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Contact splitting of symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$

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Abstract. This paper is the written and expanded version of a talk at the twenty-fourth Gökova Geometry/Topology Conference (2017). We consider splittings of a symplectic rational homology $\mathbb{CP}^2$ by a contact rational homology sphere, which is embedded as a hypersurface of contact type inside the symplectic manifold. After going over some basic properties, we give evidence of subtle obstructions for such splittings in the case of $\mathbb{CP}^2$ and explain how these are related to interesting rigidity phenomena in symplectic and algebraic geometry. Based on the existence of such obstructions, we propose a method for constructing exotic $\mathbb{CP}^2$'s and other related new small symplectic 4-manifolds.

1. Introduction

By symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$, we mean a symplectic 4-manifold (X, ω) with $b_1 = 0$ and $b_2 = b_2^+ = 1$. These are the smallest possible symplectic 4-manifolds in terms of Betti numbers and Euler characteristic. The only known examples of such symplectic 4-manifolds are $\mathbb{CP}^2$ or a fake $\mathbb{CP}^2$, the latter being an algebraic surface of general type satisfying $c_1^2 = 3c_2$. As quotient of the complex unit ball by a discrete subgroup of $PU(2,1)$, a fake $\mathbb{CP}^2$ is completely determined by its fundamental group, and there are exactly 50 fake $\mathbb{CP}^2$s as smooth 4-manifolds, cf. [62, 63, 13].

Our original and primary goal was to construct new examples of symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$, particularly the simply connected ones which give exotic smooth structures on $\mathbb{CP}^2$. However, interesting connections or applications to some other natural problems were found later, namely, the topological classification of rational cuspidal curves in $\mathbb{CP}^2$ and characterization of rationally convex domains in $\mathbb{C}^2$. These aspects will also be discussed here. There has been a surge of activities in constructing simply connected exotic smooth/symplectic 4-manifolds with small Euler characteristic; currently the best record of such small exotic manifolds are exotic $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ (cf. [3, 35]). This said, our proposed method will be completely different in nature compared with these past works. It is based on the notion of **contact splitting** which is at the center of all the discussions in this paper.

Key words and phrases. Small symplectic 4-manifolds, exotic smooth structures, rational cuspidal curves, rationally convex domains, $\mathbb{Q}$-homology spheres, hypersurface of contact type, $\mathbb{Q}$-homology ball fillings, small concave fillings, Gay's construction, open book decompositions, classification of tight contact structures, symplectic field theory, Reeb dynamical systems, finite energy spheres.

Let (X, ω) be a symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$, and M be an orientable 3-manifold which is a $\mathbb{Q}$-homology sphere. Suppose M is embedded in (X, ω) as a hypersurface of contact type, i.e., there is a contact form α on M such that $\omega|_M = d\alpha$, cf. [79]. Equivalently, there is a Liouville vector field Θ (i.e., $L_\Theta \omega = \omega$) in a neighborhood of M which is normal along M; in this connection, α and Θ are related by $i_\Theta \omega = \alpha$. Clearly, X is decomposed by M into two connected components W, V. On the other hand, M is given with a contact structure $\xi := \ker \alpha$ which is uniquely determined up to contact isotopy by the embedding $M \hookrightarrow X$; in particular, ξ is independent of the choice of α or Θ, and M will be oriented canonically by ξ. Obviously, the contact manifold (M, ξ) must be tight (cf. [25]). We shall call the decomposition $X = W \cup_M V$ a **contact splitting** of (X, ω) by the contact manifold (M, ξ), and for simplicity, when no confusion is caused or no contact structure is specified, we shall drop ξ in our notation and simply call the decomposition of X a contact splitting by M.

Throughout this paper, we shall fix the notation such that the Liouville vector field Θ is always outward-pointing with respect to W. With this convention, the contact manifold (M, ξ) is a convex (resp. concave) boundary of (W, ω) (resp. (V, ω)). We shall call W the **convex part** and V the **concave part** of the splitting. (In this paper, when we speak of convex or concave boundary, we always mean that there is a Liouville vector field in a neighborhood of the boundary, pointing outward or inward.)

In what follows, we shall discuss the various aspects of contact splitting, which are centered around, roughly, the following topics:

- Constraints on (M, ξ), particularly when $X = \mathbb{CP}^2$.
- Construction of X with a contact splitting $X = W \cup_M V$.
- Criteria for $X = W \cup_M V$ to be non-diffeomorphic to $\mathbb{CP}^2$.

In the last section, we extend our considerations to the case of Hirzebruch surfaces.

2. A preliminary lemma and some immediate corollaries

We are mostly interested in constraints on (M, ξ) which are derived from the assumption that (M, ξ) is a hypersurface of contact type in (X, ω). We begin with the following simple observation.

Lemma 2.1. *The convex part of a contact splitting is always a $\mathbb{Q}$-homology ball.*

Proof. [1] It follows from Mayer-Vietoris theorem that either W or V must be a $\mathbb{Q}$-homology ball. Assume to the contrary that V is a $\mathbb{Q}$-homology ball. Then there exists a 1-form β on V such that $\omega = d\beta$. We claim that one can choose β such that $\beta|_M = \alpha$. To see this, note that $d\beta|_M = \omega|_M = d\alpha$, so that $\beta|_M - \alpha$ is closed. Since we require M to be a $\mathbb{Q}$-homology sphere, there is a smooth function f on M such that $\beta|_M - \alpha = df$. We extend f to a neighborhood of M and let ρ be a cut-off function which equals 1 near M. Then replacing β by $\beta - d(\rho f)$, we arrived at the desired property of the 1-form β.

[1] I wish to thank Patrick Massor for showing me this simple proof; my original proof of the lemma was similar in spirit but used a neck-stretching argument from [15] which turns out to be unnecessary.

With the preceding understood, the following contradiction finishes off the proof:

$$0 < \int_V \omega \wedge \omega = \int_V d\beta \wedge d\beta = \int_{-M} \alpha \wedge d\alpha < 0,$$

where the last inequality uses the fact that (M, ξ) is a concave boundary of V, so that the orientation of M as the boundary of V is the opposite of the orientation defined by the contact structure.

□

As the first corollary, we observe the following immediate constraint on (M, ξ).

Corollary 2.2. *The contact manifold (M, ξ) must be strongly symplectically fillable by a $\mathbb{Q}$-homology ball.*

In particular, the invariant $\theta(\xi)$ introduced by Gompf [40] must be equal to -2. Indeed,

$$\theta(\xi) = c_1^2(W) - 2\chi(W) - 3\sigma(W) = 0 - 2 \cdot 1 - 3 \cdot 0 = -2.$$

The second corollary gives a criterion for X to be diffeomorphic to $\mathbb{CP}^2$ in light of a given contact splitting of X.

Corollary 2.3. *Let $X = W \cup_M V$ be a contact splitting. Then whether X is diffeomorphic to $\mathbb{CP}^2$ is entirely determined by the concave part V; more precisely,*

$$X = \mathbb{CP}^2 \text{ if and only if } c_1(K_V) \cdot [\omega] < 0.$$

Proof. By work of Taubes [73], X is diffeomorphic to $\mathbb{CP}^2$ if and only if $c_1(K_X) \cdot [\omega] < 0$. On the other hand, since M is a $\mathbb{Q}$-homology sphere, the splitting $X = W \cup_M V$ implies the decomposition $c_1(K_X) \cdot [\omega] = c_1(K_W) \cdot [\omega] + c_1(K_V) \cdot [\omega]$. By Lemma 2.1, W is a $\mathbb{Q}$-homology ball, hence $c_1(K_W) \cdot [\omega] = 0$. The corollary follows immediately.

□

Given a contact splitting $X = W \cup_M V$, in general it is not clear how to determine the sign of $c_1(K_V) \cdot [\omega]$ except in one special circumstance, namely, when the concave part V contains a pseudo-holomorphic curve (which may be singular). In this case, one can determine the sign of $c_1(K_V) \cdot [\omega]$ via the adjunction formula. We will come back to this observation in section 4 with some potential applications.

In the next section, we shall discuss some potential criteria for "$X \neq \mathbb{CP}^2$" in terms of the contact manifold (M, ξ), or even simply the $\mathbb{Q}$-homology sphere M.

3. The case of $\mathbb{CP}^2$: more subtle obstructions

When $X = \mathbb{CP}^2$, there may be more subtle constraints on (M, ξ) other than the one given in Corollary 2.2. In order to explain this, we shall consider a weaker version of contact splitting, which will be called a **weak contact splitting**.

Let (X, ω) be a symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$, and M be a $\mathbb{Q}$-homology sphere which is smoothly embedded in X, decomposing X into W and V. Suppose there is a contact structure ξ on M such that $\omega|_\xi$ is non-degenerate. We continue to fix the convention so that (M, ξ) is "convex" with respect to W, which is to say that the boundary orientation of M coincides with the orientation as a contact manifold. Finally, since it is not clear that Lemma 2.1 is true when ω is only non-degenerate on ξ, for the purpose of comparison we shall impose the condition that W is a $\mathbb{Q}$-homology ball. Under these assumptions, we shall call $X = W \cup_M V$ a **weak contact splitting**.

We remark that even though W is only a weak symplectic filling of (M, ξ), because M is a $\mathbb{Q}$-homology sphere, one can slightly modify the symplectic structure near M so that (M, ξ) is also strongly symplectically filled by W (see [59]). In particular, by our definition (M, ξ) must be strongly symplectically fillable by a $\mathbb{Q}$-homology ball as a necessary condition even for a weak contact splitting. (In what follows, we shall not distinguish between the two notions of symplectic fillings.)

With this understood, we have the following theorem, which indicates that there may be additional constraints on (M, ξ) in the case of $X = \mathbb{CP}^2$.

Let $M_0 := \mathbb{S}^3/Q(8)$, where $Q(8)$ is the subgroup of order 8 generated by the elements i, j, k of the group of unit quaternions.

Theorem 3.1. *There are no contact splittings of* $\mathbb{CP}^2$ *by* M_0*, however,* $\mathbb{CP}^2$ *admits a weak contact splitting by* M_0*.*

Proof. We note that M_0 is a small Seifert space. Under the standard notation, it is either $M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ or $M(-2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, depending on the orientation. The latter's orientation coincides with the orientation of M_0 as the link of the complex singularity in $\mathbb{C}^2/Q(8)$.

With this understood, we recall that both manifolds have a unique tight contact structure up to contactomorphisms according to the classifications in [38, 75]. In particular, the Milnor fillable contact structure (cf. [14]), denoted by ξ_{Mil}, is the only tight contact structure on $M(-2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. We shall denote by ξ_0 the unique tight structure on $M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Both contact structures are known to be Stein fillable. For ξ_{Mil}, a particular Stein filling which will be denoted by Z is given by the minimal resolution of the complex singularity in $\mathbb{C}^2/Q(8)$. We note that, since the singularity is a Du Val singularity, $c_1(Z)$ is trivial. Furthermore, Z is negative definite with $b_2(Z) = 4$, and is simply connected. This in particular implies that the Gompf invariant $\theta(\xi_{Mil}) = 2$. Indeed,

$$\theta(\xi_{Mil}) = c_1(Z)^2 - 2\chi(Z) - 3\sigma(Z) = 0 - 2 \cdot (1 + 4) - 3 \cdot (0 - 4) = 2.$$

For the contact manifold $(M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \xi_0)$, a particular Stein filling, to be denoted by W_0, is given in [58]. The manifold W_0 is diffeomorphic to a regular neighborhood of an embedded $\mathbb{RP}^2$ in $\mathbb{R}^4$. In particular, W_0 is a $\mathbb{Q}$-homology ball, and consequently, the Gompf invariant $\theta(\xi_0) = -2$.

With the preceding understood, we now give a proof of the theorem. To see that $\mathbb{CP}^2$ admits a weak contact splitting by M_0, we simply use the fact from [58] that W_0 can be realized as a Stein domain in $\mathbb{C}^2$. Embed $\mathbb{C}^2$ holomorphically in $\mathbb{CP}^2$. Then the standard Kähler form on $\mathbb{CP}^2$ is non-degenerate on ξ_0, which is identified with the 2-plane field of complex tangencies on $M_0 \subset \mathbb{CP}^2$. (Note that the standard Kähler form on $\mathbb{CP}^2$ and the canonical Kähler form $\omega_\phi := dd^c\phi$ on W_0, where ϕ is any strictly plurisubharmonic defining function on W_0, may not agree near M_0, so here we don't necessarily get a contact splitting of $\mathbb{CP}^2$ in the strong sense.)

To see that there are no contact splittings of $\mathbb{CP}^2$ by M_0, we suppose to the contrary that there is a contact splitting $\mathbb{CP}^2 = W \cup_{M_0} V$. Let ξ be the contact structure on M_0 induced from the splitting. Then by Corollary 2.2, we have $\theta(\xi) = -2$. It follows easily from the preceding discussion, particularly $\theta(\xi_{Mil}) = 2$, that $\xi = \xi_0$ must be true, and $M_0 = M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ as oriented manifolds. Now we remove the convex part W from $\mathbb{CP}^2$ and glue back the Stein filling W_0 of (M_0, ξ_0). Denote the resulting symplectic manifold by X. Then by Corollary 2.3, X is diffeomorphic to $\mathbb{CP}^2$.

To derive a contradiction, we recall the following crucial fact about W_0 from [58]: with respect to a Kähler form $\omega_\phi := dd^c\phi$ on W_0, where ϕ is a strictly plurisubharmonic defining function on W_0. There is an embedded singular Lagrangian $\mathbb{RP}^2$ in W_0. Since the symplectic structure on $X = W_0 \cup_{M_0} V$ equals ω_ϕ on the W_0 part, we conclude that $\mathbb{CP}^2$ contains an embedded singular Lagrangian $\mathbb{RP}^2$ (i.e., the one inherited from W_0). Now according to [58], one can locally resolve the singular point of the Lagrangian $\mathbb{RP}^2$ to produce an embedded Lagrangian Klein bottle in $\mathbb{CP}^2$. However, this is a contradiction, as there is no embedded Lagrangian Klein bottle in $\mathbb{CP}^2$, see [65, 57]. Hence there are no contact splittings of $\mathbb{CP}^2$ by M_0. This finishes off the proof.

□

Corollary 3.2. *Suppose (M_0, ξ_0) is the concave boundary of a symplectic 4-manifold (V, ω) with $b_1 = 0$ and $b_2 = 1$. Then $c_1(K_V) \cdot [\omega] > 0$.*

Proof. Let $X = W_0 \cup_{M_0} V$ be the symplectic 4-manifold obtained by gluing W_0 and V along the contact boundary (M_0, ξ_0). By the assumptions on the Betti numbers of V, it follows that X is a symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$. By Theorem 3.1, $X \neq \mathbb{CP}^2$, hence $c_1(K_V) \cdot [\omega] > 0$ by Corollary 2.3.

□

Non-existence results of contact splitting such as Theorem 3.1 may be used in the construction of exotic $\mathbb{CP}^2$ or other new examples of symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$.

Question 3.1. (Construction of exotic $\mathbb{CP}^2$) Does there exist a simply connected symplectic 4-manifold V with $b_2 = 1$, having (M_0, ξ_0) as its concave boundary?

If the manifold V in Question 3.1 exists, then the symplectic 4-manifold X obtained by gluing W_0 and V along (M_0, ξ_0) is an exotic $\mathbb{CP}^2$. Indeed, observe that $\pi_1(\partial W_0) \to \pi_1(W_0)$ is surjective, so X is simply connected, and hence homeomorphic to $\mathbb{CP}^2$. By Corollary 3.2, the smooth structure of X must be exotic (cf. Corollary 2.3).

Remark 3.1. The symplectic manifold V in Question 3.1 is called a concave filling of (M_0, ξ_0) (the simply-connectedness and Betti number condition are irrelevant). Concave fillings are known to exist for any contact 3-manifolds, even for the overtwisted ones (cf. [30, 37]); later a weaker version of concave fillings (also called symplectic caps), where the contact structure and symplectic structure are compatible only in the usual weak sense, are introduced and shown to exist in general (cf. [26, 29]). Although it has become a powerful tool, the fact that concave fillings exist for any given contact manifold indicates that there is no symplectic or contact rigidity in this notion. However, concave fillings which have small Betti numbers seem hard to produce with the current techniques (cf. e.g. [37]).

For a given contact $\mathbb{Q}$-homology sphere (M, ξ), a concave filling (V, ω) is called a **small concave filling** if $b_1(V) = 0$ and $b_2(V) = b_2^+(V) = 1$. A small concave filling (V, ω) is called **positive** (resp. **negative**) if $c_1(K_V) \cdot [\omega] > 0$ (resp. $c_1(K_V) \cdot [\omega] < 0$). We remark that unlike the convex fillings where one can alter the symplectic structure near the boundary to turn a weak filling into a strong filling (e.g., using Lemma 2.1 of [26]), such modification is not possible for concave fillings. Specifically, (M_0, ξ_0) admits a weak small concave filling which can not be altered into a strong one. In this paper, we shall only be concerned with small concave fillings in the strong sense.

In section 4, we shall discuss constructions of such symplectic 4-manifolds V with small Betti numbers (i.e., $b_1 = 0$ and $b_2 = b_2^+ = 1$) as well as some potential applications stemming from considerations described above.

The phenomenon revealed in Theorem 3.1 seems to be only a tip of the iceberg. In fact, it has interesting connections with a certain symplectic rigidity phenomenon expected in low dimensions.

Recall that a holomorphically convex domain $W \subset \mathbb{C}^n$ is said to be rationally convex if for every point $p \in \mathbb{C}^n \setminus W$ there is a complex algebraic hypersurface $H \subset \mathbb{C}^n$ such that $p \in H$ and $H \subset \mathbb{C}^n \setminus W$. Together with holomorphic convexity, rational convexity is one of the several notions of convexity that play an important role in several complex variables. In higher dimensions (i.e., $n > 2$), recent work of Cieliebak and Eliashberg [20] gave a complete topological characterization of rationally convex domains in $\mathbb{C}^n$, showing in particular that the above two notions of convexity are in fact equivalent. Their work relies on the recent advances [28, 56] in symplectic flexibility, which are known only in higher dimensions. In dimension two, a similar topological characterization is available only for holomorphically convex domains, see [41]. Characterization for rationally convex domains in $\mathbb{C}^2$ remains largely open, however, recent work of Nemirovski and Siegel [58] indicates that the question is more subtle in dimension two.

The connection between rational convexity and contact splitting is provided by the following symplectic characterization of rational convexity (see e.g. [20]):

A holomorphically convex domain in $\mathbb{C}^n$ is rationally convex if and only if it admits a strictly plurisubharmonic defining function ϕ such that $\omega_\phi := dd^c\phi$ extends to a Kähler form on $\mathbb{C}^n$ which is standard outside a ball.

With the preceding understood, it follows easily that the boundary of a rationally convex domain in $\mathbb{C}^2$ gives rise naturally to a hypersurface of contact type in $\mathbb{CP}^2$ (after embedding $\mathbb{C}^2$ in $\mathbb{CP}^2$); note that the boundary of a holomorphically convex domain in

$\mathbb{C}^2$ only gives a pseudoconvex hypersurface in $\mathbb{CP}^2$. In particular, let $W \subset \mathbb{C}^2$ be any holomorphically convex domain which has the rational homology of a 4-ball. Then there is an associated weak contact splitting of $\mathbb{CP}^2$, having W as its convex part. Furthermore, if W is rationally convex, then the weak contact splitting is in fact a contact splitting. With this understood, constructions in Gompf [41] give many examples of weak contact splittings of $\mathbb{CP}^2$, however, no examples of nontrivial contact splittings of $\mathbb{CP}^2$ are known from this construction. (Contact splittings by $\mathbb{S}^3$ always exist, which we regard as trivial.) Finally, we remark that our Theorem 3.1 is simply the generalization of the following result in [58] to $\mathbb{CP}^2$:

The contact manifold (M_0, ξ_0) can not be realized as a hypersurface of contact type in $\mathbb{R}^4$, but can be realized as a pseudoconvex hypersurface.

4. A construction of David Gay

In this section, we shall consider the following problem:

Construct a symplectic $\mathbb{Q}$-homology 4-ball W with convex boundary and a symplectic 4-manifold V with concave boundary, where $b_1(V) = 0$ (e.g., $\pi_1(V) = 0$), $b_2(V) = b_2^+(V) = 1$, such that the contact boundaries of W and V are contactomorphic.

Note that with W, V as above, one obtains a symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$ by gluing W and V along the contactomorphic boundaries. The resulting manifold $X = W \cup_M V$, where M denotes the boundary, comes with a natural contact splitting.

The above consideration will involve at least the following three issues in a fundamental way:

(1) Constructing V and understanding the contact boundary of V.
(2) Classifying tight contact structures on $\mathbb{Q}$-homology spheres up to contactomorphisms.
(3) Constructing or obstructing symplectic fillings of a given contact $\mathbb{Q}$-homology sphere by a $\mathbb{Q}$-homology ball.

We remark that (2) and (3) are related to or belong to some of the central problems in low-dimensional and symplectic/contact topology. More effectively implementing the ideas discussed in this section relies on further progress on these fundamental problems (see Remark 4.3 for more details).

On the other hand, as for (1), we shall revisit a beautiful construction introduced by David Gay in 1999 (cf. [36]), in which he showed how to turn a symplectic 4-manifold with convex boundary into one with concave boundary by attaching symplectic 2-handles along a certain transverse link.

More concretely, let (V', ω') be a symplectic 4-manifold with convex boundary. Suppose $L = \{K_i\} \subset \partial V'$ is a link and $p : \partial V' \setminus L \to \mathbb{S}^1$ is a fibration, such that (L, p) is compatible with the contact structure on $\partial V'$ in a suitable sense. (In the terminology of Gay [36], L is called nicely fibered with respect to the contact structure. In particular, this means that $p : \partial V' \setminus L \to \mathbb{S}^1$ is a rational open book supporting the contact structure in the sense of [4].) Then for any framing of L which is "positive" with respect to the fibration p, Gay showed in [36] how to attach a 2-handle to V' along each component K_i of the link L with the given framing, such that the resulting manifold, denoted by V, is a symplectic manifold with concave boundary.

We shall be only interested in the case where $p : \partial V' \setminus L \to \mathbb{S}^1$ defines an honest open book supporting the contact structure on $\partial V'$. In this case, the "nicely fibered" condition in [36] can be fulfilled after changing the contact structure by an isotopy (see Lemma 4.5 in [37]), and the framing of L being positive simply means that over each component K_i, the framing of K_i equals the page framing of K_i plus a positive integer k_i.

With the preceding understood, in what follows we shall be concerned with the following special situation of Gay's construction, where

- V' is a $\mathbb{Q}$-homology ball, e.g., V' is contractible,
- L has only one component and $p : \partial V' \setminus L \to \mathbb{S}^1$ is an honest open book supporting the contact structure on $\partial V'$. (When V' is Stein, there is an explicit algorithm in [2] to obtain a supporting open book for the contact boundary.)

Under these assumptions, it is clear that the resulting manifold V satisfies $b_1 = 0$ and $b_2 = b_2^+ = 1$. For our purpose here, an important issue is whether the contact structure on the boundary is fillable (in particular, tight). Since the boundary of V is concave, the filling by V does not necessarily imply the tightness. However, from Gay's construction in [36] it is known that the contact structure is supported by a canonical rational open book. One may resolve it to a honest supporting open book (cf. [4]), and then Wand's criteria [77, 78] may be used to determine whether the contact structure is tight. For fillability of the contact structure, a necessary condition is given by the non-vanishing of the Ozsváth-Szabó contact invariant (cf. [60, 47]).

We shall illustrate the idea by looking at some natural examples, where in these examples, V' is diffeomorphic to the 4-ball but the contact boundary $\partial V' = \mathbb{S}^3$ is given with some specific supporting open books with a connected binding.

A plane curve singularity $z = 0 \in C \subset \mathbb{C}^2$ is called cuspidal if the link of the singularity, i.e., $C \cap \partial B^4_\epsilon$ where B^4_ϵ is the ball of radius ϵ centered at 0, is connected for small $\epsilon > 0$. In this case, $K := C \cap \partial B^4_\epsilon$ is an algebraic knot in $\mathbb{S}^3$. It is well-known that the topological classification of the germ of C is completely determined by the isotopy class of the link K (cf. e.g. [54, 24]). Locally, up to topological equivalence a cuspidal singularity is modeled by a singularity defined by the following parametrization

$$x(t) = t^p, y(t) = t^{q_1} + t^{q_2} + \cdots + t^{q_m}.$$

The sequence of numbers $(p; q_1, q_2, \cdots, q_m)$ is called the characteristic sequence, which satisfies the following constraints: $p > 1$, $p < q_1 < q_2 < \cdots < q_m$, and if we set $r_0 = p$, $r_{i+1} = \gcd(r_i, q_{i+1})$, then the sequence r_i is strictly decreasing with $r_m = 1$. The number p is called the multiplicity of the singularity, and m is called the length of the characteristic sequence. A fundamental invariant associated to the singularity is the so-called δ-invariant, which equals the genus of the link K, i.e., $\delta = \mu/2$ where μ is the Milnor number and can be computed from the following formula (cf. [54])

$$\mu = \sum_{i=1}^{m} (q_i - 1)(r_{i-1} - r_i) = (p-1)(q_1 - 1) + \sum_{i=1}^{m-1} (r_i - 1)(q_{i+1} - q_i).$$

When $m = 1$, the link of the singularity is the (p, q)-torus knot, where $q = q_1$. While there are more subtle invariants for the analytical type of the singularity, the topological type of the singularity (the same as the isotopy class of the link) is completely determined by the characteristic sequence.

For any $n > 0$, we fix n topological types of cuspidal singularities, which will be denoted by the corresponding links, $\underline{K} = (K_1, K_2, \cdots, K_n)$, and fix an integer $k > 0$. We shall construct a symplectic 4-manifold with concave boundary, denoted by $V_{\underline{K},k}$, using Gay's construction as follows. For each $i = 1, 2, \cdots, n$, we pick a germ of plane curve singularity $z_i = 0 \in C_i \subset \mathbb{C}^2$ of type K_i, and fix a sufficiently small $\epsilon > 0$. Let B_i be the ball B^4_ϵ which contains $z_i \in C_i$, endowed with the standard symplectic structure on $\mathbb{C}^2$, and we identify K_i with the link $C_i \cap \partial B_i$. Furthermore, we shall slightly perturb C_i near the boundary of B_i so that it is tangent to the standard Liouville vector field $\frac{1}{2} r\partial r$. Note that the Milnor fibration $p_i : \partial B_i \setminus K_i \to \mathbb{S}^1$ defines an open book which supports the contact structure on ∂B_i (cf. [14]). With this understood, we attach $n-1$ symplectic 1-handles to $\sqcup_{i=1}^n B_i$, such that for each $i < n$, a 1-handle is attached to B_i, B_{i+1} with the core of the 1-handle sitting on K_i, K_{i+1}. The resulting manifold $V'_{\underline{K},k}$, a symplectic manifold with convex boundary, is diffeomorphic to the 4-ball, however, the contact boundary is equipped with a supporting open book canonically constructed from the Milnor fibrations p_i, which has a connected binding $K := K_1 \# K_2 \# \cdots \# K_n$, the connected sum of the knots K_i. The manifold $V_{\underline{K},k}$ is obtained by attaching a symplectic 2-handle along K with framing k.

We shall denote the contact boundary of $V_{\underline{K},k}$ by $(M_{\underline{K},k}, \xi_{\underline{K},k})$, where $M_{\underline{K},k}$ is canonically oriented by the contact structure. Note that since $V_{\underline{K},k}$ has a concave boundary, $M_{\underline{K},k} = -\partial V_{\underline{K},k}$ as oriented manifolds. Clearly, in our terminology $V_{\underline{K},k}$ is a small concave filling of $(M_{\underline{K},k}, \xi_{\underline{K},k})$. We note from the construction that $V_{\underline{K},k}$ contains a singular symplectic sphere, to be denoted by $C_{\underline{K},k}$, which has n cuspidal singular points with the given topological type $\underline{K}$. Clearly, $C_{\underline{K},k}$ is pseudo-holomorphic with respect to some compatible almost complex structure. Furthermore, note that $C^2_{\underline{K},k} = k > 0$. Finally, we remark that in this case, $c_1(K_{V_{\underline{K},k}})$ can be determined from the adjunction formula, from which we may compute the invariant $\theta(\xi_{\underline{K},k})$ and determine whether $V_{\underline{K},k}$ is positive or negative.

Example 4.1. (Rational cuspidal curves in $\mathbb{CP}^2$)

An algebraic curve in $\mathbb{CP}^2$ is called a rational cuspidal curve if it is topologically a two-sphere with only cuspidal singularities. It is known that rational cuspidal curves in $\mathbb{CP}^2$ exhibit severe restrictions. For example, it is conjectured that the number of singularities is at most three with the exception of a degree 5 curve with four singularities (see [61] for some experimental evidence). More generally, one is interested in what combinations of singularity types can be realized by a rational cuspidal curve in $\mathbb{CP}^2$. In this example, we shall explain an interesting connection of this problem to contact splitting of $\mathbb{CP}^2$ and related questions in symplectic and contact geometry.

Suppose we are given a topological type of cuspidal singularities $\underline{K} = (K_1, \cdots, K_n)$. Let δ_i be the δ-invariant of the singularity z_i (i.e., the genus of K_i). Assume there is an integer $d > 0$ such that $(d-1)(d-2) = \sum_{i=1}^n 2\delta_i > 0$. (Note that if the given singularity type $\underline{K}$ is realized by a rational cuspidal curve C in $\mathbb{CP}^2$, then d is the degree of C.) We shall consider the symplectic manifold $V_{\underline{K},k}$ with $k = d^2$. In this case, an easy calculation with the adjunction formula on $C_{\underline{K},k}$ shows that

$$c_1(K_{V_{\underline{K},k}}) = -\frac{3}{d} C_{\underline{K},k},$$

which implies immediately that $V_{\underline{K},k}$ is negative. Furthermore,

$$\theta(\xi_{\underline{K},k}) = -(c_1(K_{V_{\underline{K},k}})^2 - 2\chi(V_{\underline{K},k}) - 3\sigma(V_{\underline{K},k})) = -2.$$

We shall introduce the following special notations in the case of $k = d^2$: $V^0_{\underline{K}}$, $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$ and $C^0_{\underline{K}}$ for the corresponding objects $V_{\underline{K},k}$, $(M_{\underline{K},k}, \xi_{\underline{K},k})$ and $C_{\underline{K},k}$.

The negativity of $V^0_{\underline{K}}$ leads easily to the following observation, which shows that, unlike (M_0, ξ_0) in Theorem 3.1, there is no further constraint on $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$ in order to contact-split $\mathbb{CP}^2$ other than being symplectically fillable by a $\mathbb{Q}$-homology ball.

Observation 4.1: *The contact manifold $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$ is symplectically fillable by a $\mathbb{Q}$-homology ball if and only if there is a contact splitting of $\mathbb{CP}^2$ by $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$.*

Indeed, if $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$ is symplectically filled by a $\mathbb{Q}$-homology ball W, then by Corollary 2.3, $X := W \cup_{M^0_{\underline{K}}} V^0_{\underline{K}}$ is diffeomorphic to $\mathbb{CP}^2$, with a natural contact splitting by $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$. The converse implication follows from Corollary 2.2.

With the preceding understood, the following conjecture suggests an interesting connection of the rational cuspidal curve problem to symplectic and contact geometry.

Conjecture 4.1. *The following statements are equivalent.*

(1) *There is a rational cuspidal curve C in $\mathbb{CP}^2$ realizing the singularity type $\underline{K}$.*
(2) *The contact manifold $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$ is symplectically fillable by a $\mathbb{Q}$-homology ball.*

Conjecture 4.1 may be proved along the following lines (see [17] for more details): for (1)⇒(2), show that the rational cuspidal curve C has a neighborhood which is symplectically modeled by the symplectic manifold $V^0_{\underline{K}}$, and for (2)⇒(1), prove that the pseudo-holomorphic rational cuspidal curve $C^0_{\underline{K}}$ may be deformed into a genuine holomorphic rational cuspidal curve through a family of pseudo-holomorphic rational cuspidal curves, preserving the topological type of the singularities. We remark that the corresponding Gromov theory for the "symplectic isotopy problem" of singular symplectic surfaces was developed in [64, 52], see also [70, 66, 67].

Characterizing the topological types of singularities of a rational cuspidal curve of a given degree in $\mathbb{CP}^2$ is a classical problem in algebraic geometry with many interesting connections, particularly to the theory of open surfaces (cf. [33]). Crucial to the characterization problem are certain compatibility properties (not necessarily a complete set) between local invariants from the cuspidal singularities and certain global invariants such as the degree of the curve or the log Kodaira dimension of the complement of the curve in $\mathbb{CP}^2$; in fact, a complete list of rational cuspidal curves with a unique singularity whose link is a torus knot may be obtained using these compatibility properties, see [34]. A more recent breakthrough was brought about in [32] with a set of conjectured compatibility properties, which connect this old problem to the modern theories in low-dimensional topology. Further progress was made in [9] using the d-invariants from the Heegaard Floer homology theory, exploiting the fact that the complement of a regular neighborhood of the rational cuspidal curve realizing the singularity type $\underline{K}$ is a $\mathbb{Q}$-homology ball smoothly filling the 3-manifold $M^0_{\underline{K}}$. See also [5, 7]. (For a nice and relevant introduction on rational cuspidal curves, see [55].)

Let C be a rational cuspidal curve realizing singularity type $\underline{K}$. Going beyond the smooth filling of $M^0_{\underline{K}}$ by a $\mathbb{Q}$-homology ball, one may further explore the symplectic and contact aspects of the $\mathbb{Q}$-homology ball filling. In particular, consider the following:

- Any compatibility properties from the tightness or fillability of $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$.
- Compatibility properties as obstructions for $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$ to contact-split $\mathbb{CP}^2$.
- Any connection between the contact topology of $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$ and the log Kodaira dimension of the complement $\mathbb{CP}^2 \setminus C$.

Example 4.2. (A variant of Example 4.1)

In this example, we consider the construction $V_{\underline{K},k}$ with a different framing k, where we choose $k = (d-3)^2$, assuming $d > 3$ (here d is as defined in Example 4.1). With this choice of k, we have instead

$$c_1(K_{V_{\underline{K},k}}) = \frac{3}{d-3} C_{\underline{K},k}$$

from the adjunction formula on $C_{\underline{K},k}$. It is easy to see that we continue to have $\theta(\xi_{\underline{K},k}) = -2$, however, this time $V_{\underline{K},k}$ is a positive small concave filling of the contact boundary $(M_{\underline{K},k}, \xi_{\underline{K},k})$. We shall use special notations: $V_{\underline{K}}$, $(M_{\underline{K}}, \xi_{\underline{K}})$ and $C_{\underline{K}}$, for the corresponding objects in this case.

Observation 4.2: (Constructing new examples of symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$) *If the contact manifold $(M_{\underline{K}}, \xi_{\underline{K}})$ is symplectically filled by a $\mathbb{Q}$-homology ball W, then the manifold $X_{\underline{K}} := W \cup_{M_{\underline{K}}} V_{\underline{K}}$ is a new example of symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$; in particular, when $\pi_1(\partial W) \to \pi_1(W)$ is surjective, we have $X_{\underline{K}}$ an exotic $\mathbb{CP}^2$.*

Indeed, since $V_{\underline{K}}$ is positive, $X_{\underline{K}}$ is not diffeomorphic to $\mathbb{CP}^2$ by Corollary 2.3. When $\pi_1(\partial W) \to \pi_1(W)$ is surjective, $X_{\underline{K}}$ is simply connected hence an exotic $\mathbb{CP}^2$. In general, $X_{\underline{K}}$ can not be a fake $\mathbb{CP}^2$ because $\pi_2(X_{\underline{K}}) \neq 0$ (note that $C_{\underline{K}} \subset X_{\underline{K}}$ defines a nontrivial element in π_2).

Remark 4.3. For the applications in both Example 4.1 and Example 4.2, it requires construction of symplectic $\mathbb{Q}$-homology ball fillings of a given contact $\mathbb{Q}$-homology sphere, i.e., either $(M^0_{\underline{K}}, \xi^0_{\underline{K}})$ or $(M_{\underline{K}}, \xi_{\underline{K}})$. In general, such a problem is rather challenging, which involves a number of components that are all quite difficult.

First of all, given a $\mathbb{Q}$-homology sphere M, whether there exists a smooth 4-manifold W with the same $\mathbb{Q}$-homology of a 4-ball such that $\partial W = M$ (i.e., a smooth filling by a $\mathbb{Q}$-homology ball) is a fundamental problem in low-dimensional topology. Respectively, there is a related but largely independently motivated question for homology spheres, where W is required to be an integral homology ball. While there exist a number of invariants obstructing such a smooth filling, in general the problem is largely open except for some special classes of 3-manifolds, such as lens spaces, Brieskorn spheres, certain graph manifolds (which is in connection with the rational homology disk smoothings of a normal surface singularity), etc., and certain special constructions, e.g., Mazur-type 4-manifolds, or the slice-ribbon problem in knot theory. Secondly, there seem to be additional obstructions for M to be symplectically fillable by a $\mathbb{Q}$-homology ball. For example, a recent paper by Mark and Tosun [53] showed that there is an infinite family of Brieskorn spheres which are smoothly fillable but not symplectically fillable by an integral homology ball. (However, it is known that an integral homology sphere which can not be smoothly filled by an integral homology ball may be filled by a $\mathbb{Q}$-homology ball; the first such example, the Brieskorn sphere $\Sigma(2,3,7)$, was due to Fintushel and Stern, and recently two infinite families were found in [1].) Lastly, suppose M is smoothly filled by a $\mathbb{Q}$-homology ball W and suppose one actually is able to show that W carries a symplectic structure with convex contact boundary (e.g., W is Stein). It still remains to show that the induced contact structure on M is contactomorphic to the original, given contact structure on M. And this involves the fundamental problem of classification of tight contact structures, which is also largely open for a general 3-manifold.

Despite the lack of understanding in general on these fundamental questions, for the following special classes of tight contact 3-manifolds significant progress has been made in the recent past years (see [71, 72, 6, 76, 14]):

- Lens space or small Seifert space, or the connected sums.
- Link of a normal surface singularity with the Milnor fillable contact structure.

Extending Question 3.1, we consider

Problem 4.4: (Constructing small concave fillings of a given contact $\mathbb{Q}$-homology sphere) *Let (M, ξ) be a contact $\mathbb{Q}$-homology sphere from the list above which is symplectically fillable by a $\mathbb{Q}$-homology ball. Construct a small concave filling of (M, ξ) using Gay's construction described in this section.*

5. Input from the Symplectic Field Theory

The discussions in this section will be centered around the following:

Problem 5.1: *Suppose a contact $\mathbb{Q}$-homology sphere (M,ξ) contact-splits $\mathbb{CP}^2$. What can be said about the topology of M or the contact structure ξ?*

Progress on Problem 5.1 will have implications in the following problems:

- characterization of rationally convex domains in $\mathbb{C}^2$,
- classification of rational cuspidal curves in $\mathbb{CP}^2$, and most desirably,
- criteria for a symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$, $X = W \cup_M V$ (e.g., X is from the construction in Problem 4.4), to be non-diffeomorphic to $\mathbb{CP}^2$.

Regarding the last item, if $X = W \cup_M V$ where M can not contact-split $\mathbb{CP}^2$, then $X \neq \mathbb{CP}^2$. Such a criterion is in terms of the contact 3-manifold may be practically more useful than the positivity of the concave part V (cf. Corollary 2.3).

A natural approach to Problem 5.1 is by the so-called symplectic field theory techniques [27]. The method has its origin in the neck-stretching arguments in gauge theory, see also [15]. Crucial to any of this type of arguments is the corresponding Floer theory. For the case we are interested in, the foundation was laid by Hofer in [42], with additional works [43, 44, 45] joint with Wysocki and Zehnder. Later important contributions include the intersection theory of Siefring [68, 69] (see also the lecture notes by Wendl [80]). Finally, the embedded contact homology, cf. Hutchings and Taubes [48, 49, 50, 51, 74], provides the Floer homology theory in this context.

The general set-up goes as follows. Let (X,ω) be a general symplectic 4-manifold and M be a connected hypersurface of contact type, decomposing X into two connected components W and V. We assume W has the convex boundary, and let α be a contact form on M such that $\omega|_M = d\alpha$, and let $\xi = \ker\alpha$ be the corresponding contact structure. Crucial to our consideration is the Reeb dynamical system associated to the contact form α, which is generated by the Reeb vector field $Y = Y_\alpha$ on M, uniquely determined by the following equations:

$$\alpha(Y) = 1, \quad i_Y d\alpha = 0.$$

For technical reasons, we assume the contact form α is generic in the sense that all the closed orbits of the Reeb vector field Y_α are non-degenerate (cf. [46]).

A neighborhood of M in (X,ω) is modeled by $(-\epsilon,\epsilon)\times M$ in the symplectization $(\mathbb{R}\times M, d(e^s\alpha))$ for some small $\epsilon > 0$. Suppose $A \in H^2(X;\mathbb{Z})$ is a class such that the Gromov-Taubes invariant $GT(A) \neq 0$. Then for any fixed, generic choice of ω-compatible almost complex structure $\tilde{J}$ of X, which takes the following form on the neck $(-\epsilon,\epsilon)\times M$: $\tilde{J}(\frac{d}{ds}) = Y_\alpha$, $\tilde{J}|_\xi = J$ where J is a $d\alpha$-compatible complex multiplication on ξ (in particular, $\tilde{J}$ is constant in the variable s), we can, for any $R > 0$, form a 4-manifold X_R by cutting X open along M and then insert the cylinder $[-R,R]\times M$. The almost complex structure $\tilde{J}$ naturally extends to an almost complex structure on X_R, denoted by $\tilde{J}_R$, which takes the same form on $(-\epsilon-R, R+\epsilon)\times M$. With this understood, for any $R > 0$, there is an embedded $\tilde{J}_R$-holomorphic curve C_R (which may be disconnected) in X_R representing the Poincaré dual of A. Furthermore, in the limiting process $R\to\infty$, the curves C_R disintegrate into pseudo-holomorphic curves in $\mathbb{R}\times M$ and the almost complex manifolds with cylindrical ends $\widetilde{W} := W \cup [0,\infty)\times M$ and $\widetilde{V} := V \cup (-\infty,0]\times M$, where the almost complex structures are translation invariant on $\mathbb{R}\times M$ and the cylindrical ends, cf. [46, 11]. The pseudo-holomorphic curves in $\mathbb{R}\times M$ or the cylindrical end manifolds $\widetilde{W}$ and $\widetilde{V}$ are called finite energy surfaces or generalized finite energy surfaces. In particular, the finite energy surfaces in $\mathbb{R}\times M$ have deep connections with the Reeb dynamical system $\dot{y}(t) = Y_\alpha(y(t))$ on M, which in the past have been mainly exploited to show the existence of closed Reeb orbits (cf. e.g., [42, 46], see also [15]).

With the preceding understood, our purpose here is of a different nature. In the case of $X = \mathbb{CP}^2$ given with a contact splitting by (M,ξ), we are instead interested in the topological properties of the $\mathbb{Q}$-homology sphere M or the contact structure ξ, inferred from properties of the finite energy spheres in $\mathbb{R}\times M$. And to this end, we rely heavily on the following fact:

The manifold $\mathbb{CP}^2$ is foliated by embedded pseudo-holomorphic two-spheres.

Essentially, we are adapting and extending the techniques developed in the paper [46] of Hofer, Wysocki and Zehnder, but for a different purpose. We remark that the above property distinguishes $\mathbb{CP}^2$ from the other symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$s.

5.1. The concave part V contains a symplectic $(+1)$-sphere

Under this assumption, the contact $\mathbb{Q}$-homology sphere (M,ξ) can be realized as a hypersurface of contact type in $(\mathbb{R}^4,\omega_{std})$, and vice versa. In particular, if (M,ξ) is the contact boundary of a rationally convex domain in $\mathbb{C}^2$, then the corresponding contact splitting of $\mathbb{CP}^2$ by (M,ξ) satisfies this assumption.

Currently, there are no non-trivial examples, i.e., other than $M=\mathbb{S}^3$, of this type of contact splittings of $\mathbb{CP}^2$; in particular, for all the examples of rationally convex domains in $\mathbb{C}^2$ constructed in [58], the boundary is not a $\mathbb{Q}$-homology sphere. On the other hand, it is known that there are further obstructions for M to be smoothly embedded in $\mathbb{R}^4$, e.g., the lens space $L(4,1)$ can be smoothly embedded in $\mathbb{CP}^2$ but not in $\mathbb{R}^4$ (cf. [12]). Furthermore, there are also constraints from the dynamical system point of view. Indeed, if a contact $\mathbb{Q}$-homology sphere (M,ξ) is embedded in $(\mathbb{R}^4,\omega_{std})$ as a hypersurface of contact type, it is easily seen that there is a Liouville cobordism from (M,ξ) to $(\mathbb{S}^3,\xi_{std})$. Then by a theorem of Cioba and Wendl, for any non-degenerate contact form of ξ, the corresponding Reeb dynamical system on M possesses an unknotted closed orbit of Conley-Zehnder index 2 or 3, cf. [21].

We believe the following is true, cf. [18].

Conjecture 5.1. *If the concave part V contains a symplectic $(+1)$-sphere, then M must be the 3-sphere. In particular, if a rationally convex domain in $\mathbb{C}^2$ is a $\mathbb{Q}$-homology ball, then it must be diffeomorphic to the 4-ball.*

5.2. Reeb dynamical systems adapted to a given open book

Since we are interested in the topological properties of the contact manifold (M, ξ), not the Reeb dynamical systems on M, a natural strategy is to work with a Reeb dynamical system that best suits our purpose. This strategy also applies to other relevant problems which are topological in nature, e.g., gluing formula for Gromov-Taubes invariants.

This strategy is at all possible because we have complete freedom in choosing the contact form we would like to work with. More precisely, suppose α' is any other contact form which also defines the contact structure ξ. Then $\alpha' = e^f \alpha$ for some smooth function f on M. Suppose $R > 0$ is sufficiently large such that $-R < f(x) < R$ for any $x \in M$. Then the graph of f in the cylinder $(-R, R) \times M \subset X_R$ defines an embedding of M in X_R as a hypersurface of contact type, inducing the given contact form α' on M. On the other hand, X_R is diffeomorphic to the original symplectic 4-manifold X (although not necessarily symplectomorphic to X for the purpose here), which can be used to replace X in the symplectic field theory setup.

By Giroux's theorem [39], every contact structure on a 3-manifold is supported by an open book decomposition, in the sense that there is a Reeb vector field that is transversal to the pages of the open book and is tangent to the bindings. Although the supporting open book is not uniquely determined by the contact structure, this viewpoint has been extremely fruitful and fundamental in the later development of the subject. Given the central role the open book decompositions play in the 3-dimensional contact geometry, it is desirable to have this perspective in the setup of the symplectic field theory argument, particularly when the problems at hand are topological in nature.

We believe the following problem, though its formulation is somewhat "vague", would be key to success for our project and other related topological questions.

Problem 5.2: *Given an open book decomposition supporting a given contact structure, construct some specific Reeb dynamical system which is adapted to the open book in a suitable sense, for the use in the "embedded" symplectic field theory setting.*

Some initial work has been done by Colin and Honda, see [22, 23]. The basic idea goes as follows: by Thurston's theory on surface diffeomorphisms (cf. [31]), for any diffeomorphism $h : F \to F$ of an orientable surface of boundary (with negative Euler characteristic), after changing h by an isotopy, F can be decomposed along a set of disjoint circles S into subsurfaces F_i, such that $h(S) = S$, and for each i, $h|_{F_i}$ is either periodic or pseudo-Anosov up to an isotopy. With this understood, an "ideal" Reeb dynamical system would be the one whose return map will be able to "capture" these geometric pieces of the isotopy class of h.

In a forthcoming paper [19], we shall pursue this line of research in the context of contact splittings of $\mathbb{CP}^2$. A particular interesting test problem would be to give an alternative proof for Theorem 3.1 based on ideas discussed here.

6. Contact splitting in the case of Hirzebruch surfaces

In this section, we shall discuss parallel constructions and considerations in the case of Hirzebruch surfaces. We begin with the following ad hoc definition.

Definition 6.1. Given any contact $\mathbb{Q}$-homology sphere (M,ξ), a (strong) symplectic filling (W,ω) of (M,ξ) is called a **small convex filling** if the following hold:

(1) The Betti numbers of W satisfy $b_1 = 0$, $b_2 = 1$, and W is negative definite.
(2) The symplectic form is exact, i.e., $\omega = d\beta$ (e.g., W is a Stein filling).

We observe that (1) implies that $\theta(\xi) \leq -1$. Indeed,

$$\theta(\xi) = c_1(W)^2 - 2\chi(W) - 3\sigma(W) \leq 0 - 2(1+1) - 3(0-1) = -1.$$

Now let V be a small concave filling of (M,ξ). We form $X := W \cup_M V$ by gluing W and V along the contact boundary (M,ξ). Clearly, X has the $\mathbb{Q}$-homology of a Hirzebruch surface. With this understood, we make the following observation which is a consequence of Definition 6.1(2) (and the work of Taubes [73]), in analogy to Corollary 2.3.

Observation 6.1: *The manifold $X := W \cup_M V$ is diffeomorphic to a Hirzebruch surface if and only if V is a negative small concave filling of (M,ξ).*

Next we fix a topological type of cuspidal singularities $\underline{K} = (K_1, \cdots, K_n)$, and let δ_i be the corresponding δ-invariants. Suppose a positive integer k and a rational number r satisfy

$$k + rk + 2 = \sum_{i=1}^{n} 2\delta_i, \quad r^2 k \geq 8.$$

(Note that r is uniquely determined by k.) We consider the symplectic manifold $V_{\underline{K},k}$. An easy calculation with the adjunction formula on $C_{\underline{K},k}$ shows that

$$c_1(K_{V_{\underline{K},k}}) = r \cdot C_{\underline{K},k}.$$

Consequently, $V_{\underline{K},k}$ is positive (resp. negative) if and only if $r > 0$ (resp. $r < 0$). We also observe that

$$\theta(\xi_{\underline{K},k}) = -(c_1(K_{V_{\underline{K},k}})^2 - 2\chi(V_{\underline{K},k}) - 3\sigma(V_{\underline{K},k})) = 7 - r^2 k \leq -1,$$

so that the contact manifold $(M_{\underline{K},k}, \xi_{\underline{K},k})$ satisfies the necessary condition for having a small convex filling in the sense of Definition 6.1.

Example 6.1. (Rational cuspidal curves in Hirzebruch surfaces)

Suppose $V_{\underline{K},k}$ is negative. Note that in this case, one has $-1 < r < 0$. If $(M_{\underline{K},k}, \xi_{\underline{K},k})$ admits a small convex filling W, then $X := W \cup_{M_{\underline{K},k}} V_{\underline{K},k}$ is diffeomorphic to a Hirzebruch surface. Inside X there is a pseudo-holomorphic rational cuspidal curve $C_{\underline{K},k}$ realizing the given topological type of singularity $\underline{K}$. Under further unobstructedness conditions for deformation (cf. [64, 52, 70]), which in the present situation is given by

$$k + 2 - \sum_{i=1}^{n} 2\delta_i = -rk > \sum_{i=1}^{n} (p_i - 1), \text{ where } p_i \text{ is the multiplicity of singularity } z_i,$$

one may try to deform $C_{\underline{K},k}$ to a genuine (i.e., algebraic) rational cuspidal curve in the Hirzebruch surface, realizing the topological type $\underline{K}$. See [10, 8] for more discussions on rational cuspidal curves in Hirzebruch surfaces.

Example 6.2. (Constructing exotic Hirzebruch surfaces)
Likewise, if $V_{\underline{K},k}$ is positive, and $(M_{\underline{K},k}, \xi_{\underline{K},k})$ admits a small convex filling W such that $\pi_1(\partial W) \to \pi_1(W)$ is surjective, then $X \coloneqq W \cup_{M_{\underline{K},k}} V_{\underline{K},k}$ is an exotic Hirzebruch surface (i.e., homeomorphic but not diffeomorphic to a Hirzebruch surface).

We remark that regarding the construction described in section 4, in the case of Hirzebruch surfaces we still face the challenging problem of classifying tight contact structures, however, the requirement of a small convex filling for a given contact $\mathbb{Q}$-homology sphere seems to be less stringent than that of a $\mathbb{Q}$-homology ball filling (see [41] for some experimental facts in the case of Brieskorn spheres, and see also [53]). This said, one can consider the analog of Problem 4.4 as a way to construct potential examples of exotic Hirzebruch surfaces.

Finally, progress on the following problem will lead to criteria for exotic smooth structures on Hirzebruch surfaces. Furthermore, since a Hirzebruch surface is also foliated by J-holomorphic two-spheres (cf. [16]), we expect to be able to tackle the problem by the approach described in the previous section.

Problem 6.3: *Suppose $X = W \cup_M V$ is a contact splitting of a Hirzebruch surface. What can be said about the contact $\mathbb{Q}$-homology sphere M?*

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References

[1] S. Akbulut and K. Larson, *Brieskorn spheres bounding rational balls*, arXiv:1704.07739v1 [math.GT] 25 Apr 2017

[2] S. Akbulut and B. Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, Geometry & Topology **5** (2001), 319-334.

[3] A. Akhmedov and B.D. Park, *Exotic smooth structures on small 4-manifolds with odd signatures*, Invent. Math. **181** (2010), 577-603.

[4] K. L. Baker, J.B. Etnyre, and J. Van Horn-Morris, *Cabling, contact structures and mapping class monoids*, J. Diff. Geom. **90**(2012), 1-80.

[5] A. Baranowski, M. Borodzik, and J. Serano de Rodrigo, *Heegaard Floer homologies and rational cuspidal curves. Lecture Notes.* arXiv:1611.03845v1. [math.GT] 11 Nov 2016.

[6] M. Bhupal and A. Stipsicz, *Weighted homogeneous singularities and rational homology disk smoothings*, Amer. J. Math. **133** (2011), 1259-1297.

[7] J. Bodnár and A. Némethi, *Lattice cohomology and rational cuspidal curves*, Math. Res. Lett. **23** (2016), no.2, 339-375.

[8] M. Borodzik, *Rational cuspidal curves in projective surfaces. Topological versus algebraic obstructions*, arXiv:1511.05694v1 [math.GT] 18 Nov 2015

[9] M. Borodzik and C. Livingston, *Heegaard Floer homology and rational cuspidal curves*, Forum of Mathematics, Sigma (2014), Vol. 2, e28, doi:10.1017/fms.2014.28

[10] M. Borodzik and T.K. Moe, *Topological obstructions for rational cuspidal curves in Hirzebruch surfaces*, arXiv:1410.4464 v2 [math. AG] 2 Nov. 2014.

[11] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, *Compactness results in Symplectic Field Theory*, Geometry & Topology **7**(2003), 799-888.

[12] Budney, R. and Burton, B.A. *Embeddings of* 3*-manifolds in* $\mathbb{S}^4$ *from the point of view of the* 11*-tetrahedron census*, ArXiv: 0810.2346 [math. GT].

[13] D.I. Cartwright and T. Steger, *Enumeration of the* 50 *fake projective planes*, Comptes Rendus Mathematique, Elsevier Masson SAS, **348**(2010), no. 1, 11-13.

[14] C. Caubel, A. Némethi and P. Popescu-Pampu, *Milnor open books and Milnor fillable contact* 3*-manifolds*, Topology **45** (2006), 673-689.

[15] W. Chen, *Pseudo-holomorphic curves and the Weinstein conjecture*, Communications in Analysis and Geometry **8**, no.1 (2000), 115-131.

[16] W. Chen, *G-minimality and invariant negative spheres in G-Hirzebruch surfaces*, Journal of Topology **8** (2015), 621-650.

[17] W. Chen, *Rational cuspidal curves and symplectic rational homology ball filling*, in preparation.

[18] W. Chen, *Rationally convex domains in* $\mathbb{C}^2$ *which are* $\mathbb{Q}$*-acyclic*, in preparation.

[19] W. Chen, *Contact splittings of the complex projective plane*, in preparation.

[20] K. Cieliebak and Y. Eliashberg, *The topology of rationally and polynomially convex domains*, Invent. Math. **199** (2015), no.1, 215-238.

[21] A. Cioba and C. Wendl, *Unknotted Reeb orbits and nicely embedded holomorphic curves*, arXiv:1609.01660v1. [math.SG] 6 Sept 2016.

[22] V. Colin and K. Honda, *Reeb vector fields and open book decompositions*, Journal of the European Mathematical Society **15**(2013), no.2, 443-507.

[23] V. Colin, K. Honda and F. Laudenbach, *On the flux of pseudo-Anosov homeomorphisms*, Algebraic & Geometric Topology **8** (2008), 2147-2160.

[24] D. Eisenbud and W. Neumann, *Three-Dimensional Link Theory and Invariants of Plane Curve Singularities*, Annals of Math. Studies **110**, Princeton Univ. Press, Princeton, New Jersey, 1985.

[25] Y. Eliashberg, *Classification of overtwisted contact structures on* 3*-manifolds*, Invent. Math. **98** (1989), no. 3, 623-637.

[26] Y. Eliashberg, *A few remarks about symplectic filling*, Geometry & Topology **8** (2004), 277-293.

[27] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory*, GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part II, 560–673.

[28] Y. Eliashberg and E. Murphy, *Lagrangian caps*, Geom. Funct. Anal. **23**(2013), 1483-1514.

[29] J.B. Etnyre, *On symplectic fillings*, Algebraic & Geometric Topology **4** (2004), 73-80.

[30] J.B. Etnyre and K. Honda, *On symplectic cobordisms*, Math. Annalen **323** (2002), 31-39.

[31] A. Fathi, F. Laudenbach, and V. Poénaru, *Thurston's Work on Surfaces*, Translated by Djun M. Kim and Dan Margalit, Mathematical Notes **48**, Princeton Univ. Press, 2012.

[32] J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández, and A. Némethi, *On rational cuspidal projective plane curves*, Proc. London Math. Soc. (3) **92**(2006), no.1, 99-138.

[33] J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández, and A. Némethi, *On rational cuspidal plane curves, open surfaces and local singularities*, in Singularity theory, 411-442, World Sci. Publ. Hackensack, NJ (2007)

[34] J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández, and A. Némethi, *Classification of rational unicuspidal projective curves whose singularities have one Puiseux pair*, in Proceedings of Sao Carlos Workshop 2004 Real and Complex Singularities, Series Trends in Mathematics (Birkhäuser, 2007), 31-46.

[35] R. Fintushel and R. Stern, *Pinwheels and nullhomologous surgery on 4-manifolds with* $b^+ = 1$, Algebraic & Geometric Topology **11** (2011), 1649-1699.

[36] D.T. Gay, *Symplectic 2-handles and transverse links*, Trans. of A.M.S. **354** (2001), no.3, 1027-1047.

[37] D.T. Gay, *Explicit concave fillings of contact three-manifolds*, Math. Proc. Camb. Phil. Soc. **133** (2002), 431-441.

[38] P. Ghiggini, P. Lisca, and A. Stipsicz, *Tight contact structures on some small Seifert fibered 3-manifolds*, American Journal of Mathematics **129**(2007), 1403-1447.

[39] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, In: Proceedings of ICM Vol. II. (Beijing, 2002), pp. 405-414, Beijing, 2002. Higher Ed. Press.

[40] R. Gompf, *Handlebody constructions of Stein surfaces*, Ann. of Math. **148** (1998), 619-693.

[41] R. Gompf, *Smooth embeddings with Stein surface images*, Journal of Topology **6** (2013), 915-944.

[42] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. **114** (1993), 515-563.

[43] H. Hofer, K. Wysocki, and E. Zehnder, *Properties of pseudoholomorphic curves in symplectizations I. Asymptotics*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 337-379.

[44] H. Hofer, K. Wysocki, and E. Zehnder, *Properties of pseudoholomorphic curves in symplectizations II. Embedding controls and algebraic invariants*, Geom. Funct. Anal. **5**(1995), 270-328.

[45] H. Hofer, K. Wysocki, and E. Zehnder, *Properties of pseudoholomorphic curves in symplectizations III. Fredholm theory*, in Topics in Nonlinear Analysis, pp. 381-475, Progr. Nonlinear Differential Equations Appl. **35** Birkhäuser, Basel, 1999.

[46] H. Hofer, K. Wysocki and E. Zehnder, *Finite energy foliations of tight three-spheres and Hamiltonian dynamics*, Ann. of Math. **157** (2003), 125-257.

[47] K. Honda, W. Kazez and G. Matić, *On the contact class in Heegaard Floer homology*, J. Diff. Geom. **83**(2009), 289-311.

[48] M. Hutchings, *An index inequality for embedded pseudoholomorphic curves in symplectizations*, J. Eur. Math. Soc. **4** (2002), 313-361.

[49] M. Hutchings, *The embedded contact homology index revisited*, New perspectives and challenges in symplectic field theory, pp. 263-297, CRM Proc. Lecture Notes, **49**, AMS, 2009.

[50] M. Hutchings and C.H. Taubes, *Gluing pseudoholomorphic curves along branched covered cylinders I*, J. Symplectic Geometry **5** (2007), 43-137.

[51] M. Hutchings and C.H. Taubes, *Gluing pseudoholomorphic curves along branched covered cylinders II*, J. Symplectic Geometry **7** (2009), 29-133.

[52] S. Ivashkovich and V. Shevchishin, *Structure of the moduli space in a neighborhood of a cusp-curve and meromorphic hulls*, Invent. Math. **136**(1999), 571-602.

[53] T. Mark and B. Tosun, *Obstructing pseudoconvex embeddings and contractible Stein fillings for Brieskorn spheres*, arXiv:1603.07710v1 [math. GT] 24 Mar 2016

[54] J. Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Math. Studies **61**, Princeton Univ. Press, Princeton, New Jersey, 1968.

[55] T.K. Moe, *Rational cuspidal curves*, Master Thesis, University of Oslo, 2008.

[56] E. Murphy, *Loose Legendrian embeddings in high dimensional contact manifolds*, arXiv:1201.2245 [math.GT].

[57] S. Nemirovski, *Homology class of a Lagrangian Klein bottle*, Izvestiya: Mathematics **73**: 4 (2009), 689-698.

[58] S. Nemirovski and K. Siegel, *Rationally convex domains and singular Lagrangian surfaces in* $\mathbb{C}^2$, Invent. Math. **203** (2016), 333-358.

[59] H. Ohta and K. Ono, *Simple singularities and topology of symplectically filling* 4*-manifold*, Comment. Math. Helv. **74**(1999), 575-590.

[60] P. Ozsváth and Z. Szabó, *Heegaard Floer homologies and contact structures*, Duke Math. J. **129** (2005), 39-61.

[61] J. Piontkowski, *On the number of cusps of rational cuspidal plane curves*, Experimental Math. **16** (2007), no.2, 251-255.

[62] G. Prasad and S.-K. Yeung, *Fake projective planes*, Inventiones Mathematicae, **168** no. 2 (2007) 321-370.

[63] G. Prasad and S.-K. Yeung, *Addendum to "Fake projective planes"*, Inventiones Mathematicae, **182** no. 1 (2010), 213-227.

[64] V.V. Shevchishin, *Pseudoholomorphic curves and the symplectic isotopy problem*, arXiv: math/0010262v1 [math.SG] 27 Oct 2000.

[65] V.V. Shevchishin, *Lagrangian embeddings of the Klein bottle and combinatorial properties of mapping class groups*, Izvestiya: Mathematics **73**: 4 (2009), 797-859.

[66] B. Siebert and G. Tian, *On the holomorphicity of genus two Lefschetz fibrations*, Annals of Math **161** (2005), 959-1020.

[67] B. Siebert and G. Tian, *Lectures on pseudo-holomorphic curves and the symplectic isotopy problem*, Symplectic 4-manifolds and algebraic surfaces, pp. 269-341, Lecture Notes in Math., **1938**, Springer, Berlin, 2008.

[68] R. Siefring, *Relative asymptotic behavior of pseudoholomorphic half-cylinders*, Comm. Pure Appl. Math.**61** (2008), no. 12, 1631-1684.

[69] R. Siefring, *Intersection theory of punctured pseudoholomorphic curves*, Geom. Topol. **15** (2011), 2351-2457.

[70] J.-C. Sikorav, *The gluing construction for normally generic J-holomorphic curves*, in Symplectic and Contact Topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), pp. 175-199, Fields Inst. Commun. **35**, Amer. Math. Soc., Providence, RI, 2003.

[71] A. Stipsicz, *Ozsváth-Szabó invariants and* 3*-dimensional contact topology*, Proceedings of the International Congress of Mathematicians, Hyderabad 2010, vol II, 1159-1178.

[72] A. Stipsicz, Z. Szabó and J. Wahl, *Rational blowdowns and smoothings of surface singularities*, Journal of Topology **1** (2008), 477-517.

[73] C.H. Taubes, *Seiberg-Witten and Gromov Invariants for Symplectic* 4*-Manifolds*, Proceedings of the First IP Lectures Series, Volume II, R. Wentworth ed., International Press, Somerville, MA, 2000

[74] C.H. Taubes, *Embedded contact homology and Seiberg-Witten Floer cohomology I*, Geometry & Topology **14** (2010), 2497-2581.

[75] B. Tosun, *Tight small Seifert fibered manifolds with* $e_0 = -2$, arXiv: 1510.06948v2 [math. GT].

[76] J. Wahl, *On rational homology disk smoothings of valency* 4 *surface singularities*, Geom. Topol. **15** (2011), 1125-1156.

[77] A. Wand, *Detecting tightness via open book decompositions*, Proceedings of the conference 'Interactions between low dimensional topology and mapping class groups', MPIM-Bonn, June 2013.

[78] A. Wand, *Tightness is preserved by Legendrian surgery*, Annals of Mathematics (2) **182** (2015), no. 2, 723-738.

[79] A. Weinstein, *On the hypothesis of Rabinowitz's periodic orbit theorems*, J. Diff. Equ. **33** (1979), 353-358.

[80] C. Wendl, *Contact* 3*-manifolds, holomorphic curves and intersection theory*, arXiv:1706.05540v1 [math.SG] 17 Jun 2017.

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