [8]). Furthermore, λ is smooth in the complement of $\Sigma^* \setminus U$. Now if we denote by $K_{|Z|}^*$ the dual of $K_{|Z|}$, we obtain the following bundle isomorphisms

$$K_Z = \lambda^* K_{|Z|} \otimes \lambda^* K_{|Z|}^* \otimes K_Z = \lambda^* K_{|Z|} \otimes \mathrm{Hom}(\lambda^* K_{|Z|}, K_Z),$$

where $\mathrm{Hom}(\lambda^* K_{|Z|}, K_Z)$ is the bundle of endomorphisms from $\lambda^* K_{|Z|}$ to K_Z. With this understood, note that the identification of K_Z and $K_{|Z|}$ in the complement of U' defines a canonical non-zero section σ of $\mathrm{Hom}(\lambda^* K_{|Z|}, K_Z)$ in the complement of U'. It follows that the first Chern class of $\mathrm{Hom}(\lambda^* K_{|Z|}, K_Z)$ should be given by a linear combination of the Poincaré duals of Σ_i'. (Note that each Σ_i' defines a class in $H_2(|Z|, \partial |Z|, \mathbb{Q})$, so its Poincaré dual lies in $H^2(|Z|, \mathbb{Q})$.)

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In order to compute the first Chern class of $\mathrm{Hom}(\lambda^* K_{|Z|}, K_Z)$, we let U_i' be the component of U' containing Σ_i', which is taken to be a disc bundle of the normal bundle $\pi_i' : \nu_i' \to \Sigma_i'$ in $|Z|$. Let U_i be the global orbifold chart of Z over U_i', which is also a disc bundle over Σ_i' associated to the normal bundle $\pi_i : \nu_i \to \Sigma_i'$ of Σ_i' in Z (see the proof of Lemma 2.4). The $\mathbb{Z}_{m_i}$-action on U_i is given by the complex multiplication on the fibers. Let $\lambda_i : U_i \to U_i' = U_i/\mathbb{Z}_{m_i}$ be the quotient map. Then with this understood, the bundle $\mathrm{Hom}(\lambda^* K_{|Z|}, K_Z)$ over U_i' is given by the following $\mathbb{Z}_{m_i}$-equivariant bundle over U_i:

$$\mathrm{Hom}\big(\lambda_i^* \circ (\pi_i')^* (\nu_i' \otimes T\Sigma_i')^*, \pi_i^* (\nu_i \otimes T\Sigma_i')^*\big) = \mathrm{Hom}\big(\lambda_i^* (\pi_i')^* (\nu_i')^* \otimes \pi_i^* T^* \Sigma_i', \pi_i^* \nu_i^* \otimes \pi_i^* T^* \Sigma_i'\big)$$

The canonical non-zero section σ of $\mathrm{Hom}(\lambda^* K_{|Z|}, K_Z)$ in the complement of U' determines an $\mathbb{Z}_{m_i}$-equivariant non-zero section σ_i of

$$\mathrm{Hom}(\lambda_i^* (\pi_i')^* (\nu_i')^* \otimes \pi_i^* T^* \Sigma_i', \pi_i^* \nu_i^* \otimes \pi_i^* T^* \Sigma_i')$$

on $\overline{U_i} \setminus U_i$. Now observe that on the factor $\pi_i^* T^* \Sigma_i'$, σ_i is given by the identity map. In order to understand σ_i between the factors $\lambda_i^* (\pi_i')^* (\nu_i')^*$ and $\pi_i^* \nu_i^*$, we let $D_p \subset U_i$ be the disc which is the fiber of π_i over $p \in \Sigma_i'$, and let $D_p' \subset U_i'$ be the disc which is the fiber of π_i'. Denote by z and w the complex coordinates on D_p and D_p' respectively. Then over D_p, $\lambda_i^* (\pi_i')^* (\nu_i')^*$, $\pi_i^* \nu_i^*$ are trivialized by dw and dz. On the other hand, since $J = J'$ in the complement of U', dw and dz are related on ∂D_p by the equation $dw = z^{m_i-1} dz$. Now by the fact that the non-zero section σ is defined by the identification of K_Z and $K_{|Z|}$ in the complement of U', it follows easily that σ_i on ∂D_p is given by the map which assigns each $z \in \partial D_p$ the automorphism $\sigma_i(z)$ of $\mathbb{C}$ given by the multiplication by z^{m_i-1}. In other words, $\sigma_i : \partial D_p \times \mathbb{C} \to \partial D_p \times \mathbb{C}$ is given by the formula

$$\sigma_i : (z, u) \mapsto (z, z^{m_i-1} u),$$

which can be naturally extended to a map $D_p \times \mathbb{C} \to D_p \times \mathbb{C}$ given by the same formula above. It follows easily that the first Chern class of $\mathrm{Hom}(\lambda^* K_{|Z|}, K_Z)$ equals the Poincaré dual of $\sum_i \frac{m_i-1}{m_i} \Sigma_i'$. Now recall that $\lambda : Z \to |Z|$ induces the identity map on $H^2(|Z|; \mathbb{Q})$. It follows immediately that

$$c_1(K_Z) = c_1(K_{|Z|}) + \sum_i \frac{m_i - 1}{m_i} PD(\Sigma_i').$$

Finally, we observe that U is a disjoint union of $\mathbb{Q}$-homology balls. Hence there is a natural identification between $H^2(|X|, \mathbb{Q})$ and $H^2(|Z|, \mathbb{Q})$. It is easy to see that under this identification, $c_1(K_X) = c_1(K_Z)$, $c_1(K_{|X|}) = c_1(K_{|Z|})$, and $PD(\Sigma_i) = PD(\Sigma_i')$ for each i. This implies immediately

$$c_1(K_{|X|}) = c_1(K_X) + \sum_i \frac{1 - m_i}{m_i} PD(\Sigma_i),$$

and the proof of the proposition is complete.

$\square$

3.3. Symplectic resolution and equivariant blowing down

The symplectic blowing down operation (cf. [35]) can be easily extended to the equivariant setting. More concretely, let $\tilde{M}$ be a symplectic 4-manifold equipped with a finite symplectic G-action. Suppose there exists a G-invariant set of disjoint symplectic (-1)-spheres in $\tilde{M}$. Then blowing down $\tilde{M}$ along the (-1)-spheres gives rise to a symplectic 4-manifold M, which can be arranged so that the G-action descends to a symplectic G-action on M. The symplectic G-manifold $\tilde{M}$ is called *minimal* if no such set of (-1)-spheres exists. We refer the reader to [11] for more discussions on this topic.

Since it is technically more convenient to work with minimal symplectic G-manifolds, one naturally asks how the resolutions of the quotient orbifolds are related after performing an equivariant symplectic blowing-down.

Recall that for a symplectic 4-manifold M equipped with a finite symplectic G-action, the resolution of the quotient orbifold M/G is denoted by M_G. The arguments in the proof of Theorem 1.5(3) can be easily extended to give a proof of the following proposition.

Proposition 3.3. *Let $\tilde{M}$ be a symplectic 4-manifold equipped with a finite symplectic G-action. Suppose M is a G-equivariant blow-down of $\tilde{M}$. Then the resolutions $\tilde{M}_G$ and M_G are either diffeomorphic, or $\tilde{M}_G$ can be reduced to M_G by successively blowing down symplectic (-1)-spheres. In particular, $\kappa^s(\tilde{M}_G) = \kappa^s(M_G)$.*

Proof. Note that as in the proof of Theorem 1.5(3), the issue is local in nature. For this reason it suffices to consider a symplectic (-1)-sphere C in $\tilde{M}$ which is blown down to a point p in M. Let G_C be the subgroup of G which leaves C invariant. The key observation is that the G_C-action in a neighborhood of C is smoothly equivalent to a holomorphic action. To see this, note that C has a G_C-invariant regular neighborhood whose boundary is $\mathbb{S}^3$. This implies that G_C acts smoothly and effectively on $\mathbb{S}^3$, which preserves the Hopf fibration. Any such action is equivalent to a linear action preserving the Hopf fibration, hence it is by a subgroup of $U(2)$. Now we fix such a complex linear action of G_C on $\mathbb{C}^2$. Equivariantly blowing up at the origin of $\mathbb{C}^2$, we obtain a holomorphic model for the G_C-action near C.

With the preceding understood, let U_C be a G_C-invariant neighborhood of C in $\tilde{M}$. Note that the holomorphic model in the previous paragraph supports a G_C-invariant Kähler form. Hence by the equivariant Weinstein neighborhood theorem, we may assume that there is a complex structure on U_C, compatible with the symplectic structure on $\tilde{M}$, such that C is holomorphic and the G_C-action is holomorphic. Let $\pi_C : U_C \to U_p$ be the holomorphic map contracting C to a point. Then there is an induced holomorphic G_C-action on U_p, and the equivariant symplectic blowing down operation from $\tilde{M}$ to M is locally near C smoothly equivalent to $\pi_C : U_C \to U_p$. This said, if we let $\tilde{\pi} : \tilde{M}_G \to \tilde{M}/G$ and $\pi : M_G \to M/G$ be the corresponding resolutions, then $\pi^{-1}(U_p/G_C)$ is diffeomorphic to the minimal resolution of the unique isolated singularity of the complex analytic space $|U_p/G_C|$, and $\tilde{\pi}^{-1}(U_C/G_C)$ is diffeomorphic to some resolution of the singularity of $|U_p/G_C|$. By the same arguments as in the proof of Theorem 1.5(3), with the help of

Lemma 3.1, $\tilde{\pi}^{-1}(U_C/G_C)$ can be reduced to a manifold diffeomorphic to $\pi^{-1}(U_p/G_C)$ by successively blowing down symplectic (-1)-spheres. Proposition 3.3 follows easily from this local consideration.

$\square$

4. Symplectic Kodaira dimension of M_G

4.1. The case of $\kappa^s(M) = -\infty$ or 0

We begin with the following lemma.

Lemma 4.1. *Let* (M, ω) *be a symplectic* 4-*manifold equipped with a finite symplectic* G-*action. Suppose one of the following conditions is satisfied:*

- $c_1(K_M) \cdot [\omega] < 0$, *or*
- M *has torsion canonical class, and the singular set of* M/G *either contains a* 2-*dimensional component, or contains an isolated non-Du Val singularity.*

Then one can choose the symplectic structure $\tilde{\omega}$ *on* M_G *such that* $c_1(K_{M_G}) \cdot [\tilde{\omega}] < 0$. *In particular,* M_G *is rational or ruled.*

Proof. Let $X = M/G$ be the quotient orbifold. For simplicity, we continue to denote by ω the symplectic structure on X. We let $\{\Sigma_i\}$ be the 2-dimensional components of the singular set of X, regarded as symplectic surfaces in the orbifold $(|X|, \omega')$, and let $\{p_j\}$ be the set of singular points of $(|X|, \omega')$. With this understood, let $m_i > 1$ be the order of the isotropy groups along Σ_i, and for each j, let $\{E_{j,k} | k \in I_j\}$ be the set of exceptional divisors in the minimal resolution of the singular point p_j.

By Proposition 3.2, we have

$$c_1(K_{|X|}) = c_1(K_X) + \sum_i \frac{1 - m_i}{m_i} PD(\Sigma_i),$$

and

$$c_1(K_{M_G}) = \pi^* c_1(K_{|X|}) + \sum_j \sum_{k \in I_j} a_{j,k} E_{j,k}$$

where $a_{j,k} \in \mathbb{Q}$ and $a_{j,k} \leq 0$. With this understood, we compute $c_1(K_{M_G}) \cdot [\tilde{\omega}]$.

To this end, we fix a closed 2-form η representing $c_1(K_{|X|})$. Then recall that in the definition of $\tilde{\omega}$ in the proof of Theorem 1.5, $\tilde{\omega} := \pi_Z^* \omega' + \epsilon d(\rho \gamma)$ on $\tilde{U}_p$. It follows easily that

$$|\pi^* c_1(K_{|X|}) \cdot [\tilde{\omega}] - c_1(K_{|X|}) \cdot [\omega']| \leq \epsilon \cdot |\eta| \cdot C$$

for some constant $C > 0$ which is independent of the choice of ϵ but may depend on the choice of the neighborhoods U_p, the Kähler form Ω, the cutoff function ρ as well as the 1-form γ in the construction of $\tilde{\omega}$. With this understood, note that by taking $\epsilon > 0$ sufficiently small, we can make the above difference arbitrarily close to zero, once we fix the other various choices. On the other hand, by choosing the neighborhood U in Theorem

1.1 sufficiently small, we can also arrange so that the difference $|c_1(K_X)\cdot[\omega']-c_1(K_X)\cdot[\omega]|$ is arbitrarily close to zero. With this understood, the difference

$$|c_1(K_{M_G})\cdot[\tilde\omega] - (c_1(K_X)\cdot[\omega] + \sum_i \frac{1-m_i}{m_i}\omega'(\Sigma_i) + \sum_j \sum_{k\in I_j} a_{j,k}\tilde\omega(E_{j,k}))|$$

can be made arbitrarily close to zero by choosing $\tilde\omega$ properly. We claim that under the assumptions of the lemma, $c_1(K_{M_G})\cdot[\tilde\omega] < 0$ for such an $\tilde\omega$. To see this, consider first the case where $c_1(K_M)\cdot[\omega] < 0$. In this case, $c_1(K_X)\cdot[\omega] = \frac{1}{|G|}c_1(K_M)\cdot[\omega] < 0$. With the other two terms $\sum_i \frac{1-m_i}{m_i}\omega'(\Sigma_i)$ and $\sum_j \sum_{k\in I_j} a_{j,k}\tilde\omega(E_{j,k}))$ being non-positive, it follows immediately that $c_1(K_{M_G})\cdot[\tilde\omega] < 0$. On the other hand, when M has torsion canonical class, $c_1(K_X)\cdot[\omega] = \frac{1}{|G|}c_1(K_M)\cdot[\omega] = 0$. If the singular set of M/G contains a 2-dimensional component, then the term $\sum_i \frac{1-m_i}{m_i}\omega'(\Sigma_i)$ is negative, which implies that $c_1(K_{M_G})\cdot[\tilde\omega] < 0$. If the singular set of M/G contains no 2-dimensional components, then by Proposition 3.2,

$$c_1(K_{M_G})\cdot[\tilde\omega] = \sum_j \sum_{k\in I_j} a_{j,k}\tilde\omega(E_{j,k}),$$

which is also negative as in this case, the singular set of M/G must contain an isolated non-Du Val singularity, so that one of the coefficients $a_{j,k}$ is negative. This finishes off the proof of the lemma.

$\square$

Proof of Theorem 1.9:

By Proposition 3.3, we may assume without loss of generality that the symplectic G-manifold M is minimal. We consider first the case where $\kappa^s(M) = 0$. Then by Theorem 1.0 in [11], the above assumption means that M is minimal as a smooth 4-manifold. With this understood, the assumption $\kappa^s(M) = 0$ is equivalent to M having a torsion canonical class (cf. [30]).

If the G-action is free, then $M_G = M/G$, which also has torsion canonical class. Hence in this case, $\kappa^s(M_G) = 0$. If M/G has only isolated singular points which are all Du Val singularities, then $c_1(K_{M_G})$ is torsion by Proposition 3.2, and we have $\kappa^s(M_G) = 0$ as well. In the remaining case, M_G is rational or ruled by Lemma 4.1.

We claim that when M_G is irrational ruled, $b_1(M_G) = 2$ so that M_G is a ruled surface over T^2. Suppose to the contrary that $b_1(M_G) > 2$. Then

$$b_1(M) \geq b_1(M/G) = b_1(M_G) \geq 4.$$

On the other hand, it is known that $b_1(M) \leq 4$ (cf. [30, 31, 3]), so that

$$b_1(M) = b_1(M/G) = 4.$$

In particular, the induced G-action on $H^1(M, \mathbb{Q})$ is trivial. Now we note that $b_1(M) = 4$ implies that M is a $\mathbb{Q}$-homology T^4 (cf. [30]), and furthermore, the work of Ruberman-Strle in [43] implies that M has the same $\mathbb{Q}$-cohomology ring of T^4. This gives a contradiction, because the triviality of the induced G-action on $H^1(M, \mathbb{Q})$ implies that the action is also trivial on $H^2(M, \mathbb{Q})$, and consequently, $b_2^+(M/G) = b_2^+(M) = 3$. But this violates $b_2^+(M/G) = b_2^+(M_G) = 1$. Hence the claim.

It remains to consider the case where M is a rational surface. Denote by ω_0 the symplectic structure on M. Suppose M is $\mathbb{CP}^2$ or a Hirzebruch surface. Then one has in this case $c_1(K_M) \cdot [\omega_0] < 0$, so that by Lemma 4.1, M_G must be rational (note that $b_1(M) = 0$, so $b_1(M_G) = 0$ as well). For the remaining case, i.e., $M = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ for $N \geq 2$, we need to recall some relevant results from [16] first.

There are two possibilities for M: either M is monotone, i.e., the class $[\omega_0]$ is a multiple of $c_1(K_{\omega_0})$, or M is a symplectic G-conic bundle. In the former case, it follows easily that $c_1(K_{\omega_0}) \cdot [\omega_0] < 0$, so that by Lemma 4.1, M_G is rational. In the latter case, we need further information about the equivariant symplectic cone of M. Roughly speaking, let F be the fiber class of the symplectic G-conic bundle on M. Then it is shown in [16] that for any sufficiently large $\delta > 0$, the class $-c_1(K_{\omega_0}) + \delta F$ can be realized by a G-invariant symplectic structure ω on M, where $c_1(K_\omega) = c_1(K_{\omega_0})$. With this understood, observing that $c_1(K_{\omega_0}) \cdot F = -2$, we have

$$c_1(K_\omega) \cdot [\omega] = -c_1(K_{\omega_0})^2 - 2\delta,$$

which is negative for sufficiently large $\delta > 0$. Hence by Lemma 4.1, M_G is rational (note that the diffeomorphism type of M_G depends on the smooth orbifold M/G alone, not on the symplectic structure on it). This finishes the proof of Theorem 1.9.

4.2. The case of $\kappa^s(M) = 1$

Let (M, ω) be a symplectic 4-manifold with $\kappa^s = 1$, equipped with a finite symplectic G-action such that $b_2^+(M/G) > 1$. To simplify the situation, we note that by Proposition 3.3, we may assume the G-manifold M is minimal as far as $\kappa^s(M_G)$ is concerned. By Theorem 1.0 in [11], this condition is equivalent to the smooth 4-manifold M being minimal. Then note that with this assumption, the condition $\kappa^s(M) = 1$ is equivalent to $c_1(K_M) \cdot [\omega] > 0$ and $c_1(K_M)^2 = 0$. Finally, for simplicity we assume that $G = \mathbb{Z}_p$ is cyclic of prime order p.

With the preceding understood, for any given G-invariant ω-compatible almost complex structure J on M, there is a finite set of J-holomorphic curves $\{C_k\}$ with multiplicities $n_k > 0$, which has the following significance (see [13], Theorem 3.2):

- $c_1(K_M) = \sum_k n_k C_k$, where the set $\cup_k C_k$ (as well as $\{n_k\}$) is G-invariant.
- Any 2-dimensional fixed component of G is contained in $\cup_k C_k$, and if a fixed point m of G is not contained in $\cup_k C_k$, then the G-action on the tangent space $T_m M$ is contained in $SL_2(\mathbb{C})$, in particular, m is an isolated fixed point.

- If C_k is a fixed component of G, then $n_k \geq p-1$ (cf. [10], Lemma 1.6). Moreover, C_k is either a torus of self-intersection zero or a (-2)-sphere. In the latter case, C_k must intersect another (-2)-sphere in $\cup_k C_k$ which is not fixed by G.

Proof of Theorem 1.10:

We set $X := M/G$ to be the quotient orbifold.

For (1), we pick a G-invariant ω-compatible almost complex structure J, so that $c_1(K_M) = \sum_k n_k C_k$ for a finite set of J-holomorphic curves $\{C_k\}$ with multiplicities $n_k > 0$. Then since any 2-dimensional fixed component of G is contained in $\cup_k C_k$, and moreover, if C_k is a fixed component of G, then $n_k \geq p-1$, it follows easily from Proposition 3.2 that $c_1(K_{|X|})$ is torsion if and only if every C_k is fixed by G and $n_k = p-1$ for every k. Note that in particular, each C_k must be a torus of self-intersection zero. Furthermore, if every C_k is fixed by G, then any isolated fixed point of G must be in the complement of $\cup_k C_k$, so that the induced G-action on the tangent space of the fixed point is contained in $SL_2(\mathbb{C})$. This in particular implies that $c_1(K_{M_G})$ is torsion. Part (1) follows easily.

For (2), we first note that $c_1(K_{M_G})^2 = c_1(K_{|X|})^2 + \sum_m K_m^2$, where the second term is the sum over all isolated fixed points m of G. Thus it suffices to show that

$$c_1(K_{|X|})^2 = -\frac{2(p-1)^2}{p} \cdot s,$$

where s is the number of (-2)-spheres fixed by G. To this end, we denote by Y any 2-dimensional fixed component of G, and notice that by Proposition 3.2,

$$c_1(K_X) = c_1(K_{|X|}) + \sum_Y \frac{p-1}{p} Y,$$

where Y is regarded as a surface in $|X|$. Then since $c_1(K_X)^2 = \frac{1}{p} c_1(K_M)^2 = 0$, we have

$$c_1(K_{|X|})^2 + \sum_Y \frac{(p-1)^2}{p} Y^2 + \sum_Y \frac{2(p-1)}{p} c_1(K_{|X|}) \cdot Y = 0,$$

where Y^2 denotes the self-intersection of Y as a surface in M. Here we used the fact that as a surface in $|X|$, the self-intersection of Y equals $p \cdot Y^2$. Now by the adjunction formula, $c_1(K_{|X|}) \cdot Y + p \cdot Y^2 = 2g_Y - 2$, where g_Y denotes the genus of Y. It follows easily that

$$c_1(K_{|X|})^2 = \sum_Y \frac{2(p-1)}{p}(2 - 2g_Y) + \sum_Y \frac{p^2-1}{p} Y^2.$$

Finally, recall that as a surface in M, Y is either a torus of self-intersection zero or a (-2)-sphere, which gives immediately that $c_1(K_{|X|})^2 = -\frac{2(p-1)^2}{p} \cdot s$.

For (3), we first note that

$$\chi(M_G) = \chi(M/G) + \sum_m \chi_m,$$

and

$$\sigma(M_G) = \sigma(M/G) - \sum_m \chi_m,$$

where χ_m is the number of exceptional divisors in the minimal resolution of the singular point of M/G corresponding to the isolated fixed point m of G. Combining these two equations, we obtain

$$c_1(K_{M_G})^2 = 2\chi(M_G) + 3\sigma(M_G) = 2\chi(M/G) + 3\sigma(M/G) - \sum_m \chi_m.$$

On the other hand, when the G-action is homologically trivial, $\chi(M/G) = \chi(M)$ and $\sigma(M/G) = \sigma(M)$, so that

$$2\chi(M/G) + 3\sigma(M/G) = 2\chi(M) + 3\sigma(M) = c_1(K_M)^2 = 0.$$

It follows immediately that $c_1(K_{M_G})^2 = -\sum_m \chi_m$.

For (4), we fix a symplectic structure $\tilde{\omega}$ on M_G as constructed in Theorem 1.5. With this understood, first note that if $c_1(K_{M_G})^2 = 0$, then the only 2-dimensional fixed components are tori of self-intersection zero, and for all the isolated fixed points m, $K_m = 0$. This gives immediately $c_1(K_{M_G}) = \pi^* c_1(K_{|X|})$ by Proposition 3.2. On the other hand, if M_G is not minimal, then for any $\tilde{\omega}$-compatible almost complex structure J, there exists a J-holomorphic (-1)-sphere C in M_G (cf. [10], Lemma 2.3). We will get a contradiction by showing $\pi^* c_1(K_{|X|}) \cdot C \geq 0$, because $c_1(K_{M_G}) \cdot C = -1$.

To this end, we shall particularly choose an $\tilde{\omega}$-compatible almost complex structure on M_G as follows. We begin by fixing a G-invariant J on M with the following property: near each of the isolated fixed points m, we may identify ω, J with the standard symplectic structure and complex structure on $\mathbb{C}^2$, such that the G-action is given by a complex linear action (this is possible by the equivariant Darboux theorem). In particular, J is integrable near each m. With this understood, we have $c_1(K_M) = \sum_k n_k C_k$ for a finite set of J-holomorphic curves $\{C_k\}$ as shown in [13]. Since $\cup_k C_k$ is G-invariant, its image in $X = M/G$ under the G-action is also a set of J-holomorphic curves, which will be denoted by $\{C_i'\}$, with multiplicities n_i'. We remark that if C_i' is the image of some C_k which is not fixed under G, then $n_i' = n_k$, and if C_k is fixed under G, then $n_i' = n_k/p$. With this understood, note that $c_1(K_X) = \sum_i n_i' C_i'$. Furthermore, since $c_1(K_{M_G})^2 = 0$, C_i' must be a torus with self-intersection zero if it is the image of some C_k which is fixed by G, and such a C_i' is disjoint from the other components in $\cup_i C_i'$. In particular, this allows us to modify J near each of such C_i' and obtain an ω'-compatible J' on $|X|$, such that each C_i' in $\cup_i C_i'$ is J'-holomorphic. With this understood, note that by Proposition 3.2,

$$c_1(K_{|X|}) = \sum_i \hat{n}_i C_i',$$

where $\hat{n}_i = n_i'$ if C_i' is not the image of a fixed component, and $\hat{n}_i = n_i' - (p-1)/p$ otherwise. Note that $\hat{n}_i \geq 0$ for all i.

Now with $|X|$ equipped with ω' and J', we proceed to the construction of M_G with a symplectic structure $\tilde{\omega}$ as in Theorem 1.5. The key observation here is that since $J' = J$

is integrable near each singular point of $|X|$, it can be naturally extended to an almost complex structure, still denoted by J', on M_G which is integrable near the exceptional divisors. Furthermore, J' is also $\tilde{\omega}$-compatible except in the interpolation regions A, where it is only $\tilde{\omega}$-tame. With this understood, we modify J' in the regions A to obtain an $\tilde{\omega}$-compatible J on M_G, such that (the proper transform of) each C_i' is J-holomorphic (this is possible because $(\cup_i C_i') \cap A$ is a disjoint union of embedded symplectic surfaces). This J is the particular $\tilde{\omega}$-compatible almost complex structure on M_G we choose to work with.

With the preceding understood, we now consider the J-holomorphic (-1)-sphere C in M_G. Since J is integrable near the exceptional divisors and the map $\pi : M_G \to |X|$ is simply holomorphically contracting the exceptional divisors to the singular points in $|X|$, it follows easily that J descends to $|X|$ such that each C_i' in $|X|$ is J-holomorphic. Moreover, the image of C under the map π, which is denoted by C', is also J-holomorphic. With this understood, we note that

$$\pi^* c_1(K_{|X|}) \cdot C = c_1(K_{|X|}) \cdot C' = (\sum_i \hat{n}_i C_i') \cdot C'.$$

To see that $\pi^* c_1(K_{|X|}) \cdot C \geq 0$, we first consider the case where for any i, $C' \neq C_i'$. In this case, $\pi^* c_1(K_{|X|}) \cdot C = (\sum_i \hat{n}_i C_i') \cdot C' \geq 0$ by the positivity of intersection of J-holomorphic curves. If $C' = C_i'$ for some i, we will need to recall some additional information from [13] about the curves $\{C_k\}$ in $c_1(K_M) = \sum_k n_k C_k$. It is shown in Lemma 3.3 of [13] that $c_1(K_M) \cdot C_k = 0$ for each k. From this it follows easily that $c_1(K_X) \cdot C_i' = 0$ for each i, where $c_1(K_X) = \sum_i n_i' C_i'$. Now recall that $c_1(K_{|X|}) = \sum_i \hat{n}_i C_i'$ where $\hat{n}_i = n_i'$ if C_i' is not the image of a fixed component, and if C_i' is the image of a fixed component, we have $\hat{n}_i = n_i' - (p-1)/p$. Since in the latter case, C_i' is a torus of self-intersection zero which is disjoint from the other components of $\{C_i'\}$, it follows easily that $c_1(K_{|X|}) \cdot C_i' = 0$ for each i as well. This shows that if $C' = C_i'$ for some i, then $\pi^* c_1(K_{|X|}) \cdot C = c_1(K_{|X|}) \cdot C' = 0$. This finishes the proof of (4), and the proof of Theorem 1.10 is complete.

Remark 4.2. We shall point out that the expression

$$c_1(K_{M_G})^2 = -\frac{2(p-1)^2}{p} \cdot s + \sum_m K_m^2$$

in Theorem 1.10(2) can be also derived independently from the Lefschetz fixed-point theorem and the G-signature theorem; in particular, the integrability of the right-hand side does not give any new constraints to the fixed-point set structure. To see this, note that we have already seen that

$$c_1(K_{M_G})^2 = 2\chi(M/G) + 3\sigma(M/G) - \sum_m \chi_m.$$

Now by the Lefschetz fixed-point theorem, we have

$$p \cdot \chi(M/G) = \chi(M) + (p-1) \cdot (\sum_m 1 + \sum_Y (2 - 2g_Y)),$$

and by the G-signature theorem, we have

$$p \cdot \sigma(M/G) = \sigma(M) + \sum_m def_m + \sum_Y def_Y,$$

where def_m and def_Y stand for the signature defect at m and Y respectively (cf. [26]). With this understood, it follows easily that

$$c_1(K_{M_G})^2 = -\sum_m \chi_m + \sum_m \frac{2(p-1)}{p} + \sum_Y \frac{2(p-1)}{p}(2-2g_Y) + \frac{3}{p}(\sum_m def_m + \sum_Y def_Y).$$

On the other hand, it is known that if an isolated fixed point m is of type $(1,q)$ for some $0 < q < p$, then $K_m^2 + \chi_m = \frac{2(p-1)}{p} - 12 \cdot s(q,p)$, where $s(q,p)$ denotes the corresponding Dedekind sum (see e.g. [38], §7.1 in page 304). Now with the fact that $def_m = -4p \cdot s(q,p)$ and $def_Y = \frac{p^2-1}{3} Y^2$ (cf. [26]), we see easily that

$$c_1(K_{M_G})^2 = \sum_m K_m^2 + \sum_Y \frac{2(p-1)}{p}(2-2g_Y) + \sum_Y \frac{p^2-1}{p} Y^2,$$

which easily implies the expression $c_1(K_{M_G})^2 = -\frac{2(p-1)^2}{p} \cdot s + \sum_m K_m^2$.

Example 4.3. In this example, we shall examine homologically trivial symplectic G-actions on a symplectic homotopy $K3$ surface M (we assume $\kappa^s(M) = 1$; otherwise there are no such actions, cf. [13]). This is one of the main rigidity problems concerning symplectic finite group actions, so it is interesting to test the strength of our new approach on this question. For simplicity we assume $G = \mathbb{Z}_3$ (for $G = \mathbb{Z}_2$, there are no homologically trivial actions, cf. [32, 42]).

First of all, we recall that the 2-dimensional fixed components of G are either tori with self-intersection zero or (-2)-spheres. The former type of components make no contributions in any of the G-index theorems' calculation; in particular, they can not be detected by these theorems. For simplicity, we shall ignore the 2-dimensional toroidal fixed components in the discussion.

With this understood, let x be the number of (-2)-spheres fixed by G, and let y, z be the number of isolated fixed points of type $(1,1)$ and $(1,2)$ respectively. Notice that if m is an isolated fixed point of type $(1,1)$, one has $K_m^2 = -1/3$ and $\chi_m = 1$, and if m is of type $(1,2)$, one has $K_m^2 = 0$ and $\chi_m = 2$. Now since the G-action is homologically trivial, the Lefschetz fixed-point theorem gives $\chi(M^G) = \chi(M) = 24$. On the other hand, by Theorem 1.10(2) and (3), we have

$$-\frac{2(p-1)^2}{p} \cdot s + \sum_m K_m^2 = -\sum_m \chi_m.$$

It follows easily that x, y, z satisfy the following equations

$$2x + y + z = 24, \quad \frac{8}{3}x + \frac{1}{3}y = y + 2z.$$

There are four possible solutions, which are listed below:

(1) $x = 4$, $y = 16$, $z = 0$,
(2) $x = 5$, $y = 11$, $z = 3$,
(3) $x = 6$, $y = 6$, $z = 6$,
(4) $x = 7$, $y = 1$, $z = 9$.

Case (4) can be further eliminated using the constraints from [13]. More concretely, recall that $c_1(K_M)$ is represented by J-holomorphic curves $\{C_k\}$ with multiplicities n_k. In order to eliminate (4), we appeal to the following properties of the curves $\{C_k\}$: the connected components of $\cup_k C_k$ fall into five different types, and when $p = 3$, one can easily see that whenever a connected component of $\cup_k C_k$ contains a (-2)-sphere fixed by G, it also contains an isolated fixed point of type $(1, 1)$ (cf. [13], Proposition 3.7). In particular, this implies $y \geq x$. The remaining three cases can not be eliminated by the constraints from [13], neither do they violate any known obstructions for smoothable $\mathbb{Z}_p$-actions (e.g. as listed in §3 of [14]).

With the preceding understood, we look at the resolution M_G of these remaining actions listed in (1)-(3). From the proof of Theorem 1.10, we have seen how to determine $c_1(K_{M_G})$ from a concrete J-holomorphic representative of $c_1(K_M)$, i.e. $c_1(K_M) = \sum_k n_k C_k$ from [13]. For illustration we shall examine one particular case in detail; all other possibilities (there are finitely many of them) can be similarly analyzed and they all give the same conclusion.

Let's assume $c_1(K_M) = \sum_k n_k C_k$, where the connected components of $\cup_k C_k$ consist of four tori of self-intersection zero, denoted by T_i, $i = 1, 2, 3, 4$, with multiplicities α_i, and four other connected components, denoted by Λ_j, $j = 1, 2, 3, 4$, where each Λ_j is a union of three (-2)-spheres whose intersection graph forms a cycle (it is denoted by $\tilde{A}_2$ in [13]). We denote by β_j the multiplicity of each (-2)-sphere in Λ_j. Concerning the fixed-point set, each T_i contains three isolated fixed points of type $(1, 1)$, and each Λ_j contains a fixed (-2)-sphere and an isolated fixed point of type $(1, 1)$, which is the intersection point of the two (-2)-spheres in Λ_j that are not fixed by G. So totally, we have sixteen isolated fixed points of type $(1, 1)$, four fixed (-2)-spheres, and no isolated fixed points of type $(1, 2)$ (i.e., we are in case (1), where $x = 4$, $y = 16$, $z = 0$).

With this understood, we observe that each T_i is mapped to an orbifold two-sphere in M/G which becomes a (-1)-sphere in M_G (i.e., the proper transform) after resolving the singularities of M/G. We denote this (-1)-sphere by S_i. It follows easily that each T_i gives rise to a configuration of four two-spheres in M_G: one (-1)-sphere, S_i, and three (-3)-spheres, denoted by $E_{i,1}, E_{i,2}, E_{i,3}$, which are the corresponding exceptional divisors in the resolution. On the other hand, each Λ_j also gives rise to a configuration of four two-spheres in M_G: one (-6)-sphere, denoted by A_j which is the fixed (-2)-sphere in Λ_j, two (-1)-spheres, denoted by $B_{j,1}, B_{j,2}$, which are the image of the two (-2)-spheres in

Λ_j that are not fixed by G, and one (-3)-sphere, denoted by E_j, which is the exceptional sphere from the isolated fixed point in Λ_j. With this understood, we note that

$$c_1(K_{M_G}) = \sum_{i=1}^{4}(\alpha_i S_i + \frac{\alpha_i - 1}{3}\sum_{k=1}^{3}E_{i,k}) + \sum_{j=1}^{4}(\frac{\beta_j - 2}{3}A_j + \beta_j(B_{j,1} + B_{j,2}) + \frac{2\beta_j - 1}{3}E_j).$$

We immediately see twelve disjoint (-1)-spheres in M_G, which are S_i, $i = 1, 2, 3, 4$, and $B_{j,1}, B_{j,2}$, $j = 1, 2, 3, 4$. Blow down these (-1)-spheres and denote the images of $E_{i,k}$, E_j by $E'_{i,k}$, E'_j respectively. Then it is easy to see that each $E'_{i,k}$ is a (-2)-sphere and each E'_j is a (-1)-sphere. Further blow down each E'_j, we arrive at a symplectic 4-manifold, denoted by M'_G. We observe that each (-6)-sphere A_j becomes a nodal (-2)-sphere in M'_G, which we denote by A'_j. With this understood, we note that

$$c_1(K_{M'_G}) = \sum_{i=1}^{4}\frac{\alpha_i - 1}{3}(E'_{i,1} + E'_{i,2} + E'_{i,3}) + \sum_{j=1}^{4}\frac{\beta_j - 2}{3}A'_j.$$

Since each $E'_{i,k}$, A'_j is J-holomorphic for some J on M'_G, and none of them is a (-1)-sphere, it follows that M'_G is minimal (see [10], Lemma 2.3). On the other hand, $c_1(K_{M_G})^2 = -16$, and we have blown down successively sixteen (-1)-spheres in M_G to reach M'_G, it follows easily that $c_1(K_{M'_G})^2 = 0$. This shows that $\kappa^s(M_G) \leq 1$.

In conclusion, one can verify that Conjecture 1.7 is true for homologically trivial symplectic $\mathbb{Z}_3$-actions on a symplectic homotopy $K3$ surface. On the other hand, for these group actions our new approach does not give any new constraints to the fixed-point set structure.

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