PART 2: PSEUDO-HOLOMORPHIC CURVES

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Contents

1. Properties of J-holomorphic curves

The main reference of this section is McDuff-Salamon [18].

1.1. **Basic definitions.** Let M be a smooth manifold of dimension $2n$, and ω be a non-degenerate 2-form on M. An almost complex structure $J : TM \rightarrow TM$ is called *ω-tame* if

$$v \neq 0 \Rightarrow \omega(v, Jv) > 0, \forall v \in T_xM, x \in M.$$

It is called *ω-compatible* if it is ω-tame and

$$\omega(Jv, Jw) = \omega(v, w), \forall v, w \in T_xM, x \in M.$$

Every ω-tame almost complex structure J determines a Riemannian metric

$$g_J(v, w) := \langle v, w \rangle_J := \frac{1}{2}(\omega(v, Jw) + \omega(w, Jv)).$$

In the ω-compatible case, this metric is simply $\langle\cdot,\cdot\rangle_J = \omega(\cdot, J\cdot)$. Throughout we shall denote by $\mathcal{J}(M,\omega)$ the space of ω-compatible almost complex structures and $\mathcal{J}_\tau(M,\omega)$ the space of ω-tame almost complex structures. Note that $\mathcal{J}(M,\omega) \subset \mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}_\tau(M,\omega)$ is open in the space of all almost complex structures.

Proposition 1.1. *The space $\mathcal{J}_\tau(M,\omega)$ is contractible.*

Proof. Let ω_0 be the standard symplectic structure on $\mathbb{R}^{2n}$ and let $\mathcal{J}_\tau(\omega_0)$ be the space of complex structures on $\mathbb{R}^{2n}$ which are ω_0-tame. Then $\mathcal{J}_\tau(M,\omega)$ is the space of smooth sections of a certain fiber bundle over M with fiber $\mathcal{J}_\tau(\omega_0)$. Therefore, the contractibility of $\mathcal{J}_\tau(M,\omega)$ follows from the contractibility of $\mathcal{J}_\tau(\omega_0)$, which we shall prove next. The proof is due to Sévennec, cf. [3].

Let J_0 be the standard complex structure on $\mathbb{R}^{2n}$ such that $\omega_0(\cdot, J_0\cdot)$ is the standard Euclidean metric. Let

$$X := \{S | S \text{ is } 2n\times 2n \text{ matrix }, ||S|| < 1, SJ_0 + J_0 S = 0\},$$

where $||S|| := \max_{0\neq v\in\mathbb{R}^{2n}} |Sv|/|v|$. Clearly, X is contractible. We will show that for any $J \in \mathcal{J}_\tau(\omega_0)$, the map

$$\Phi : J \mapsto (J + J_0)^{-1}(J - J_0)$$

defines a diffeomorphism between $\mathcal{J}_\tau(\omega_0)$ and X.

First of all, we show that $J + J_0$ is invertible, so that Φ is defined. To see this, note that for any $0 \neq v \in \mathbb{R}^{2n}$, $\omega_0(v,(J + J_0)v) = \omega_0(v, Jv) + \omega_0(v, J_0v) > 0$, which implies that $\ker(J + J_0) = \{0\}$. Hence $J + J_0$ is invertible.

Next we show that $\Phi(J) \in X$ for each $J \in \mathcal{J}_\tau(\omega_0)$. Let $S := \Phi(J)$ and $A := -J_0J$. Then $S = (A + Id)^{-1}(A - Id)$. It is easy to check that $||S|| < 1$ is equivalent to $||A - Id|| < ||A + Id||$. To see the latter, let $0 \neq v \in \mathbb{R}^{2n}$. Then

$$\begin{aligned} |Av + v|^2 - |Av - v|^2 &= \omega_0(Av + v, J_0(Av + v)) - \omega_0(Av - v, J_0(Av - v)) \\ &= 4\omega_0(Av, J_0v) \\ &= 4\omega_0(v, Jv) > 0. \end{aligned}$$

Hence $||S|| < 1$. To see $SJ_0 + J_0S = 0$, note that

$$\begin{aligned} SJ_0 &= (J + J_0)^{-1}(J - J_0)J_0 = (J + J_0)^{-1}(JJ_0 + Id) = (J + J_0)^{-1}J(J_0 - J) \\ &= -(J(J + J_0))^{-1}(J_0 - J) = ((J_0 + J)J_0)^{-1}(J - J_0) = -J_0(J + J_0)^{-1}(J - J_0) \\ &= -J_0S. \end{aligned}$$

Hence $\Phi(J) \in X$ for any $J \in \mathcal{J}_\tau(\omega_0)$. It remains to show that Φ^{-1} exists. For any $S \in X$, note that $||S|| < 1$ so that $Id - S$ is invertible, and

$$J := \Phi^{-1}(S) := J_0(Id + S)(Id - S)^{-1}$$

is defined. It is straightforward to check that $SJ_0 + J_0S = 0$ implies that $J^2 = -Id$, and moreover, J is ω_0-tame. (Check that for any $v \in \mathbb{R}^{2n}$, $\omega_0(v, Jv) = |w|^2 - |Sw|^2$ where $w = (Id - S)^{-1}v$.) ∎

Exercise 1.2. Prove that the space $\mathcal{J}(M,\omega)$ is also contractible.

The *Nijenhuis tensor* $N = N_J \in \Omega^2(M, TM)$ of an almost complex structure J is defined by

$$N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \ \ \forall X, Y \in \mathrm{Vect}(M).$$

An almost complex structure J is called *integrable* if (M, J) is a complex manifold. Then a theorem of Newlander and Nirenberg says that J is integrable iff $N_J \equiv 0$. In particular, $N_J \equiv 0$ for any 2-dimensional manifold, and the Newlander-Nirenberg theorem recovers the classical result that any almost complex structure on a Riemann surface is integrable.

Exercise 1.3. Suppose $J \in \mathcal{J}(M,\omega)$. Let ∇ be the Levi-Civita connection for the metric g_J. Show that
(1) $\nabla J \equiv 0$ iff J is integrable and ω is closed.
(2) Suppose ω is closed (i.e. symplectic). Define

$$\widetilde{\nabla}_X Y := \nabla_X Y - \frac{1}{2} J(\nabla_X J)Y, \ \ \forall X, Y \in \mathrm{Vect}(M).$$

Show that $\widetilde{\nabla}$ is a connection which preserves J as well as g_J. Moreover, show that its torsion $T(X,Y) := \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y]$ is given by $\frac{1}{4} N_J(X,Y)$.

Let (Σ, j) be a Riemann surface with a fixed complex structure j. (In this chapter, Σ will be either a closed Riemann surface or a disc of radius $R > 0$ in $\mathbb{C}$.) Let J be an almost complex structure on M. A smooth map $u : \Sigma \to M$ is called a (J, j)*-holomorphic curve*, or simply a J*-holomorphic curve* if

$$J \circ du = du \circ j.$$

By introducing

$$\bar{\partial}_J(u) := \frac{1}{2}(du + J \circ du \circ j),$$

one can re-write the J-holomorphic curve equation in the form $\bar{\partial}_J(u) = 0$. In local coordinates, the above equation can be written as follows. Suppose $z = s + it$ be a local coordinate on Σ, and $u = (u_\alpha)$ in a local coordinates of M. Then the J-holomorphic curve equation is given by the following system

$$\partial_s u_\alpha + \sum_{\beta=1}^{2n} J_{\alpha\beta}(u) \partial_t u_\beta = 0, \ \ \alpha = 1, \cdots, 2n = \dim M,$$

where $J(u) = (J_{\alpha\beta}(u))$. In particular, when J is integrable, i.e., $J = J_0$ the standard complex structure and $u = f + ig : D \subset \mathbb{C} \to \mathbb{C}^n$, the above system reduces to the Cauchy-Riemann equations

$$\partial_s f = \partial_t g, \ \ \partial_s g = -\partial_t f.$$

We end this section with a discussion on the energy of a smooth map from Σ to M. To this end, let g_j be a metric on Σ whose conformal structure is the complex structure j. Then for any smooth map $u:\Sigma\rightarrow M$, the *energy density* of u is defined to be the square of the norm of its differential $du\in\Omega^1(\Sigma, TM)$,

$$e(u):=|du|^2=|du(e_1)|^2_{g_J}+|du(e_2)|^2_{g_J},$$

where (e_1,e_2) is a local orthonormal frame with respect to the metric g_j, and the *energy* of u is defined to be

$$E(u):=\frac{1}{2}\int_\Sigma e(u)dvol_\Sigma,$$

where $dvol_\Sigma$ is the volume form of the metric g_j. The following fact is of fundamental importance in Gromov compactness.

Exercise 1.4. Show that even though the energy density $e(u)$ depends on the metric g_j, the energy $E(u)$ depends only on the complex (or conformal) structure j. Moreover, for any e_1, $e_2:=j(e_1)$, if the dual of (e_1,e_2) is denoted by (e^1,e^2), then

$$E(u)=\frac{1}{2}\int_\Sigma(|du(e_1)|^2_{g_J}+|du(e_2)|^2_{g_J})e^1\wedge e^2.$$

Proposition 1.5. *(Energy identity) Let ω be a non-degenerate 2-form on M.*
(1) Suppose $J\in\mathcal{J}_\tau(M,\omega)$ and $u:\Sigma\rightarrow M$ is J-holomorphic. Then

$$E(u)=\int_\Sigma u^*\omega.$$

(2) Suppose $J\in\mathcal{J}(M,\omega)$. Then for any smooth map $u:\Sigma\rightarrow M$,

$$E(u)=\int_\Sigma|\bar\partial_J(u)|^2_{g_J}dvol_\Sigma+\int_\Sigma u^*\omega.$$

Proof. Choose a local conformal coordinate $z=s+it$ on Σ, and note the following local expression for $\bar\partial_J(u)$:

$$\bar\partial_J(u)=\frac{1}{2}(\partial_s u+J\partial_t u)ds+\frac{1}{2}(\partial_t u-J\partial_s u)dt.$$

Then

$$\begin{aligned}\frac{1}{2}e(u)dvol_\Sigma &= \frac{1}{2}(|\partial_s u|^2_{g_J}+|\partial_t u|^2_{g_J})ds\wedge dt\\ &= \frac{1}{2}|\partial_s u+J\partial_t u|^2_{g_J}ds\wedge dt-g_J(\partial_s u,J\partial_t u)ds\wedge dt\\ &= |\bar\partial_J(u)|^2_{g_J}dvol_\Sigma+\frac{1}{2}(\omega(\partial_s u,\partial_t u)+\omega(J\partial_s u,J\partial_t u))ds\wedge dt.\end{aligned}$$

If $J\in\mathcal{J}(M,\omega)$, then the second term in the bottom of the right hand side is $u^*\omega$. If $J\in\mathcal{J}_\tau(M,\omega)$ and u is J-holomorphic, then $\partial_s u+J\partial_t u=0$ so that the bottom of the right hand side is $u^*\omega$. □

1.2. **Unique continuation and critical points.** This section concerns some fundamental local properties of J-holomorphic curves. Suppose $u, v : \Sigma \to M$ are J-holomorphic curves and there is a $z_0 \in \Sigma$ such that $u(z_0) = v(z_0) \in M$. We would like to describe the behavior of the difference of u, v near z_0. Since the problem is local, we may assume without loss of generality that $\Sigma = D \subset \mathbb{C}$ is a disc in $\mathbb{C}$ and $M = \mathbb{R}^{2n}$ after fixing a coordinate chart on M.

First of all, an integrable function $w : D \subset \mathbb{C} \to \mathbb{R}^{2n}$ is said to *vanish to the infinite order* at $z = 0 \in D$ if

$$\int_{|z|\leq r} |w(z)| dxdy = O(r^k)$$

for every integer $k > 0$. Note that if w is smooth, this is saying that the ∞-jet of w (i.e. all partial derivatives of w) vanishes at $z = 0$.

Theorem 1.6. *Suppose that u, v are C^1-maps and are J-holomorphic for some almost complex structure J which is of $C^{1,1}$-class. If u, v agree to infinite order at a point, then $u \equiv v$.*

Proof. When J is integrable, u, v are holomorphic maps and the result is classical. For non-integrable J, the maps u, v still resemble some of the properties of holomorphic maps (see the Micallef-White theorem in §1.4), and the result should be understood from this perspective. The proof given here is based on a theorem of Aronszajn ([2]).

Theorem 1.7. *(Aronszajn Theorem) Let $\Omega \subset \mathbb{C}$ be a connected open set. Suppose a function $w \in L^{2,2}_{loc}(\Omega; \mathbb{R}^m)$ satisfies the following differential inequality for $z = s+it \in \Omega$ almost everywhere*

$$|(\partial_s^2 + \partial_t^2)w| \leq C(|w| + |\partial_s w| + |\partial_t w|), \text{ where } C > 0 \text{ is a constant.}$$

Then $w \equiv 0$ if w vanishes to infinite order at some $z_0 \in \Omega$.

Recall that in local coordinates, the J-holomorphic curve equation is given the a system

$$\partial_s u_\alpha + \sum_{\beta=1}^{2n} J_{\alpha\beta}(u)\partial_t u_\beta = 0, \quad \alpha = 1, \cdots, 2n.$$

Differentiating both sides (as if u is of C^2-class), one obtains

$$(\partial_s^2 + \partial_t^2)u_\alpha = \sum_{\beta,\gamma=1}^{2n} \partial_\gamma J_{\alpha\beta}(u)(\partial_t u_\gamma \partial_s u_\beta - \partial_s u_\gamma \partial_t u_\beta).$$

In the current situation where u is only of C^1-class, the above equations hold true weakly.

Exercise: Let u be a J-holomorphic curve where J is of C^1-class and u is of $L^{1,p}$-class for some $p > 2$. Show that in local coordinates the following equations are satisfied weakly:

$$(\partial_s^2 + \partial_t^2)u_\alpha = \sum_{\beta,\gamma=1}^{2n} \partial_\gamma J_{\alpha\beta}(u)(\partial_t u_\gamma \partial_s u_\beta - \partial_s u_\gamma \partial_t u_\beta).$$

Note that when J and u are of C^1-class, the right-hand side is continuous, so that it lies in L^p_{loc} for any $p>1$. By elliptic regularity for the Laplacian $-(\partial_s^2+\partial_t^2)$, u is of class $L^{2,p}_{loc}$ for any $p>1$.

Now set $w:=u-v$. Then the above argument shows that $w\in L^{2,2}_{loc}$. Since $\partial_\gamma J_{\alpha\beta}$ are Lipschitz functions, and $\partial_s u_\alpha$, $\partial_t u_\alpha$ are continuous hence bounded, w satisfies almost everywhere

$$|(\partial_s^2+\partial_t^2)w|\leq C(|w|+|\partial_s w|+|\partial_t w|), \text{ where } C>0 \text{ is a constant,}$$

and Theorem 1.6 follows from Aronszajn Theorem. □

The following theorem of Hartman-Wintner gives an even more useful local description for $w=u-v$ when u,v are not identical. See [18] for a proof of the Hartman-Wintner theorem.

Theorem 1.8. *(Hartman-Wintner) Assume $0<\alpha<1$. Let $a,b,c: D\subset\mathbb{C}\rightarrow\mathbb{R}$ be three $C^{1,\alpha}$-functions such that*

$$a>0,\;\; c>0,\;\; ac-b^2>0.$$

Let $u: D\subset\mathbb{C}\rightarrow\mathbb{R}^N$ be a C^2-map satisfying $u(0)=0$ and the estimate

$$|a\partial_s^2 u+2b\partial_s\partial_t u+c\partial_t^2 u|\leq C(|u|+|du|).$$

Then if u is not identically zero, there exists an integer $m\geq 1$ and a nonzero homogeneous polynomial $h:\mathbb{C}\rightarrow\mathbb{R}^N$ of degree m such that

$$u(z)=h(z)+o(|z|^m),\;\; du(z)=dh(z)+o(|z|^{m-1}).$$

Now back to Theorem 1.6, since u is of class $L^{2,p}_{loc}$ for any $p>1$, u is a $C^{1,\alpha}$-map for some $0<\alpha<1$ by Morrey's embedding theorem. This then implies that the right hand side of

$$(\partial_s^2+\partial_t^2)u_\alpha=\sum_{\beta,\gamma=1}^{2n}\partial_\gamma J_{\alpha\beta}(u)(\partial_t u_\gamma\partial_s u_\beta-\partial_s u_\gamma\partial_t u_\beta)$$

is of class $C^{0,\alpha}$. By elliptic regularity, u is of $C^{2,\alpha}$-class. In particular, $w=u-v$ is a C^2-map, and by Hartman-Wintner, there exists a nonzero homogeneous polynomial h of degree m for some $m\geq 1$ such that

$$w(z)=h(z)+o(|z|^m),\;\; dw(z)=dh(z)+o(|z|^{m-1}).$$

In particular, by taking v to be a constant map, one obtains a local description for J-holomorphic maps.

Proposition 1.9. *Let $u: D\subset\mathbb{C}\rightarrow\mathbb{R}^{2n}$ be a non-constant, C^1 J-holomorphic curve with $u(0)=0$, where J is of $C^{1,\alpha}$-class. Then there exists an integer $m\geq 1$ and a non-zero vector $a\in\mathbb{C}^n$, such that after suitable identification $\mathbb{R}^{2n}=\mathbb{C}^n$,*

$$u(z)=az^m+o(|z|^m),\;\; du(z)=maz^{m-1}dz+o(|z|^{m-1}).$$

Proof. By Hartman-Wintner, there exists an integer $m\geq 1$ and a nonzero homogeneous polynomial $h:\mathbb{C}\rightarrow\mathbb{R}^{2n}$ of degree m such that

$$u(z)=h(z)+o(|z|^m),\;\; du(z)=dh(z)+o(|z|^{m-1}).$$

Now we fix an identification $C^n = \mathbb{R}^{2n}$ such that $J(0) = J_0 = i$. Then the above local description of u implies that $J(u(z)) = J_0 + O(|z|^m)$. On the other hand, u satisfies the J-holomorphic curve equation $\partial_s u + J(u)\partial_t u = 0$, which gives rise to

$$\partial_s h + J_0 \partial_t h = 0.$$

Hence $h(z)$ is holomorphic. Being non-zero and homogeneous of degree m, $h(z) = az^m$ for some non-zero $a \in \mathbb{C}^n$. $\square$

This has the following corollary. First, a point $z \in \Sigma$ is called a *critical point* of a J-holomorphic curve $u : \Sigma \to M$ if $du(z) = 0$, in which case, $u(z) \in M$ is called a *critical value*.

Corollary 1.10. *Let Σ be a compact Riemann surface without boundary, let J be an almost complex structure of $C^{1,\alpha}$-class, and $u : \Sigma \to M$ be a non-constant J-holomorphic curve. Then $u^{-1}(x)$ is a finite set for any $x \in M$. In particular, the preimage of the set of critical values of u is finite.*

Proof. By Proposition 1.9, a point of Σ is a critical point of u iff near that point the local description of u as given in Prop. 1.9 has $m \geq 2$. But this implies that in a small neighborhood there is no other critical point as $du(z) = maz^{m-1} + o(|z|^{m-1})$. In other words, critical points are isolated. Since Σ is compact, there are only finitely many critical points of u, and therefore, there are only finitely many critical values. To see that $u^{-1}(x)$ is a finite set for any $x \in M$, we assume that to the contrary it is infinite. Since Σ is compact, $u^{-1}(x)$ as a closed subset is also compact. Hence there exists a sequence of distinct points $z_i \in u^{-1}(x)$ such that z_i converges to a $z_0 \in u^{-1}(x)$. Clearly z_0 is a critical point, because otherwise u would be a local embedding near z_0. We write u locally near z_0 as

$$u(z) = az^m + o(|z|^m), \quad \text{where } 0 \neq a \in \mathbb{C}^n, m \geq 2.$$

If there is a sequence z_i converging to 0 but $z_i \neq 0$, and $u(z_i) = u(0) = 0$, then it would imply that $a = 0$, which is a contradiction. Hence $u^{-1}(x)$ is a finite set for any $x \in M$. $\square$

Exercise 1.11. Let J be a $C^{2,1}$ almost complex structure on M and $u, v : D \to M$ be C^1 J-holomorphic curves such that $u(0) = v(0)$ and $du(0) \neq 0$. Moreover, assume there are sequences $z_i, w_i \in D$ such that

$$u(z_i) = v(w_i), \quad \lim_{i\to\infty} z_i = \lim_{i\to\infty} w_i = 0, \quad w_i \neq 0.$$

Prove that there exists a holomorphic function ϕ defined near 0 such that $v = u \circ \phi$.

Hints: (1) Show that there is a $C^{1,1}$ coordinate chart centered at $u(0)$ such that

$$u(z) = (z, 0, \cdots, 0), \quad J(z, 0, \cdots, 0) = J_0 = i.$$

(2) In this coordinate chart, write $v(z) = (v_1(z), v_2(z))$, where $v_1(z) \in \mathbb{C}$, $v_2(z) \in \mathbb{C}^{n-1}$. (Here $2n = \dim M$.) Then $v_2(z)$ satisfies

$$|(\partial_s^2 + \partial_t^2)v_2| \leq C(|v_2| + |\partial_s v_2| + |\partial_t v_2|) \text{ for some } C > 0.$$

1.3. **Simple curves.** Let (Σ, j) be a connected Riemann surface and (M, J) be an almost complex manifold. A J-holomorphic curve $u : \Sigma \to M$ is said to be *multiply covered* if there exists a Riemann surface (Σ', j'), a J-holomorphic curve $u' : \Sigma' \to M$, and a non-trivial holomorphic branched covering $\phi : \Sigma \to \Sigma'$ such that $u = u' \circ \phi$. The curve u is called *simple* if it is not multiply covered. A point $z \in \Sigma$ is called an *injective point* if

$$du(z) \neq 0, \;\; u^{-1}(u(z)) = \{z\}.$$

A J-holomorphic curve is called *somewhere injective* if there is an injective point. Note that a somewhere injective curve must be simple.

Theorem 1.12. *Let J be a $C^{2,1}$ almost complex structure, Σ be a compact, connected Riemann surface without boundary.*

(1) For any J-holomorphic curve $u : \Sigma \to M$, there is a somewhere injective curve $u' : \Sigma' \to M$ and a holomorphic branched covering such that $u = u' \circ \phi$.

(2) Let $u : \Sigma \to M$ be a simple J-holomorphic curve. Then the complement of the set of injective points in Σ is at most countable and can only accumulate at critical points of u. In particular, u is somewhere injective.

Proof. Denote by $u(\Sigma)$ the image of u in M. Let $X' \subset u(\Sigma)$ be the set of critical values of u, and let $X := u^{-1}(X') \subset \Sigma$ be the preimage. Then both X, X' are finite sets by Corollary 1.10. Note that u is an immersion on $\Sigma \setminus X$.

Consider the subset Q of $u(\Sigma) \setminus X'$, which consists of points x such that there exist $z_1, z_2 \in \Sigma \setminus X$, $z_1 \neq z_2$, with the following properties:

$$u(z_1) = u(z_2) = x, \text{ and there are no neighborhoods } U_1, U_2 \text{ of } z_1, z_2 \text{ s.t. } u(U_1) = u(U_2).$$

By the result in Exercise 1.11, there are neighborhoods U_1, U_2 of z_1, z_2 respectively, such that $u(U_1) \cap u(U_2) = \{x\}$. It follows that $u^{-1}(Q)$ is a discrete subset of $\Sigma \setminus X$, which has no accumulation points in $\Sigma \setminus X$. Note that $u^{-1}(Q) \cup X$ is at most countable and can only accumulate at the critical points of u.

Now consider the subset $S := u(\Sigma) \setminus (Q \cup X')$. It is clear that S is an embedded, 2-dimensional submanifold of M (of class $C^{3,\alpha}$). The tangent space at each point of S is invariant under J, which gives rise to an almost complex structure on S. This makes S into an open Riemann surface with ends (by the Newlander-Nirenberg theorem), such that the inclusion map $i : S \hookrightarrow M$ is J-holomorphic. Moreover, one can compactify S into a compact, connected Riemann surface Σ' without boundary such that $i : S \hookrightarrow M$ extends to a J-holomorphic curve $u' : \Sigma' \to M$. Clearly, u' is somewhere injective. In fact, from the construction, the complement of the set of injective points in Σ' is at most countable and can only accumulate at critical points of u'. Finally, when restricted to $u^{-1}(Q) \cup X$, u is a local biholomorphism onto the open Riemann surface S. By removable singularity theorem in one complex variable, this extends to a holomorphic branched covering $\phi : \Sigma \to \Sigma'$. It is clear that $u = u' \circ \phi$. This proves (1).

For (2), if one assumes that u is simple, then ϕ must be a biholomorphism. The claim concerning the complement of injective points must be true for u since it has been shown to be true for u'. $\square$

Corollary 1.13. *Let J be a $C^{2,1}$ almost complex structure, and Σ_1, Σ_2 be compact, connected Riemann surfaces without boundary. Suppose $u_i : \Sigma_i \to M$, $i = 1, 2$, are simple J-holomorphic curves such that $u(\Sigma_1) = u(\Sigma_2)$. Then there exists a biholomorphism $\phi : \Sigma_1 \to \Sigma_2$ such that $u_1 = u_2 \circ \phi$.*

Remark 1.14. A connected subset C of an almost complex manifold (M, J) is called a *J-holomorphic curve* if C is the image of a J-holomorphic curve $u : \Sigma \to M$. The map u is called a *parametrization* of C if u is simple. With this understood, we have just shown above that any J-holomorphic curve C has a parametrization and any two distinct parametrizations of C differ by a reparametrization of biholomorphisms between the domains.

1.4. **Adjunction inequality.** The purpose of this section is to discuss singularity and intersection properties of J-holomorphic curves in an almost complex 4-manifold (M, J). These results play a fundamental role in the study of symplectic 4-manifolds via the pseudo-holomorphic curve theory. The exposition of this material here follows the paper of Micallef and White [19], sections 6 and 7.

The key technical input is the following theorem of Micallef and White (see Theorems 6.1 and 6.2 in [19]).

Theorem 1.15. *(Micallef-White) Let M be a smooth manifold of dimension $2n$ and J be an almost complex structure on M of C^2-class, and let Σ be a compact Riemann surface (with or without boundary). Let $u : \Sigma \to M$ be a simple, non-constant J-holomorphic curve of C^1-class. Then for any point $x \in u(\Sigma) \setminus u(\partial\Sigma)$, there are neighborhoods $U_i \subset \Sigma$ of p_i, where $u^{-1}(x) = \{p_i\} \subset \Sigma$, and a neighborhood V of x, such that there exist a C^1 coordinate chart $\Psi : V \to \mathbb{C}^n$ at x, $C^{2,\alpha}$ coordinate charts $\psi_i : U_i \to \mathbb{C}$ at p_i, so that each $\Psi \circ u \circ \psi_i^{-1}$ is a holomorphic map, and moreover, it can be written in the form*

$$\Psi \circ u \circ \psi_i^{-1}(z) = (z^{Q_i}, f_i(z))$$

where $Q_i \geq 1$ and $f_i(z) \in \mathbb{C}^{n-1}$ vanishes to an order $\geq Q_i$ at $z = 0$ (one can even assume at least one of the f_i's vanishes to an order $> Q_i$.)

Remark 1.16. One corollary of this theorem is that by choosing the U_i's sufficiently small, $u(U_i) \cap u(U_j) = \{x\}$ for any pair of i, j with $i \neq j$. This is an improvement over the result in Exercise 1.11 in that the assumption $du(0) \neq 0$ therein may be dropped. This in turn implies that the complement of the set of injective points in Theorem 1.12 for a simple curve is actually finite.

In what follows, M is a compact closed almost complex 4-manifold. The Micallef-White theorem allows us to introduce the following definition.

Definition 1.17. (1) Let $u_i : U_i \to M$, $i = 1, 2$, be two simple J-holomorphic curves such that $u_1(U_1) \cap u_2(U_2) = \{x\}$. Moreover, u_i are embedded on $U_i \setminus \{p_i\}$, where $u_1(p_1) = u_2(p_2) = x$. By Micallef-White theorem, there exists a C^1 coordinate chart $\Psi : V \to \mathbb{C}^n$ at x and $C^{2,\alpha}$ coordinate charts ψ_i at p_i, such that $\Psi \circ u_i \circ \psi_i^{-1}(z) = (z^{Q_i}, f_i(z))$ where $f_i(z)$ vanishes to an order $\geq Q_i$ at $z = 0$. With this understood, define the local intersection number of u_1, u_2 at (p_1, p_2) by

$$\delta(p_1,p_2) := \frac{1}{m_1 m_2}\sum_{\nu^Q=1} \text{the order of vanishing of } f_1(\nu z^{m_1}) - f_2(z^{m_2}) \text{ at } z=0,$$

where Q is the least common multiple of Q_1, Q_2 and $Q = m_i Q_i$. Note that $\delta(p_1,p_2) = \delta(p_2,p_1)$.

(2) Let $u : U \to M$ be a simple J-holomorphic curve such that for some critical point $p \in U$, u is embedded on $U \setminus \{p\}$. Let $x = u(p)$. By Micallef-White theorem, there exists a C^1 coordinate chart $\Psi : V \to \mathbb{C}^n$ at x and a $C^{2,\alpha}$ coordinate chart ψ at p, such that $\Psi \circ u \circ \psi^{-1}(z) = (z^Q, f(z))$ for some $Q > 1$, where $f(z)$ vanishes to an order $Q' > Q$ at $z = 0$. Define

$$\delta(p) := \sum_{\nu^Q=1,\nu\neq 1} \text{the order of vanishing of } \frac{f(\nu z) - f(z)}{z} \text{ at } z=0.$$

If a J-holomorphic curve is locally given in the form $(z^Q, f(z))$ where $f(z)$ vanishes to an order $> Q$ at $z = 0$, then we define the tangent plane of the curve at $z = 0$ to be $\mathbb{C} \times \{0\}$.

Lemma 1.18. *(1) $\delta(p_1,p_2)$ is an integer and is always $\geq Q_1Q_2$, with equality iff u_1, u_2 have distinct tangent planes at x. In particular, $\delta(p_1,p_2) = 1$ iff both u_1, u_2 are embedded at x and they intersect transversely at x.*

(2) $\delta(p)$ is always even and $\geq (Q-1)(Q'-1)$, with equality iff Q, Q' are relatively prime. In particular, $\delta(p) \geq 2$ with equality iff $x = u(p)$ is a cusp singularity, i.e., u is given in local form $z \mapsto (z^2, z^3 + \cdots)$.

Proof. (1) Without loss of generality, we assume $f_1(z)$ vanishes to an order $Q_1' > Q_1$, and $f_2(z) = az^{Q_2} + g(z)$ where $a \in \mathbb{C}$ and $g(z)$ vanishes to an order $Q_2' > Q_2$. Note that u_1, u_2 have distinct tangent planes at x iff $a \neq 0$.

Let $\nu^Q = 1$. Then

$$f_1(\nu z^{m_1}) - f_2(z^{m_2}) = -az^Q + h_\nu(z),$$

where the order of vanishing of $h_\nu(z)$ is $> Q$, and is either divisible by m_1 or by m_2. This implies that $\delta(p_1,p_2)$ is always integral and

$$\delta(p_1,p_2) \geq \frac{1}{m_1m_2}Q^2 = Q_1Q_2,$$

with equality iff $a = 0$.

(2) First of all, $\delta(p) \geq (Q-1)(Q'-1)$ is obvious, because for any $\nu \neq 1$, $\nu^Q = 1$, the order of vanishing of $f(\nu z) - f(z)$ is $\geq Q'$. Moreover, it equals Q' for every such ν iff Q, Q' are relatively prime. Hence $\delta(p) = (Q-1)(Q'-1)$ iff Q, Q' are relatively prime. To see that $\delta(p)$ is always even, we write $f(z) = \sum_{i=1}^\infty a_i z^{n_i}$, where $n_{i+1} > n_i$ for all i and $n_1 = Q'$. Note that Q and $n_1, n_2, \cdots$, have no common factor. Now consider an $\nu \neq 1$ such that $\nu^Q = 1$. Suppose the order of ν is R_ν. Then there exists an $i(\nu)$ such that R_ν divides n_j for any $j < i(\nu)$ but R_ν does not divides $n_{i(\nu)}$. It follows that the order of vanishing of $f(\nu z) - f(z)$ is $n_{i(\nu)}$. If R_ν is odd, then the number of such ν's with the same order is even, with each contributing $n_{i(\nu)} - 1$ to $\delta(p)$, so that the

total contribution from these ν's is even no matter what is the parity of $n_{i(\nu)}$. If R_ν is even, then there are two possibilities: $n_{i(\nu)}$ odd or $n_{i(\nu)}$ even. If $n_{i(\nu)}$ is odd, then ν makes a contribution of $n_{i(\nu)} - 1$ to $\delta(p)$ which is even. If $n_{i(\nu)}$ is even, then $R_\nu \neq 2$, so that such ν's come in pairs: ν, $-\nu$, and the total contribution is also even. hence $\delta(p)$ is even. □

Theorem 1.19. *(Positivity of Intersections,* [19]*, Theorem 7.1) Let $u_i : \Sigma_i \to M$, $i = 1, 2$, be distinct simple J-holomorphic curves, where Σ_i is a compact Riemann surface without boundary. Let $C_i := u_i(\Sigma_i)$. Then $S(u_1, u_2) := \{(p_1, p_2) | p_i \in \Sigma_i, u_1(p_1) = u_2(p_2)\}$ is a finite set, and the algebraic intersection number $C_1 \cdot C_2$ of C_1, C_2 is given by*

$$C_1 \cdot C_2 = \sum_{(p_1,p_2)\in S(u_1,u_2)} \delta(p_1, p_2).$$

In particular, $C_1 \cdot C_2 \geq 0$ with equality iff C_1, C_2 are disjoint.

Theorem 1.20. *(Adjunction Formula,* [19]*, Theorem 7.3) Let $u : \Sigma \to M$ be a simple J-holomorphic curve where Σ is a compact connected Riemann surface without boundary. Let $C := u(\Sigma)$. Then*

$$C^2 - c_1(TM) \cdot C = 2 \ genus(\Sigma) - 2 + \sum_{(p,q)\in\Sigma\times\Sigma, p\neq q, u(p)=u(q)} \delta(p, q) + \sum_{p\in\Sigma, du(p)=0} \delta(p).$$

Corollary 1.21. *(Adjunction Inequality) Let $u : \Sigma \to M$ be a simple J-holomorphic curve where Σ is a compact connected Riemann surface without boundary. Let $C := u(\Sigma)$. Then*

$$C^2 - c_1(TM) \cdot C + 2 \geq 2 \ genus(\Sigma)$$

where equality holds iff C is embedded.

Remark 1.22. The positivity of intersections was known to Gromov in his seminar paper on pseudoholomorphic curves. The adjunction inequality was first stated by McDuff. The most rigorous proof of these results are in the Macallef-White paper. Extensions of these results to almost complex 4-orbifolds can be found in [4].

Example 1.23. (1) Consider the cubic curve $C := \{[z, w, u] | z^3 + w^2 u = 0\} \subset \mathbb{CP}^2$, which is irreducible. Since C has degree 3,

$$C^2 - c_1(T\mathbb{CP}^2) \cdot C = 3^2 - 3 \cdot 3 = 0.$$

On the other hand, C contains a cusp singularity at $[0, 0, 1]$, which contributes $\delta(p) = 2$ to the right hand side of the adjunction formula. This implies that C can be parametrized by a simple holomorphic curve $u : \Sigma \to \mathbb{CP}^2$, such that Σ has genus 0 and u is embedded off the cusp singularity.

(2) Let C_1, C_2 be the cubic curves in $\mathbb{CP}^2$, where $C_1 := \{[z, w, u] | z^3 + w^2 u = 0\}$ and $C_2 := \{[z, w, u] | w^3 + z^2 u = 0\}$. Then it is easy to check that the intersection $C_1 \cap C_2$ consists of six points: the cusp singularity $[0, 0, 1]$, and $[\lambda, \lambda^{-1}, 1]$ where $\lambda^5 + 1 = 0$. The cusp singularity contributes 4 to the algebraic intersection number $C_1 \cdot C_2$, which equals 9, and each $[\lambda, \lambda^{-1}, 1]$, $\lambda^5 + 1 = 0$, contributes 1 to $C_1 \cdot C_2$.

Exercise 1.24. (1) Let $C := \{[z,w,u] | z^5+w^3u^2=0\} \subset \mathbb{CP}^2$. (Check it is irreducible!) Suppose C is parametrized by a simple holomorphic curve $u : \Sigma \to \mathbb{CP}^2$. What is the genus of Σ and what kind of singularities does u has?

(2) Let $C_1 := \{[z,w,u] | z^5+w^3u^2=0\} \subset \mathbb{CP}^2$, $C_2 := \{[z,w,u] | w^5+z^3u^2=0\} \subset \mathbb{CP}^2$. Find out the set of intersection of C_1, C_2, and compute the local contribution from each of the intersection points in $C_1 \cap C_2$ to $C_1 \cdot C_2$.

2. Gromov compactness

2.1. **Gromov compactness theorem.** Let (M,ω) be a compact closed, connected symplectic manifold of dimension $2n$, let J be an ω-tame almost complex structure of C^l-class, where $l \geq 1$. Let Σ be a compact connected Riemann surface without boundary, and j be a complex structure on Σ. We shall fix a compatible metric g_j on Σ. Let $u : \Sigma \to M$ be a (J,j)-holomorphic curve, of least C^1-class. Then by Proposition 1.5(1), the energy $E(u) = \int_\Sigma u^*\omega = [\omega] \cdot u_*([\Sigma])$, where $[\omega] \in H^2_{dR}(M)$ is the deRham cohomology class of ω and $u_*([\Sigma])$ is the push-forward of the fundamental class of Σ under the map u. In particular, this shows that $E(u)$, or equivalently the L^2-norm of du, is a topological invariant, hence bounded if the homology class $u_*([\Sigma])$ is fixed.

In this chapter, we shall investigate the following question:

Question: Let $u_n : \Sigma \to M$ be a sequence of (J,j_n)-holomorphic curves such that the homology class $(u_n)_*([\Sigma]) = A \in H_2(M;\mathbb{Z})$ is fixed (so that the energy $E(u_n)$ or the L^2-norm of du_n is uniformly bounded by $[\omega] \cdot A$). Does (u_n) admit a subsequence which may converge in a suitable sense?

First, we look at some good news.

Proposition 2.1. *Suppose in addition $j_n = j$ is fixed, and for some fixed metric g_j on Σ, the L^p-norm of du_n is uniformly bounded for some $p > 2$, then there exists a subsequence of (u_n) which converges in $C^{1,\alpha}$-topology for some $0 < \alpha < 1$.*

Proof. For simplicity of arguments, we embed (M, g_J) isometrically into an Euclidean space $\mathbb{R}^N$, and regard u_n as maps into $\mathbb{R}^N$. Then since M is compact, $|u_n|$ is uniformly bounded. By the assumption du_n has uniformly bounded L^p-norm, we see immediately that, by regarding $u_n \in L^{1,p}(\Sigma, \mathbb{R}^N)$, the sequence (u_n) has a uniformly bounded $L^{1,p}$-norm.

Recall that J-holomorphic curves satisfy the following equations weakly, which are given in local coordinates:

$$(\partial_s^2 + \partial_t^2) u_\alpha = \sum_{\beta,\gamma=1}^{2n} \partial_\gamma J_{\alpha\beta}(u)(\partial_t u_\gamma \partial_s u_\beta - \partial_s u_\gamma \partial_t u_\beta).$$

By Hölder inequality, the right hand side is in $L^{p/2}_{loc}$ and is uniformly bounded, so that by elliptic regularity and interior estimates, $u_n \in L^{2,p/2}(\Sigma, \mathbb{R}^N)$ and has uniformly bounded $L^{2,p/2}$-norm. If $p > 4$ is true, then by Morrey's embedding theorem, $u_n \in C^{1,\beta}(\Sigma, \mathbb{R}^N)$ and has uniformly bounded $C^{1,\beta}$-norm, where $2 - 4/p = 1 + \beta$. This implies that u_n has a subsequence which converges in $C^{1,\alpha}$ for any $\alpha < \beta$.

Suppose $p \leq 4$. If $p = 4$, then since Σ is compact, we know by Hölder inequality that $u_n \in L^{1,q}$ and has a uniformly bounded $L^{1,q}$-norm for some $q < p = 4$. So without loss of generality we assume $p < 4$. In this case, Sobolev embedding theorem tells us that $u_n \in L^{1,p_1}(\Sigma, \mathbb{R}^N)$ and has uniformly bounded L^{1,p_1}-norm, where $2 - 4/p = 1 - 2/p_1$, or equivalently $p_1 = 2p/(4-p)$. If $p_1 > 4$, then we are done, if not, we continue with this process with p_1 replaced by $p_2 = 2p_1/(4-p_1)$. (This process is called *elliptic bootstrapping.*)

Computing $\frac{p_1}{p} - 1$, we obtain

$$\frac{p_1}{p} - 1 = \frac{2p}{p(4-p)} - 1 = \frac{p-2}{4-p} > 0.$$

On the other hand, $\frac{d}{dp}(\frac{p-2}{4-p}) = \frac{2}{(4-p)^2} > 0$. It follows that there exists a $\nu > 0$ (independent of i), such that for any $i \geq 1$,

$$p_i \geq (1+\nu)p_{i-1}, \text{where } p_0 = p.$$

Hence there exists an i such that $p_i > 4$, and the proposition follows. $\square$

Remark: Let u be a (J,j)-holomorphic curve of $L^{1,p}$-class for some $p > 2$, where J is C^l-smooth, $l \geq 1$. The same argument shows that u is of $L^{l+1,p}$-class, and $||u||_{l+1,p,D} \leq C(l, ||J||_{C^l}, ||j||_{C^l}, ||u||_{1,p,D'}, dist(D, \partial D'))$ for any open subsets $D, D' \subset \Sigma$ with $\overline{D} \subset D'$.

Next, some bad news.

Example 2.2. (1) (Degeneration of complex structures) Consider the sequence of holomorphic curves $C_n := \{[z,w,u] | z^3 + w^2 u = n^{-1}u^3 + n^{-1}zu^2\} \subset \mathbb{CP}^2$ where $n = 1, 2, \cdots$, which are a family of embedded tori with a fixed homology class. When $n \to \infty$, it "converges" to the cusp curve $C_\infty = \{[z,w,u] | z^3 + w^2 u = 0\} \subset \mathbb{CP}^2$, which is an singular 2-sphere with a cusp singularity at $[0,0,1]$. If we parametrize C_n by $u_n : \Sigma \to \mathbb{CP}^2$, then Σ has genus 1, and if we parametrize C_∞ by $u_\infty : \Sigma \to \mathbb{CP}^2$, then Σ has genus 0. Hence no subsequences of u_n can converge to u_∞ in the sense of Proposition 2.1 above. What happened is that the complex structure j_n on C_n does not converge in the space of complex structures on a genus 1 Riemann surface Σ, rather, the Riemann surfaces (Σ, j_n) degenerates to a Riemann surface of lower genus.

(2) (Re-parametrizations) Consider an embedded J-holomorphic 2-sphere $u : \mathbb{S}^2 \to M$. Recall that the 2-sphere $\mathbb{S}^2$ has a unique complex structure j_0, and there is a large group of biholomorphisms $\phi : \mathbb{S}^2 \to \mathbb{S}^2$, i.e., the group of Möbius transformations $PSL(2,\mathbb{C})$, where

$$\phi(z) = \frac{az+b}{cz+d}, \;\; a,b,c,d \in \mathbb{C}, \; ad - bc = 1.$$

The problem is that $PSL(2,\mathbb{C})$ is not compact. For example, if we let $\phi_n(z) = nz$, where $n = 1, 2, \cdots$. Then $u_n := u \circ \phi_n : (\mathbb{S}^2, j_0) \to (M, J)$ can not converge in the sense of Proposition 2.1 above. This example shows that (with $j_n = j_0$ being fixed, hence the metric on $\mathbb{S}^2$ fixed), uniform boundedness of du_n in the L^2-norm does not necessarily imply uniform boundedness of du_n in a L^p-norm for some $p > 2$. The problem is that the L^2-norm of du is conformally invariant while a L^p-norm with $p > 2$ is not (cf. Exercise 1.4).

(3) (Bubbling off of 2-spheres) Consider the sequence of holomorphic curves $C_n := \{[z, w, u] | zw = n^{-1}u^2\} \subset \mathbb{CP}^2$ where $n = 1, 2, \cdots$, which are a family of embedded 2-spheres by the adjunction formula. As $n \to \infty$, C_n "converges" to $C_\infty := \{[z, w, u] | zw = 0\} \subset \mathbb{CP}^2$, which is the union of two lines $z = 0$ and $w = 0$. If one parametrizes C_n by holomorphic maps $u_n : (\mathbb{S}^2, j_0) \to (\mathbb{CP}^2, J_0)$, even with suitable re-parametrizations, no subsequences of u_n can converge in the sense of Proposition 2.1 for the following simple reason: there is no simple J_0-holomorphic curve $u : \mathbb{S}^2 \to \mathbb{CP}^2$ such that $u(\mathbb{S}^2) = C_\infty$. (Explain why!) This phenomenon is called *bubbling*, where a J-holomorphic 2-sphere is split off during the limiting process.

Fortunately, the problems regarding compactness which are illustrated in the example above are the only ones.

Theorem 2.3. *(Gromov Compactness, cf. eg.* [3, 24, 21, 26, 11, 18]*) Let (M, ω) be a compact closed, connected symplectic manifold of dimension $2n$, and let $(J_n) \subset \mathcal{J}_\tau(M, \omega)$ be a sequence which converges to $J \in \mathcal{J}_\tau(M, \omega)$ in C^∞-topology. Let Σ be a compact, connected Riemann surface without boundary, and let (j_n) be a sequence of complex structures on Σ. Suppose $u_n : \Sigma \to M$ is a sequence of (J_n, j_n)-holomorphic curves such that $(u_n)_*([\Sigma]) = A \in H_2(M; \mathbb{Z})$, $A \neq 0$. Then up to a re-parametrization of each u_n, there are*

(1) *finitely many simple closed loops $\{\gamma_l\}$ in Σ,*
(2) *a finite union of Riemann surfaces $\Sigma' = \cup_\nu \Sigma_\nu$ which is obtained from Σ by collapsing each of the simple closed curves γ_l to a point, i.e., there is a continuous, surjective map $\phi : \Sigma \to \Sigma'$ which maps each γ_l to a point and in the complement of any open neighborhood of $\cup_l \gamma_l$, ϕ is a diffeomorphism,*
(3) *a continuous map $u : \Sigma' \to M$ such that $u|_{\Sigma_\nu}$ is a (J, j')-holomorphic curve, where j' stands for the complex structure on (each component of) Σ'*

such that a subsequence of (u_n), still denoted by (u_n) for simplicity, converges to u in the following sense:

(i) *in the complement of any fixed open neighborhood of $\cup_l \gamma_l$, j_n converges to j' in C^∞-topology and u_n converges to $u \circ \phi$ in C^∞-topology,*
(ii) $\sum_\nu u_*([\Sigma_\nu]) = A \in H_2(M; \mathbb{Z})$.

Gromov called the limiting curve $u : \Sigma' \to M$ a *cusp-curve*.

Remark 2.4. (1) Statement (ii) is equivalent to energy preservation: $E(u_n) = \sum_\nu E(u|_{\Sigma_\nu})$.

(2) If γ_l bounds a disc in Σ, then collapsing γ_l corresponds to a bubbling off of a non-constant J-holomorphic 2-sphere; otherwise, it corresponds to degeneration of complex structures j_n. In particular,

Corollary 2.5. *Suppose $\Sigma = \mathbb{S}^2$ and $j_n = j_0$, and there are no classes $B \in H_2(M; \mathbb{Z})$ which lies in the image of $\pi_2(M) \to H_2(M; \mathbb{Z})$, such that $0 < \omega(B) < \omega(A)$. Then a subsequence of u_n converges in C^∞-topology after possible re-parametrizations.*

In the remaining sections, we shall explain the various analytical issues involved in the proof of the Gromov Compactness Theorem.

2.2. **Energy estimate and bubbling.** Let (Σ, j) be a Riemann surface with a Kähler metric ν, and (M, J) be an almost complex manifold with a Hermitain metric h. Let $u : \Sigma \to M$ be a (J, j)-holomorphic curve. Recall that the energy density $e(u)$ and energy $E(u)$ are defined by (using metrics ν and h)

$$e(u) = |du|^2, \ \ E(u) = \frac{1}{2}\int_\Sigma e(u) dvol_\Sigma.$$

The first main result is the following theorem, where we follow the discussion in [24, 21].

Theorem 2.6. *(Energy Estimate) There exist constants $C > 0$ and $\epsilon_0 > 0$, depending on the geometry of (M, J, h) and the metric ν, such that for any geodesic disc $D(2r) \subset \Sigma$ of radius $2r$ with $E(u)|_{D(2r)} := \frac{1}{2}\int_{D(2r)} e(u) \leq \epsilon_0$, one has the estimate*

$$\sup_{x\in D(r)} e(u)(x) \leq \frac{CE(u)|_{D(2r)}}{r^2}.$$

Proof. The proof follows by a standard argument once we established the following lemma, whose proof is postponed to the end of this section.

Lemma 2.7. *There exist constants $C_1 > 0, C_2 > 0$ depending on the geometry of (M, J, h) and the metric ν, such that*

$$\Delta e(u) \leq C_1 e(u) + C_2 e(u)^2,$$

where $\Delta := d^*d$ is the Laplacian on (Σ, ν).

Assume Lemma 2.7 momentarily. Set $f(\rho) := \rho^2 \sup_{D(2r-2\rho)} e(u)$, $\rho \in (0, r]$, and let ρ_0 be the maximum of $f(\rho)$. Set $e_0 = \sup_{D(2r-2\rho_0)} e(u)$, and let $x_0 \in \overline{D(2r - 2\rho_0)}$ be the point such that $e_0 = e(u)(x_0)$. Note that $D(x_0, \rho_0)$, the disc centered at x_0 of radius ρ_0, is contained in $D(2r - \rho_0)$, so that

$$(\frac{\rho_0}{2})^2 \sup_{D(x_0,\rho_0)} e(u) \leq (\frac{\rho_0}{2})^2 \sup_{D(2r-\rho_0)} e(u) \leq \rho_0^2 \sup_{D(2r-2\rho_0)} e(u),$$

which gives

$$\sup_{D(x_0,\rho_0)} e(u) \leq 4e_0.$$

Now by Lemma 2.7, the function $e(u)$ satisfies on $D(x_0, \rho_0)$ the following inequality

$$\Delta e(u) \leq (C_1 + 4C_2e_0)e(u).$$

From the proof of Theorem 9.20 in [10] (a mean value theorem), there is a constant $C_3 > 0$ depending on the geometry of (M, J, h) and the metric ν, such that

$$e_0 = e(u)(x_0) \leq C_3(1 + e_0\rho_0^2)\frac{1}{\rho_0^2}\int_{D(x_0,\rho_0)} e(u).$$

Set $\epsilon_0 = 1/4C_3$. Then if $E(u)|_{D(2r)} \leq \epsilon_0$, $\int_{D(x_0,\rho_0)} e(u) \leq 1/2C_3$, so that

$$(\frac{r}{2})^2 \sup_{D(2r-r)} e(u) \leq \rho_0^2 e_0 \leq 2C_3E(u)|_{D(2r)},$$

which gives rise to

$$\sup_{x\in D(r)} e(u)(x) \leq \frac{CE(u)|_{D(2r)}}{r^2}, \text{ where } C := 8C_3.$$

□

Using a covering argument due to Sacks and Uhlenbeck, one obtains a "partial" compactness theorem as a consequence of the Energy Estimate. But first, one needs the following Removable Singularity Theorem (cf. [3]). A proof is given in the next section.

Theorem 2.8. *(Removable Singularity Theorem) Let (M,ω) be symplectic with $J \in \mathcal{J}_\tau(M,\omega)$. Then any smooth, finite energy, J-holomorphic curve $u : D\setminus\{0\} \to M$ extends to a smooth J-holomorphic curve on D.*

For simplicity, we assume $J_n = J$ and $j_n = j$. The same argument works for the more general case $J_n \to J$ and $j_n \to j$ as $n \to \infty$. (In particular, the complex structures j_n are assumed to NOT degenerate.)

Theorem 2.9. *Let (Σ, j) be a compact Riemann surface with a Kähler metric ν, and (M,J) be a compact almost complex manifold with a Hermitain metric h. Let $u_n : \Sigma \to M$ be a (J,j)-holomorphic curves such that $E(u_n) \leq E_0$ for some constant $E_0 > 0$ independent of n. Then there is a finite set of points $\{x_1, \cdots, x_l\} \subset \Sigma$, a subsequence of (u_n) (still denoted by (u_n) for simplicity), and a (J,j)-holomorphic curve $u_0 : \Sigma \to M$ such that in the complement of any neighborhood of $\{x_1,\cdots,x_l\}$, $u_n \to u_0$ in C^∞-topology. Moreover, for each i, $1 \leq i \leq l$, there is a non-constant J-holomorphic curve $u^i : \mathbb{S}^2 \to M$ bubbling off at x_i, with the total energy of u_0 and u^i satisfying $E(u_0) + \sum_i E(u^i) \leq E_0$.*

Proof. Choose $r_0 > 0$ and set $r_m := 2^{-m} r_0$, $m \in \mathbb{Z}_+$. For each m take a finite covering $\mathcal{U}_m = \{D_{2r_m}(y_\alpha)\}$ of Σ by geodesic discs of radius $2r_m$ centered at y_α which has the following properties: (1) each point in Σ is covered at most h times where h is a constant depending on (Σ,ν) only, (2) the discs of only half of the radius, $\{D_{r_m}(y_\alpha)\}$, is also a covering of Σ. Now we fix an m. Then for each n,

$$\sum_\alpha \frac{1}{2}\int_{D_{2r_m}(y_\alpha)} e(u_n) \leq hE_0,$$

so that there are at most hE_0/ϵ_0 many discs in the set $\mathcal{U}_m$, such that

$$E(u_n)|_{D_{2r_m}(y_\alpha)} \geq \epsilon_0. \quad \text{(Here } \epsilon_0 \text{ is the constant in Theorem 2.6.)}$$

The center points of these discs make at most hE_0/ϵ_0 sequences of points of Σ (by letting $n = 1,2,3,\cdots$). By passing to a subsequence, we may assume these center points are fixed, which we denote by $\{x_{1,m},\cdots,x_{l(m),m}\}$. By further passing to a subsequence, we may assume $l(m) = l$ which is independent of $m \in \mathbb{Z}_+$. Now for each fixed m, u_n has a uniform bound on the C^1-norm outside the discs $D_{r_m}(x_{i,m})$, $1 \leq i \leq l$ (cf. Theorem 2.6). By elliptic bootstrapping as in Proposition 2.1, for any fixed $k \geq 1$, u_n has a uniform bound on $C^{k,\alpha}$-norm and by Arzéla-Ascoli theorem, a subsequence of u_n converges in C^k-topology. Letting $m \to \infty$ and by passing to a subsequence, $x_{1,m}, \cdots, x_{l,m}$ converges to $x_1, \cdots, x_l$. Choosing a diagonal subsequence of (u_n) finishes the proof. Note that the limit J-holomorphic curve u_0 is only defined on $\Sigma \setminus \{x_1, \cdots, x_l\}$, but by the Removable Singularity Theorem, u_0 can be extended to the whole Σ because by the nature of construction, $E(u_0) \leq E_0 < \infty$.

Now for each x_i, $1 \leq i \leq l$, we fix a sufficiently small $\delta > 0$, and for each n, set

$$b_n^2 = \max_{x \in D_\delta(x_i)} e(u_n)(x), \quad y_n \in D_\delta(x_i) \text{ such that } e(u_n)(y_n) = b_n^2.$$

If b_n is bounded, then u_n converges to u in C^k-topology over $D_\delta(x_i)$, and we remove x_i from the set $\{x_1, \cdots, x_l\}$. So without loss of generality, we assume $b_n \to \infty$ as $n \to \infty$. Then it follows that $y_n \to x_i$ as $n \to \infty$. We define $u_n^i : D_{2^{-1}b_n\delta}(0) \subset \mathbb{C} \to M$ by setting $u_n^i(x) := u_n(y_n + x/b_n)$. Then

$$\max_{x \in D_{2^{-1}b_n\delta}(0)} e(u_n^i)(x) = e(u_n^i)(0) = 1, \quad \forall n = 1, 2, \cdots.$$

(Note that $D_{2^{-1}b_n\delta}(0)$ is given with the pull-back metric $\phi_n^*\nu$ under $\phi_n : D_{2^{-1}b_n\delta}(0) \to D_\delta(x_i)$, which converges to the flat metric on $\mathbb{C}$ as $n \to \infty$.) For any $R > 0$, u_n^i converges in C^k-topology over the disc $D_R(0) \subset \mathbb{C}$. Taking $R = 1, 2, \cdots$, and passing to the diagonal subsequence, we obtained a J-holomorphic curve $u^i : \mathbb{C} \to M$ with $e(u^i)(0) = 1$, so that u^i is non-constant. By the nature of construction, it is clear that

$$E(u_0) + \sum_i E(u^i) \leq E_0.$$

In particular, by thinking $\mathbb{C} = \mathbb{S}^2 \setminus \{\infty\}$ and using the Removable Singularity Theorem, we obtain $u^i : \mathbb{S}^2 \to M$ via extension of $u^i : \mathbb{C} \to M$. $\square$

Remark 2.10. (1) Corollary 2.5 is a direct consequence of Theorem 2.9.

(2) In the case of Theorem 2.3, the energy $E(u_n) = E_0 = [\omega] \cdot A$ is fixed. It is not clear that in Theorem 2.9, $E(u_0) + \sum_i E(u^i) = E_0$ holds true or not. In other words, there might be some energy loss during the limiting process. Related issues are that the total homology class $(u_0)_*([\Sigma]) + \sum_i u_*^i([\mathbb{S}^2])$ might not be equal to A, and the subset $u_0(\Sigma) \cup (\cup_i u^i(\mathbb{S}^2))$ might not be connected. These issues will be dealt with in Section 2.4.

It remains to prove Lemma 2.7, which will be given in a set of exercises.

Exercise 2.11. (1) Let (M, g) be an oriented Riemannian manifold, and let E be a smooth vector bundle equipped with a metric $\langle , \rangle$. Let ∇ be a connection on E which is compatible with the metric $\langle , \rangle$, and let $\Delta = - * d * d : C^\infty(M) \to C^\infty(M)$ be the Laplacian defined by the metric g. Show that for any smooth section ξ of E,

$$\Delta |\xi|^2 = \langle \nabla^* \nabla \xi, \xi \rangle - 2|\nabla \xi|^2 + \langle \xi, \nabla^* \nabla \xi \rangle.$$

(2) Let (M, g) be an oriented Riemannian manifold, and let E be a smooth vector bundle equipped with a metric $\langle , \rangle$. Let ∇ be a connection on E which is compatible with the metric $\langle , \rangle$. Together with the Levi-Civita connection of (M, g), ∇ extends uniquely to a metric compatible connection on $\Lambda^k M \otimes E$, $1 \leq k \leq \dim M$, which is

also denoted by ∇. Skew-symmetrization of the connection then defines a first order p.d.o. $D : \Omega^k(E) \to \Omega^{k+1}(E)$, which satisfies that

$$D(\xi \otimes \omega) = \nabla\xi \wedge \omega + \xi \otimes d\omega, \quad \forall \xi \in C^\infty(E), \omega \in \Omega^k(M).$$

Let D^* be the formal adjoint of D. Show that on $\Omega^1(E)$,

$$D^*D + DD^* = \nabla^*\nabla + R,$$

where R is a p.d.o. of order 0, which is defined as follows: $\forall \xi \in C^\infty(E), \omega \in \Omega^1(M)$,

$$R(\xi \otimes \omega) = \Omega(I(\omega), \cdot)\xi + \xi \otimes (I^{-1} \circ \mathrm{Ric} \circ I(\omega)).$$

Here $I : T^*M \to TM$ is the isomorphism induced by the metric g, Ric is the Ricci tensor, and $\Omega \in \Omega^2(\mathrm{End}E)$ is the curvature of the connection ∇ on E.

(3) (Continuation of (2) above.) Suppose (M, g) is Kähler and E is a complex vector bundle with a Hermitian metric $\langle , \rangle$ and a Hermitian connection ∇. Then for any p, q, $k = p + q$, the operator $D : \Omega^k(E) \to \Omega^{k+1}(E)$, when restricted to $\Omega^{p,q}(E)$, has a decomposition $D = D' + \overline{D}'$, where $D' : \Omega^{p,q}(E) \to \Omega^{p+1,q}(E)$ and $\overline{D}' : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$. Let $(D')^*$ be the formal adjoint of D'. Prove that

$$2((D')^*D' + D'(D')^*) = D^*D + DD^*.$$

(Similarly, $2((\overline{D}')^*\overline{D}' + \overline{D}'(\overline{D}')^*) = D^*D + DD^*$.)

(4) Let (Σ, j) be a Riemann surface with Kähler metric ν, and let (M, J) be an almost complex manifold with a Hermitian metric h. Choose a Hermitian connection ∇ on (M, J, h). Let $u : \Sigma \to M$ be a (J, j)-holomorphic curve. Then we have a special case of (3) above, where (Σ, ν) is the Kähler manifold, $E = u^*TM$ is the complex vector bundle with the pull-back Hermitian metric u^*h and the pull-back connection $u^*\nabla$. With this understood, note that $du = \partial_J u \in \Omega^{1,0}(u^*TM)$, where

$$\partial_J u := \frac{1}{2}(du - J \circ du \circ j).$$

Prove that $\partial_J u$ satisfies the following system of equations:

$$D'(\partial_J u) = 0, \quad (D')^*(\partial_J u) = q(\partial_J u, \partial_J u),$$

where $q(\partial_J u, \partial_J u)$ is quadratic in $\partial_J u$ whose coefficients depends linearly on the torsion of ∇ and ∇J. As a consequence, show that

$$((D')^*D' + D'(D')^*)\partial_J u = t(\partial_J u, \partial_J u, \partial_J u),$$

where the coefficients of t depends on $J, \nabla J$.

(5) Combine (2), (3), (4) to show that

$$\nabla^*\nabla\partial_J u = L_1(\partial_J u) + L_3(\partial_J u, \partial_J u, \partial_J u),$$

and then use (1) to give a proof of Lemma 2.7.

2.3. **The isoperimetric inequality.** In this section we derive an isoperimetric inequality for J-holomorphic curves in a symplectic manifold (M,ω) with $J \in \mathcal{J}_\tau(M,\omega)$. This together with the Energy Estimate forms the technical back bones in the Gromov compactness theorem. We follow the discussion in [18]. When J is ω-compatible, J-holomorphic curves are absolute minimizing minimal surfaces and the isoperimetric inequality follows from that for minimal surfaces in $\mathbb{R}^n$ (cf. [21]). Ye derived an isoperimetric inequality for J-holomorphic curves in almost complex manifolds ([26]).

Theorem 2.12. *(Isoperimetric Inequality) Let (M,ω) be a compact symplectic manifold and $J \in \mathcal{J}_\tau(M,\omega)$. Fix the Hermitian metric $g_J(v,w) := \frac{1}{2}(\omega(v,Jw)+\omega(w,Jv)$ on (M,J). Then for any constant $c > 1/4\pi$ there exists a constant $\delta > 0$ such that*

$$|\int_D u^*\omega| \leq c \cdot \mathit{length}^2(u(\partial D))$$

for every simple C^1-map from a disc $u : D \to M$ such that $\mathit{length}(u(\partial D)) < \delta$ and $u(D)$ lies in a Darboux chart. In particular,

$$\mathit{area}(u(D)) \leq c \cdot \mathit{length}^2(u(\partial D))$$

if u is J-holomorphic.

In the last part we make use of the following observation.

Exercise 2.13. Let (M,J) be an almost complex manifold and h be a Hermitian metric. Set $\Omega := h(J\cdot,\cdot)$. Show that for any $p \in M$ and $X, Y \in T_pM$ where $X, Y \neq 0$,

$$\sqrt{h(X,X)h(Y,Y) - h^2(X,Y)} \geq \Omega(X,Y)$$

with equality iff $JX = cY$ for some $c > 0$. In the special case where $h = g_J$, this implies that

$$\mathrm{area}(u(\Sigma)) = \int_\Sigma u^*\omega = E(u)$$

for any simple J-holomorphic curve $u : \Sigma \to M$.

For a proof of Theorem 2.12, we first consider the case where $M = \mathbb{C}^n$ with J the standard complex structure, and $w = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$. We introduce the 1-form

$$\lambda := \frac{i}{4}\sum_{j=1}^n (z_j \wedge d\bar{z}_j - \bar{z}_j \wedge dz_j)$$

and notice that $w = d\lambda$. We set $D := \{re^{i\theta} | 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ and $\mathbb{S}^1 = \partial D = \mathbb{R}/2\pi\mathbb{Z}$. Let $\gamma(\theta)$, $\gamma : \mathbb{S}^1 \to \mathbb{C}^n$, be any smooth loop, and $u : D \to \mathbb{C}^n$ be any smooth map such that $u|_{\partial D} = \gamma$. Then

$$\int_D u^*\omega = \int_{\partial D} \gamma^*\lambda.$$

We shall first prove by a direct calculation that

$$|\int_{\partial D} \gamma^*\lambda| \leq \frac{1}{2}\int_0^{2\pi} |\gamma'(\theta)|^2 d\theta.$$

To this end, write $\gamma(\theta)=(z_j(\theta))$, and let $z_j(\theta)=\sum_k a_k^j e^{ik\theta}$ be the Fourier expansion. Then

$$\begin{aligned}\int_{\partial D}\gamma^*\lambda &= \frac{i}{4}\int_0^{2\pi}\sum_{j=1}^n (z_j(\theta)\overline{z_j'(\theta)}-\overline{z_j(\theta)}z_j'(\theta))d\theta\\ &= \frac{i}{4}\int_0^{2\pi}\sum_{j=1}^n\sum_{k,l}(-ila_k^j\bar{a}_l^j e^{i(k-l)\theta}-il\bar{a}_k^j a_l^j e^{i(-k+l)\theta})\\ &= \pi\sum_{j=1}^n\sum_k k|a_k^j|^2.\end{aligned}$$

A similar calculation shows that

$$\frac{1}{2}\int_0^{2\pi}|\gamma'(\theta)|^2 d\theta=\pi\sum_{j=1}^n\sum_k k^2|a_k^j|^2$$

which implies the claimed inequality $|\int_{\partial D}\gamma^*\lambda|\leq\frac{1}{2}\int_0^{2\pi}|\gamma'(\theta)|^2d\theta$. Finally, we reparametrize $\gamma(\theta)$ such that $|\gamma'(\theta)|\equiv L/2\pi$ where $L=\text{length}(\gamma)$. This gives us

$$|\int_D u^*\omega|=|\int_{\partial D}\gamma^*\lambda|\leq\frac{1}{2}\int_0^{2\pi}|\gamma'(\theta)|^2d\theta=\frac{1}{4\pi}L^2.$$

Now consider the general case. Assume $u:D\rightarrow M$ is a simple J-holomorphic curve which lies in a Darboux chart. Given any constant $c>1/4\pi$, note that $1/4\pi c<1$, so that we can choose $\delta>0$ sufficiently small, such that if $\text{length}(u(\partial D))<\delta$, the variation of J along $u(\partial D)$ is so small that

$$\frac{\text{length}_{g_J}(u(\partial D))}{\text{length}_{g_{J_0}}(u(\partial D))}\geq\sqrt{\frac{1}{4\pi c}}.$$

Then

$$\text{area}(u(D))=\int_D u^*\omega\leq c\cdot\text{length}^2(u(\partial D)).$$

This finishes the proof of Theorem 2.12.

Proof of Removable Singularity Theorem:

Let (M,ω) be symplectic with $J\in\mathcal{J}_\tau(M,\omega)$, and let $u:D\setminus\{0\}\rightarrow M$ be any smooth, finite energy, J-holomorphic curve. First we introduce some notations. Let $D_r\subset D$ be the disc of radius r, and $\epsilon(r):=E(u)|_{D_r\setminus\{0\}}$. Then there exists $r_0>0$ such that $\epsilon(r_0)\leq\epsilon_0$ because $E(u)<\infty$, where $\epsilon_0>0$ is the constant in the Energy Estimate (with the metric ν being the standard metric). Note that ϵ_0 depends only on the geometry of (M,ω,J). Then by the Energy Estimate, for any $0<r\leq r_0/2$,

$$|du|^2(re^{i\theta})\leq\frac{C}{r^2}\epsilon(2r)$$

for some constant $C>0$ depending on the geometry of (M,ω,J) only. (We will continue to denote such a constant by C.)

Now for any $0 < r \leq r_0/2$, let $\gamma_r(\theta) := u(re^{i\theta})$ be the loop. Then

$$|\gamma_r'(\theta)| = r|\partial_\theta u|(re^{i\theta}) = \frac{r}{\sqrt{2}}|du|(re^{i\theta}) \leq C\sqrt{\epsilon(2r)},$$

and hence $\text{length}(\gamma_r) \leq C\sqrt{\epsilon(2r)}$. Therefore, fix any $c > 1/4\pi$, there exists a $r_1 > 0$ such that for all $0 < r \leq r_1$, $\text{length}(\gamma_r) < \delta$ where δ is the constant in the Isoperimetric Inequality. Moreover, γ_r lies in a Darboux chart. For each $0 < r \leq r_1$, pick a smooth map $u_r : D \to M$ such that $(u_r)|_{\partial D} = \gamma_r$ and $u_r(D)$ lies in the Darboux chart. Note that for any $0 < \rho < r \leq r_1$,

$$E(u)|_{D_r \setminus D_\rho} + \int_D u_\rho^*\omega = \int_D u_r^*\omega.$$

(This is true when ρ, r are close enough. For the general case use continuity.) Let $\rho \to 0$, we obtain

$$\epsilon(r) = \int_D u_r^*\omega$$

because by the isoperimetric inequality, $|\int_D u_\rho^*\omega| \leq c \cdot \text{length}^2(\gamma_\rho) \leq C\epsilon(2\rho)$.

On the other hand,

$$\epsilon(r) = \frac{1}{2}\int_0^r \rho(\int_0^{2\pi} |du|^2(\rho e^{i\theta})d\theta)d\rho,$$

from which it follows that

$$\epsilon'(r) = \frac{r}{2}\int_0^{2\pi} |du|^2(re^{i\theta})d\theta = \frac{1}{r}\int_0^{2\pi} |\gamma_r'(\theta)|^2 d\theta \geq \frac{1}{2\pi r}\text{length}^2(\gamma_r).$$

It follows by the isoperimetric inequality that $\epsilon(r) \leq c \cdot \text{length}^2(\gamma_r) \leq 2\pi c r\epsilon'(r)$. Set $\mu := 1/4\pi c < 1$, then

$$\frac{\epsilon'(r)}{\epsilon(r)} \geq \frac{2\mu}{r}.$$

Integrating both sides, we obtain $\epsilon(r) \leq \epsilon(r_1)r_1^{-2\mu}r^{2\mu}$, and consequently,

$$|du|(re^{i\theta}) \leq \frac{C}{r}\sqrt{\epsilon(2r)} \leq \frac{C}{r^{1-\mu}}, \quad \forall 0 < r < r_1.$$

As an immediate corollary we obtain for any $0 < \rho_1, \rho_2 < r_1$, θ_1, θ_0,

$$\text{dist } (u(\rho_1 e^{i\theta_1}), u(\rho_2 e^{i\theta_2}) \leq C \cdot r \cdot \frac{1}{r^{1-\mu}} \leq Cr^\mu,$$

where $r := \max(\rho_1, \rho_2)$. Hence u can be uniquely extended to a continuous map $\tilde{u} : D \to M$.

Exercise 2.14. Show that $d\tilde{u}$ exists weakly on D and $d\tilde{u} = du$ on $D \setminus \{0\}$.

With the above understood, we will show $d\tilde{u} \in L^p(D_{r_1})$ for any p satisfying $p < 2/(1-\mu)$. In particular, one can choose a p such that $p > 2$ because $0 < \mu < 1$. Consequently, $\tilde{u} \in L^{1,p}(D_{r_1})$ for some $p > 2$ (since $\tilde{u}$ is continuous), and by the elliptic bootstrapping as in Proposition 1.2, $\tilde{u}$ is smooth.

To see $d\tilde{u} \in L^p(D_{r_1})$ for any p satisfying $p < 2/(1-\mu)$,

$$\int_{D_{r_1}} |du|^p \rho d\rho d\theta \leq C \int_0^{r_1} \rho^{1-(1-\mu)p} d\rho < \infty \text{ if } p < 2/(2-\mu).$$

The proof of the Removable Singularity Theorem is completed.

A similar argument allows us to prove the following lemma concerning behavior of a "neck-region" J-holomorphic curve, which plays a critical role in the discussion in the next section. Set $A(r,R) = \{\rho e^{i\theta} | r < \rho < R, 0 \leq \theta \leq 2\pi\}$.

Lemma 2.15. *(Neck-region Behavior) Let (M,ω) be a compact symplectic manifold and $J \in \mathcal{J}_\tau(M,\omega)$. Then, $\forall 0 < \mu < 1$, there is a constant $\delta = \delta(M,J,\omega,\mu)$ and a constant $C = C(M,J,\omega,\mu) > 0$ such that every simple J-holomorphic curve $u : A(r,R) \to M$ with*

$$E(u) = E(u, A(r,R)) \leq \delta$$

satisfies

$$E(u, A(e^T r, e^{-T} R)) \leq C e^{-2\mu T} E(u),$$

and

$$\sup_{z,z' \in A(e^T r, e^{-T} R)} \textit{dist } (u(z), u(z')) \leq C e^{-\mu T} \sqrt{E(u)}$$

for any T with $\ln 2 \leq T \leq \ln \sqrt{R/r}$.

Exercise 2.16. Work out the details of the proof of Lemma 2.15.

2.4. **Bubbles connect.** One of the issues in Theorem 2.9 is that the bubbling off J-holomorphic curves u^i may not be connected to the limiting curve u_0. In this section we describe a different way of extracting bubbles which connect. The key idea is to more carefully control the energy so that there is no energy loss during the process. First, two preliminary lemmas.

Lemma 2.17. *There exists a constant $\hbar > 0$ depending on the geometry of (M,ω,J) only, such that for any non-constant J-holomorphic curve $u : \mathbb{S}^2 \to M$, $E(u) \geq \hbar$.*

Proof. Regard $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$, and let $v := u|_{\mathbb{C}}$. We give $\mathbb{C}$ the standard metric. Then if $E(u) \leq \epsilon_0$ where ϵ_0 is the constant in the Energy Estimate, we have

$$\sup_{|z| \leq r} e(v)(z) \leq \frac{C}{r^2}$$

for any $r > 0$, where C is a constant independent of r. Letting $r \to \infty$, we see that v is constant. The lemma follows easily. □

Let $D \subset \mathbb{C}$ be the unit disc centered at 0. Consider a sequence of J-holomorphic curves $u_n : D \to M$ such that (1) $E(u_n) \leq E_0$, (2) u_n converges to $u_0 : D \to M$ in C^∞-topology on any compact subset of $D \setminus \{0\}$, (3) there is a bubbling off $u : \mathbb{S}^2 \to M$ at $0 \in D$. Let $D_\rho \subset D$ be a disc of radius ρ centered at 0.

Lemma 2.18. *After passing to a subsequence, the limit*

$$m = \lim_{\rho \to 0} \lim_{n \to \infty} E(u_n, D_\rho)$$

exists, and moreover, $m \geq E(u) \geq \hbar$, where $u : \mathbb{S}^2 \to M$ is the bubbling off at $0 \in D$.

We remark that $\lim_{n\to\infty} E(u_n) = E(u_0) + m$, so m measures the amount of energy lost at $0 \in D$ during the convergence $u_n \to u_0$. If $m > E(u)$, then the bubble u does not capture all the energy loss.

Proof. Since $E(u_n) \leq E_0$, after passing to a subsequence, $\lim_{n\to\infty} E(u_n)$ exists. Now for any disc D_ρ, since u_n converges to u_0 on $D \setminus D_\rho$, $\lim_{n\to\infty} E(u_n, D_\rho)$ also exists. The function $\lim_{n\to\infty} E(u_n, D_\rho)$ is decreasing in ρ, so that

$$m = \lim_{\rho\to 0}\lim_{n\to\infty} E(u_n, D_\rho)$$

exists. From the construction of the bubble u, it is clear that $\lim_{n\to\infty} E(u_n, D_\rho) \geq E(u)$ for any $\rho > 0$. Hence $m \geq E(u)$. □

Now we state the main result of this section.

Theorem 2.19. *There exist a sequence of Möbius transformations (ϕ_n), a J-holomorphic curve $v : \mathbb{S}^2 \to M$, and finitely many distinct points $Z := \{z_1, \cdots, z_l\} \subset \mathbb{S}^2 \setminus \{\infty\}$ such that*

(a) *the sequence ϕ_n converges in C^∞-topology to the constant map to 0 on any compact subset of $\mathbb{S}^2 \setminus \{\infty\}$,*
(b) *the sequence $v_n := u_n \circ \phi_n$ converges to v in C^∞-topology over any compact subset of $\mathbb{S}^2 \setminus \{z_1, \cdots, z_l, \infty\}$, moreover, the limits*

$$m_j := \lim_{\rho\to 0}\lim_{n\to\infty} E(v_n, D_\rho(z_j)),\ \ 1 \leq j \leq l, z_j \in Z,$$

exist and are positive.

Furthermore, the limit v satisfies the following conditions:

(i) *(no energy loss) $m = E(v) + \sum_{j=1}^{l} m_j$;*
(ii) *(bubbles connect) $v(\infty) = u_0(0)$;*
(iii) *(stability) $\#Z = l \geq 2$ if v is constant.*

Proof. First, without loss of generality, we assume that $|du_n|(0) = \sup_{x\in D} |du_n|(x)$ for all n. We shall begin by describing the Möbius transformations ϕ_n and how the J-holomorphic curve $v : \mathbb{S}^2 \to M$ is obtained.

Let $\hbar > 0$ be the constant in Lemma 2.17. Choose a $\delta > 0$ with $\delta \leq \hbar$ such that Lemma 2.15 holds with it. Now since $m = \lim_{\rho\to 0}\lim_{n\to\infty} E(u_n, D_\rho)$ and the function $\lim_{n\to\infty} E(u_n, D_\rho)$ is decreasing in ρ, for any fixed $\rho > 0$, $E(u_n, D_\rho) \geq m - \delta/2$ for all sufficiently large n. For each such fixed n, since $\lim_{\rho\to 0} E(u_n, D_\rho) = 0$, we can choose a $\rho_n > 0$ such that

$$E(u_n, D_{\rho_n}) = m - \delta/2.$$

By passing to a subsequence, we may assume without loss of generality that for each n there exists a $\rho_n > 0$ such that the above equation holds. Clearly, $\rho_n \to 0$ as $n \to \infty$.

With this understood, the Möbius transformations ϕ_n are defined by $\phi_n(z) = \rho_n z$. Moreover, $v_n := u_n \circ \phi_n : D_{1/\rho_n} \subset \mathbb{C} \to M$ is J-holomorphic and has uniformly bounded energy, hence by Theorem 2.9, there exist a finite set of points $Z := \{z_1, \cdots, z_l\} \subset \mathbb{C}$, such that a subsequence of (v_n) (still named (v_n)) converges in C^∞-topology to a J-holomorphic curve $v : \mathbb{S}^2 \to M$ on any compact subset of

$\mathbb{S}^2 \setminus \{z_1, \cdots, z_l, \infty\}$. Moreover, we assume bubbles do occur at $z_1, \cdots, z_l$ so that by Lemma 2.18, the limits

$$m_j := \lim_{\rho\to 0}\lim_{n\to\infty} E(v_n, D_\rho(z_j)) \;\; 1 \le j \le l, z_j \in Z,$$

exist and are positive after passing further to a subsequence. We should point out that by the assumption $|du_n|(0) = \sup_{x\in D} |du_n|(x)$ for all n, if $Z \neq \emptyset$, then $0 \in Z$. This observation will play a role in proving the stability condition (iii). At this point we have finished the part on ϕ_n and v.

Next, we verify the three conditions (i-iiii). For (i) no energy loss, we first show that the claim

$$\lim_{R\to\infty}\lim_{n\to\infty} E(u_n, D_{R\rho_n}) = m$$

would imply $m = E(v) + \sum_{j=1}^l m_j$. To see this, first note that

$$E(v_n, D_1) = E(u_n, D_{\rho_n}) = m - \delta/2 \ge m - \hbar/2,$$

and $\lim_{R\to\infty}\lim_{n\to\infty} E(v_n, D_R) = \lim_{R\to\infty}\lim_{n\to\infty} E(u_n, D_{R\rho_n}) = m$, so that all the bubbles occur within the unit circle $|z| = 1$. Now fix any $s > 1$, we have

$$\begin{aligned} m &= \lim_{R\to\infty}\lim_{n\to\infty} E(v_n, D_R) \\ &= \lim_{R\to\infty}\lim_{n\to\infty} E(v_n, D_R \setminus D_s) + \lim_{n\to\infty} E(v_n, D_s) \\ &= E(v, \mathbb{C} \setminus D_s) + E(v, D_s) + \sum_{j=1}^l m_j \\ &= E(v) + \sum_{j=1}^l m_j. \end{aligned}$$

Now for $\lim_{R\to\infty}\lim_{n\to\infty} E(u_n, D_{R\rho_n}) = m$, suppose it is not true. Then there exists a $\rho_0 > 0$ such that for every $R \ge 1$, and for a subsequence of (u_n) (still named u_n),

$$\lim_{n\to\infty} E(u_n, D_{R\rho_n}) \le m - \rho_0.$$

This implies that for every $R \ge 1$, $\lim_{n\to\infty} E(u_n, A(\rho_n, R\rho_n)) \le \delta/2 - \rho_0$. We will show this leads to a contradiction.

After passing to a subsequence, there exists a sequence $\epsilon_n > 0$ such that

$$\lim_{n\to\infty} E(u_n, D_{\epsilon_n}) = m, \quad \lim_{n\to\infty} \epsilon_n = 0, \quad \lim_{n\to\infty} \epsilon_n/\rho_n = \infty.$$

We first show that for any $T > 0$, $\lim_{n\to\infty} E(u_n, A(e^{-T}\epsilon_n, \epsilon_n)) = 0$. To see this, introduce $w_n(z) := u_n(\epsilon_n z)$, then for any $T > 0$,

$$\lim_{n\to\infty} E(w_n, A(e^{-T}, e^T)) = \lim_{n\to\infty} E(u_n, A(e^{-T}\epsilon_n, e^T\epsilon_n)) \le \lim_{n\to\infty} E(u_n, A(\rho_n, e^T\epsilon_n)) \le \hbar/2,$$

so that w_n converges in C^∞-topology to a constant J-holomorphic curve $w : \mathbb{S}^2 \to M$ on any compact subset of $\mathbb{S}^2 \setminus \{0, \infty\}$. This implies $\lim_{n\to\infty} E(u_n, A(e^{-T}\epsilon_n, \epsilon_n)) = 0$ for any fixed $T > 0$.

On the other hand, by Lemma 2.15 (with $\mu = 1/2$), there exists a constant $c > 0$ such that for any $T > \ln 2$,

$$\lim_{n\to\infty} E(u_n, A(e^T\rho_n, e^{-T}\epsilon_n)) \le c\cdot e^{-T}\frac{\delta}{2}.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} E(u_n, A(\rho_n, e^T\rho_n)) &\ge \lim_{n\to\infty} E(u_n, A(\rho_n,\epsilon_n)) - \lim_{n\to\infty} E(u_n, A(e^T\rho_n, e^{-T}\epsilon_n))\\ &\ge (1-c\cdot e^{-T})\frac{\delta}{2} > \frac{\delta}{2} - \rho_0\end{aligned}$$

for a sufficiently large choice of $T > 0$. This contradicts to $\lim_{n\to\infty} E(u_n, A(\rho_n, R\rho_n)) \le \delta/2 - \rho_0$ for every fixed $R \ge 1$. This proves that there is no energy loss.

For condition (ii) that $v(\infty) = u_0(0)$, we argue as follows. For any given $\epsilon > 0$, choose large $R > 1$ and small $0 < \rho < 1$ such that for sufficiently large $n > 0$,

$$dist(v(\infty), u_n(2\rho_n Re^{i\theta})) \le \epsilon/5,\ dist(u_0(0), u_n(2^{-1}\rho e^{i\theta})) \le \epsilon/5.$$

On the other hand, since $\lim_{R\to\infty}\lim_{n\to\infty} E(u_n, D_{R\rho_n}) = m = \lim_{\rho\to 0}\lim_{n\to\infty} E(u_n, D_\rho)$, we have

$$\lim_{R\to\infty,\rho\to 0}\lim_{n\to\infty} E(u_n, A(R\rho_n,\rho)) = 0.$$

By Lemma 2.15 (with $\mu = 1/2$, $T = \ln 2$), this implies that for sufficiently large $R > 1$, sufficiently small $0 < \rho < 1$, and large $n > 0$,

$$dist(u_n(2\rho_n Re^{i\theta}), u_n(2^{-1}\rho e^{i\theta})) \le \epsilon/5.$$

Putting the estimates together, we obtain $dist(v(\infty), u_0(0)) \le 3\epsilon/5$ for any $\epsilon > 0$. Hence $v(\infty) = u_0(0)$.

Finally, we show that if v is constant, $\#Z = l \ge 2$. The point is that if v is constant, since v_n converges to v in C^∞-topology on any compact subset of $|z| > 1$, we see that $\lim_{n\to\infty} E(v_n, D_R) = m$ for any $R > 1$. On the other hand, $E(v_n, D_1) = m - \delta/2$, so that $\lim_{n\to\infty} E(v_n, A(D_1, D_R)) = \delta/2$. If there is no bubbling off on the unit circle $|z| = 1$, then $\lim_{n\to\infty} E(v_n, A(D_1, D_R)) = E(v, A(D_1, D_R)) = 0$, a contradiction. This shows that $Z \ne \emptyset$. But we have shown that if $Z \ne \emptyset$, then $0 \in Z$. Together with the bubbling off on $|z| = 1$, we see that $\#Z = l \ge 2$. □

What we have discussed so far gives a proof to the Gromov compactness theorem provided that the complex structures j_n on Σ do not degenerate. For the general case, see discussions in [26, 11].

3. Moduli spaces of J-holomorphic curves

3.1. **The Fredholm setup.**

We shall warm up by a discussion on two different but equivalent definitions of a connection on a smooth vector bundle E over a smooth manifold M.

Lemma 3.1. *Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M. The following are true.*

(1) *Let ∇ be a connection on E, i.e., $\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M)$ such that $\nabla(f\xi) = \xi \otimes df + f\nabla\xi$, $\forall f \in C^\infty(M)$, $\xi \in C^\infty(E)$. Then for any smooth path $\gamma : [0,a) \to M$, there exists a family of parallel transport $P_t : E|_{\gamma(0)} \to E|_{\gamma(t)}$, $t \in [0,a)$, where P_t are isomorphisms satisfying $P_0 = Id$, $P_{t_1+t_2} = P_{t_2} \circ P_{t_1}$, $\forall t_1, t_2, t_1+t_2 \in [0,a)$, such that for any $\xi \in E|_{\gamma(0)}$, $\nabla_X(P_t\xi) \equiv 0$, i.e., $P_t\xi$ is parallel along γ, where X stands for the tangent vector of γ.*

(2) *Suppose for any smooth path $\gamma : [0,a) \to M$, there exists a family of isomorphisms $P_t : E|_{\gamma(0)} \to E|_{\gamma(t)}$, $t \in [0,a)$, where P_t satisfy $P_0 = Id$, $P_{t_1+t_2} = P_{t_2} \circ P_{t_1}$, $\forall t_1, t_2, t_1+t_2 \in [0,a)$. Then there exists a connection ∇ on E such that for any smooth path $\gamma : [0,a) \to M$ and any $\xi \in E|_{\gamma(0)}$, $\nabla_X(P_t\xi) \equiv 0$, i.e., $P_t\xi$ is parallel along γ with respect to ∇. Here X stands for the tangent vector of γ.*

Proof. (1) Let ∇ be a connection on E. Note that the definition of P_t is a local problem. Let ξ_i be a local frame of E. Suppose A_{ij} are the connection 1-forms with respect to ξ_i, i.e., $\nabla\xi_i = \sum_j A_{ji}\xi_j$. Then for any $\xi = \sum_i f_i\xi_i$, one has

$$\nabla\xi = \sum_i (df_i + \sum_j A_{ij}f_j)\xi_i.$$

Now for any given smooth path $\gamma : [0,a) \to M$, and any $\xi \in E|_{\gamma(0)}$, where $\xi = \sum_i f_i\xi_i$, we define $P_t\xi := \sum_i f_i(t)\xi_i$, where $f_i(t)$ satisfy the following system of ODEs

$$f_i'(t) + \sum_j A_{ij}(X)(\gamma(t))f_j(t) = 0, \;\; f_i(0) = f_i,$$

with X being the tangent vector of γ.

(2) Suppose P_t are given. Then for any $p \in M$, $X \in T_pM$, and any $\xi \in C^\infty(E)$, we pick a smooth path $\gamma : [0,a) \to M$ such that $\gamma(0) = p$, $\gamma'(0) = X$, and let P_t be the family of isomorphisms associated to γ. With this understood, we define (note that $P_t^{-1}\xi(\gamma(t)) \in E_p = E_{\gamma(0)}$)

$$(\nabla_X\xi)(p) := \frac{d}{dt}(P_t^{-1}\xi(\gamma(t)))|_{t=0}.$$

To verify that ∇ is a connection, $\forall f \in C^\infty(M)$,

$$\begin{aligned}
(\nabla_X f\xi)(p) &= \frac{d}{dt}(P_t^{-1}(f\xi)(\gamma(t)))|_{t=0} \\
&= \frac{d}{dt}(f(\gamma(t))P_t^{-1}\xi(\gamma(t)))|_{t=0} \\
&= (Xf)(\gamma(t))P_t^{-1}\xi(\gamma(t))|_{t=0} + f(\gamma(t))\frac{d}{dt}(P_t^{-1}\xi(\gamma(t)))|_{t=0} \\
&= (Xf \cdot \xi)(p) + (f\nabla_X\xi)(p).
\end{aligned}$$

It remains to show that for any $\xi_0 \in E_{\gamma(0)}$, $\xi := P_t\xi_0$ is parallel along γ with respect to ∇. To see this, let X be the tangent vector along γ, then

$$(\nabla_X\xi)(\gamma(t)) = \frac{d}{ds}(P_s^{-1}P_{t+s}\xi_0)|_{s=0} = \frac{d}{ds}(P_t\xi_0)|_{s=0} = 0, \;\; \forall t \in [0,a).$$

□

Lemma 3.2. *Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M.*

(1) Suppose a connection ∇ on E is given. Then for any $\xi \in E$, there is a decomposition

$$T_\xi E = E_{\pi(\xi)} \oplus H_\xi,$$

where H_ξ depends on ξ smoothly and $d\pi|_{H_\xi} : H_\xi \to T_{\pi(\xi)}M$ is isomorphic.

(2) For any smooth section $s : M \to E$ and any point $p \in M$, define $D_p s : T_pM \to E_p$ by $D_p s := \Pi_{s(p)} \circ ds_p$, where $ds_p : T_pM \to T_{s(p)}E$ is the differential of s at p, and $\Pi_{s(p)} : T_{s(p)}E \to E_p$ is the projection given by the decomposition $T_{s(p)}E = E_p \oplus H_{s(p)}$ from part (1), then

$$D_p s(X) = (\nabla_X s)(p) \in E_p, \quad \forall X \in T_pM.$$

Remark 3.3. In some sense $D_p s : T_pM \to E_p$ is the "vertical part" of $ds_p : T_pM \to T_{s(p)}E$, and the above lemma shows that in order to define it, one can specify a connection ∇ on E and compute ∇s at p. Note also, as we will see in the proof, that if $p \in s^{-1}(0)$, then $D_p s$ is in fact independent of the choice of ∇.

Proof. (1) We define H_ξ as follows. Fix a local trivialization of E in a neighborhood U of $\pi(\xi)$, $E|_U = U \times V$ (V is a vector space). For any $X \in T_{\pi(\xi)}M$, pick a smooth path $\gamma : [0, a) \to U \subset M$ with $\gamma(0) = \pi(\xi)$ and $\gamma'(0) = X$. Let $P_t : V \to V$ be the parallel transport along γ. Then we define H_ξ to be the graph of the linear map

$$X \in T_{\pi(\xi)}M \mapsto \frac{d}{dt}(P_t\xi)|_{t=0} \in V.$$

Clearly H_ξ depends on ξ smoothly and $d\pi|_{H_\xi} : H_\xi \to T_{\pi(\xi)}M$ is isomorphic. What remains to verify is that H_ξ is independent of the choice of the local trivialization of E. The point is that if the change of trivialization is given by $q \in U \mapsto \Omega(q)$ where each $\Omega(q)$ is an automorphism of V, then P_t is changed to $\Omega(\gamma(t))P_t\Omega(\gamma(t))^{-1}$. It is easy to verify that

$$\frac{d}{dt}(\Omega(\gamma(t))P_t\Omega(\gamma(t))^{-1})|_{t=0} = \Omega(\pi(\xi)))(\frac{d}{dt}P_t|_{t=0})\Omega(\pi(\xi)))^{-1},$$

from which it follows that H_ξ is well-defined.

(2) Let ξ_i be a local frame which gives the local trivialization $E|_U = U \times V$. We write $s(q) = \sum_i s_i(q)\xi_i$, $\forall q \in U$. Then $ds_p : T_pM \to T_pM \times V$ is given by $X \in T_pM \mapsto (X, (Xs_i)(p))$. On the other hand, for any $X \in T_pM$, pick a smooth path $\gamma : [0, a) \to U$ with $\gamma(0) = p$ and $\gamma'(0) = X$. Let $P_t : V \to V$ be the parallel transport along γ. Then $P_t(s(p)) = \sum_i f_i(t)\xi_i$ where $f_i(t)$ satisfy the following system of ODEs

$$f_i'(t) + \sum_j A_{ij}(\tilde{X})(\gamma(t))f_j(t) = 0, \;\; f_i(0) = s_i(p),$$

with $\tilde{X}$ being the tangent vector of γ. (Here A_{ij} are the connection 1-forms with respect to ξ_i, i.e., $\nabla\xi_i = \sum_j A_{ji}\xi_j$.) By part (1), $H_{s(p)}$ is the graph of the linear map

$$X \in T_pM \mapsto (f_i'(0)) = (-\sum_j A_{ij}(X)(p)s_j(p)).$$

Hence $D_p s(X) = ((Xs_i)(p) + \sum_j A_{ij}(X)(p)s_j(p)) = (\nabla_X s)(p)$. □

In what follows the above discussion will be adopted to the infinite dimensional setting.

Let (M,ω) be a compact closed symplectic manifold of dimension $2n$, and let J be a ω-tame almost complex structure of class C^l. We endow M with the Hermitian metric g_J, where $g_J(v,w) := \frac{1}{2}(\omega(v,Jw)+\omega(w,Jv))$. Let Σ be a compact connected Riemann surface without boundary, and let j be any complex structure on Σ.

For a fixed choice of k,p satisfying $k \geq 2$ and $p > 2$ (we also assume $l \gg k$), we let $\mathcal{B}$ be the Banach manifold of locally $L^{k,p}$-maps from Σ to M, and let $\mathcal{E}$ be the Banach bundle over $\mathcal{B}$, where $\mathcal{E}_u = L^{k-1,p}(\Lambda^{0,1}\otimes u^*TM)$, $\forall u \in \mathcal{B}$. (cf. Exercise 2.15 in Lecture 1.) Note that J is of C^l-class so that the metric g_J is also of class C^l. By the C^l-version of Proposition 2.3 of Lecture 1 (i.e. with the assumption $H \in C^l(\mathbb{R})$), both $\mathcal{B}$ and $\mathcal{E}$ are of only C^l-class. Moreover, the section $s : \mathcal{B} \to \mathcal{E}$ defined by $s : u \in \mathcal{B} \mapsto (u, \bar{\partial}_J u) \in \mathcal{E}$ is a C^l-section.

With the preceding understood, we will introduce a connection on $\mathcal{E}$ and as explained in Lemma 3.2(2), use it to define the "vertical part" of the differential $ds : T\mathcal{B} \to T\mathcal{E}$. To this end, let ∇ be the Levi-Civita connection associated to g_J, and let $\widetilde{\nabla}$ be the connection defined by

$$\widetilde{\nabla}_X Y := \nabla_X Y - \frac{1}{2}J(\nabla_X J)Y, \ \ \forall X, Y \in \mathrm{Vect}(M).$$

Direct verification shows that $\widetilde{\nabla} J \equiv 0$, or equivalently, $\widetilde{\nabla}_X JY = J\widetilde{\nabla}_X Y$, which means that $\widetilde{\nabla}$ is complex linear with respect to J.

Denote by P_t the parallel transport with respect to $\widetilde{\nabla}$. We define a connection on $\mathcal{E}$ by defining the corresponding parallel transport Ψ_t as follows. For any smooth path $\gamma : [0,a) \to \mathcal{B}$, where we will write $\gamma(t) = u_t$, $u_t : \Sigma \to M$, and for any $\xi \in \mathcal{E}_{u_0} = L^{k-1,p}(\Lambda^{0,1}\otimes u_0^*TM)$, we define $\Psi_t\xi \in \mathcal{E}_{u_t} = L^{k-1,p}(\Lambda^{0,1}\otimes u_t^*TM)$ by

$$(\Psi_t\xi)(z)(X) = P_t(\xi(z)(X)), \ \ \forall z \in \Sigma, X \in T_z\Sigma.$$

Here P_t is the parallel transport along the path $\gamma_z(t) = u_t(z)$ in M. Note that $\xi(z)(X) \in u_0^*TM$, so that $P_t(\xi(z)(X)) \in u_t^*TM$. Since P_t commutes with J, we see that $\Psi_t\xi \in \mathcal{E}_{u_t}$.

For any $u \in \mathcal{B}$, we denote by $D_u : T_u\mathcal{B} \to \mathcal{E}_u$ the corresponding "vertical part" of ds_u. Recall $T_u\mathcal{B} = L^{k,p}(u^*TM)$.

Lemma 3.4. *For any* $\xi \in T_u\mathcal{B} = L^{k,p}(u^*TM)$,

$$D_u\xi = \frac{1}{2}(\nabla\xi + J(u)\circ\nabla\xi\circ j) - \frac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J u,$$

where $\partial_J u := \frac{1}{2}(du - J(u)\circ du\circ j)$.

Proof. Let $\gamma : [0,a) \to \mathcal{B}$ be the path, $\gamma(t) = u_t$ where $u_t(z) = \exp_{u(z)}(t\xi(z))$, $\forall z \in \Sigma$. Then $\gamma(0) = u$ and $\gamma'(0) = \xi$. Let Ψ_t be the parallel transport along γ, and let P_t be the parallel transport along the path $\gamma_z(t) = u_t(z)$ in M. Then as we showed in

Lemma 3.1(2), and by Lemma 3.2(2),

$$D_u\xi = \frac{d}{dt}(\Psi_t^{-1}\bar{\partial}_J(u_t))|_{t=0}.$$

With this understood, for any $z \in \Sigma$ and $X \in T_z\Sigma$, we have

$$(D_u\xi)(z)(X) = \frac{d}{dt}(P_t^{-1}\bar{\partial}_J(u_t)(z)(X))|_{t=0}.$$

With $\bar{\partial}_J(u_t)(z)(X) = \frac{1}{2}((u_t)_*(X) + J(u_t(z))(u_t)_*(jX))$, we have

$$\begin{aligned}
(D_u\xi)(z)(X) &= \frac{1}{2}\widetilde{\nabla}_{\xi(z)}((u_t)_*(X) + J(u_t(z))(u_t)_*(jX)) \\
&= \frac{1}{2}(\widetilde{\nabla}_{\xi(z)}(u_t)_*(X) + J(u(z))\widetilde{\nabla}_{\xi(z)}(u_t)_*(jX)) \\
&= \frac{1}{2}(\nabla_{\xi(z)}(u_t)_*(X) + J(u_t(z))\nabla_{\xi(z)}(u_t)_*(jX)) \\
&\quad -\frac{1}{4}(J(u(z))(\nabla_{\xi(z)}J)(u(z))u_*(X) - (\nabla_{\xi(z)}J)(u(z))u_*(jX))
\end{aligned}$$

Now observe that $\nabla_{\xi(z)}(u_t)_*(X) = \nabla_{u_*(X)}\xi(z)$, $\nabla_{\xi(z)}(u_t)_*(jX) = \nabla_{u_*(jX)}\xi(z)$ because ∇ is torsion free. This gives

$$D_u\xi = \frac{1}{2}(\nabla\xi + J(u)\circ\nabla\xi\circ j) - \frac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J u.$$

(Note that in above ∇ is actually the pull-back of the Levi-Civita connection.) $\square$

Exercise 3.5. (1) Introduce $\widehat{\nabla}_X Y := \widetilde{\nabla}_X Y - \frac{1}{4}(\nabla_{JY}J)X - \frac{1}{4}J(\nabla_Y J)X$. Show that $\widehat{\nabla}$ commutes with J. Furthermore, if we denote by $\widehat{\nabla}^{0,1}$ the $(0,1)$-component of $\widehat{\nabla}$, show that

$$D_u\xi = \widehat{\nabla}^{0,1}\xi - \frac{1}{4}N_J(\xi, \partial_J u) + \frac{1}{4}(J(\nabla_{\bar{\partial}_J u}J) + \nabla_{J\bar{\partial}_J u}J)\xi.$$

Note that both $T_u\mathcal{B}$ and $\mathcal{E}_u$ are complex Banach spaces (induced by J), and with this understood, $\widehat{\nabla}^{0,1}$ is complex linear while K_u is anticomplex linear, where

$$K_u\xi := -\frac{1}{4}N_J(\xi, \partial_J u) + \frac{1}{4}(J(\nabla_{\bar{\partial}_J u}J) + \nabla_{J\bar{\partial}_J u}J)\xi.$$

(2) Show that when J is ω-compatible, $D_u\xi = \widetilde{\nabla}^{0,1}\xi - \frac{1}{4}N_J(\xi, \partial_J u)$, and in particular, when (M, ω, J) is Kähler, $D_u\xi = \nabla^{0,1}\xi$ (cf. Exercise 1.3(1)).

Proposition 3.6. *For each $\lambda \in [0,1]$ we define $D_u^\lambda := \widehat{\nabla}^{0,1} + \lambda K_u$. Then D_u^λ is a generalized Cauchy-Riemann operator (not necessarily with smooth coefficients, only in the sense of principal symbol), and $D_u^\lambda : L^{k,p}(u^*TM) \to L^{k-1,p}(\Lambda^{0,1}\otimes u^*TM)$ is a Fredholm operator.*

Proof. Since K_u is of order 0, we only need to compute the principal symbol of $\widehat{\nabla}^{0,1}$. For any $f \in C^\infty(\Sigma)$, $[\widehat{\nabla}^{0,1}, f] = (df)^{0,1}\otimes$, where $(df)^{0,1}$ is the $(0,1)$-component of df. This shows that D_u^λ has the same principal symbol of a Cauchy-Riemann operator.

To see D_u^λ is Fredholm, we first show that K_u is a compact operator. In fact this follows easily from the estimate $||K_u\xi||_{k-1,p} \le C||\xi||_{C^{k-1}}$, because the embedding

$L^{k,p} \hookrightarrow C^{k-1}$ is compact by Morrey's theorem (here we use the assumption that $p > 2$). This reduces to show that D_u is Fredholm. Recall that

$$D_u\xi = \frac{1}{2}(\nabla\xi + J(u) \circ \nabla\xi \circ j) - \frac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J u.$$

By the same argument, $\xi \mapsto -\frac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J u$ is also compact. So we further reduce the problem to showing that the operator

$$L_u\xi := \frac{1}{2}(\nabla\xi + J(u) \circ \nabla\xi \circ j)$$

is Fredholm. We simply repeat the proof of Theorem 1.75 in Lecture 1, with the help of the following lemma.

Lemma 3.7. *(1) Let $1 \leq s \leq k$ and $q > 1$. There exists a constant $C > 0$ such that*

$$||\xi||_{s,q} \leq C(||L_u\xi||_{s-1,q} + ||\xi||_q).$$

The same holds for the formal adjoint L_u^*.

(2) Let $q > 1$. If $L_u\xi = 0$ weakly and ξ is of L^q-class, then ξ is of $L^{k+1,p}$-class. Moreover, the same regularity of weak L^q-solutions holds for L_u^*.

□

Exercise 3.8. (1) Prove Lemma 3.7. Hint: for part (1), generalize the arguments of Prop. 1.60 and Ex. 1.61 in Lecture 1 (here we use the assumption $p > 2$); for part (2), generalize the arguments in Lemma 1.68 of Lecture 1 (here we use the assumption $k \geq 2$ and $p > 2$), then use the elliptic bootstrapping as in Prop. 2.1 (of Lecture 2).

(2) Finish the proof of Proposition 3.6.

Remark 3.9. By the Atiyah-Singer index theorem, the index of D_u can be computed via Riemann-Roch formula, and is given by

$$\text{Index } D_u = 2c_1(TM) \cdot u_*([\Sigma]) + \dim M \cdot (1 - \text{genus}(\Sigma)).$$

Note that indexD_u is always an even number.

For any integer $g \geq 0$, let $\mathcal{M}_g$ be the moduli space of complex structures of a Riemann surface Σ of genus g. It is known that when $\Sigma = \mathbb{S}^2$ of genus 0, $\mathcal{M}_0 = \{j_0\}$, the standard complex structure on $\mathbb{S}^2$, and when $\Sigma = \mathbb{T}^2$ of genus 1, $\mathcal{M}_1$ is the quotient of the upper half plane by $PSL(2;\mathbb{Z})$. For $g \geq 2$, $\mathcal{M}_g$ is a complex orbifold of dimension $3g - 3$.

Let a homology class $0 \neq A \in H_2(M)$ be given. Fix a $k \geq 2$ and $p > 2$, we let $\mathcal{B}$ be the Banach manifold of locally $L^{k,p}$-maps u from Σ to M such that $u_*([\Sigma]) = A$. Denote by $\mathcal{B}^*$ the subset of $\mathcal{B}$ which consists of u that are somewhere injective, i.e., there exists a point $z \in \Sigma$ such that $du(z) \neq 0$ and $u^{-1}(u(z)) = \{z\}$. Since $L^{k,p} \hookrightarrow C^1$, it follows that $\mathcal{B}^*$ is an open submanifold of $\mathcal{B}$. We fix a $l \gg k$, and denote by $\mathcal{J}_\tau^l = \mathcal{J}_\tau^l(M,\omega)$ the Banach manifold of ω-tame almost complex structures on M which are of C^l-class. Even though $\mathcal{M}_g$ is generally an orbifold, we shall assume it is a manifold for technical simplification. Hence the Banach manifold $\mathcal{B} \times \mathcal{J}_\tau^l \times \mathcal{M}_g$, and the open submanifold $\mathcal{B}^* \times \mathcal{J}_\tau^l \times \mathcal{M}_g$. We let $\mathcal{E}$ be the Banach bundle over $\mathcal{B} \times \mathcal{J}_\tau^l \times \mathcal{M}_g$ or $\mathcal{B}^* \times \mathcal{J}_\tau^l \times \mathcal{M}_g$, where at each (u, J, j), the fiber $\mathcal{E}_{(u,J,j)} = L^{k-1,p}((\Lambda^{0,1}, j) \otimes (u^*TM, J))$, where $(\Lambda^{0,1}, j)$ is the holomorphic line bundle of $(0,1)$-forms over (Σ, j) and (u^*TM, J) is the pull-back complex vector bundle over Σ with complex structure given by J. We have discussed the trivialization of $\mathcal{E}$ along the $\mathcal{B}$ factor; the trivialization along the other two factor $\mathcal{J}^l_\tau$ and $\mathcal{M}_g$ is elementary.

The section $s : \mathcal{B} \times \mathcal{J}^l_\tau \times \mathcal{M}_g :\rightarrow \mathcal{E}$, $(u, J, j) \mapsto ((u, J, j), \bar{\partial}_J u)$ where $\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j)$, is a C^l-smooth section. By Proposition 3.6, s is a family of Fredholm sections parametrized by $\mathcal{J}^l_\tau \times \mathcal{M}_g$. Since $\mathcal{M}_g$ is finite dimensional, we can consider as well s as a family of Fredholm sections parametrized by $\mathcal{J}^l_\tau$. With this understood, we define

$$\mathcal{M}(g, A) := s^{-1}(0), \mathcal{M}(g, A, J) := s^{-1}(0) \cap (\mathcal{B} \times \{J\} \times \mathcal{M}_g), \text{ where } J \in \mathcal{J}^l_\tau.$$

Roughly speaking, $\mathcal{M}(g, A)$ is the set of (J, j)-holomorphic curves for some $j \in \mathcal{M}_g$ and $J \in \mathcal{J}^l_\tau$ from a genus g Riemann surface into M carrying homology class A, $\mathcal{M}(g, A, J)$ is the subset of J-holomorphic curves for that given J. By elliptic bootstrapping as in Prop 2.1, elements of $\mathcal{M}(g, A)$ or $\mathcal{M}(g, A, J)$ are of $L^{l+1,p}$-class, particularly, it is independent of k. We let $\mathcal{M}^*(g, A)$, $\mathcal{M}^*(g, A, J)$ be the corresponding open subset consisting of simple curves.

In the next section, we will use Sard-Smale theorem to show that $\mathcal{M}^*(g, A, J)$ is a finite dimensional manifold for a generic J. We shall end here with a discussion on the orientation.

Proposition 3.10. *$\mathcal{M}^*(g, A, J)$ is naturally given a coherent orientation. When (M, ω, J) is Kähler, $\mathcal{M}^*(g, A, J)$ is complex analytic, and the coherent orientation is the same as the canonical orientation from the complex analytic structure.*

Proof. Consider the family of Fredholm operators $\mathcal{L} = \{L_x | x = (u, J, j) \in \mathcal{B} \times \mathcal{J}^l_\tau \times \mathcal{M}_g\}$, where $L_x = (D_u, D_j) : T_u\mathcal{B} \times T_j\mathcal{M}_g \rightarrow \mathcal{E}_{(u,J,j)}$. Here D_j is the "vertical part" of ds that comes from the factor $\mathcal{M}_g$. Since $T_j\mathcal{M}_g$ is finite dimensional and D_u is Fredholm, L_x is Fredholm. The point here is that the determinant line bundle $\det \mathcal{L}$ over $\mathcal{B} \times \mathcal{J}^l_\tau \times \mathcal{M}_g$ is trivial and carries a natural trivialization. To see this, we consider a larger family $\hat{\mathcal{L}} = \{L_{x,\lambda}\}$ over $\mathcal{B} \times \mathcal{J}^l_\tau \times \mathcal{M}_g \times [0, 1]$, where $L_{x,\lambda} = (D^\lambda_u, \lambda D_j)$. Since D_j is a compact operator, $L_{x,\lambda}$ continued to be Fredholm. We note that $\det \mathcal{L}$ is the restriction of $\det \hat{\mathcal{L}}$ to the subspace $\mathcal{B} \times \mathcal{J}^l_\tau \times \mathcal{M}_g \times \{1\}$, which is isomorphic to the restriction of $\det \hat{\mathcal{L}}$ to the subspace $\mathcal{B} \times \mathcal{J}^l_\tau \times \mathcal{M}_g \times \{0\}$. Since when $\lambda = 0$, $L_{x,\lambda} = (D^0_u, 0) = (\widehat{\nabla}^{0,1}, 0)$ which is complex linear, the corresponding determinant line bundle is trivial with a canonical trivialization coming from the complex structure. Hence $\det \mathcal{L}$ is trivial and carries a natural trivialization. The proposition follows by standard construction. The Kähler case follows from the fact that $D_u = \nabla^{0,1}$ in this case. $\square$

Let $\mathcal{J}^l$ denote the Banach manifold of ω-compatible almost complex structures of C^l-class. Then the above discussion goes through verbatim with $\mathcal{J}^l_\tau$ replaced by $\mathcal{J}^l$.

3.2. **Transversality.** Let $J \in \mathcal{J}^l_\tau$ or $\mathcal{J}^l$. We shall first find out the tangent space of $\mathcal{J}^l_\tau$ or $\mathcal{J}^l$ at J. It is given by the space of C^l-smooth sections of a certain vector bundle over M; for the purpose here, it is important to identify the fiber of the bundle.

To this end, we consider $\mathbb{R}^{2n}$ equipped with the standard symplectic structure ω_0 and the standard complex structure J_0. If $J(t) := J_0 e^{tB}$, where B is a $2n \times 2n$ matrix, is a complex structure, i.e., $J(t)^2 = -Id$, for all small $t > 0$, then B satisfies

$$J_0 B J_0 - B = 0, \text{ or equivalently } J_0 B = -B J_0.$$

If furthermore, $J(t)$ is ω_0-compatible, which means $\omega_0(J(t)v, J(t)w) = \omega_0(v, w)$, $\forall v, w \in \mathbb{R}^{2n}$, then B will further satisfy

$$\omega_0(J_0 Bv, J_0 w) + \omega_0(J_0 v, J_0 Bw) = 0, \forall v, w \in \mathbb{R}^{2n},$$

or equivalently, the matrix $J_0 B$ is symmetric.

With the preceding understood, it follows easily that the tangent space of $\mathcal{J}^l_\tau$ at J is given by the space of C^l-smooth sections of endomorphisms of TM which anti-commutes with J, and the tangent space of $\mathcal{J}^l$ at J is the subspace of $T_J\mathcal{J}^l_\tau$ consisting of sections B such that JB is symmetric with respect to the metric g_J.

Theorem 3.11. *Given $g \geq 0$ and $A \in H_2(M)$, there are subsets of Baire's second category $\mathcal{J}^l_{\tau,reg} \subset \mathcal{J}^l_\tau$, $\mathcal{J}^l_{reg} \subset \mathcal{J}^l$, such that for any $J \in \mathcal{J}^l_{\tau,reg}$ or $\mathcal{J}^l_{reg}$, the operator D_u is surjective for every $u \in \mathcal{M}^*(g, A, J)$. Consequently, for such a J, $\mathcal{M}^*(g, A, J)$ is a C^l-smooth manifold of dimension $d(g, A)$, where*

$$d(g, A) = \begin{cases} 2c_1(TM) \cdot A + \dim M & \text{if } g = 0 \\ 2c_1(TM) \cdot A + 2 & \text{if } g = 1 \\ 2c_1(TM) \cdot A + (6 - \dim M)(g - 1) & \text{if } g \geq 2. \end{cases}$$

Moreover, for any two such J_0, J_1, and any path $\gamma(t) = J_t, t \in [0, 1]$, connecting them, one can slightly perturb $\gamma(t)$ such that

$$W := \cup_{t \in [0,1]} \mathcal{M}^*(g, A, J_t) \times \{t\} \subset \mathcal{M}(g, A) \times [0, 1]$$

is a C^l-smooth manifold of dimension $d(g, A)+1$ with boundary $\mathcal{M}^*(g, A, J_0) \sqcup \mathcal{M}^*(g, A, J_1)$.

Proof. It suffices to consider the case $\mathcal{J}^l$ only. To this end, we will study the family of Fredholm sections $s : \mathcal{B} \times \mathcal{J}^l \times \mathcal{M}_g \to \mathcal{E}$, where $(u, J, j) \mapsto ((u, J, j), \bar{\partial}_J u)$, $\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j)$, and show that s is transverse to the zero section. The theorem then follows from Theorem 2.29 in Lecture 1.

The point of the said transversality is that the space $\mathcal{J}^l$ is large enough such that the "vertical part" of ds along the factor of $T_J\mathcal{J}^l$, $\forall J \in \mathcal{J}^l$, will generate enough elements to cover the cokernel of D_u for any $u \in \mathcal{M}^*(g, A, J)$. More precisely, for any $x := (u, J, j) \in \mathcal{M}^*(g, A)$, denote by D_x the "vertical part" of ds at x. Then the cokernel of D_x is a quotient of the cokernel of D_u, which is finite dimensional. Thus its dual space $(\text{coker} D_x)^*$ is naturally identified as a subspace of the dual space of the cokernel of D_u, which by Theorem 1.57 in Lecture 1, is given by $\ker D_u^*$ via the L^2-product. With this understood, the theorem follows readily from the following lemma. ◻

Lemma 3.12. *For any $0 \neq \xi \in \ker D_u^*$, there exists an $\eta \in T_J\mathcal{J}^l$ such that*

$$\int_\Sigma \langle D_x \eta, \xi \rangle_{g_J} dvol_\Sigma \neq 0.$$

Proof. Since u is a simple curve, by Theorem 1.12(2) the set of injective points is open and dense in Σ. If $\xi \in \ker D_u^*$ is nonzero, then there must exist an injective point $z \in \Sigma$ such that ξ is non-vanishing at z. Recall that for z as an injective point, there exists a neighborhood V of $q := u(z)$ and a neighborhood U of z such that $U = u^{-1}(V)$ and u is embedded on U. We could further identify V as an open neighborhood of $0 \in \mathbb{R}^{2n}$, with coordinates $x_1, y_1, \cdots, x_n, y_n$, such that $u(U)$ lies in the x_1y_1-plane. Furthermore, $z = 0 \in U$ and $q = 0 \in \mathbb{R}^{2n}$, $J = J_0$ at 0, and if $z = s + it$, then $\partial_s u = \partial_{x_1}$ and $\partial_t u = \partial_{y_1}$ at 0.

With this understood, we write $\xi(0) = \xi_1 ds + \xi_2 dt$, where $\xi_1, \xi_2 \in \mathbb{R}^{2n}$ satisfying $J_0\xi_1 = -\xi_2$. On the other hand, for any $\eta \in T_J\mathcal{J}^l$, $D_x\eta(0)$ takes the form

$$D_x\eta(0) = B(\partial_{x_1})ds - J_0B(\partial_{x_1})dt$$

for some $2n \times 2n$ matrix B satisfying $J_0B = -BJ_0$ and J_0B is symmetric. (In fact, $B = \eta(0)/2$.)

It is straightforward to check that there is a B satisfying the above conditions, such that $B(\partial_{x_1}) = \xi_1$. For example, take B to be the matrix whose first two columns are ξ_1, ξ_2 as column vectors and whose first two rows are also ξ_1, ξ_2 but as row vectors, and the rest entries are all zero. We extend B to a smooth section η on M by multiplying B by a bump function which is supported in a sufficiently small neighborhood of q. Then with this η, it is easy to see that the claim of the lemma holds. (We remark that the condition $\xi \in \ker D_u^*$ was only used here to ensure that ξ is continuous at z.) $\square$

We remark that the almost complex structures in $\mathcal{J}^l_{\tau,reg}$ or $\mathcal{J}^l_{reg}$, even though forming a dense subset, are only known to be C^l-smooth. Next we will show that in fact, both $\mathcal{J}^l_{\tau,reg}$ and $\mathcal{J}^l_{reg}$ contain enough of smooth almost complex structures.

First, we introduce some definitions. We let $\mathcal{J}_{reg} \subset \mathcal{J}(M,\omega)$ be the subset of ω-compatible almost complex structures J such that D_u is surjective for every $u \in \mathcal{M}^*(g, A, J)$. More generally, for any $K > 0$, we let $\mathcal{J}_{reg,K} \subset \mathcal{J}(M,\omega)$ be the subset of J such that for every $u \in \mathcal{M}^*(g, A, J)$ satisfying (1) $|du| \leq K$, and (2) there exists a $z \in \Sigma$ such that $\inf_{w \neq z} dist(u(w), u(z))/dist(w, z) \geq 1/K$. Clearly,

$$\mathcal{J}_{reg} = \cap_{K=1}^{\infty} \mathcal{J}_{reg,K}.$$

Similarly, we can define $\mathcal{J}_{\tau,reg}$ and $\mathcal{J}_{\tau,reg,K}$ for the ω-tame case.

Proposition 3.13. *For each $K > 0$, $\mathcal{J}_{\tau,reg,K}$, $\mathcal{J}_{reg,K}$ are open and dense subset of $\mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}(M,\omega)$ respectively. In particular, $\mathcal{J}_{\tau,reg}$ and $\mathcal{J}_{reg}$ are subsets of Baire's second category.*

We remark that $\mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}(M,\omega)$ are Fréchet manifolds, which are locally modeled on a complete, quasi-normed space. The Baire-Hausdorff theorem implies that $\mathcal{J}_{\tau,reg}$, $\mathcal{J}_{reg}$ are dense subsets, in particular, nonempty.

Proof. We will only consider $\mathcal{J}_{reg,K}$; the case of $\mathcal{J}_{\tau,reg,K}$ is the same. The argument is due to Taubes.

To see that $\mathcal{J}_{reg,K}$ is open, let $J \in \mathcal{J}_{reg,K}$ be any element, and suppose $(J_n) \subset \mathcal{J}(M,\omega)$ is a sequence converging to J in C^∞-topology. If $u_n \in \mathcal{M}^*(g, A, J_n)$ such that $|du_n| \leq K$, then a subsequence of u_n, still denoted by u_n, will converge in C^∞-topology to a $u_0 \in \mathcal{M}(g, A, J)$ with $|du_0| \leq K$ satisfied. If furthermore, there exists a $z_n \in \Sigma$ such that $\inf_{w \neq z_n} dist(u_n(w), u_n(z_n))/dist(w, z_n) \geq 1/K$, $\forall n$, then there is a $z_0 \in \Sigma$ such that $\inf_{w \neq z_0} dist(u_0(w), u_0(z_0))/dist(w, z_0) \geq 1/K$. In particular, $u_0 \in \mathcal{M}^*(g, A, J)$. This shows that for sufficiently large n, D_{u_n} must be surjective. It follows easily that $\mathcal{J}_{reg,K}$ is open.

Note that for any l, one can similarly define $\mathcal{J}^l_{reg,K}$, and the same argument also shows that $\mathcal{J}^l_{reg,K}$ is open in $\mathcal{J}^l$. With this understood, we will show $\mathcal{J}_{reg,K}$ is dense in $\mathcal{J}(M, \omega)$. Let a $J \in \mathcal{J}(M, \omega)$ be given. First, note that by Theorem 3.11, $\mathcal{J}^l_{reg,K}$ is dense in $\mathcal{J}^l$ for any l. Hence for any l, there is a $J'_l \in \mathcal{J}^l_{reg,K}$ such that $||J'_l - J||_{C^l} \leq 1/2l$. On the other hand, since $\mathcal{J}^l_{reg,K}$ is open, there exists a $J_l \in \mathcal{J}^l_{reg,K} \cap \mathcal{J}(M, \omega) = \mathcal{J}_{reg,K}$, such that $||J_l - J'_l||_{C^l} \leq 1/2l$. This gives $||J_l - J||_{C^l} \leq 1/l$, $\forall l$, where $J_l \in \mathcal{J}_{reg,K}$. Our claim that $\mathcal{J}_{reg,K}$ is dense in $\mathcal{J}(M, \omega)$ follows by showing that the sequence (J_n) converges to J in C^∞-topology.

To see this, for any fixed l, let $\epsilon > 0$ be given. Then when $n \geq \max(l, 1/\epsilon) + 1$, we have

$$||J_n - J||_{C^l} \leq ||J_n - J||_{C^n} \leq 1/n < \epsilon.$$

□

We observe that the group of biholomorphisms acts on the space of J-holomorphic curves by re-parametrization, and the action is free on the subset of simple curves. We denote by $\widetilde{\mathcal{M}^*}(g, A, J)$ the quotient of $\mathcal{M}^*(g, A, J)$ under the action. Note that only when $g = 0$ or 1, the group of biholomorphisms has a positive dimension, which is 6 and 2 respectively. This gives

Corollary 3.14. *For any $J \in \mathcal{J}^l_{\tau,reg}$ or $\mathcal{J}^l_{reg}$ ($l = \infty$ included), $\widetilde{\mathcal{M}^*}(g, A, J)$ is a C^l-smooth manifold of dimension*

$$\tilde{d}(g, A) := 2c_1(TM) \cdot A + (6 - \dim M)(g - 1).$$

In particular, if $\tilde{d}(g, A) < 0$, then $\widetilde{\mathcal{M}^*}(g, A, J) = \emptyset$ for any $J \in \mathcal{J}^l_{\tau,reg}$ or $\mathcal{J}^l_{reg}$.

Example 3.15. Let (M, ω) be a symplectic 4-manifold. Then for a generic choice of J, i.e., $J \in \mathcal{J}^l_{\tau,reg}$ or $\mathcal{J}^l_{reg}$ ($l = \infty$ included), there exists no embedded J-holomorphic sphere C in M with $C^2 \leq -2$. To see this, note that by the Adjunction Formula,

$$C^2 - c_1(TM) \cdot C = 0 - 2.$$

This implies that $\tilde{d}(g, A) = 2c_1(TM) \cdot C + (6 - \dim M)(0 - 1) = 2(C^2 + 2) - 2 < 0$.

This example should be compared with the following proposition.

Proposition 3.16. *Let (M, ω) be a symplectic 4-manifold, and let $J \in \mathcal{J}_\tau(M, \omega)$ be any element. For any embedded J-holomorphic sphere $C = u(\mathbb{S}^2)$ in M satisfying $C^2 \geq -1$, the operator D_u is surjective.*

The following exercise is designed to give a proof of Proposition 3.16.

Exercise 3.17. (1) Note that $u^*TM = T\mathbb{S}^2 \oplus E$ where E is the normal bundle (naturally a complex line bundle) of C. This gives rise to decompositions

$$L^{k,p}(u^*TM) = L^{k,p}(T\mathbb{S}^2)\oplus L^{k,p}(E), L^{k-1,p}(\Lambda^{0,1}\otimes u^*TM) = L^{k-1,p}(\Lambda^{0,1}\otimes T\mathbb{S}^2)\oplus L^{k-1,p}(\Lambda^{0,1}\otimes E).$$

Let $\pi : L^{k-1,p}(\Lambda^{0,1} \otimes u^*TM) \to L^{k-1,p}(\Lambda^{0,1} \otimes E)$ be the projection. Show that (i) D_u maps $L^{k,p}(T\mathbb{S}^2)$ into $L^{k-1,p}(\Lambda^{0,1} \otimes T\mathbb{S}^2)$, (ii) one can arrange so that

$$\pi \circ D_u : L^{k,p}(E) \to L^{k-1,p}(\Lambda^{0,1} \otimes E)$$

is a generalized Cauchy-Riemann operator, and (iii) D_u is surjective if both $D_u : L^{k,p}(T\mathbb{S}^2) \to L^{k-1,p}(\Lambda^{0,1}\otimes T\mathbb{S}^2)$ and $\pi\circ D_u : L^{k,p}(E) \to L^{k-1,p}(\Lambda^{0,1}\otimes E)$ are surjective.

(2) Let E be any complex line bundle over a Riemann surface Σ, and let $D : C^\infty(E) \to \Omega^{0,1}(E)$ be any generalized Cauchy-Riemann operator. Show that D is surjective if $c_1(E) \cdot [\Sigma] + (2 - 2\text{genus}(\Sigma)) > 0$.

Hints: (i) Let D^* be the formal adjoint of D. Then any $0 \neq \xi \in \ker D^*$ satisfies

$$|\Delta\xi| \leq C(|\xi| + |d\xi|).$$

(ii) Use Hartman-Wintner to show that near any zero point, ξ can be written locally as

$$\xi(z) = a\bar{z}^m + o(|z|^m), \text{ for some } m > 0 \text{ and } 0 \neq a \in \mathbb{C}.$$

(iii) Show that if $c_1(\Lambda^{0,1} \otimes E) \cdot [\Sigma] > 0$, $\ker D^* = 0$.

Finally, we mention a transversality result which follows from Bochner's technique.

Proposition 3.18. *Let (M, ω, J) be Kähler with non-negative holomorphic bisectional curvature. Then for any holomorphic sphere $u : \mathbb{S}^2 \to M$ (not necessarily simple), D_u is surjective.*

Exercise 3.19. Prove Proposition 3.18. Hint: use Exercise 2.11(2), (3) and Bochner's argument.

4. Gromov-Witten invariants

4.1. **Stable maps and axioms of Gromov-Witten invariants.** In this section we will give a general review of Gromov-Witten invariants. A key concept in the theory, which we will explain first, is Kontsevich's notion of a *stable map* [14, 15]. We follow the discussion in [11].

Definition 4.1. (Mumford [20]) A *semistable curve* with m marked points is a pair $(\Sigma, \mathbf{z})$ of a space $\Sigma = \cup\pi_\nu(\Sigma_\nu)$ where Σ_ν is a Riemann surface and $\pi_\nu : \Sigma_\nu \to \Sigma$ is a continuous map, and $\mathbf{z} = (z_1, z_2, \cdots, z_m)$ are m points in Σ ($m \geq 0$) with the following properties.

1. For each $p \in \Sigma_\nu$ there exists a neighborhood of it such that the restriction of π_ν to this set is a homeomorphism onto its image.
2. For each $p \in \Sigma$, we have $\sum_\nu \#\pi_\nu^{-1}(p) \leq 2$.
3. $\sum_\nu \#\pi_\nu^{-1}(z_i) = 1$ for each $z_i \in \mathbf{z}$.
4. Σ is connected.
5. $z_i \neq z_j$ for $i \neq j$.
6. The number of Riemann surfaces Σ_ν is finite.

(7) The set $\{p|\sum_\nu \#\pi_\nu^{-1}(p)=2\}$ is finite.

A point $p\in\Sigma_\nu$ is *singular* if $\sum_\mu \#\pi_\mu^{-1}(\pi_\nu(p))=2$, and is *marked* if $\pi_\nu(p)=z_j$ for some $z_j\in\mathbf{z}$. Each Σ_ν is called a *component* of Σ.

A map $\theta:\Sigma\to\Sigma'$ between two semistable curves is called an *isomorphism* if it is a homeomorphism and it can be lifted to biholomorphisms $\theta_{\nu\mu}:\Sigma_\nu\to\Sigma'_\mu$ between the components. If Σ,Σ' have marked points $(z_1,z_2,\cdots,z_m)$, $(z'_1,z'_2,\cdots,z'_m)$, we require $\theta(z_i)=z'_i$ also. Let $\mathrm{Aut}(\Sigma,\mathbf{z})$ denote the group of all automorphisms of $(\Sigma,\mathbf{z})$.

In order to define the genus of a semistable curve Σ, we associate it with a graph T_Σ as follows. The vertices of T_Σ correspond to the components of Σ and for each pair of singular points we draw an edge between the corresponding vertices (or vertex). The graph T_Σ is connected because Σ is connected.

Definition 4.2. The *genus* of a semistable curve $\Sigma=\cup\pi_\nu(\Sigma_\nu)$ is defined to be

$$g:=\sum_\nu g_\nu+\mathrm{rank}H_1(T_\Sigma;\mathbb{Q}),$$

where g_ν is the genus of Σ_ν.

Now let (M,ω) be a compact symplectic manifold of dimension $2n$ and let $J\in\mathcal{J}_\tau(M,\omega)$. Let $A\in H_2(M)$. A map $h:\Sigma\to M$ from a semistable curve is called *J-holomorphic* if it is continuous and $h\circ\pi_\nu:\Sigma_\nu\to M$ is J-holomorphic for each ν. We define the homology class of h to be $h_*([\Sigma])=\sum_\nu(h\circ\pi_\nu)_*[\Sigma_\nu]\in H_2(M)$.

Definition 4.3. A pair $((\Sigma,\mathbf{z}),h)$ of a semistable curve with m marked points and a J-holomorphic map $h:\Sigma\to M$ is said to be *stable* if for each ν one of the following conditions hold.

(1) $h\circ\pi_\nu:\Sigma_\nu\to M$ is not constant.
(2) Let m_ν be the number of points on Σ_ν which are either singular or marked. Then $m_\nu+2g_\nu\geq 3$.

The automorphism group of $((\Sigma,\mathbf{z}),h)$, denoted by $\mathrm{Aut}((\Sigma,\mathbf{z}),h)$, is the subgroup of $\mathrm{Aut}(\Sigma,\mathbf{z})$ consisting of elements θ such that $h\circ\theta=h$.

Exercise 4.4. Show that $((\Sigma,\mathbf{z}),h)$ is stable if and only if $\mathrm{Aut}((\Sigma,\mathbf{z}),h)$ is finite.

Definition 4.5. We define the *moduli space of stable maps of genus g, m marked points and of homology class A*, which is denoted by $\overline{\mathcal{M}}_{g,m}(M,J,A)$, to be the space of equivalence classes of stable pairs $((\Sigma,\mathbf{z}),h)$, where $((\Sigma,\mathbf{z}),h)$, $((\Sigma',\mathbf{z}'),h')$ are equivalent if and only if there exists an isomorphism $\theta:(\Sigma,\mathbf{z})\to(\Sigma',\mathbf{z}')$ such that $h'\circ\theta=h$. (Without confusion, we will denote the equivalence class of $((\Sigma,\mathbf{z}),h)$ by the same notation.)

A semistable curve $\Sigma=\cup\pi_\nu(\Sigma_\nu)$ with m marked points $\mathbf{z}$ is called *stable* if $((\Sigma,\mathbf{z}),h)$ is stable where h is a constant map. The moduli space $\overline{\mathcal{M}}_{g,m}$ of stable curves of genus g and m marked points is called the *Deligne-Mumford compactification* of the moduli space of curves of genus g and m marked points, where $2g+m\geq 3$. It is known to be a compact, complex orbifold of complex dimension $3g-3+m$.

We assume $2g+m-3\geq 0$. Then there are two maps, $\pi:\overline{\mathcal{M}}_{g,m}(M,J,A)\to\overline{\mathcal{M}}_{g,m}$ and $ev:\overline{\mathcal{M}}_{g,m}(M,J,A)\to M^m$, which we will explain below. The map π is defined as follows. Given a $((\Sigma, \mathbf{z}), h) \in \overline{\mathcal{M}}_{g,m}(M, J, A)$, we shrink any unstable component Σ_ν of Σ, i.e., $2g_\nu + m_\nu - 3 < 0$, to a point. Then the quotient space Σ' can be made into a stable curve of genus g and m marked points as follows. Note that Σ_ν is unstable if and only if $g_\nu = 0$ and $0 < m_\nu < 3$, where Σ_ν contains at least 1 singular point. It follows that Σ_ν contains at most 1 marked point. With this understood, we will let the image of Σ_ν be a marked point in Σ' if and only if Σ_ν contains 1 marked point. During this process, both g and m are unchanged, and since $2g + m - 3 \geq 0$, there is at least one component of Σ is stable. Hence Σ' is a stable curve. The equivalence class of Σ' in $\overline{\mathcal{M}}_{g,m}$ is defined to be the image of $((\Sigma, \mathbf{z}), h)$ under the map π. The map ev (called the evaluation map) is defined by

$$ev : ((\Sigma, \mathbf{z}), h) \mapsto (h(z_1), h(z_2), \cdots, h(z_m)) \in M^m.$$

Note that if $2g + m - 3 < 0$, then $\overline{\mathcal{M}}_{g,m}$ is an empty set, and the map π is not defined.

With the preceding understood, the Gromov compactness theorem can be transformed into the following theorem.

Theorem 4.6. *The moduli space $\overline{\mathcal{M}}_{g,m}(M, J, A)$ is a compact space such that both maps $\pi : \overline{\mathcal{M}}_{g,m}(M, J, A) \to \overline{\mathcal{M}}_{g,m}$ and $ev : \overline{\mathcal{M}}_{g,m}(M, J, A) \to M^m$ are continuous.*

The key issue is whether $\overline{\mathcal{M}}_{g,m}(M, J, A)$, being a compact space, carries a "fundamental class". The proof of the following theorem is beyond the scope here. See McDuff-Salamon [18] for complete references.

Theorem 4.7. *The space $\overline{\mathcal{M}}_{g,m}(M, J, A)$ carries a "fundamental class" $[\overline{\mathcal{M}}_{g,m}(M, J, A)]$ of degree $d = d(g, m, A)$ over $\mathbb{Q}$, which is called the "virtual fundamental cycle", where*

$$d = d(g, m, A) := 2c_1(TM) \cdot A + (6 - \dim M)(g - 1) + 2m.$$

Moreover, $[\overline{\mathcal{M}}_{g,m}(M, J, A)]$ is independent of J and invariant under smooth deformations of the symplectic structure ω.

With this understood, the Gromov-Witten invariants (GW invariants, also called GW classes in [15]), as linear maps

$$GW^M_{g,m,A} : H^*(M; \mathbb{Q})^{\otimes m} \to H^*(\overline{\mathcal{M}}_{g,m}; \mathbb{Q}),$$

are defined as follows. Given any $\alpha_1, \alpha_2, \cdots, \alpha_m \in H^*(M; \mathbb{Q})$ and $\beta \in H_*(\overline{\mathcal{M}}_{g,m}; \mathbb{Q})$, we define $GW^M_{g,m,A}(\alpha_1, \alpha_2, \cdots, \alpha_m) \in H^*(\overline{\mathcal{M}}_{g,m}; \mathbb{Q})$ such that

$$\langle GW^M_{g,m,A}(\alpha_1, \alpha_2, \cdots, \alpha_m), \beta\rangle := \int_{[\overline{\mathcal{M}}_{g,m}(M,J,A)]} ev^*(\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m) \cup \pi^* PD(\beta).$$

Note that since $[\overline{\mathcal{M}}_{g,m}(M, J, A)]$ has degree $d = 2c_1(TM) \cdot A + (6 - \dim M)(g-1) + 2m$, the classes $\alpha_1, \alpha_2, \cdots, \alpha_m$ and β satisfy

$$\sum_{i=1}^m \deg\, \alpha_i = 2c_1(TM) \cdot A + (1 - g) \dim M + \deg\, \beta.$$

Note that when $\overline{\mathcal{M}}_{g,m} = \emptyset$, $H^*(\overline{\mathcal{M}}_{g,m}; \mathbb{Q})$ is taken to be $\mathbb{Q}$ and $\beta = 1$ with $\deg\, \beta = 0$.

The geometric meaning of Gromov-Witten invariants is as follows. Suppose the cohomology classes α_i are represented by cycles $\Gamma_i \subset M$. Consider the subspace of stable maps of genus g and m marked points and of homology class A, $(\Sigma, \mathbf{z}, h)$, such that $h(z_i) \in \Gamma_i$ (here z_i stands for the i-th marked point), then the image of this subspace under the map π is a cycle representing the class $GW^M_{g,m,A}(\alpha_1, \alpha_2, \cdots, \alpha_m)$. In other words, $\langle GW^M_{g,m,A}(\alpha_1, \alpha_2, \cdots, \alpha_m), \beta\rangle$ is the intersection product of this cycle with β, meaning that it is the"count" of stable maps with pointwise constraints given by Γ_i and stable curve constraints given by β. In particular, the case where this cycle is 0-dimensional corresponds to varying marked points, and these classes are called *zero-codimensional*. For zero-codimensional classes, introduce

$$\begin{aligned}\langle GW^M_{g,m,A}\rangle(\alpha_1, \alpha_2, \cdots, \alpha_m) &= \int_{\overline{\mathcal{M}}_{g,m}} GW^M_{g,m,A}(\alpha_1, \alpha_2, \cdots, \alpha_m) \\ &= \int_{[\overline{\mathcal{M}}_{g,m}(M,J,A)]} ev^*(\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m).\end{aligned}$$

Based on this geometric intuition, Kontsevich and Manin [15] write down the following axioms for Gromov-Witten invariants (assuming $2g + m - 3 \geq 0$).

(0) Effectivity: $GW^M_{g,m,A} = 0$ if $[\omega] \cdot A < 0$.

(1) S_m-Covariance: The symmetric group S_m acts on $H^*(M;\mathbb{Q})^{\otimes m}$ via permutations (as superspace with $\mathbb{Z}$ mod 2 grading) and upon $\overline{\mathcal{M}}_{g,m}$ via renumbering marked points. The maps $GW^M_{g,m,A}$ must be compatible with the actions.

(2) Grading: for any $\alpha_1, \alpha_2, \cdots, \alpha_m \in H^*(M;\mathbb{Q})$,

$$\deg\, GW^M_{g,m,A}(\alpha_1, \alpha_2, \cdots, \alpha_m) = \sum_{i=1}^m \deg\, \alpha_i - 2c_1(TM) \cdot A - (1-g)\dim M.$$

(3) Fundamental Class: Let $e^0_M \in H^0(M;\mathbb{Q})$ be the PD of the fundamental class $[M]$, and let $\pi_m : \overline{\mathcal{M}}_{g,m} \to \overline{\mathcal{M}}_{g,m-1}$ be the map of forgetting the last marked point. Then

$$GW^M_{g,m,A}(\alpha_1, \cdots, \alpha_{m-1}, e^0_M) = \pi_m^* GW^M_{g,m-1,A}(\alpha_1, \cdots, \alpha_{m-1}).$$

(4) Divisor: If $\deg\, \alpha_m = 2$, then

$$(\pi_m)_* GW^M_{g,m,A}(\alpha_1, \cdots, \alpha_m) = GW^M_{g,m-1,A}(\alpha_1, \cdots, \alpha_{m-1}) \cdot \int_A \alpha_m.$$

(5) Mapping to Point: for $A = 0$ and $g = 0$,

$$GW^M_{0,m,0}(\alpha_1, \alpha_2, \cdots, \alpha_m) = (\int_M \alpha_1 \cup \cdots \cup \alpha_m) \cdot e^0_{\overline{\mathcal{M}}_{0,m}}$$

is the only possible non-zero class where $\sum_{i=1}^m \deg\, \alpha_i = \dim M$, and $e^0_{\overline{\mathcal{M}}_{0,m}} \in H^0(\overline{\mathcal{M}}_{0,m};\mathbb{Q})$ is the identity. (There are similar statements for the case $g \neq 0$ which we omit here.)

(6) Splitting: Fix g_1, g_2 and m_1, m_2 such that $g = g_1 + g_2$, $m = m_1 + m_2$, $2g_i + m_i - 2 \geq 0$. Fix also two complementary subsets $S = S_1, S_2$ of $\{1, 2, \cdots, m\}$ with $|S_i| = m_i$. Denote by $\phi_S : \overline{\mathcal{M}}_{g_1,m_1+1} \times \overline{\mathcal{M}}_{g_2,m_2+1} \to \overline{\mathcal{M}}_{g,m}$ the canonical map which assigns to a pair of stable curves $(\Sigma_i, \mathbf{z}_i)$ the stable curve $(\Sigma, \mathbf{z})$,

where Σ is the union of Σ_1 and Σ_2 with the last marked point identified with the first marked point of Σ_2, and $\mathbf{z}$ is the set of the remaining marked points renumbered by $\{1, 2, \cdots, m\}$ in such a way that their relative order is kept intact, and points on Σ_i are numbered by S_i. Finally, choose a homogeneous basis $\{\Delta_a\}$ of $H^*(M; \mathbb{Q})$ and put $g_{ab} = \int_M \Delta_a \cup \Delta_b$, $(g^{ab}) = (g_{ab})^{-1}$. Then the Splitting Axiom reads:

$$\begin{aligned} & \phi_S^* GW^M_{g,m,A}(\alpha_1, \cdots, \alpha_m) \\ = \ & \epsilon(S) \sum_{A=A_1+A_2} \sum_{a,b} GW^M_{g_1,m_1+1,A_1}(\{\alpha_j | j \in S_j\}, \Delta_a) g^{ab} \otimes GW^M_{g_2,m_2+1,A_2}(\Delta_b, \{\alpha_j | j \in S_2\}) \end{aligned}$$

where $\epsilon(S)$ is the sign of permutation induced by S on $\{\alpha_j\}$ with odd degrees. Note that $\sum_{a,b} \Delta_a \otimes \Delta_b g^{ab}$ is PD of the diagonal of $M \times M$ in $H^*(M \times M; \mathbb{Q})$.

(7) Genus Reduction: Denote by $\psi : \overline{\mathcal{M}}_{g-1,m+2} \to \overline{\mathcal{M}}_{g,m}$ the map corresponding to gluing the last two marked points. Then

$$\psi^* GW^M_{g,m,A}(\alpha_1, \cdots, \alpha_m) = \sum_{a,b} GW^M_{g-1,m+2,A}(\alpha_1, \cdots, \alpha_m, \Delta_a, \Delta_b).$$

We remark that most of the axioms (or properties) of Gromov-Witten invariants follow easily from the very definition of the invariants, except for the last two axioms, Splitting and Genus Reduction, where the proofs involve gluing theorems of pseudo-holomorphic curves.

Note that if one extends the coefficients from $\mathbb{Q}$ to $\mathbb{C}$, then any system of GW invariants possesses a scaling transformation $GW^M_{g,m,A} \mapsto e(A) GW^M_{g,m,A}$, where $e : H_2(M) \to \mathbb{C}^*$ is a homomorphism, for example, $e_t(A) = \exp(-t\omega(A))$. On the other hand, for the genus zero case (i.e. $g = 0$), the following system satisfies the axioms: $GW^M_{0,m,A} = 0$ if $A \neq 0$, and

$$GW^M_{0,m,0}(\alpha_1, \alpha_2, \cdots, \alpha_m) = (\int_M \alpha_1 \cup \cdots \cup \alpha_m) \cdot e^0_{\overline{\mathcal{M}}_{0,m}}.$$

Under the scaling transformation with $e_t(A) = \exp(-t\omega(A))$, any initial system converges to this one as $t \to \infty$.

The theory of Gromov-Witten invariants was rooted in topological quantum field theory and has applications in enumerative problems in algebraic geometry. We will briefly explain this aspect of the story below. Only genus zero GW invariants will be considered.

First of all, the GW invariants $\langle GW^M_{0,3,A} \rangle$ are involved in the definition of *small quantum cohomology* of M. Let $T_0 = 1, T_1, \cdots, T_s$ be a basis of $H^*(M; \mathbb{Q})$ consisting of homogeneous elements. Introduce $g_{ij} := \int_M T_i \cup T_j$ and $(g^{ij}) = (g_{ij})^{-1}$, and a formal variable q. For any $a, b \in H^*(M; \mathbb{Q})$, we define the *small quantum product*

$$a * b := \sum_{A \in H_2(M)} \sum_{i,j} \langle GW^M_{0,3,A} \rangle (a, b, T_i) g^{ij} q^A T_j.$$

Then under the product $H^*(M; \mathbb{Q})$ (after adjoining q) becomes a ring, called the *small quantum cohomology ring* of M. Note that the term in front of q^A with $A = 0$ is the usual cup product because of the Mapping to Point Axiom. So in this sense the quantum cohomology ring is a deformation of the usual cohomology ring of M.

The associativity $(a * b) * c = a * (b * c)$ follows from the Splitting Axiom. Indeed,

$$\begin{aligned}
& (a * b) * c \\
= \; & \Big(\sum_{A \in H_2(M)} \sum_{i,j} \langle GW^M_{0,3,A}\rangle(a,b,T_i) g^{ij} q^A T_j\Big) * c \\
= \; & \sum_A \sum_{k,l} \Big(\sum_{A=A_1+A_2} g^{ij} \langle GW^M_{0,3,A_1}\rangle(a,b,T_i)\langle GW^M_{0,3,A_2}\rangle(T_j,c,T_k)\Big) g^{kl} q^A T_l \\
= \; & \sum_A \sum_{k,l} \phi_S^* \langle GW^M_{0,4,A}\rangle(a,b,c,T_k) g^{kl} q^A T_l
\end{aligned}$$

where in the last step Splitting Axiom is used, with $\phi_S : \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{0,4} = \mathbb{CP}^1$ for a certain partition S of $\{1,2,3,4\}$. One can similarly express $a * (b * c)$ and the claim follows.

Exercise 4.8. Compute the small quantum cohomology ring of $\mathbb{CP}^2$. Hint: the only non-zero GW invariant is $\langle GW^{\mathbb{CP}^2}_{0,3,A}\rangle(T_2,T_2,T_1) = 1$, where T_1 is the PD of a complex line, $T_2 = T_1^2$ the PD of a point, and A is the class of a complex line. Note that this number is the "count" of complex lines in $\mathbb{CP}^2$ which passes two given points in a general position and intersects with a complex line. Of course the number of such complex lines is 1 !

The definition of small quantum cohomology ring only involves part of the genus zero GW invariants, i.e., where $m = 3$. One can define the *big quantum cohomology ring* by using the full genus zero GW invariants. To this end, we introduce for each T_i an variable t_i given with the same degree as T_i, and introduce relations

$$t_i t_j = (-1)^{\deg t_i \deg t_j} t_j t_i.$$

Then we introduce the *GW potential* (of genus zero)

$$\Phi(\gamma) := \sum_m \sum_A \frac{1}{m!} \langle GW^M_{0,m,A}\rangle(\gamma^m) q^A,$$

where $\gamma = \sum_i t_i T_i$, and $\gamma^m = (\sum_i t_i T_i)^m$. With this understood, one defines the *big quantum product* by

$$T_i * T_j := \sum_{k,l} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k} g^{kl} T_l.$$

Exercise 4.9. Show that if one replaces Φ by the following function

$$\Psi(\gamma) := \sum_A \frac{1}{6} \langle GW^M_{0,3,A}\rangle(\gamma^3) q^A,$$

then $T_i * T_j := \sum_{k,l} \frac{\partial^3 \Psi}{\partial t_i \partial t_j \partial t_k} g^{kl} T_l$ recovers the small quantum product.

The associativity of big quantum product is equivalent to the following so-called WDVV equation satisfied by the GW potential Φ: for any indices i, j, k, l,

$$\sum_{a,b} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_a} g^{ab} \frac{\partial^3 \Phi}{\partial t_b \partial t_k \partial t_l} = (-1)^{\deg t_i (\deg t_i + \deg t_k)} \sum_{a,b} \frac{\partial^3 \Phi}{\partial t_j \partial t_k \partial t_a} g^{ab} \frac{\partial^3 \Phi}{\partial t_b \partial t_i \partial t_l},$$

which also follows from the Splitting Axiom.

In the following example, we shall determine the big quantum cohomology ring of $\mathbb{CP}^2$, and explain how this is related to the problem of enumeration of rational curves of degree d in $\mathbb{CP}^2$ passing through $3d-1$ points in general position.

Example 4.10. An irreducible algebraic curve in $\mathbb{CP}^2$ of degree d is described by the zero set of an irreducible homogeneous polynomial of degree d in three variables. The set of such curves is embedded in a projective space $\mathbb{CP}^D$ where $D = d(d+3)/2$. The smooth (i.e. non-singular) curves form an open subset, and the subset of curves containing exactly δ double points form a subspace of dimension $d(d+3)/2 - \delta$. On the other hand, by the Adjunction Formula,

$$d^2 - 3d + 2 = 2g + 2\delta,$$

one can substitute δ by g, and we find the set of irreducible algebraic curves in $\mathbb{CP}^2$ of degree d, genus g and having only double point singularities form a space of complex dimension $3d - 1 + g$. In particular, the set of rational curves (i.e., $g = 0$) depends on $3d-1$ parameters. If we specify $3d-1$ points in a general position, then we expect that there are only finitely many such rational curves. The number is denoted by N_d. Finding a formula for N_d is a classical problem in enumerative geometry. The first few values of N_d are: $N_1 = 1, N_2 = 1, N_3 = 12, N_4 = 620$, and relatively recently $N_5 = 87304$.

Next let's compute the GW potential of $\mathbb{CP}^2$. Recall the basis of $H^*(\mathbb{CP}^2; \mathbb{Q})$, $T_0 = 1, T_1, T_2$, each is associated with a variable, t_0, t_1, t_2, of degree $0, 2, 4$ respectively. Moreover, $H_2(\mathbb{CP}^2) = \mathbb{Z}$, so we write $A = d$ meaning the class of degree d. With this understood, we will write the GW potential $\Phi = \Phi_{cl} + \Phi_{qu}$, where Φ_{cl} is the "classical part", i.e., contributions from $A = 0$, and Φ_{qu} is the "quantum part", i.e., contributions from $A = d > 0$. By the Mapping to Point Axiom, the classical part is given by cup product, so that $\Phi_{cl} = \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2)$. As for the quantum part, recall that the only non-zero GW invariants with $m = 3$ is $\langle GW^{\mathbb{CP}^2}_{0,3,1} \rangle (T_1 T_2^2)$, and for $m \geq 4$, by the Fundamental Class Axiom, $\langle GW^{\mathbb{CP}^2}_{0,m,d} \rangle$ vanishes if the entries contain a T_0. Hence

$$\Phi_{qu} = \sum_{a+b} \sum_{d=1}^{\infty} \frac{1}{a!b!} \langle GW^{\mathbb{CP}^2}_{0,a+b,d} \rangle (T_1^a T_2^b) t_1^a t_2^b q^d.$$

Furthermore, by the Divisor Axiom $\langle GW^{\mathbb{CP}^2}_{0,a+b,d} \rangle (T_1^a T_2^b) = \langle GW^{\mathbb{CP}^2}_{0,b,d} \rangle (T_2^b) d^a$, and by the Degree Axiom, $b = 3d - 1$. Finally, observe that the geometric meaning of GW invariants suggests that

$$\langle GW^{\mathbb{CP}^2}_{0,3d-1,d} \rangle (T_2^{3d-1}) = N_d.$$

Putting everything together, we obtain a formula for Φ:

$$\Phi = \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + \sum_{d=1}^{\infty} \frac{N_d}{(3d-1)!} e^{dt_1} t_2^{3d-1} q^d.$$

Now we can have some fun. For indices $(i,j,k,l) = (1,1,2,2)$, the WVDD equation for the GW potential Φ of $\mathbb{CP}^2$ reads

$$\Phi_{222} + \Phi_{111}\Phi_{122} = \Phi_{112}^2,$$

where Φ_{ijk} stands for $\frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k}$. To exploit the above equation, we note that N_d is contained in the term

$$\frac{N_d}{(3d-4)!} t_2^{3d-4} q^d$$

in Φ_{222}. Looking for terms containing $t_2^{3d-4} q^d$ in $\Phi_{111}\Phi_{122}$ and Φ_{112}^2, we find

$$\sum_{d_1+d_2=d} \frac{N_{d_1}}{(3d_1-1)!} \cdot \frac{N_{d_2}}{(3d_2-3)!} d_1^3 d_2 t_2^{3d-4} q^d$$

and

$$\sum_{d_1+d_2=d} \frac{N_{d_1}}{(3d_1-2)!} \cdot \frac{N_{d_2}}{(3d_2-2)!} d_1^2 d_2^2 t_2^{3d-4} q^d$$

respectively. From here, one obtains a recursion relation for N_d:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \begin{pmatrix} 3d-4 \\ 3d_1-2 \end{pmatrix} - d_1^3 d_2 \begin{pmatrix} 3d-4 \\ 3d_1-1 \end{pmatrix}\right).$$

This beautiful argument is due to Kontsevich [15].

Exercise 4.11. Use the recursion relation to verify the first few values for N_d, with the initial value $N_1 = 1$.

The Gromov-Witten invariants of symplectic orbifolds were defined in [7], where a new cohomology ring (called orbifold cohomology) was found [8]. (See also [1] in the algebraic geometry setting.) Motivations came from string theory and McKay correspondence (cf. e.g. [22]).

4.2. **Genus zero invariants of semipositive manifolds.** A compact closed symplectic manifold (M,ω) of dimension $2n$ is called *semipositive* if for any spherical class A, i.e., $A \in \operatorname{Im}(\pi_2(M) \to H_2(M))$,

$$\omega(A) > 0, \;\; c_1(TM) \cdot A \geq 3-n \quad \Rightarrow \quad c_1(TM) \cdot A \geq 0.$$

In particular, semipositive manifolds include symplectic manifolds of dimension ≤ 6, and Kähler manifolds which are either Calabi-Yau (i.e., $c_1(TM) = 0$) or Fano (i.e., $c_1(TM)$ positive). Note that by the transversality result we established in the last chapter (cf. Corollary 3.14), there are no J-holomorphic 2-spheres with negative Chern number (i.e., $c_1(TM) \cdot A < 0$) in a semipositive manifold for a generic $J \in \mathcal{J}_\tau$.

The purpose of this section is to explain that under the semipositivity condition, the genus zero GW invariants $\langle GW_{0,m,A}^M \rangle$ can be constructed via the traditional transversality arguments, and moreover, the GW invariants are integer-valued in this case.

First, we prove a transversality theorem extending Theorem 3.11 (and Corollary 3.14), where we allow disconnected domain Σ and pointwise constraints to the J-holomorphic curves. More precisely, we consider $(\Sigma, \mathbf{z})$, where $\Sigma = \sqcup_\nu \Sigma_\nu$ is a disjoint union of connected Riemann surfaces and $\mathbf{z} = \{z_1, z_2, \cdots, z_m\} \subset \Sigma$ is a set of marked points on Σ. Let $Y \subset M^m$ be any given embedded submanifold. We denote by $\mathcal{M}^*((\Sigma,\mathbf{z}), A, J, Y)$ the space of simple J-holomorphic curves $u : \Sigma \to M$ modulo re-parametrizations fixing the set $\mathbf{z}$ of marked points, where each u satisfies the pointwise constraints $ev(u) := (u(z_1), u(z_2), \cdots, u(z_m)) \in Y$. Here, by definition, u is simple means that each $u|_{\Sigma_\nu}$ is simple and for $\nu \neq \mu$, $u(\Sigma_\nu) \neq u(\Sigma_\mu)$. Finally, $A \in H_2(M)$ is a given homology class and $A = u_*([\Sigma])$.

Theorem 4.12. *There is a subset $\mathcal{J}^l_{reg}$ of $\mathcal{J}^l_\tau$ or $\mathcal{J}^l$ of Baire's second category, such that for any $J \in \mathcal{J}^l_{reg}$, $\mathcal{M}^*((\Sigma,\mathbf{z}), A, J, Y)$ is a C^l-smooth manifold of dimension*

$$d = 2c_1(TM)\cdot A + (\dim M - 6)\cdot \frac{\chi(\Sigma)}{2} + 2m - \mathit{codim}\ Y.$$

Proof. We give a sketch of the proof. Let $\mathcal{B}$ be the Banach manifold of locally $L^{1,p}$-maps for a fixed $p > 2$ from Σ into M, such that each component Σ_ν contains an injective point of u, i.e., an $z \in \Sigma$ such that $u^{-1}(u(z)) = \{z\}$, and that $A = u_*([\Sigma])$. Let $\mathcal{M}_\Sigma$ be the moduli space of complex structures on Σ. Then $\mathcal{M}^*((\Sigma,\mathbf{z}), A, J, Y)$ is the space $(s\times ev)^{-1}(\{0\}\times Y) \cap \mathcal{B}\times\{J\}\times\mathcal{M}_\Sigma$ modulo re-parametrizations, where $s \times ev : \mathcal{B}\times\mathcal{J}^l\times\mathcal{M}_\Sigma \to \mathcal{E}\times M^m$, $(u, J, j) \mapsto (\bar\partial_J u, ev(u))$. It is clear that the theorem follows by Sard-Smale, if we prove that $s\times ev$ is transverse to $\{0\}\times Y \subset \mathcal{E}\times M^m$. The calculation of the dimension of $\mathcal{M}^*((\Sigma,\mathbf{z}), A, J, Y)$ is straightforward.

To see that $s \times ev$ is transverse to $\{0\}\times Y \subset \mathcal{E}\times M^m$, we first note that for any $\xi \in T_u\mathcal{B}$, $d(ev)(\xi) = (\xi(z_1), \xi(z_2), \cdots, \xi(z_m))$, from which it follows that $ev : \mathcal{B} \to M^m$ is transverse to Y. Set $\mathcal{B}_Y := ev^{-1}(Y)$, which is an embedded Banach submanifold of $\mathcal{B}$. Then the problem becomes showing that the restriction of s to $\mathcal{B}_Y\times\mathcal{J}^l\times\mathcal{M}_\Sigma$, $s_Y : \mathcal{B}_Y\times\mathcal{J}^l\times\mathcal{M}_\Sigma \to \mathcal{E}$, is transverse to the 0-section. It is clear that s_Y is a family of Fredholm sections parametrized by $\mathcal{J}^l$ (because M^m is finite dimensional), and moreover, for any $u \in s_Y^{-1}(0)$, the dual space of the cokernel of the "vertical part" of ds_u is a subspace (via the L^2-product) of the space of $\xi \in L^p(\Lambda^{0,1}\otimes u^*TM)$ such that $D_u^*\xi = 0$ holds on the complement of the marked points $z_1, \cdots, z_m$. Finally, we observe that the set of injective points of u is open and dense, and the claim that s_Y is transverse to the zero section follows from Lemma 3.12. $\square$

Definition 4.13. A stable map $((\Sigma,\mathbf{z}), h)$ is called *simple* if every non-constant component is simple and for any $\nu \neq \mu$, $h(\Sigma_\nu) \neq h(\Sigma_\mu)$ if both are non-constant.

Recall that every stable map is associated with a graph where the vertices of the graph correspond to the components of Σ and for each pair of singular points we draw an edge between the corresponding vertices (or vertex). Denote by $\mathcal{M}^*_{g,m,T}(M,J,A) \subset \overline{\mathcal{M}}_{g,m}(M,J,A)$ the subset of equivalence classes of stable maps associated to a given graph type T. Then as a corollary of Theorem 4.12, we have

Corollary 4.14. *For a generic J, $\mathcal{M}^*_{g,m,T}(M,J,A)$ is a smooth manifold of dimension*

$$d(g,m,A) - 2e(T) = 2c_1(TM)\cdot A + (6-\dim M)(g-1) + 2m - 2e(T),$$

where $e(T)$ is the number of edges of T.

Corollary 4.15. *Let $\Gamma_1, \cdots, \Gamma_m \subset M$ be embedded submanifolds which intersect transversely. Then for a generic J, the set $ev^{-1}(\Gamma_1 \times \cdots \times \Gamma_m)$ is a smooth manifold of dimension*

$$\dim \mathcal{M}^*_{g,m,T}(M, J, A) - \sum_{i=1}^{m} \text{codim } \Gamma_i,$$

where $ev : \mathcal{M}^*_{g,m,T}(M, J, A) \to M^m$ is the evaluation map.

Note that the above results show that if we stay within the realm of *simple* stable maps, the Gromov compactification procedure only introduces strata of lower dimensions, of co-dimension at least 2, and as far as homology is concerned, these strata can be neglected.

Exercise 4.16. Prove Corollary 4.14 and Corollary 4.15.

Finally, we need the following technical lemma, see [18], Prop. 6.1.2, for a proof.

Lemma 4.17. *For any stable map $((\Sigma, \mathbf{z}), h) \in \overline{\mathcal{M}}_{0,m}(M, J, A)$, there is a simple stable map $((\Sigma', \mathbf{z}'), h') \in \mathcal{M}^*_{0,m,T}(M, J, A')$ such that $h(\Sigma) = h'(\Sigma')$ and $ev((\Sigma, \mathbf{z}), h) = ev((\Sigma', \mathbf{z}'), h') \in M^m$, where if $A'_{\nu'}$ denote the homology classes of the components of $((\Sigma', \mathbf{z}'), h')$, then $A' = \sum_{\nu'} A'_{\nu'}$ and $A = \sum_{\nu'} n_{\nu'} A'_{\nu'}$ for some $n_{\nu'} \geq 1$.*

Roughly speaking, as far as evaluation map is concerned, one can always replace a stable map by a simple stable map. The issue is to control the dimension of these simple stable maps obtained. This is where the semipositivity condition comes in.

Assume (M, ω) is semipositive, and let $0 \neq A \in H_2(M)$. We shall make the following technical assumption first:

$(*)$ If there is a B with $A = mB$ for some $m > 1$ and B is the homology class of a J-holomorphic 2-sphere for a generic J, then $c_1(TM) \cdot B > 0$.

With this understood, let $\mathcal{M}^*_{0,m}(M, J, A) \subset \overline{\mathcal{M}}_{0,m}(M, J, A)$ be the open subset consisting of simple stable maps $(\Sigma, \mathbf{z}), h)$ where $\Sigma = \mathbb{S}^2$ (note that it may be empty). Then for a generic J, it is a smooth manifold of dimension

$$d(m, A) = 2c_1(TM) \cdot A + \dim M - 6 + 2m.$$

For any $(\Sigma, \mathbf{z}), h) \in \overline{\mathcal{M}}_{0,m}(M, J, A) \setminus \mathcal{M}^*_{0,m}(M, J, A)$, there is a simple stable map $((\Sigma', \mathbf{z}'), h') \in \mathcal{M}^*_{0,m,T}(M, J, A')$ by Lemma 4.17. By the semipositivity condition, $c_1(TM) \cdot A'_{\nu'} \geq 0$, so that by the assumption $(*)$, $((\Sigma', \mathbf{z}'), h')$ is contained in a moduli space of simple stable maps, $\mathcal{M}^*_{0,m,T}(M, J, A')$, which for a generic J is a smooth manifold of dimension (cf. Corollary 4.14)

$$2c_1(TM) \cdot A' + \dim M - 6 + 2m - 2e(T) \leq d(m, A) - 2.$$

Now let $\alpha_1, \cdots, \alpha_m \in H^*(M)$ be given, with $\sum_{i=1}^m \deg\alpha_i = d(m, A)$, which are represented by submanifolds $\Gamma_1, \cdots, \Gamma_m$ intersecting transversely. Then for a generic J, $ev(\mathcal{M}^*_{0,m,T}(M, J, A'))$ does not intersect $\Gamma_1 \times \cdots \times \Gamma_m \subset M^m$ for all possible A', T (cf. Corollary 4.15). Consequently, $ev^{-1}(\Gamma_1 \times \cdots \times \Gamma_m) = \{x_\alpha\} \subset \mathcal{M}^*_{0,m}(M, J, A)$

consists of finitely many points. The Gromov-Witten invariant $\langle GW^M_{0,m,A}\rangle(\alpha_1,\cdots,\alpha_m)$ is defined to be

$$\langle GW^M_{0,m,A}\rangle(\alpha_1,\cdots,\alpha_m)=\sum_{x_\alpha\in ev^{-1}(\Gamma_1\times\cdots\times\Gamma_m)}\text{sign } x_\alpha.$$

It is clear that the invariant is independent of the choice of J and $\Gamma_1,\cdots,\Gamma_m$.

Without assumption $(*)$, one has to introduce a larger class of perturbations, either using inhomogeneous equations $\bar\partial_J u=\mu$ as in the original paper of Ruan and Tian [23], or using more general "almost complex structures" $J=J(\cdot,z)$ which depend also on the parameter $z\in\Sigma$ (see [18]). Note that both types of perturbations do not work for bubbling components. But the bubbling components can be perturbed away using the technique we explained in this section under the semipositivity condition.

4.3. **Gromov invariants of symplectic 4-manifolds.** The pseudoholomorphic curve theory is particularly interesting in dimension 4, partly because of the tools such as Adjunction Formula for dimensional reasons, partly because (perhaps more importantly) of the deep work of Cliff Taubes on the equivalence of Gromov invariant (obtained by counting J-holomorphic curves) and Seiberg-Witten invariant (defined using gauge theory) for symplectic 4-manifolds [25], and it has many geometric applications (see e.g. [18]). In particular, Taubes' work has brought enormous progresses to our understanding of symplectic 4-manifolds. See [16] for a survey on the classifications and [6] for a survey about finite group actions on symplectic 4-manifolds. The purpose of this section is to give an overview on Taubes' work and illustrate with a few simple results its power in understanding symplectic 4-manifolds.

Let X be a smooth 4-manifold. Given any Riemannian metric on X, a $Spin^{\mathbb{C}}$ structure is a principal $Spin^{\mathbb{C}}(4)$ bundle over X which descends to the principal $SO(4)$ bundle of oriented orthonormal frames under the canonical homomorphism $Spin^{\mathbb{C}}(4)\to SO(4)$. There are two associated $U(2)$ vector bundles (of rank 2) S_+,S_- with $\det(S_+)=\det(S_-)$, and a Clifford multiplication which maps T^*X into the skew adjoint endomorphisms of $S_+\oplus S_-$.

The Seiberg-Witten equations associated to the $Spin^{\mathbb{C}}$ structure (there is always one) are equations for a pair (A,ψ), where A is a connection on $\det(S_+)$ and ψ is a section of S_+. The Levi-Civita connection together with A defines a covariant derivative ∇_A on S_+. On the other hand, there are two maps $\sigma: S_+\otimes T^*X\to S_-$ and $\tau:\text{End}(S_+)\to\Lambda_+\otimes\mathbb{C}$ induced by the Clifford multiplication, with the latter being the adjoint of $c_+:\Lambda_+\to\text{End}(S_+)$, where Λ_+ is the bundle of self-dual 2-forms. With this understood, the Seiberg-Witten equations read

$$D_A\psi=0 \text{ and } P_+F_A=\frac{1}{4}\tau(\psi\otimes\psi^*)+\mu,$$

where $D_A\equiv\sigma\circ\nabla_A$ is the Dirac operator, $P_+:\Lambda^2T^*X\to\Lambda_+$ is the orthogonal projection, and μ is a fixed, imaginary valued, self-dual 2-form which is added in as a perturbation term.

The Seiberg-Witten equations are invariant under the gauge transformations $(A,\psi)\mapsto(A-2\varphi^{-1}d\varphi,\varphi\psi)$, where $\varphi\in C^\infty(X;S^1)$ are circle-valued smooth functions on X. The space of solutions modulo gauge equivalence, denoted by $\mathcal{M}$, is compact, and when $b_2^+(X) \geq 1$ and when it is nonempty, $\mathcal{M}$ is a smooth orientable manifold for a generic choice of (g, μ), where g is the Riemannian metric and μ is the self-dual 2-form of perturbations. Furthermore, $\mathcal{M}$ contains no classes of reducible solutions (ie., those with $\psi \equiv 0$), and if let $\mathcal{M}^0$ be the space of solutions modulo the based gauge group, ie., those $\varphi \in C^\infty(X; S^1)$ such that $\varphi(p_0) = 1$ for a fixed base point $p_0 \in X$, then $\mathcal{M}^0 \to \mathcal{M}$ defines a principal $\mathbb{S}^1$-bundle. Let c be the first Chern class of $\mathcal{M}^0 \to \mathcal{M}$, $d = \dim \mathcal{M}$, and fix an orientation of $\mathcal{M}$. Then the Seiberg-Witten invariant associated to the $Spin^{\mathbb{C}}$ structure is defined as follows.

- When $d < 0$ or $d = 2n + 1$, the Seiberg-Witten invariant is zero.
- When $d = 0$, the Seiberg-Witten invariant is a signed count of points in $\mathcal{M}$.
- When $d = 2n > 0$, the Seiberg-Witten invariant equals $c^n[\mathcal{M}]$.

The Seiberg-Witten invariant of X is well-defined when $b_2^+(X) \geq 2$, depending only on the diffeomorphism class of X. Moreover, there is an involution on the set of $Spin^{\mathbb{C}}$ structures which preserves the Seiberg-Witten invariant up to a change of sign. When $b_2^+(X) = 1$, there is a chamber structure and the Seiberg-Witten invariant also depends on the chamber which the pair (g, μ) is in. Moreover, the change of the Seiberg-Witten invariant when crossing a wall of the chambers can be analyzed using wall-crossing formula.

Now suppose (X, ω) is a symplectic 4-manifold. We orient X by $\omega \wedge \omega$, and fix any ω-compatible almost complex structure J. Then the almost complex structure J gives rise to a canonical $Spin^{\mathbb{C}}$ structure where the associated $U(2)$ bundles are $S_+^0 = \mathbb{I} \oplus K_X^{-1}$, $S_-^0 = T^{0,1}X$. Here $\mathbb{I}$ is the trivial complex line bundle and K_X is the canonical bundle $\det(T^{1,0}X)$. Moreover, the set of $Spin^{\mathbb{C}}$ structures is canonically identified with the set of complex line bundles where each complex line bundle E corresponds to a $Spin^{\mathbb{C}}$ structure whose associated $U(2)$ bundles are $S_+^E = E \oplus (K_X^{-1} \otimes E)$ and $S_-^E = T^{0,1}X \otimes E$. The involution on the set of $Spin^{\mathbb{C}}$ structures which preserves the Seiberg-Witten invariant up to a change of sign sends E to $K_X \otimes E^{-1}$.

There is a canonical (up to gauge equivalence) connection A_0 on $K_X^{-1} = \det(S_+^0)$ such that the fact $d\omega = 0$ implies that $D_{A_0} u_0 = 0$ for the section $u_0 \equiv 1$ of $\mathbb{I}$ which is considered as the section $(u_0, 0)$ in $S_+^0 = \mathbb{I} \oplus K_X^{-1}$. Furthermore, by fixing such an A_0, any connection A on $\det(S_+^E) = K_X^{-1} \otimes E^2$ is canonically determined by a connection a on E. With these understood, there is a distinguished family of the Seiberg-Witten equations on X, which is parametrized by a real number $r > 0$ and is for a triple (a, α, β), where in the equtions, the section ψ of S_+^E is written as $\psi = \sqrt{r}(\alpha, \beta)$ and the perturbation term μ is taken to be $-\sqrt{-1}(4^{-1} r\omega) + P_+ F_{A_0}$. (Here α is a section of E and β a section of $K_X^{-1} \otimes E$.) Note that when $b_2^+(X) = 1$, this distinguished family of Seiberg-Witten equations belongs to a specific chamber for the Seiberg-Witten invariant. This particular chamber is usually referred to as the Taubes' chamber.

Here is a fundamental theorem of Taubes.

Theorem 4.18. *Let (X, ω) be a symplectic 4-manifold. Then the following are true.*

(1) *The Seiberg-Witten invariant (in the Taubes' chamber when $b_2^+(X) = 1$) associated to the canonical $Spin^{\mathbb{C}}$ structure equals ± 1. In particular, the Seiberg-Witten invariant corresponding to the canonical bundle K_X equals ± 1 when $b_2^+(X) \geq 2$.*
(2) *Let E be a complex line bundle. Suppose there is an unbounded sequence of values for the parameter r such that the corresponding Seiberg-Witten equations have a solution (a, α, β). Then for any ω-compatible almost complex structure J, there are J-holomorphic curves $C_1, C_2, \cdots, C_k$ in X and positive integers $n_1, n_2, \cdots, n_k$ such that $c_1(E) = \sum_{i=1}^k n_i PD(C_i)$. Moreover, if a closed subset $\Omega \subset X$ is contained in $\alpha^{-1}(0)$ throughout, then $\Omega \subset \cup_{i=1}^k C_i$ also.*

We give some application of this theorem. Suppose $b_2^+(X) > 1$. Then the canonical class $c_1(K_X)$ is represented by J-holomorphic curves, $c_1(K_X) = \sum_{i=1}^k n_i PD(C_i)$. This implies that $c_1(K_X) \cdot [\omega] \geq 0$ with "=" if and only if $c_1(K_X) = 0$. Suppose $b_2^+(X) = 1$. Then if we assume $b_1(X) = 0$, the wall-crossing formula plus the above theorem implies that $c_1(2K_X)$ is represented by J-holomorphic curves provided that $c_1(K_X) \cdot [\omega] \geq 0$.

It turns out that the genus zero GW invariants of symplectic 4-manifolds are very easy to understand.

Proposition 4.19. *Suppose (X, ω) is a symplectic 4-manifold, either $b_2^+(X) > 1$ or $b_2^+(X) = 1$, $b_1(X) = 0$ and $c_1(K_X) \cdot [\omega] \geq 0$. Then the only genus zero GW invariant of (X, ω) counts embedded J-holomorphic 2-spheres of self-intersection -1.*

Proof. Under our assumption, we know that $c_1(2K)$ is represented by J-holomorphic curves for any given J. On the other hand, by the semipositivity of (X, ω), for a generic J, $c_1(K_X) \cdot C < 0$ for any J-holomorphic 2-sphere C. Combining these two, we see that C must be a component of $c_1(K_X)$ and that $C^2 \leq \lambda c_1(K) \cdot C < 0$ (where $\lambda \in \mathbb{Q}^+$). Noticing that in the Adjunction Inequality

$$C^2 + c_1(K_X) \cdot C + 2 \geq 0,$$

$C^2 \leq -1$ and $c_1(K_X) \cdot C \leq -1$, we see that both $C^2 = -1$ and $c_1(K_X) \cdot C = -1$, and C is embedded. By the regularity result in Proposition 3.16, the GW invariant $\langle GW^X_{0,0,C} \rangle = \pm 1$. □

If X contains an embedded symplectic 2-sphere C, then one can perform a symplectic blow-down $\pi : X \to X'$ to get another symplectic 4-manifold X' where C is sent to a point under π (on the complement π is diffeomorphic). This process ends in finitely many steps, and the end result is called a *minimal symplectic* 4*-manifold*.

The case of $c_1(K_X) \cdot [\omega] < 0$ is covered by the following theorem of Ai-Ko Liu.

Theorem 4.20. *Let (X, ω) be a symplectic 4-manifold with $c_1(K_X) \cdot [\omega] < 0$. Then X can be symplectically blow-down to either $\mathbb{CP}^2$ or a $\mathbb{S}^2$-bundle over a Riemann surface.*

Combing these two results, the genus zero GW invariants of symplectic 4-manifolds are very well understood (at least for the case of $b_1 = 0$).

Note that the argument in Proposition 4.19 shows that if (X, ω) is minimal and $c_1(K_X) \cdot [\omega] \geq 0$, then every J-holomorphic curve (or embedded symplectic surface) C

in X satisfies $c_1(K_X) \cdot C \geq 0$. (This is saying that $c_1(K_X)$ is NEF in the language of algebraic geometry.)

Taubes defined a version of Gromov invariant which counts the number of embedded J-holomorphic curves (maybe disconnected) which are PD to $c_1(E)$ for a given complex line bundle E. Moreover, Taubes proved

Theorem 4.21. *Given any complex line bundle E, the Seiberg-Witten and Gromov invariants associated to E equal.*

Part of the Taubes' work was extended to symplectic 4-orbifolds and has found many applications [5].

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