SYMPLECTIC RATIONAL G-SURFACES AND EQUIVARIANT SYMPLECTIC CONES

Weimin Chen, Tian-Jun Li, and Weiwei Wu

Abstract

We give characterizations of a finite group G acting symplectically on a rational surface ($\mathbb{CP}^2$ blown up at two or more points). In particular, we obtain a symplectic version of the dichotomy of G-conic bundles versus G-del Pezzo surfaces for the corresponding G-rational surfaces, analogous to a classical result in algebraic geometry. Besides the characterizations of the group G (which is completely determined for the case of $\mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, $N = 2, 3, 4$), we also investigate the equivariant symplectic minimality and equivariant symplectic cone of a given G-rational surface.

1. Introduction

In this paper we study symplectic 4-manifolds (X, ω) equipped with a finite symplectomorphism group G, where X is diffeomorphic to a rational surface. We shall call such a pair, i.e., $((X, \omega), G)$, a **symplectic rational G-surface**.

They are the symplectic analog of **(complex) rational G-surfaces** studied in algebraic geometry, which are rational surfaces equipped with a holomorphic G-action. These rational G-surfaces played a central role in the classification of finite subgroups of the plane Cremona group, a problem dating back to the early 1880s, see [**6**].

Note that any rational G-surface can be regarded as a symplectic rational G-surface – simply endowing it with a G-invariant Kähler form which always exists. Our work shows that a large part of the story regarding the classification of rational G-surfaces can be recovered by techniques from 4-manifold theory and symplectic topology. Furthermore, we also add some new, interesting symplectic geometry aspect to the study of rational G-surfaces; in particular, in regard to the equivariant symplectic minimality and equivariant symplectic cone of the

underlying smooth action of a rational G-surface. In addition, we also obtain some result which does not seem previously known in the algebraic geometry literature (cf. Theorem 2.8).

We begin with a discussion on the notion of minimality (i.e., G-minimality) in the equivariant context. Let (X, ω) be a symplectic 4-manifold with a finite symplectomorphism group G. Suppose there exists a G-invariant set of disjoint union of symplectic (-1)-spheres in X. Then blowing down X along the (-1)-spheres gives rise to a symplectic 4-manifold (X', ω'), which can be arranged so that G is natually isomorphic to a finite symplectomorphism group of (X', ω'). The symplectic G-manifold X is called *minimal* if no such set of (-1)-spheres exists. It was shown in [**5**] that when X is neither rational nor ruled, the symplectic G-manifold is minimal if and only if the underlying smooth manifold is minimal. However, in the case considered in the present paper, the underlying rational surface is often not minimal even though the corresponding symplectic rational G-surface is minimal. Furthermore, it is not known whether the notion of G-minimality is the same for the various different categories, i.e., the holomorphic, symplectic, or smooth categories. In general it is a difficult problem to establish the equivalence of G-minimality in the different categories, and we refer the reader to [**5**] for a more thorough discussion on this topic. For our purpose in this paper, it suffices to only study minimal symplectic rational G-surfaces.

The most fundamental problem in our study is to classify symplectic rational G-surfaces up to equivariant symplectomorphisms. However, work in [**2**, **3**, **4**, **5**] showed that even in the simple case where X is $\mathbb{CP}^2$ or a Hirzebruch surface and G is a cyclic or meta-cyclic group, such a classification is already quite involved. In fact, in one circumstance where G is meta-cyclic, a weaker classification, i.e., classification up to equivariant diffeomorphisms, still remains open.

With the preceding understood, the main objectives of this paper are more basic: for symplectic rational G-surfaces X in general, we would like to

(1) classify the possible symplectic structures;
(2) describe the induced action of G on $H_2(X)$;
(3) give a list of possible finite groups for G;
(4) understand the equivariant minimality and equivariant symplectic cones.

These problems, however, are still highly non-trivial and not completely settled. In particular, part of our determination of G and the induced action on $H_2(X)$ relies on the Dolgachev-Iskovskikh's solution of the corresponding problems in algebraic geometry, with new inputs from Gromov-Witten theory and a detailed analysis of the symplectic structures.

1.1. The setup. In this paper, we shall be focusing on the case where the rational surface, denoted by X, is $\mathbb{CP}^2$ blown up at 2 or more points. More concretely, we shall consider minimal symplectic rational G-surfaces (X, ω) where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, for $N \geq 2$. (Note that the minimality assumption implies in particular that G is a nontrivial group.) The case where the rational surface is $\mathbb{CP}^2$ or a Hirzebruch surface had been previously studied, cf. [**2**, **3**, **4**, **5**]; we point out that the J-holomorphic curve techniques employed in this paper are drastically different in flavor from those developed in these previous works.

For convenience, we shall fix some notations and terminology, which will be frequently used throughout the paper. We will denote by $H, E_1, E_2, \cdots, E_N$ a basis of $H^2(X, \mathbb{Z})$, under which the intersection matrix takes its standard form, i.e., $H^2 = 1$, $E_i^2 = -1$, $H \cdot E_i = 0$, $\forall i$, and $E_i \cdot E_j = 0, \forall i \neq j$. The canonical class of (X, ω) will be denoted by $K_\omega \in H^2(X)$, in order to emphasize its dependence on the symplectic form ω. Another frequently used notation is $H^2(X)^G$, which denotes the subset of $H^2(X, \mathbb{Z})$ consisting of elements fixed under the induced action of G, and is called the *invariant lattice*.

Recall that a symplectic rational surface (X, ω) is called *monotone* if $K_\omega = \lambda[\omega]$ is satisfied in $H^2(X; \mathbb{R})$ for some $\lambda \in \mathbb{R}$. In this case, we have $\lambda < 0$, and N must be in the range $N \leq 8$. Such a symplectic rational surface is the symplectic analog of Del Pezzo surface in algebraic geometry. Another important notion, given in the following definition and called a *symplectic G-conic bundle*, corresponds to a conic bundle structure on a rational G-surface in algebraic geometry.
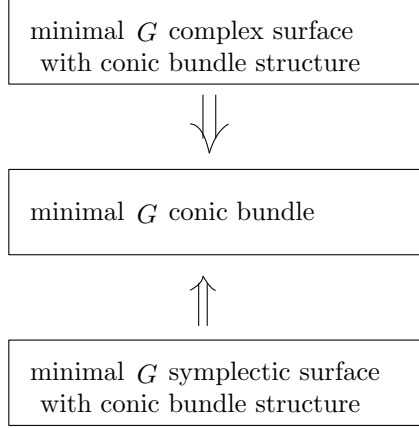
Definition 1.1. Let (X, ω) be a symplectic 4-manifold equipped with a finite symplectomorphism group G. A symplectic G-conic bundle structure on (X, ω) is a genus-0 smooth Lefschetz fibration $\pi : X \to B$ which obeys the following conditions:

- each singular fiber of π contains exactly one critical point;
- there exists a G-invariant, ω-compatible almost complex structure J such that the fibers of π are J-holomorphic;
- the group action of G preserves the Lefschetz fibration.

Remark 1.2. Although the above definition looks more rigid than it should be (in particular, it is always an almost complex fibration), Theorem 1.3 shows that this is a purely symplectic notion in the case of minimal symplectic rational G-surfaces.

A symplectic G-conic bundle is called *minimal* if for any singular fiber there is an element of G whose action switches the two components of the singular fiber.

Here are some immediate consequences from the definition:

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│     minimal G complex surface     │
│    with conic bundle structure    │
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│      minimal G conic bundle       │
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              ⇑
┌─────────────────────────────┐
│   minimal G symplectic surface    │
│    with conic bundle structure    │
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- X is a rational surface if and only if $B = \mathbb{S}^2$; in this case, note that the number of singular fibers of π equals $N - 1$, where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$;
- the Lefschetz fibration is symplectic with respect to ω;
- the fiber class lies in the invariant lattice $H^2(X)^G$ as G preserves the Lefschetz fibration;
- if the underlying symplectic G-manifold is minimal, then the symplectic G-conic bundle must be also minimal.

1.2. The symplectic structures. Our first theorem is concerned with the symplectic structure of a minimal symplectic rational G-surface.

Theorem 1.3. *Let (X, ω) be a minimal symplectic rational G-surface, where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ for some $N \geq 2$. Then $N \neq 2$, and one of the following holds true:*

(1) *The invariant lattice $H^2(X)^G$ has rank 1. In this case, $3 \leq N \leq 8$ and (X, ω) must be monotone.*

(2) *The invariant lattice $H^2(X)^G$ has rank 2. In this case, $N = 5$ or $N \geq 7$, and there exists a symplectic G-conic bundle structure on $((X, \omega), G)$.*

Remark 1.4. (a) An analog of Theorem 1.3 for minimal rational G-surfaces is a classical theorem in algebraic geometry (see e.g. Theorem 3.8 in [**6**]), a proof of which can be given using the equivariant Mori theory (see §4 of [**8**]). With this understood, we remark that our proof of Theorem 1.3 gives an independent proof of the corresponding result in algebraic geometry by taking ω to be a G-invariant Kähler form. Consequently, a significant portion of the theory of rational G-surfaces (e.g. as described in [**6**]) can be recovered (see Theorems 1.5 1.8).

(b) In case (1) of Theorem 1.3 the invariant lattice $H^2(X)^G$ is spanned by K_ω, and in case (2), $H^2(X)^G$ is spanned by K_ω and the fiber

class of the symplectic G-conic bundle. We should point out that the analysis on the type of the group G, the induced action on $H^2(X)$, as well as the structure of the equivariant symplectic cone, depends on the rank of the invariant lattice $H^2(X)^G$.

(c) In case (2), the proof of Theorem 1.3 reveals the following additional information about the symplectic G-conic bundle structure: there exists a basis $H, E_1, E_2, \cdots, E_N$ of $H^2(X)$ with standard intersection matrix such that

  (i) the fiber class is given by $H - E_1$ and the pair of (-1)-spheres in a singular fiber are given by the classes E_j and $H - E_1 - E_j$, where $j = 2, \cdots, N$;

  (ii) the symplectic areas satisfy $\omega(E_j) = \frac{1}{2}\omega(H - E_1)$, $\omega(E_1) \geq \omega(E_j)$ for $j = 2, \cdots, N$;

  (iii) the canonical class $K_\omega = -3H + E_1 + E_2 + \cdots + E_N$.

(d) In complex geometry, it was known that a minimal (complex) rational G-surface which is diffeomorphic to $\mathbb{CP}^2$ blown up at 6 points must be Del Pezzo (cf. [**6**], Theorem 3.8, Proposition 5.2). However, it seemed new that the invariant Picard group $\mathrm{Pic}(X)^G$ must be of rank 1.

Comparing the three minimality assumptions:

The reader should be noted that a minimal symplectic G-conic bundle is different from *a minimal symplectic G-surface with G-conic bundle structure*: a minimal symplectic G-conic bundle may still contain a G-invariant disjoint union of symplectic (-1)-spheres. However, Lemma 4.2 implies this cannot happen when $N \geq 5$ and $N \neq 6$.

The minimal G-conic bundles is an intermediate notion between minimal symplectic G-surfaces and minimal complex G-surfaces. There is always a G-invariant symplectic form compatible with the given complex structure on a minimal complex G-surface. Although we do not know whether this symplectic form is always G-minimal, if we assume a symplectic G-conic bundle structure underlies this action, this conic bundle structure is always minimal. Therefore, proving our results in the more general minimal G-conic bundle context plays an important role in bridging the complex and symplectic G-surfaces as well as in the study of the equivariant symplectic cones.

1.3. The homological action and the groups. Our next task is to describe possible candidates for G.

We begin with case (1) in Theorem 1.3 where (X, ω) is a minimal symplectic rational G-surface such that the invariant lattice $H^2(X)^G$ has rank 1. In this case (X, ω) is monotone, $H^2(X)^G$ is spanned by K_ω, and $3 \leq N \leq 8$. With this understood, we note that the orthogonal complement of K_ω in $H^2(X)$ (with respect to the intersection product), denoted by R_N, is a G-invariant root lattice of type E_N ($N = 6, 7, 8$),

D_5 $(N = 5)$, $A_4 (N = 4)$, and $A_2 + A_1$ $(N = 3)$ respectively. We denote by W_N the corresponding Weyl group.

Theorem 1.5. *Let (X, ω) be a minimal symplectic rational G-surface such that $H^2(X)^G$ has rank 1. There are two cases:*

(1) *Suppose $4 \leq N \leq 8$. Then the induced action of G on $H^2(X)$ is faithful, which gives rise to a monomorphism $\rho : G \to W_N$. Moreover, the image $\rho(G)$ in W_N satisfies*

$$\sum_{g \in G} trace\{\rho(g) : R_N \to R_N\} = 0.$$

(2) *Suppose $N = 3$. Let Γ be the subgroup of G which acts trivially on $H^2(X)$ and let $K := G/\Gamma$ be the quotient group. Then Γ is isomorphic to a subgroup of the 2-dimensional torus and K is isomorphic to $\mathbb{Z}_6$ or the dihedral group D_{12}. Furthermore, G is a semi-direct product of Γ by K. As a corollary, G can be written as a semi-direct product of an imprimitive finite subgroup of $PGL(3)$ by $\mathbb{Z}_2$.*

Remark 1.6. 1. For $N = 4, 5$, the subgroups of the Weyl group W_N which satisfy the condition $\sum_{g \in G} trace\{\rho(g) : R_N \to R_N\} = 0$ are determined, see Theorem 6.4 and Theorem 6.9 in [**6**] respectively. All such groups can be realized by a minimal G-Del Pezzo surface, which is also minimal as a symplectic rational G-surface with respect to any G-invariant Kähler form (cf. Theorem 1.10(2)).

2. For $N = 3$, Theorem 1.3 (2) and Theorem 1.5 completely determined all possible G that acts minimally on X. The statement in Theorem 1.5 implies the corresponding statement in [**6**], Theorem 6.3. In fact, the semi-direct product structure of G in our statement is an improvement upon the corresponding theorem in algebraic geometry. For a list of imprimitive finite subgroups of $PGL(3)$ up to conjugacy, see Theorem 4.7 in [**6**].

Now we consider case (2) of Theorem 1.3. In fact, we will work in a slightly more general situation, where the symplectic rational G-surface (X, ω) is not assumed to be minimal, but only admits a minimal symplectic G-conic bundle $\pi : (X, \omega) \to \mathbb{S}^2$. Furthermore, we assume $N \geq 4$ (instead of the fact that $N \geq 5$ when (X, ω) is a minimal symplectic rational G-surface).

We make the following definition.

Definition 1.7.
- $Q \lhd G$ is the subgroup that acts trivially on the base $\mathbb{S}^2$ of the G-conic bundle;
- $G_0 \lhd Q$ is the subgroup that acts trivially on $H^2(X)$ (in the case we consider, there are $N - 1 \geq 3$ critical values on the base that was fixed, hence $G_0 \lhd Q$);

- $P = G/Q$, so that G decomposes as

(1) $$1 \to Q \to G \to P \to 1.$$

We denote by Σ the subset of $\mathbb{S}^2$ which parametrizes the singular fibers of the symplectic G-conic bundle π. Note that $\#\Sigma = N - 1$, and the induced action of P on $\mathbb{S}^2$ leaves the subset Σ invariant. The action of P on $\mathbb{S}^2$ is effective, so P is isomorphic to a polyhedral group, i.e., a finite subgroup of $SO(3)$.

Therefore, up to an extension problem, the description of G boils down to the following theorem which describes the subgroups G_0 and Q.

Theorem 1.8. *Let (X, ω) be a symplectic rational G-surface equipped with a minimal symplectic G-conic bundle structure with at least three singular fibers (i.e, $N \geq 4$). Let G_0, Q be given as in Definition 1.7. Then one of the following is true.*

1. *$G_0 = \mathbb{Z}_m$, $m > 1$, and Q is either the dihedral group D_{2m} containing G_0 as an index 2 subgroup, or $Q = G_0$ and m is even. Moreover, N must be odd, and any element $\tau \in Q \setminus G_0$ switches the two (-1)-spheres in each singular fiber.*
2. *G_0 is trivial and $Q = \mathbb{Z}_2$ or $(\mathbb{Z}_2)^2$. In the latter case, let τ_1, τ_2, τ_3 be the distinct involutions in Q. Then Σ is partitioned into subsets Σ_1, Σ_2, Σ_3, where Σ_i parametrizes those singular fibers of which τ_i leaves each (-1)-sphere invariant, and $\#\Sigma_i \equiv N - 1 \pmod{2}$, for $i = 1, 2, 3$.*

Remark 1.9. (a) Note that when G_0 is trivial and $Q = (\mathbb{Z}_2)^2$, each element of P acts on Q as an automorphism, permuting the three involutions τ_1, τ_2, τ_3. Consequently, the action of P on the base $\mathbb{S}^2$ preserves the partition $\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3$. For the corresponding result in algebraic geometry, see [**6**], Theorem 5.7.

(b) When the fiber class of the symplectic G-conic bundle is unique, Q is uniquely determined as a subgroup of G, see Proposition 4.5 for more details.

1.4. Minimality and equivariant symplectic cones. In this subsection, we are concerned with the underlying smooth action of a minimal symplectic rational G-surface. In particular, our consideration here offers an interesting symplectic geometry perspective to the study of rational G-surfaces in algebraic geometry.

We begin with the setup of our study here. Let $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, $N \geq 2$, which is equipped with a smooth action of a finite group G. Suppose there is a G-invariant symplectic form ω_0 on X such that the corresponding symplectic rational G-surface (X, ω_0) is minimal. With this understood, we denote

(2) $\quad \Omega(X, G) := \{\omega : \omega \text{ is symplectic on } X, g^*\omega = \omega, \text{ for any } g \in G\}.$

Note that $\Omega(X, G)$ is non-empty as $\omega_0 \in \Omega(X, G)$. The part (2) of the following theorem shows that the underlying smooth action of a minimal (complex) rational G-surface satisfies the above assumption, where we can take ω_0 to be any G-invariant Kähler form.

Theorem 1.10. *(1) Let $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, $N \geq 2$, which is equipped with a smooth action of a finite group G. Suppose there is a G-invariant symplectic form ω_0 on X such that $((X, \omega_0), G)$ is minimal. Then for any $\omega \in \Omega(X, G)$, the canonical class $K_\omega = K_{\omega_0}$ or $-K_{\omega_0}$, and the symplectic rational G-surface (X, ω) is minimal.*
(2) Let X be any minimal (complex) rational G-surface which is $\mathbb{CP}^2$ blown up at 2 or more points. Then for any symplectic form ω which is invariant under the underlying smooth action of G (e.g., any G-invariant Kähler form), the corresponding symplectic rational G-surface (X, ω) is minimal.

At the time of writing, it is not known whether the symplectic minimality would imply the smooth minimality of the underlying group action. The symplectic minimality in Theorem 1.10 is a weaker statement that symplectic minimality is determined by the underlying smooth action, but the proof is still quite non-trivial. For related (stronger) results in the case of G-Hirzebruch surfaces, see [**5**].

With the minimality in Theorem 1.10(1) in place, we now turn our attention to the equivariant symplectic cone of the G-manifold (X, G).

Definition 1.11. The equivariant symplectic cone of (X, G) is defined as

$$\tilde{C}(X, G) = \{\Omega : \Omega = [\omega],\ \omega \text{ is a } G\text{-invariant symplectic form}\} \subset H^2(X; \mathbb{R})^G.$$

Note that $K_{-\omega} = -K_\omega$. Therefore it suffices to consider the subset

$$C(X, G) := \{[\omega] | \omega \in \Omega(X, G), K_\omega = K_{\omega_0}\} \subset H^2(X; \mathbb{R})^G.$$

Furthermore, we observe that if $H^2(X)^G$ has rank 1, $C(X, G) = \{\lambda K_{\omega_0} | \lambda \in \mathbb{R}, \lambda < 0\}$. In what follows, we shall assume that $H^2(X)^G$ has rank 2. Note that under this assumption, $N \neq 6$ by part (2) of Theorem 1.3.

In order to describe $C(X, G)$, it is helpful to introduce the following terminology. A class $F \in H^2(X)^G$ is called a **fiber class** if there exists an $\omega \in \Omega(X, G)$ such that F is the class of the regular fibers of a symplectic G-conic bundle on $((X, \omega), G)$. Since we will focus on the set $C(X, G)$, we shall assume further that $[\omega] \in C(X, G)$, i.e., $K_\omega = K_{\omega_0}$.

We observe that since rank $H^2(X)^G = 2$, the class of such an ω can be written as

$$[\omega] = -aK_{\omega_0} + bF, a > 0.$$

With this understood, we consider the following subset of $C(X,G)$ and its projective classes

(3)
$$C(X, G, F) = \{[\omega] \in C(X,G) : \omega = -aK_{\omega_0} + bF, a > 0, b \geq 0\},$$

$$\hat{C}(X, G, F) = \{[\omega] \in C(X,G,F) : \omega(F) = 2\} \text{ (equivalently, } a = 1).$$

Now $[\omega] \in \hat{C}(X,G,F)$ can be written as $[\omega] = -K_{\omega_0} + \delta_{\omega,F}F$. Then the one to one correspondence $[\omega] \mapsto \delta_{\omega,F}$ identifies $\hat{C}(X,G,F)$ with a subset of $\mathbb{R}$. With this understood, we introduce

$$\delta_{X,G,F} := \inf_{[\omega] \in \hat{C}(X,G,F)} \delta_{\omega,F} \in [0, \infty).$$

Note that (X, ω) is monotone if and only if $\delta_{\omega,F} = 0$, so $\delta_{\omega,F}$ may be thought of as a sort of gap function which measures how far away (X, ω) is from being monotone.

Theorem 1.12. *Let $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, $N \geq 2$, be equipped with a smooth finite G-action which is symplectic and minimal with respect to some symplectic form ω_0. Furthermore, assume rank $H^2(X)^G = 2$.*
(1) If $N \geq 9$ or G_0 is nontrivial, there is a unique fiber class F, and $C(X,G) = C(X,G,F)$.
(2) For $N = 5, 7, 8$, either there is a unique fiber class F, or there are two distinct fiber classes F, F'. In the former case, $C(X,G) = C(X,G,F)$, and in the latter case, $C(X,G) = C(X,G,F) \cup C(X,G,F')$, with $C(X,G,F) \cap C(X,G,F')$ being either empty or consisting of $[\omega]$ such that (X, ω) is monotone.
(3) For any fiber class F, $\hat{C}(X,G,F)$ is identified with either $[0,\infty)$ or $(\delta_{X,G,F}, \infty)$ under $[\omega] \mapsto \delta_{\omega,F}$. (In particular, $\delta_{X,G,F}$ can not be attained unless it equals 0.)

We conjecture that when there are two distinct fiber classes, the equivariant symplectic cone must contain the class of a monotone form. Furthermore, it is an interesting problem to determine the gap functions $\delta_{X,G,F}$ for a given minimal rational G-surface X with $\text{Pic}(X)^G = \mathbb{Z}^2$. We shall leave these studies for a future occasion.

The organization of this paper is as follows. Section 2 is concerned with the proof of the structural theorem, Theorem 1.3. In section 2.1 we collect some preliminary lemmas on minimal symplectic G-conic bundles. In section 2.2 we review a reduction process for the exceptional classes of a rational surface which plays an essential role in the proof of Theorem 1.3. Sections 2.3 and 2.4 are devoted to the proof of Theorem 1.3. Section 3 is concerned with the analysis of the structure of group G. Proofs of Theorems 1.5 and 1.8 are presented here. Finally, section 4 is devoted to the discussion on equivariant symplectic minimality and equivariant symplectic cones. In particular, we prove Theorems 1.10

and 1.12. At the end of section 4, we also include a uniqueness result on the subgroup Q in Definition 1.7.

2. The structure of (X, ω)

2.1. Preliminary lemmas on symplectic G-conic bundles. Let (X, ω) be a symplectic rational G-surface, where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, and let $\pi : X \to \mathbb{S}^2$ be a symplectic G-conic bundle on (X, ω). We first observe that each singular fiber of π consists of a pair of (-1)-spheres. To see this, let C_1, C_2 be the components of a singular fiber, which are embedded J-holomorphic spheres. Since C_1, C_2 are isolated J-curves, their self-intersection must be negative. On the other hand, $C_1 \cdot C_2 = 1$, so it follows easily from $(C_1 + C_2)^2 = 0$ that both of C_1, C_2 are (-1)-spheres.

There are $N - 1$ singular fibers. We shall pick a (-1)-sphere from each singular fiber and name the homology classes by $E_2, \cdots, E_N$. Then there is a unique pair of line class and exceptional class H and E_1 such that

(i) $H - E_1$ is the fiber class of π,
(ii) $H, E_1, E_2, \cdots, E_N$ form a basis of $H_2(X)$ with standard intersection matrix.

With this understood, observe that

(1) for each (-1)-sphere E_j, $j = 2, \cdots, N$, the other (-1)-sphere lying in the same singular fiber has homology class $H - E_1 - E_j$,
(2) the canonical class $K_\omega = -3H + E_1 + \cdots + E_N$.

There are further consequences if the symplectic G-conic bundle is minimal. In this case, for each (-1)-sphere E_j, $j = 2, \cdots, N$, there exists a $g \in G$ such that $g \cdot E_j = H - E_1 - E_j$. This implies

$$\omega(E_j) = \frac{1}{2}\omega(H - E_1), \ \ j = 2, \cdots, N.$$

Thus a minimal symplectic G-conic bundle falls into three cases:

$$(i) \ \omega(E_1) = \omega(E_j), \ (ii) \ \omega(E_1) > \omega(E_j), \ (iii) \ \omega(E_1) < \omega(E_j).$$

Case (i) occurs iff (X, ω) is monotone. Since E_1 is a section class, we shall call case (ii) (resp. case (iii)) a symplectic G-conic bundle with *small fiber area* (resp. *large fiber area*).

The following lemma is the J-holomorphic analog of Lemma 5.1 in [**6**].

Lemma 2.1. *Let $\pi : X \to \mathbb{S}^2$ be a symplectic G-conic bundle on (X, ω), which comes with a G-invariant, ω-compatible almost complex structure J. Suppose E, E' are two distinct J-holomorphic sections of self-intersection $-m, -m'$, and let r be the number of singular fibers where E, E' intersect the same component. Then*

$$N - 1 = r + m + m' + 2E \cdot E'.$$

Moreover, if the symplectic G-manifold (X, ω) is minimal, then $N \geq 5$ must be true.

Proof. Since $E_1, F = H - E_1, E_s, s > 1$ generate $H^2(X)$, both E and E' have the form

(4) $$E_1 + cF + \sum_{t>1} c_t E_t,$$

where $c \in \mathbb{Z}$, and $c_t = 0$ or 1 depending on which component they intersect on each singular fiber.

Since E, E' are sections, we have

$$E' - E = bF + \sum_s b_s E_s,$$

where $b \in \mathbb{Z}$ and $b_s = \pm 1$ with s running over the set of singular fibers at where E', E intersect *different* components. Note that the number of s is exactly $N - 1 - r$. Now

$$(E' - E)^2 = \sum_s b_s^2 E_s^2 = -(N - 1 - r),$$

which gives

$$N - 1 = r + m + m' + 2E \cdot E'.$$

Now suppose (X, ω) is minimal. We will show that $N \geq 5$ in this case. First, we claim that the sections E, E' in the lemma do exist. This is because the class E_1 can be represented by a J-holomorphic stable curve. By checking the intersection with $H - E_1$, the E_1 stable representative contains a unique J-holomorphic section E. Note that $E, F = H - E_1, E_s, s > 1$ also form a basis of $H_2(X)$, and E_1 can be written as

$$E_1 = a(H - E_1) + \sum_{s>1} a_s E_s + E,$$

where $a \geq 0$, and $a_s = 0$ or 1 depending on which component that E intersects at each singular fiber. Note that

$$E^2 = -2a - 1 - \sum_{s>1} a_s.$$

We take $E' = g \cdot E$ for some $g \in G$ such that $E' \neq E$. Such a $g \in G$ exists because E must intersect a singular fiber and there is a $g \in G$ which switches the two (-1)-spheres in that singular fiber. Note that E' is a section, as the fiber class is G-invariant so that $E' \cdot (H - E_1) = E \cdot (H - E_1) = 1$. Furthermore, note that $(E')^2 = E^2$. Consequently, if $E^2 \leq -2$, we must have

$$N = 1 + r + m + m' + 2E \cdot E' \geq 1 + 0 + 2 + 2 + 0 = 5.$$

If $E^2 > -2$, then $a = a_s = 0$ and $E = E_1$ must be a (-1)-sphere. The minimality assumption then implies that $E' = g \cdot E$ for some $g \in G$ can be chosen such that E' intersects E. In this case, we have

$$N = 1 + r + m + m' + 2E \cdot E' \geq 1 + 0 + 1 + 1 + 2 = 5.$$

This finishes off the proof of Lemma 2.1.

<div align="right">q.e.d.</div>

Lemma 2.2. *Let (X, ω) be a symplectic rational G-surface which admits a minimal symplectic G-conic bundle structure. Then the invariant lattice $H^2(X)^G$ has rank 2 which is spanned by K_ω and the fiber class of the symplectic G-conic bundle.*

Proof. First, we show that $H^2(X; \mathbb{R})^G$ is 2-dimensional. To see this, we first note that $K_\omega, H - E_1, E_2, \cdots, E_N$ form a basis of $H^2(X, \mathbb{R})$. We set

$$V = H^2(X; \mathbb{R})/\mathrm{Span}\,_{\mathbb{R}}(K_\omega, H - E_1).$$

Then it suffices to show that $V^G = \{0\}$ because $K_\omega, H - E_1 \in H^2(X)^G$.

We let $e_2, \cdots, e_N$ be the image of $E_2, \cdots, E_N$ under the quotient map, which form a basis of V. Suppose to the contrary, there is a $x \neq 0$ in V^G. We write $x = \sum_{k=2}^N a_k e_k$. Then there exists a k_0 such that $a_{k_0} \neq 0$. With this understood, we note that by the minimality assumption, there is a $g \in G$ such that $g \cdot E_{k_0} = H - E_1 - E_{k_0}$, which means that $g \cdot e_{k_0} = -e_{k_0}$. Now we set $I = \{k | g \cdot e_k = -e_k\}$. Then $k_0 \in I$; in particular, $I \neq \emptyset$. We let J be the complement of I in the set $\{2, \cdots, N\}$. Then it follows easily that if $k \in J$, then $g \cdot e_k = \pm e_l$ for some $l \in J$ ($g \cdot E_k = E_l$ or $H - E_1 - E_l$ for some l).

With this understood, we write $x = y + z$, where $y = \sum_{k \in I} a_k e_k$ and $z = \sum_{k \in J} a_k e_k$. Then $g \cdot x = -y + z'$ for some $z' \in \mathrm{Span}\,_{\mathbb{R}}(e_k | k \in J)$. Since $g \cdot x = x$, we have $2y = z' - z \in \mathrm{Span}\,_{\mathbb{R}}(e_k | k \in J)$. Since $e_2, \cdots, e_N$ form a basis of V, this clearly contradicts the fact that $y \in \mathrm{Span}\,_{\mathbb{R}}(e_k | k \in I)$ and $y \neq 0$. Hence the claim that $H^2(X; \mathbb{R})^G$ is 2-dimensional.

Now for any $\alpha \in H^2(X)^G$, we write $\alpha = aK_\omega + b(H - E_1)$ for some $a, b \in \mathbb{Q}$. Then $\alpha \cdot E_N = -a$, implying $a \in \mathbb{Z}$. On the other hand, $\alpha \cdot E_1 = -a + b$, we have $b \in \mathbb{Z}$ also. Hence $H^2(X)^G$ is spanned by K_ω and $H - E_1$.

<div align="right">q.e.d.</div>

Lemma 2.3. *Let $\pi : X \to \mathbb{S}^2$ be a minimal symplectic G-conic bundle where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ with $N \geq 6$ and even. Let J be any G-invariant, compatible almost complex structure, and let*

$$m_J = max\{m \in \mathbb{Z} : \text{there is a J-holomorphic section of π of self-intersection $-m$}\}.$$

Then $m_J \leq (N - 4)/2$. In particular, when $N = 6$, $m_J \leq 1$.

Proof. First, note that if E is a J-holomorphic section of self-intersection $-m$, then there is a $g \in G$ such that $g \cdot E \neq E$, where $g \cdot E$ is also of self-intersection $-m$. This is because E must intersect a singular fiber, and by the minimality assumption there is a $g \in G$ which switches the two components of that singular fiber. Clearly for this g, $g \cdot E \neq E$. Now apply Lemma 2.1 to E and $E' = g \cdot E$, we see that $m \leq (N - 2)/2$ as N is even. Hence we reduced the lemma to showing $m \neq \frac{N-2}{2}$.

Suppose to the contrary that $m = (N - 2)/2$, and let E, E' be a pair of sections whose self-intersection equals $-m$. Then by Lemma 2.1, we see that E, E' are disjoint and $r = 1$. Let F be the singular fiber where E, E' intersect the same (-1)-sphere. Then again by the minimality assumption there is a $h \in G$ which switches the two (-1)-spheres in F. It follows easily that $h \cdot E$, E, and E' are distinct. Let r, r' be the number of singular fibers where $h \cdot E$, E and $h \cdot E$, E' intersect the same (-1)-sphere, respectively. Then it follows easily that $r + r' = N - 2$. On the other hand, Lemma 2.1 implies that $r = r' = 1$, contradicting the fact that $N \geq 6$. Hence the lemma.

<div align="right">q.e.d.</div>

2.2. Reduction of exceptional classes. Momentarily we let ω be any symplectic structure on $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, and denote

$$\mathcal{E}_\omega = \{e \in H^2(X) | e^2 = -1, K_\omega \cdot e = -1, \omega(e) > 0\}$$

which may depend on ω. The following fact is crucial in our considerations.

Lemma 2.4. *(cf. [9]) Assume $N \geq 2$. Then for any ω-compatible almost complex structure J on X, each class $E \in \mathcal{E}_\omega$ with minimal area, i.e., $\omega(E) = \min_{e \in \mathcal{E}_\omega} \omega(e)$, is represented by an embedded J-holomorphic sphere.*

The key ingredient in the proof of Theorem 1.3 is a reduction procedure which involves a certain type of standard basis of $H^2(X)$, called a *reduced basis*. We shall begin by a brief digression and refer the reader to [11, Proposition 4.14] or independently [1] for more details.

Recall that a *reduced basis* is a basis $H, E_1, \cdots, E_N$ of $H^2(X)$ with standard intersection matrix, where $E_i \in \mathcal{E}_\omega$, such that $\omega(E_N) = \min_{e \in \mathcal{E}_\omega} \omega(e)$, and for any $i < N$, E_i satisfies the following inductive condition: let $\mathcal{E}_i = \{e \in \mathcal{E}_\omega | e \cdot E_j = 0 \quad \forall i < j\}$, then $\omega(E_i) = \min_{e \in \mathcal{E}_i} \omega(e)$. Furthermore, the canonical class $K_\omega = -3H + E_1 \cdots + E_N$. Reduced basis always exists.

If $N = 2$, then $\mathcal{E}_\omega = \{E_1, E_2, H - E_1 - E_2\}$. For $N \geq 3$, a reduction procedure can be introduced, which requires the following discussions.

1. Introduce $H_{ijk} = H - E_i - E_j - E_k$ for $i < j < k$ and $H_{ij} = H - E_i - E_j$ for $i < j$. Then $H_{ij} \in \mathcal{E}_j$, which implies that $\omega(H_{ijk}) \geq 0$.

2. For any $E \in \mathcal{E}_\omega$, write $E = aH - \sum_s b_s E_s$. Then

- $a \geq 0$;
- if $a > 0$, then $b_s \geq 0$ for all s;
- if $a = 0$, then $E = E_l$ for some l;
- assume $a > 0$, and let b_i, b_j, b_k be the largest three coefficients (here we use the assumption $N \geq 3$), then $b_i \leq a < b_i + b_j + b_k$, which is equivalent to $E \cdot H_{ijk} < 0$ and $E \cdot (H - E_i) \geq 0$.

3. The classes H_{ijk} are represented by embedded (-2)-spheres, hence for each H_{ijk} there is a diffeomorphism of X inducing an automorphism $R(H_{ijk})$ on $H^2(X)$:

$$R(H_{ijk})\alpha = \alpha + (\alpha \cdot H_{ijk}) \cdot H_{ijk}, \quad \forall \alpha \in H^2(X).$$

Moreover, each $R(H_{ijk})$ has the following properties: (1) $R(H_{ijk})K_\omega = K_\omega$, (2) $R(H_{ijk})E \in \mathcal{E}_\omega, \quad \forall E \in \mathcal{E}_\omega$.

With the preceding understood, consider any $E \in \mathcal{E}_\omega$, where $E = aH - \sum_s b_s E_s$ with $a > 0$. Let b_i, b_j, b_k be the largest three coefficients for some $i < j < k$. Then it follows easily from the last bullet of item 2 that

- $R(H_{ijk})E = a'H - \sum_s b'_s E_s$ for some $a' < a$, and
- $\omega(R(H_{ijk})E) \leq \omega(E)$ with " $=$ " iff $\omega(H_{ijk}) = 0$.

Set $E' := R(H_{ijk})E$. We say that E *is reduced to* E' *by* H_{ijk}.

The operations $R(H_{ijk})$ has the following properties of our interests:

1) (finite termination) By [**13**, Proposition 1.2.12], one may find a sequence of finitely many H_{ijk} for any $E \in \mathcal{E}_\omega$, such that after performing $R(H_{ijk})$, E is reduced to E_l for some l.
2) (monotonicity) The symplectic area is **monotonically decreasing** during the above reduction procedure.

Therefore, after the reduction procedure, $\omega(E) \geq \omega(E_l)$, with " $=$ " iff $\omega(H_{ijk}) = 0$ for all the H_{ijk}'s involved. In particular, when E has the minimal area in $\mathcal{E}_\omega$, i.e., $\omega(E) = \omega(E_N)$, then we have

(5)
$$\omega(E_l) = \omega(E_{l+1}) = \cdots = \omega(E_N), \text{ and } \omega(H_{ijk}) = 0 \text{ for all the } H'_{ijk}s \text{ involved.}$$

We first rule out the case of $N = 2$.

Lemma 2.5. *There are no minimal symplectic rational $G-$surface with $N = 2$.*

Proof. Suppose $((X,\omega), G)$ is a minimal symplectic rational $G-$surface with $N = 2$. Let $\{H, E_1, E_2\}$ be a reduced basis. Then $\mathcal{E}_\omega = \{E_1, E_2, H - E_1 - E_2\}$.

Fix a G-invariant J, and let C be the J-holomorphic (-1)-sphere representing E_2 (cf. Lemma 2.4). We set $\Lambda := \cup_{g \in G} g \cdot C$. Then Λ is a union of finitely many J-holomorphic (-1)-spheres, containing at least two distinct (-1)-spheres intersecting each other because of the minimality assumption. Since $\mathcal{E}_\omega = \{E_1, E_2, H - E_1 - E_2\}$, there are only two possibilities: (1) Λ is a union of three (-1)-spheres, representing the classes E_1, E_2 and $H - E_1 - E_2$, and (2) Λ is a union of two (-1)-spheres representing the classes E_2 and $H - E_1 - E_2$. In either case, $H - E_1 - E_2$ is represented by a (-1)-sphere. Since $H - E_1 - E_2$ is the only characteristic element in $\mathcal{E}_\omega$, it must be fixed by the G-action, which contradicts the minimality of the symplectic G-manifold (X, ω). Hence there are no minimal symplectic rational $G-$surface with $N = 2$.

<div align="right">q.e.d.</div>

2.3. Reduced basis for symplectic rational G-surfaces. Due to Lemma 2.5, in what follows we assume that (X, ω) is a minimal symplectic rational G-surface, where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ with $N \geq 3$.

Lemma 2.6. *Suppose $N \geq 3$. Then one of the following must be true.*

(i) *(X, ω) is monotone.*
(ii) *The reduced basis of X satisfies $\omega(E_1) > \omega(E_2) = \cdots = \omega(E_N)$, $\omega(H - E_1) = 2\omega(E_j)$. Moreover, if $E \in \mathcal{E}_\omega$ has minimal area, then either $E = E_j$ or $E = H_{1j} = H - E_1 - E_j$ for some $j > 1$.*

Proof. Assume (X, ω) is not monotone. We shall first prove a slightly weaker statement that is independent of the G-action.

Claim: If $E \in \mathcal{E}_\omega$ has minimal area in $\mathcal{E}_\omega$ and $E \neq E_s$ for any s, then $E = H - E_1 - E_j$ for some $j > 1$, and furthermore, if such an E exists and $E = H - E_1 - E_j$ for some $j > 2$, then we must have

$$\omega(E_2) = \cdots = \omega(E_N).$$

Proof of Claim. Let $E \in \mathcal{E}_\omega$ be such a class, i.e., E has minimal area in $\mathcal{E}_\omega$ and $E \neq E_s$ for any s, We reduce E to E_l for some l by a

sequence of H_{ijk}'s. Since E has minimal area in $\mathcal{E}_\omega$, it follows that $\omega(E_l) = \cdots = \omega(E_N)$. Note that $l > 1$. Otherwise, $\omega(E_i) = \omega(E_j)$ and $\omega(H_{ijk}) = 0$ for some i, j, k from property (3) in the reduction process in Section 2.2. This violates that (X, ω) is not monotone.

Suppose the reduction from E to E_l takes n steps, let $E' \in \mathcal{E}$ be the class obtained at the $(n-1)^{\text{th}}$ step. Then $E' = aH - \sum_s b_s E_s$, where $a > 0$ and $b_s \geq 0$. The equation $E_l = E' + (E' \cdot H_{ijk}) \cdot H_{ijk}$ reads

$$E_l = (2a - b_i - b_j - b_k)H - \sum_{s=i,j,k} (b_s + a - b_i - b_j - b_k)E_s - \sum_{s\neq i,j,k} b_s E_s,$$

which implies that l must be one of i, j or k, and without loss of generality assuming that $l = k$, then

$$2a = b_i + b_j + b_k, \ a = b_j + b_k = b_i + b_k, \ a - b_i - b_j = -1.$$

It follows easily that $E' = H_{ij} = H - E_i - E_j$. Here we assume $i < j$, but do not require any condition on $k = l > 1$. Note that E' has minimal area in $\mathcal{E}_\omega$, which implies that $\omega(H) = \omega(E_i) + \omega(E_j) + \omega(E_l)$ from (5).

Next we prove that $i = 1$. Suppose to the contrary that $i > 1$. Then $\omega(H_{1ij}) = \omega(H) - \omega(E_1) - \omega(E_i) - \omega(E_j) \geq 0$. On the other hand, $\omega(H) = \omega(E_i) + \omega(E_j) + \omega(E_l)$ and $\omega(E_1) \geq \omega(E_l)$, which implies that $\omega(E_1) = \cdots = \omega(E_N) = \frac{1}{3}\omega(H)$. This is a contradiction because we assume (X, ω) is not monotone. Hence $i = 1$ and $E' = H - E_1 - E_j$.

We claim that $E = E' = H - E_1 - E_j$. Suppose this is not true. Then there must be a class $\tilde{E}$ which is reduced to E' by some H_{vrt} for $v < r < t$. We assert $v > 1$. To see this, write $\tilde{E} = aH - \sum_s b_s E_s$, where $a > 0$ and $b_s \geq 0$. Then similarly we have

$$H - E_1 - E_j = (2a - b_v - b_r - b_t)H - \sum_{s=v,r,t} (b_s + a - b_v - b_r - b_t)E_s - \sum_{s\neq v,r,t} b_s E_s.$$

If $v = 1$, then $2a - b_v - b_r - b_t = 1$ and $a - b_r - b_t = 1$, implying $a = b_v$. Now for $s = r, t$, the coefficient of E_s on the right hand side is b_t, b_r respectively, which are non-negative. It follows that $b_r = b_t = 0$ from the property 2 in Section 2.2, which contradicts the fact that $a < b_v + b_r + b_t$. Hence $v > 1$.

We now get a contradiction as follows. Note that $\omega(H_{vrt}) = 0$ and $\omega(H_{1vr}) \geq 0$ implies that

$$\omega(E_1) = \cdots = \omega(E_t).$$

Then with $\omega(H_{vrt}) = 0$ again, we have $\omega(H) = 3\omega(E_1)$. On the other hand, $\omega(E_l) = \omega(E') = \omega(H) - \omega(E_1) - \omega(E_j)$, from which it follows that

$$\omega(E_1) = \cdots = \omega(E_l) = \cdots = \omega(E_N).$$

This contradicts the assumption that (X, ω) is not monotone. Hence $E = H - E_1 - E_j$ is proved. Finally, if $j > 2$, then $\omega(H_{12j}) \geq 0$ and

$\omega(H) - \omega(E_1) - \omega(E_j) = \omega(E_l)$ implies that $\omega(E_2) = \omega(E_l)$, hence $\omega(E_2) = \cdots = \omega(E_N)$. This concludes the claim.

<div align="right">q.e.d.</div>

To obtain Lemma 2.6, it remains to show that if (X, ω) is not monotone, then there exists some $j > 2$, such that $E = H - E_1 - E_j$ attains the minimal area within $\mathcal{E}_\omega$. To this end, we fix a G-invariant ω-compatible J. Then by Lemma 2.4, there exists a J-holomorphic (-1)-sphere C representing the class E_N. Note that for any $g \in G$, $g \cdot C \in \mathcal{E}_\omega$. Now by the assumption that (X, ω) is minimal, there must be a $g \in G$ such that $g \cdot C \neq C$ and $g \cdot C$ intersects with C. The class $g \cdot C$ has minimal area in $\mathcal{E}_\omega$ and $g \cdot C \neq E_s$ for any s. Let $g \cdot C = H - E_1 - E_j$. Then $j = N \geq 3$ must be true because $g \cdot C$ intersects with C. This finishes the proof of the lemma.

<div align="right">q.e.d.</div>

2.4. Proof of Theorem 1.3. In what follows, we still assume that $N \geq 3$. We will discuss according to the following three possibilities:

i) (X, ω) *is not monotone*: By Lemma 2.6, there is a reduced basis $\{H, E_1, \cdots, E_N\}$ such that

$$\omega(E_1) > \omega(E_2) = \cdots = \omega(E_N),$$

and moreover, if $E \in \mathcal{E}_\omega$ has minimal area, then either $E = E_j$ or $E = H_{1j} = H - E_1 - E_j$ for some $j > 1$.

Let J be any G-invariant ω-compatible almost complex structure. For each fixed $j > 1$, E_j has minimal area so by Lemma 2.4, so there is an embedded J-holomorphic sphere C representing E_j. Since (X, ω) is minimal, there must be a $g \in G$ such that $g \cdot C \neq C$ and $g \cdot C \cap C \neq \emptyset$. It follows easily that $g \cdot C$ must be the J-holomorphic (-1)-sphere representing the class $H - E_1 - E_j$, since it also has the minimal area. Furthermore, $g \cdot C$ and C intersect transversely and positively at a single point. Standard gluing construction in J-holomorphic curve theory yields a J-holomorphic sphere $\hat{C}$, carrying the class $C + g \cdot C = H - E_1$. It follows that $\hat{C}$ has self-intersection 0, and by the adjunction formula it must be embedded. By a standard Gromov-Witten index computation, the moduli space of J-spheres in class $[\hat{C}]$ gives rise to a fibration (which contains singular fibers) structure on X, where each fiber is homologous to $\hat{C}$. Since X is a rational surface, the base of the fibration must be $\mathbb{S}^2$. We denote the fibration by $\pi_j : X \to \mathbb{S}^2$.

Next we show that π_j is G-invariant. To see this, note that for any $h \in G$, $h \cdot (C \cup g \cdot C)$ must be a pair of J-holomorphic (-1)-spheres representing E_k and $H - E_1 - E_k$ for some k (see Lemma 2.6). It follows that the class of $\hat{C}$, which is $H - E_1$, is invariant under the G-action. This implies that the fibration π_j is G-invariant.

Finally, we note that the fibration π_j is independent of j, because the fiber class, which is $H - E_1$, is independent of j. We will denote the fibration by $\pi : X \to \mathbb{S}^2$. Note that the same argument shows also that π contains at least $N - 1$ singular fibers, consisting of a pair of (-1)-spheres whose classes are $E_j, H - E_1 - E_j$ for $j = 2, \cdots, N$. There are no other singular fibers by an Euler number count. Note that Lemma 2.2 asserts rank $H^2(X)^G = 2$ in this case.

ii) (X, ω) *is monotone and* $H^2(X)^G$ *has rank* 1: In this case, note that $K_\omega \in H^2(X)^G$ and is a primitive class, hence $H^2(X)^G$ is spanned by K_ω. The constraint $N \leq 8$ is an easy consequence of (X, ω) being monotone.

iii) (X, ω) *is monotone and rank* $H^2(X)^G > 1$: Note that the de Rham class $[\omega] = \lambda K_\omega \in H^2(X; \mathbb{R})^G$ for some $\lambda \in \mathbb{R}$. Since rank $H^2(X)^G > 1$, we may pick a G-invariant closed 2-form η such that $[\eta]$ lies in a different direction in $H^2(X; \mathbb{R})^G$. Let $\omega' := \omega + \epsilon\eta$ for some very small ϵ. Then ω' is a G-invariant symplectic structure such that (X, ω') is not monotone.

Claim: (X, ω') is minimal as a symplectic G-manifold for sufficiently small ϵ.

Suppose there is a disjoint union of ω'-symplectic (-1)-spheres $\{C_i\}$, such that for any $g \in G$, $g \cdot C_i = C_j$ for some j. Note that for sufficiently small ϵ, we have $\mathcal{E}_\omega = \mathcal{E}_{\omega'}$ because $K_\omega = K_{\omega'}$. Let e_i be the class of C_i. It follows easily that for each i, $e_i \in \mathcal{E}_\omega$. Now pick a G-invariant J compatible with ω. Since (X, ω) is monotone, each e_i is represented by an embedded J-holomorphic (-1)-sphere $\hat{C}_i$. Notice that $\{e_i\}$ has the following properties: $e_i \cdot e_j = 0$ for $i \neq j$, and for any $g \in G$, $g \cdot e_i = e_j$ for some j. It follows that $\{\hat{C}_i\}$ is a disjoint union of ω-symplectic (-1)-spheres which is invariant under the G-action. This contradicts the minimality assumption on (X, ω), hence the claim.

We apply the argument for case i) to (X, ω'). Consequently there is a reduced basis $\{H, E_1, \cdots, E_N\}$ (w.r.t. ω'), and a G-invariant fibration $\pi' : X \to \mathbb{S}^2$, whose fiber class is $H - E_1$ and the singular fibers are pairs of (-1)-spheres representing $E_j, H - E_1 - E_j$ for $j > 1$. Note that by taking ϵ small, we have $E_i \in \mathcal{E}_{\omega'} = \mathcal{E}_\omega$. Finally, observe the following crucial property $(\sum_{g \in G} g \cdot E_j)^2 = 0$ for any $j > 1$ because E_k and $H - E_1 - E_k$ must appear in pairs in the sum.

Now let J be any G-invariant ω-compatible almost complex structure. For each fixed $j > 1$, since (X, ω) is monotone and $E_j \in \mathcal{E}_\omega$, E_j is represented by an embedded J-holomorphic (-1)-sphere C. Let $\Lambda :=$

$\cup_{g \in G}\ g \cdot C$. Then the set Λ is a union of finitely many distinct J-holomorphic (-1)-spheres. Since $(\sum_{g \in G} g \cdot E_j)^2 = 0$ for any $j > 1$, it follows easily that $\Lambda^2 = 0$.

Let $\{\Lambda_i\}$ be the set of connected components of Λ. Since G acts on the connected components of Λ transitively, it follows that for any $i \neq k$, $\Lambda_i^2 = \Lambda_k^2$. Clearly $\Lambda^2 = \sum_i \Lambda_i^2$, which implies that $\Lambda_i^2 = 0$ for any i. Now $\Lambda_i^2 = 0$, together with the fact that Λ_i is a union of finitely many distinct J-holomorphic (-1)-spheres and Λ_i is connected, implies that Λ_i consists of two (-1)-spheres intersecting transversely at a single point. We claim that the pair of (-1)-spheres in each Λ_i have classes $E_k, H - E_1 - E_k$ for some $k > 1$. This is because for each $g \in G$, $g \cdot E_j$ is either E_k or $H - E_1 - E_k$ for some $k > 1$, and for each $k > 1$, there is a $g \in G$ such that $g \cdot E_k = H - E_1 - E_k$. By the same argument as in case i), we obtain a G-invariant fibration on X as desired, independent of the choice of j.

The statement that $N \geq 5$ was proved in Lemma 2.1. The statement that $N \neq 6$ follows from Lemma 2.7 below. This completes the proof of Theorem 1.3.

Lemma 2.7. *Let $\pi : X \to \mathbb{S}^2$ be a minimal symplectic G-conic bundle where $X = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$. Then for any G-invariant, compatible almost complex structure J, there is a G-invariant J-holomorphic (-1)-sphere.*

Proof. Let $H, E_1, \cdots, E_6$ be a basis of $H^2(X)$ with standard interaction matrix such that $H - E_1$ is the fiber class and $E_2, \cdots, E_6$ are (-1)-spheres contained in singular fibers. Note that the canonical class $K_X = -3H + E_1 + \cdots + E_6$.

With this understood, note that $C = -K_X - (H - E_1) = 2H - \sum_{6 \geq i \geq 2} E_i \in H^2(X)^G$, and C is an exceptional class. Hence for any given G-invariant, compatible almost complex structure J, C has a J-holomorphic representative, which admits a decomposition of irreducible components

$$(6) \qquad\qquad C = \hat{F} + \hat{E},$$

where $\hat{F}$ is the sum of components contained in the fibers of the G-conic bundle (the *vertical class*), and $\hat{E}$ is the sum of other components (the *horizontal class*).

By Lemma 2.3, E_1 must have a J-holomorphic representative: otherwise, it has a stable curve representative where one of the component has to be a section since $E_1 \cdot F = 1$. Such a section has self-intersection less than -1, a contradiction.

To continue with our proof, we first show that $\hat{E}$ does not contain E_1-components. Suppose the multiplicity of the E_1-component is k, since all irreducible components pairs with $H - E_1$ non-negatively, $(C - kE_1) \cdot (H - E_1) \geq 0$, hence $k \leq 2$.

Furthermore, if the multiplicity is 2 (meaning either there is a doubly covered component or two components), $\hat{E} = 2E_1$. Otherwise, $\hat{E} \cdot (H - E_1) \geq 3 > C \cdot (H - E_1)$, violating the positivity of intersection with $H - E_1$.

For the multiplicity 2 case, $\hat{F} = 2H - 2E_1 - \sum_{6 \geq i \geq 2} E_i$, while all possible irreducible components are of the form $H - E_1 - E_i$, E_i or $H - E_1$ for $6 \geq i \geq 2$. A simple check shows this cannot be consistent with (6).

For the multiplicity 1 case, $\hat{E} = E_1 + E'$. E' has $E' \cdot (H - E_1) = 1$ hence a section. Therefore, $E' \geq -1$ from Lemma 2.3. Since $E' \neq E_1$, its coefficient of H under the reduced basis must be positive. One may again easily check $\hat{F}$ cannot be represented as a combination of classes of forms $H - E_1 - E_i$, E_i or $H - E_1$.

Now since the equivariant J-holomorphic representative of C does not contain E_1-components, it is disjoint from the E_1-section. Therefore, $\hat{F} \cdot E_1 = 0$, which implies the sum of vertical components $\hat{F} = \sum m_s E_s$ for some $s \neq 1$. However, there is always an element $g \in G$ which sends E_s to $H - E_1 - E_s$ for any $s \neq 1$. This forces $\hat{F} = 0$, and $C = \hat{E}$.

At last, again note that $\hat{E}$ has at most 2 components by positivity of intersection with $H - E_1$. We assume $\hat{E}$ has two components, which are both sections S_1 and S_2.

$$C^2 = (S_1 + S_2)^2 = -1 = S_1^2 + S_2^2 + 2S_1S_2 \geq -2 + 2S_1 \cdot S_2$$

from Lemma 2.3. Therefore, $2S_1 \cdot S_2 \leq 1$, which implies $S_1 \cdot S_2 = 0$. This implies these are disjoint sections $S_1^2 = -1$ and $S_2^2 = 0$. A simple check on class C shows there are no such decomposition of section classes (recall that $[S_1] \neq E_1$).

Summarizing, we have showed that the J-representative of C is indeed irreducible, which is a G-invariant J-holomorphic (-1)-sphere. Hence the lemma.

<div align="right">q.e.d.</div>

To compare with known results in algebraic geometry, it seems worth to record the following easy consequence.

Theorem 2.8. *Let (X, ω) be a minimal symplectic rational G-surface where $X = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$. Then the invariant lattice $H^2(X)^G$ must be of rank 1. In particular, a minimal complex rational G-surface X which is $\mathbb{CP}^2$ blown up at 6 points must be Del Pezzo with $Pic(X)^G = \mathbb{Z}$.*

Proof. Under the above assumptions, Lemma 2.7 implies (X, ω) is not a G-conic bundle, hence $rank(H^2(X)^G) = 1$ by Theorem 1.3.

For the second statement, notice that a conic bundle on a minimal complex rational G-surface defines a minimal symplectic G-conic bundle

with respect to any G-invariant Kähler form. Theorem 3.8 of [**6**] asserts then X must be a del Pezzo surface if $rank H^2(X)^G = 1$.

<div align="right">q.e.d.</div>

3. The structure of G

3.1. Proof of Theorem 1.5, $4 \leq N \leq 8$**.** We start with the following observation.

Lemma 3.1. *Suppose* (X, ω) *is monotone. If* $N \geq 4$*, then the representation of* G *on* $H^2(X)$ *is faithful.*

Proof. Fix a G-invariant J. Let $g \in G$ be any element acting trivially on $H^2(X)$. Then g fixes every element $E \in \mathcal{E}_\omega$, which implies that all the J-holomorphic (-1)-spheres are invariant under g. Now let C_1 be the J-holomorphic (-1)-sphere representing E_1, and for each $1 < j \leq N$, let C_j be the J-holomorphic (-1)-sphere representing $H - E_1 - E_j$. Then it is clear that C_1 intersects each C_j, $j > 1$, transversely at one point and the C_j's are mutually disjoint. It follows that the cardinality of the set $C_1 \cap (\cup_j C_j)$ is $N - 1$. On the other hand, each point in $C_1 \cap (\cup_j C_j)$ is fixed under g, hence the action of g on C_1 contains at least $N - 1$ fixed points. When $N \geq 4$, it follows that C_1 must be fixed by g. A similar argument shows that every J-holomorphic (-1)-sphere is fixed by g. Since there are J-holomorphic (-1)-spheres intersecting transversely at a point, the action of g on the tangent space at the intersection point must be trivial, which shows that g must be trivial. Hence the lemma.

<div align="right">q.e.d.</div>

Assume (X, ω) is a symplectic rational G-surface with $rank H^2(X)^G = 1$.

Since $K_\omega \in H^2(X)$ is fixed under the action of G, there is an induced representation of G on the orthogonal complement R_N, which is faithful by Lemma 3.1. This gives rise to a monomorphism $\rho : G \to W_N$. On the other hand, $H^2(X)^G$ is spanned by K_ω, so that $R_N^G = \{0\}$. This implies that

$$\frac{1}{|G|} \sum_{g \in G} trace\{\rho(g) : R_N \to R_N\} = \text{rank } R_N^G = 0.$$

The proof for Case 1 is completed.

3.2. Proof of Theorem 1.5, $N = 3$**.** Our main objective in this case is to show that G contains an index 2 subgroup which is isomorphic to an imprimitive finite subgroup of $PGL(3)$.

We first describe the list of subgroups of $PGL(3)$ involved. For simplicity, we will adapt the following convention from [**6**]: we denote an element $T \in PGL(3)$ by the image of $[z_0, z_1, z_2] \in \mathbb{CP}^2$ under T.

Here is the list of imprimitive finite subgroups of $PGL(3)$ up to conjugacy (see Theorem 4.7 in [**6**]), where $\mu_r = \exp(2\pi i/r)$ is the r-th root of unity.

- G_n, generated by the following elements of $PGL(3)$:

$$[\mu_n z_0, z_1, z_2], \quad [z_0, \mu_n z_1, z_2], \quad [z_2, z_0, z_1]$$

  The group G_n is isomorphic to a semi-direct product of $(\mathbb{Z}_n)^2$ and $\mathbb{Z}_3$.

- $\tilde{G}_n$, generated by the following elements of $PGL(3)$:

$$[\mu_n z_0, z_1, z_2], \quad [z_0, \mu_n z_1, z_2], \quad [z_0, z_2, z_1], \quad [z_2, z_0, z_1]$$

  The group $\tilde{G}_n$ is isomorphic to a semi-direct product of $(\mathbb{Z}_n)^2$ and S_3.

- $G_{n,k,s}$, where $k > 1$, $k|n$, and $s^2 - s + 1 = 0 \pmod{k}$. It is generated by the following elements of $PGL(3)$:

$$[\mu_{n/k} z_0, z_1, z_2], \quad [\mu_n^s z_0, \mu_n z_1, z_2], \quad [z_2, z_0, z_1]$$

  The group $G_{n,k,s}$ is isomorphic to a semi-direct product of $\mathbb{Z}_n \times \mathbb{Z}_{n/k}$ and $\mathbb{Z}_3$.

- $\tilde{G}_{n,3,2}$, generated by the following elements of $PGL(3)$:

$$[\mu_{n/3} z_0, z_1, z_2], \quad [\mu_n^2 z_0, \mu_n z_1, z_2], \quad [z_0, z_2, z_1], \quad [z_1, z_0, z_2]$$

  The group $\tilde{G}_{n,3,2}$ is isomorphic to a semi-direct product of $\mathbb{Z}_n \times \mathbb{Z}_{n/3}$ and S_3.

3.2.1. Preliminaries: factorization of G by exceptional spheres.
Let $\{H, E_1, E_2, E_3\}$ be a reduced basis. Then

$$\mathcal{E}_\omega = \{E_1, E_2, E_3, H - E_1 - E_2, H - E_1 - E_3, H - E_2 - E_3\}.$$

Since (X, ω) is monotone, the classes in $\mathcal{E}_\omega$ have the same area, and consequently, each class in $\mathcal{E}_\omega$ is represented by a J-holomorphic (-1)-sphere for any fixed J, which we assume to be G-invariant. Let Λ be the union of these six (-1)-spheres. The intersection pattern of these curves can be described by a hexagon, where each edge represents a (-1)-sphere and each vertex represents an intersection point (See Figure 1). For simplicity, for each $E \in \mathcal{E}_\omega$ we shall use the same notation to denote the corresponding (-1)-sphere.

Obviously there is an induced G-action on Λ.

Lemma 3.2. *The action of G on the components of Λ is transitive, and consequently, there is a short exact sequence*

(7) $$1 \to \Gamma \to G \to K \to 1,$$

where Γ is isomorphic to a subgroup of $\mathbb{S}^1 \times \mathbb{S}^1$, and K is either D_{12}, the full automorphism group of the hexagon, or the cyclic subgroup of order 6.
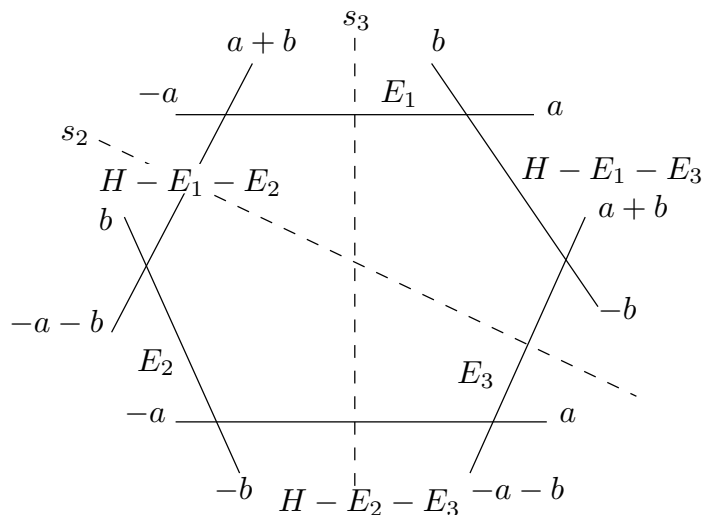
Figure 1. Exceptional configurations and action weights in $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$

Proof. Let C be any of the (-1)-spheres. Then the class of the union $\cup_{g\in G}g\cdot C$ must lie in $H^2(X)^G$. Since $H^2(X)^G = \mathrm{Span}\,(K_\omega)$, the class of $\cup_{g\in G}g\cdot C$ must be a multiple of K_ω. On the other hand, the class of Λ equals $-K_\omega$, from which it follows easily that $\Lambda = \cup_{g\in G}g\cdot C$. This proves that the action of G on components of Λ is transitive.

The action of G on Λ gives rise to a short exact sequence

$$1 \to \Gamma \to G \to K \to 1,$$

where Γ is the normal subgroup of G consisting of elements which leave each (-1)-sphere invariant. The quotient group $K = G/\Gamma$ has an effective, transitive action on the hexagon Λ, which must be either the automorphism group D_{12} of the hexagon, or the cyclic subgroup of order 6.

To see that Γ is a subgroup of $\mathbb{S}^1 \times \mathbb{S}^1$, we look at the action of Γ on the tangent space of any intersection point of two adjacent (-1)-spheres in Λ. The action preserves a pair of complex lines intersecting transversely, giving a natural isomorphism of Γ to a subgroup of $\mathbb{S}^1 \times \mathbb{S}^1$.

q.e.d.

In fact, we may identify K geometrically as follows. There is a natural isomorphism $D_{12} = \mathbb{Z}_2 \times S_3$, where the latter is the Weyl group W_N of the corresponding root lattice R_N, which is generated by $H - E_1 - E_2 - E_3$, $E_1 - E_2$ and $E_2 - E_3$. In this sense, $\mathbb{Z}_2 = \langle s_1 \rangle$ where s_1 is the rotation of 180 degrees of the hexagon, and $S_3 = \langle s_2, s_3 \rangle$ where s_2, s_3 are the reflections of the hexagon which switches E_1 and E_2, E_2 and E_3 respectively. Note that s_2s_3, which is s_3 followed by s_2, is

a counter-clockwise rotation of 120 degrees of the hexagon. It follows that the short exact sequence $1 \to \Gamma \to G \to K \to 1$ is the same as the one obtained from the induced action of G on the root lattice R_N, with Γ being the subgroup of G acting trivially on $H^2(X)$. Under this identification, one of the following is true by Lemma 3.2

- K is generated by s_1, s_2 and s_3 if it is the cyclic subgroup of order 12.
- K is generated by s_1 and $s_2 s_3$ if it is the cyclic subgroup of order 6.

In order to understand the structure of G, we begin by getting more information about the subgroup Γ. To this end, we fix a monomorphism $\Gamma \to \mathbb{S}^1 \times \mathbb{S}^1$ induced from the action of Γ on the tangent space of the intersection point of E_1 and $H - E_1 - E_3$, where the first $\mathbb{S}^1$-factor is from the action on E_1. We summarize the main objects in consideration as follows for readers' convenience:

- $\Gamma \lhd G$ is the subgroup with trivial homological action,
- $K = G/\Gamma$,
- $\rho_i : \Gamma \to \mathbb{S}^1$, $i = 1, 2$, are the projection to the two $\mathbb{S}^1$-factors,
- $\Gamma_i = \ker \rho_i$,
- $\Gamma_i' = \text{image } \rho_i \subset \mathbb{S}^1$.
- $\tilde{\Gamma}_i$ is a subgroup of Γ such that $\rho_i : \tilde{\Gamma}_i \to \Gamma_i'$ is isomorphic.

Lemma 3.3.

1) Γ_i, Γ_i' are cyclic,
2) ord (Γ_1)=ord (Γ_2) and ord (Γ_1')=ord (Γ_2')
3) ord $(\Gamma_i)|$ord (Γ_i')

We will hence denote $n := ord(\Gamma_i')$ and $k := ord(\Gamma_i')/ord(\Gamma_i)$

Proof. It is clear that Γ_i' is cyclic. Γ_i is also cyclic because both $\rho_2|_{\Gamma_1}$ and $\rho_1|_{\Gamma_2}$ are injective. Note that this also shows that the order of Γ_1' (resp. Γ_2') is divisible by the order of Γ_2 (resp. Γ_1).

Finally, Γ_1 and Γ_2 have the same order. This is because if we let $g \in G$ be an element whose action on the hexagon is a counter-clockwise rotation of 60 degrees, then $g\Gamma_2 g^{-1} = \Gamma_1$. Consequently, the order of $\Gamma_i' = \Gamma/\Gamma_i$ is independent of i.

q.e.d.

Lemma 3.4. *The subgroup $\tilde{\Gamma}_i < \Gamma$ exists. In particular, since Γ is Abelian, $\Gamma \cong \Gamma_i \times \tilde{\Gamma}_i$ for $i = 1, 2$.*

Proof. Let $h \in \Gamma$ be an element such that $\rho_1(h)$ is a generator of Γ_1'. Since $\rho_1(h)^n = 1$, we have $h^n \in \Gamma_1$. We claim $h^n = 1$. This is because $\rho_2|_{\Gamma_1}$ is injective, so that if $h^n \neq 1$ in Γ_1, then $\rho_2(h)^n = \rho_2(h^n) \neq 1$ in Γ_2'. But this contradicts the fact that the order of Γ_2' equals n. Hence $h^n = 1$. With this understood, we simply take $\tilde{\Gamma}_1$ to be the subgroup generated by h.

q.e.d.

3.2.2. Rotation numbers. Next we recall some basic facts about rotation numbers. Let $h \in \Gamma$ be any element and $E \in \mathcal{E}$ be any (-1)-sphere which is invariant under h. Then either h fixes E or h acts on E nontrivially. In the latter case, h fixes two points on E. The following fact is a straightforward local computation:

Fact: *if we let (a, b) be the rotation numbers of h at one of the fixed point, where a is the tangential weight and b is the weight in the normal direction, then the rotation numbers at the other fixed point are $(-a, b + a)$, with the second number being the weight in the normal direction.*

Explicitly, the tangential weight being a means that the action of h is given by multiplication of $\exp(2a\pi\sqrt{-1}/ord(h))$ and similarly for the normal weight. Here we orient the (-1)-sphere by the almost complex structure J and orient the normal direction accordingly, so there is no sign ambiguity on a, b. Note that even when E is fixed by h, this continues to make sense, with the understanding that $a = 0$ in this case.

With the preceding understood, we order the exceptional curves and their intersections according to the counter-clockwise orientation of the hexagon in Figure 1, and ask E_1 to be the first exceptional curve.

For example, we say $H - E_1 - E_3$ is **before** E_1 and $H - E_1 - E_2$ is **after** E_1, and the intersection point of $H - E_1 - E_3$ and E_1 is the **first** fixed point on E_1, etc.

Now for any $h \in \Gamma$, we denote by (a, b) the rotation numbers at the intersection point of $H - E_1 - E_3$ and E_1, with a being the weight in the direction tangent to E_1. According to the orientation of the hexagon, this is the first fixed point of h on E_1. With this understood, the rotation numbers at the fixed points of h on (-1)-spheres are given below according to the orientation of the hexagon,

$$(a + b, -a), (b, -a - b), (-a, -b), (-a - b, a), (-b, a + b).$$

Finally, we remark that h is completely determined by the rotation numbers at the six vertices of the hexagon.

Throughout the rest of the proof of Theorem 1.5,

$g \in G$ will denote those elements which act on the hexagon by a counter-clockwise rotation of 60 degrees,

and we shall investigate the action of g on $\Gamma = \Gamma_1 \times \tilde{\Gamma}_1$ given by conjugation, i.e., $h \mapsto ghg^{-1}$, $\forall h \in \Gamma$.

Here is the first corollary of the analysis on rotation numbers: note that the action of g^3, which is a rotation of 180 degrees, sends every pair of rotation numbers to its negative, i.e., $(a, b) \mapsto (-a, -b)$, $(a+b, -a) \mapsto$

$(-a - b, a)$ and $(b, -a - b) \mapsto (-b, a + b)$. This implies that

$$g^3 h g^{-3} = h^{-1}, \quad \forall h \in \Gamma.$$

Since g^3 is sent to $s_1 \in K$ under the homomorphism $G \to K$ in (7), we obtain the following lemma:

Lemma 3.5. *(1) For any element of G which is sent to $s_1 \in K$ under $G \to K$, its action on Γ by conjugation sends $h \in \Gamma$ to h^{-1}. (2) $g^6 = 1$.*

Proof. It remains to show that $g^6 = 1$. First, note that $g^6 \in \Gamma$. Secondly, the action of g^3 on g^6 by conjugation is trivial, but the action of g^3 on Γ sends h to h^{-1} as we just showed, from which we see that either $g^6 = 1$ or g^6 is an involution. We rule out the latter by showing that the action of g on any involution of Γ is nontrivial, and G is a semi-direct product of Γ and K follows.

Let $\tau \in \Gamma$ be any involution. Without loss of generality, we assume τ acts nontrivially on E_1. Then in the rotation numbers of the two fixed points, (a, b) and $(-a, b + a)$, $a = 1$ must be true because τ has order 2, and furthermore, either $b = 0$ or $b = 1$. If $b = 0$, the rotation numbers for the action of τ at the six vertices are

$$(1, 0), (1, -1), (0, -1), (-1, 0), (-1, 1), (0, 1).$$

If $b = 1$, the rotation numbers for the action of τ at the six vertices are

$$(1, 1), (0, -1), (1, 0), (-1, -1), (0, 1), (-1, 0).$$

In the former case, the rotation numbers for the action of $g\tau g^{-1}$ are

$$(0, 1), (1, 0), (1, -1), (0, -1), (-1, 0), (-1, 1),$$

which shows $g\tau g^{-1} \neq \tau$. In the latter case, the rotation numbers for the action of $g\tau g^{-1}$ are

$$(-1, 0), (1, 1), (0, -1), (1, 0), (-1, -1), (0, 1),$$

which also shows $g\tau g^{-1} \neq \tau$. This finishes the proof of the lemma.

<div align="right">q.e.d.</div>

With these preparation, we may now continue our proof according to the alternative by Lemma 3.2

- Case A: $K = \mathbb{Z}_6$, and
- Case B: $K = D_{12}$.

3.2.3. Case A: $K = \mathbb{Z}_6$. We begin with the following observation which follows immediately from the fact that $g^6 = 1$.

Proposition 3.6. *G is a semi-direct product of Γ and K.*

Let $\langle \Gamma, g^2 \rangle$ denote the index 2 subgroup of G generated by Γ and g^2.

Proposition 3.7. *If $K = \mathbb{Z}_6$, then $\langle \Gamma, g^2 \rangle$ is isomorphic to G_n when $k = 1$, and is isomorphic to $G_{n,k,s}$ when $k > 1$, where $n = |\Gamma_1'| = |\tilde{\Gamma}_1|$ and $n/k = |\Gamma_1|$.*

Proof. Let $h_1 \in \Gamma_1$ be the generator whose rotation numbers at the six vertices of the hexagon are

$$(0,1),(1,0),(1,-1),(0,-1),(-1,0),(-1,1),$$

and let $\tilde{h}_1 \in \tilde{\Gamma}_1$ be the generator whose rotation numbers at the six vertices of the hexagon are

$$(1,b),(1+b,-1),(b,-1-b),(-1,-b),(-1-b,1),(-b,1+b).$$

This is can be shown by examining the weight on the first intersection (the one between $H - E_1 - E_3$ and E_1), because on the base it should project to that of Γ_1'. Then $g^2 h_1 g^{-2}$ has rotation numbers

$$(-1,0),(-1,1),(0,1),(1,0),(1,-1),(0,-1),$$

and $g^2 \tilde{h}_1 g^{-2}$ has rotation numbers

$$(-1-b,1),(-b,1+b),(1,b),(1+b,-1),(b,-1-b),(-1,-b).$$

Let $g^2 h_1 g^{-2} = \tilde{h}_1^{kl} h_1^u$ for some l and u (note that $k \cdot ord(\Gamma_i) = ord(\Gamma_i')$). Then comparing the rotation numbers we have

$$(-1,0) = (l, bl) + (0, u),$$

which implies that $l = -1$ and $u = b$. Similarly, let $g^2 \tilde{h}_1 g^{-2} = \tilde{h}_1^m h_1^v$ for some m and v, we have

$$(-1-b,1) = (m, mb) + (0, kv),$$

which implies that $b^2 + b + 1 = kv \pmod{n}$ and $m = -1 - b$. Putting together, we have

$$g^2 h_1 g^{-2} = \tilde{h}_1^{-k} h_1^b, \quad g^2 \tilde{h}_1 g^{-2} = \tilde{h}_1^{-b-1} h_1^v.$$

We will need a different presentation of $\langle \Gamma, g^2 \rangle$. To this end, we let $t_1 = h_1$, $t_2 = \tilde{h}_1^{-1}$, and moreover, we set $s = -b$. Then we have

$$g^2 t_1 g^{-2} = t_2^k t_1^{-s}, \quad g^2 t_2 g^{-2} = t_2^{s-1} t_1^{-v},$$

where $s^2 - s + 1 = kv \pmod{n}$. With this presentation, one can identify the subgroup $\langle \Gamma, g^2 \rangle$ with an imprimitive finite subgroup of $PGL(3)$. More precisely, when $k = 1$, i.e., $\tilde{\Gamma}_1$ and Γ_1 have the same order, we can actually take $\tilde{\Gamma}_1$ to be Γ_2, which corresponds to $b = 0$. Then $s = -b = 0$, and with $k = 1$, we have $v = 1$. In this case. $\langle \Gamma, g^2 \rangle$ is isomorphic to G_n by identifying $t_1 = [\mu_n z_0, z_1, z_2]$, $t_2 = [z_0, \mu_n z_1, z_2]$, and $g^2 = [z_2, z_0, z_1]$. When $k > 1$, $\langle \Gamma, g^2 \rangle$ is isomorphic to the group $G_{n,k,s}$, by identifying $t_1 = [\mu_{n/k} z_0, z_1, z_2]$, $t_2 = [\mu_n^s z_0, \mu_n z_1, z_2]$, and $g^2 = [z_2, z_0, z_1]$. This finishes off the proof of Proposition 3.7.

q.e.d.

3.2.4. Case B: $K = D_{12}$**.** We will first show that G is a semi-direct product of Γ and K.

Lemma 3.8. *There exists an involution in G which is sent to the reflection s_3 under $G \to K$. Moreover,*

- *for any $h \in \Gamma$ and any such involution τ, $h\tau h^{-1} = \hat{h}\tau$, where $h, \hat{h}$ are related as follows: if the rotation numbers of h at the first fixed point on E_1 (according to the orientation of the hexagon which is counter-clockwise) are (a, b), then the rotation numbers of $\hat{h}$ at the first fixed point on E_1 are $(2a, -a)$.*
- *for any two such involutions τ, τ', $\tau' = h\tau$ for some $h \in \Gamma$ whose rotation numbers at the first fixed point on E_1 are $(2a, -a)$ for some a.*

We note that analogous statements hold for the reflections s_2 and $s_2 s_3 s_2$ in K.

Proof. Suppose $\tau \in G$ is sent to s_3 under $G \to K$, then τ leaves the (-1)-spheres E_1 and $H - E_2 - E_3$ invariant. Let $h_1 \in \Gamma_1$ be the generator with rotation numbers $(0, 1)$ at the first fixed point. By examining the rotation numbers, one can check easily that $\tau h_1 \tau^{-1} = h_1$. On the other hand, it is easily seen that $\tau^2 \in \Gamma$ fixes 4 points (two fixed points from τ and two from intersections with other exceptional curves) hence the whole (-1)-sphere E_1. Therefore, $\tau^2 \in \Gamma_1$.

The key observation is that $\tau^2 = h_1^{2b}$ is an even power of h_1. To see this, note that τ has two fixed points on E_1 and their rotation numbers are (a, b) and $(-a, b + a)$ for some a, b. Furthermore, the order of τ must be even, say $2m$. Since τ^2 fixes the (-1)-sphere E_1, we must have $2a = 2m$, and the rotation numbers of τ^2 at the two fixed-points are $(0, 2b)$ and $(0, 2b)$. Comparing with the rotation numbers with h_1, we see easily that $\tau^2 = h_1^{2b}$. With $\tau h_1 \tau^{-1} = h_1$, we see easily that τh_1^{-b} is an involution which is sent to s_3 under $G \to K$.

Now we consider any involution τ which is sent to s_3. For any $h \in \Gamma$, let x, y be the first and second fixed points on E_1, and suppose the rotation numbers of h at x are (a, b). Then $\tau(x) = y$ and $\tau : T_x E_1 \to T_y E_1$. It is easily seen that $h\tau h^{-1} : T_x E_1 \to T_y E_1$ equals $(-a)\tau(-a) = -2a\tau$ (here $(-a)$ means multiplication by $\exp(2(-a)\pi\sqrt{-1}/ord(h))$). Similarly, $h\tau h^{-1} = (a + b)\tau(-b) = a\tau$ in the normal direction. Hence $h\tau h^{-1} = \hat{h}\tau$ for some $\hat{h}$ whose rotation numbers at y are $(-2a, a)$. It follows easily that the rotation numbers of $\hat{h}$ at x are $(2a, -a)$.

Finally, let τ, τ' be any two involutions sent to s_3. We let

$$f = d\tau : T_x X \to T_y X$$

$$g = d\tau' : T_y X \to T_x X.$$

Then

$$g \circ f = d(\tau'\tau) : T_x X \to T_x X$$

$$g^{-1} \circ f^{-1} = d(\tau\tau') : T_y X \to T_y X$$

Note that $\tau'\tau = h \in \Gamma$. Since $g^{-1} \circ f^{-1} = (f \circ (g \circ f) \circ f^{-1})^{-1}$, we see easily that the rotation numbers of h at y are the negative of the rotation numbers at x. If the rotation numbers at x are (c, d), then the rotation numbers at y are $(-c, c + d)$ (the second number in each pair stands for the weight in the normal direction). This gives rise to the relation $c + d = -d$, so that the rotation numbers of h at x are $(-2d, d)$. Setting $d = -a$ we proved that $\tau' = h\tau$ for some $h \in \Gamma$ whose rotation numbers at the first fixed point on E_1 are $(2a, -a)$ for some a, as we claimed.

<div align="right">q.e.d.</div>

Proposition 3.9. *The group G is a semi-direct product of Γ and K.*

Proof. Recall that we fixed an element $g \in G$ which is sent to a counter-clockwise rotation of 60 degrees under $G \to K$. We set $\tau_1 = g^3$, which is an involution by Lemma 3.5. We pick another involution $\tau_3 \in G$ which is sent to $s_3 \in K$ from Lemma 3.8. We shall first show that we can always arrange to have $\tau_1 \tau_3 \tau_1 = \tau_3$.

By Lemma 3.5, $\tau_1 \tau_3 \tau_1 = h'\tau_3$ for some $h' \in \Gamma$ whose rotation numbers at the first fixed point on E_1 are $(2a_3, -a_3)$ for some a_3. On the other hand, if we replace τ_3 by $h\tau_3 h^{-1} = \hat{h}\tau_3$, then from Lemma 3.5

$$\tau_1(\hat{h}\tau_3)\tau_1 = \hat{h}^{-1}\tau_1\tau_3\tau_1 = \hat{h}^{-2}h'(\hat{h}\tau_3).$$

Hence if there exists an $\hat{h} \in \Gamma$ such that $\hat{h}^2 = h'$, which is equivalent to a_3 being even, then we can replace τ_3 by $h\tau_3 h^{-1}$ for an $h \in \Gamma$ to achieve the commutativity property.

To show that a_3 is even, we pick an involution τ_2 which is sent to $s_2 \in K$ (by an analog of Lemma 3.5) and consider $\tau_1 \tau_2 \tau_1$. By the corresponding version of Lemma 3.5, we see that $\tau_1 \tau_2 \tau_1 = \tilde{h}\tau_2$ for some $\tilde{h} \in \Gamma$ whose rotation numbers at the second fixed point on E_1 are $(2a_2, -a_2)$ for some a_2. It follows easily that the rotation numbers of $\tilde{h}$ at the first fixed point on E_1 are (a_2, a_2).

Now $\tau_2 \tau_3$ is sent to a counter-clockwise rotation of 120 degrees under $G \to K$, so that there exists an $h \in \Gamma$ such that $\tau_2 \tau_3 = hg^2$. Now

$$\tau_1(\tau_2\tau_3)\tau_1 = \tilde{h}\tau_2 h'\tau_3 = \tilde{h}h'k\tau_2\tau_3$$

for some $k \in \Gamma$ whose rotation numbers can be determined as follows: since the rotation numbers of h' at the second fixed point on E_1 are $(a_3, -2a_3)$, by the analog of Lemma 3.5 the rotation numbers of k at the second fixed point on E_1 are $(-2a_3, a_3)$. It follows that the rotation

numbers of k at the first fixed point are $(-a_3, -a_3)$. With this under-
stood, note that the rotation numbers of $\tilde{h}h'k$ at the first fixed point on
E_1 are $(a_2 + a_3, a_2 - 2a_3)$. On the other hand,

$$(8) \qquad \tau_1(\tau_2\tau_3)\tau_1 = \tau_1(hg^2)\tau_1 = h^{-1}\tau_1 g^2 \tau_1 = h^{-1}g^2 = h^{-2}\tau_2\tau_3,$$

which implies that both $a_2 + a_3$, $a_2 - 2a_3$ are even. It follows that
a_3 is even, and hence there is an involution τ_3 sent to $s_3 \in K$ with the
property that $\tau_1\tau_3\tau_1 = \tau_3$.

We set $\tau_2 := g\tau_3 g^{-1}$. Then τ_2 is an involution sent to $s_2 \in K$ which
naturally satisfies $\tau_1\tau_2\tau_1 = \tau_2$. We will show that τ_2, τ_3 satisfy the
relation $\tau_2\tau_3\tau_2 = \tau_3\tau_2\tau_3$. Note that with this relation, the subgroup
generated by τ_2, τ_3 is isomorphic to S_3. Together with the involution τ_1,
we obtain a lifting of $K = \mathbb{Z}_2 \times S_3$ in G, proving that G is a semi-direct
product of Γ and K.

As we have seen earlier, $\tau_2\tau_3 = hg^2$ for some $h \in \Gamma$, where $h^2 = 1$
from (8) because both τ_2, τ_3 commute with τ_1. The rotation numbers of
h at the first fixed point on E_1 must be one of the following: (i) $(0, 1)$,
(ii) $(1, 0)$, (iii) $(1, 1)$. We claim that in case (i), we have $\tau_2\tau_3\tau_2 = \tau_3\tau_2\tau_3$.
To see this,

$$\tau_2\tau_3\tau_2 = hg^2\tau_2 = h\tau_3 g^2 = \tau_3 hg^2 = \tau_3\tau_2\tau_3,$$

where we use the fact $h\tau_3 = \tau_3 h$ because the rotation numbers of h are
$(0, 1)$.

It remains to rule out the cases (ii) and (iii). To this end, we compute

$$g\tau_2\tau_3 g^{-1} = g^2\tau_3 g^{-2}\tau_2 = (h\tau_2\tau_3)\tau_3(h\tau_2\tau_3)^{-1}\tau_2 = h\tau_2\tau_3\tau_2 h\tau_2 = hhk\tau_2\tau_3\tau_2\tau_2 = k\tau_2\tau_3,$$

where the rotation numbers of $k \in \Gamma$ can be determined as follows. Note
that $\tau_2\tau_3\tau_2$ is sent to the reflection $s_2 s_3 s_2 \in K$, so the rotation numbers
of k at the first fixed point on $H - E_1 - E_3$ can be determined by an
analog of Lemma 3.5. In case (ii), the rotation numbers of h at the first
fixed point on $H - E_1 - E_3$ are $(0, 1)$, so by an analog of Lemma 3.5, the
rotation numbers of k at the first fixed point on $H - E_1 - E_3$ are $(0, 0)$,
i.e., k is trivial in this case. In case (iii), the rotation numbers of h at
the first fixed point on $H - E_1 - E_3$ are $(1, 0)$, so by an analog of Lemma
3.5, the rotation numbers of k at the first fixed point on $H - E_1 - E_3$
are $(0, 1)$. It follows that the rotation numbers of k at the first fixed
point on E_1 are $(1, 0)$ in this case.

On the other hand, $g(hg^2)g^{-1} = ghg^{-1}g^2 = h'g^2$ for some $h' \in \Gamma$,
where the rotation numbers of h' at the first fixed point on E_1 are $(0, 1)$
in case (ii) and $(1, 0)$ in case (iii). Since in both cases, $h'h \neq k$, so we
reached a contradiction. This ruled out the cases (ii) and (iii), and the
proposition is proved.

<div align="right">q.e.d.</div>

Finally, we show that G contains an index 2 subgroup which is iso-
morphic to an imprimitive subgroup of $PGL(3)$. To this end, we fix a

lifting K' of K to G, and let $g \in K'$ be an element of order 6 and $\tau \in K'$ be the involution sent to $s_3 \in K$. We denote by $\langle \Gamma, g^2, \tau \rangle$ the subgroup generated by the elements in Γ, g^2 and τ.

Proposition 3.10. *Suppose $K = D_{12}$. Then $\langle \Gamma, g^2, \tau \rangle$ is isomorphic to the imprimitive finite subgroup $\tilde{G}_n$ of $PGL(3)$ when $k = 1$, and is isomorphic to $\tilde{G}_{n,3,2}$ when $k > 1$, where $n = |\Gamma_1'| = |\tilde{\Gamma}_1|$ and $n/k = |\Gamma_1|$.*

Proof. Let $h_1 \in \Gamma_1$ be the generator whose rotation numbers at the six vertices of the hexagon are

$$(0, 1), (1, 0), (1, -1), (0, -1), (-1, 0), (-1, 1),$$

and let $\tilde{h}_1 \in \tilde{\Gamma}_1$ be the generator whose rotation numbers at the six vertices of the hexagon are

$$(1, b), (1 + b, -1), (b, -1 - b), (-1, -b), (-1 - b, 1), (-b, 1 + b).$$

Then the rotation numbers of $\tau \tilde{h}_1 \tau$ are

$$(-1, 1 + b), (b, 1), (1 + b, -b), (1, -1 - b), (-b, -1), (-1 - b, b).$$

Writing $\tau \tilde{h}_1 \tau = \tilde{h}_1^l h_1^u$ for some l, u, we get

$$(-1, 1 + b) = (l, lb) + (0, ku),$$

which implies that $l = -1$ and $2b + 1 = ku \pmod{n}$.

(i) Assume $\tilde{\Gamma}_1$ and Γ_1 have the same order n, i.e. $k = 1$. Recall from the proof of Proposition 3.7 that in this case, $b = 0$, so that $u = 1$ and $\tau \tilde{h}_1 \tau = \tilde{h}_1^{-1} h_1$. Renaming $t_1 = h_1$, $t_2 = \tilde{h}_1^{-1}$ as in the proof of Proposition 3.7, we get

$$g^2 t_1 g^{-2} = t_2, \quad g^2 t_2 g^{-2} = t_2^{-1} t_1^{-1}, \quad \tau t_1 \tau = t_1, \quad \tau t_2 \tau = t_2^{-1} t_1^{-1}.$$

With this presentation, the subgroup $\langle \Gamma, g^2, \tau \rangle$ can be identified with $\tilde{G}_n$ by

$$g^2 = [z_2, z_0, z_1], \quad \tau = [z_0, z_2, z_1], \quad t_1 = [\mu_n z_0, z_1, z_2], \quad t_2 = [z_0, \mu_n z_1, z_2].$$

(ii) Assume $\tilde{\Gamma}_1$ and Γ_1 have different orders, i.e., $|\tilde{\Gamma}_1| = n$, $|\Gamma_1| = n/k$, with $k > 1$. In this case, we first note that $2b + 1 = 0 \pmod{k}$. On the other hand, recall from the proof of Proposition 3.7 that $b^2 + b + 1 = 0 \pmod{k}$. It follows that $b = 1 \pmod{k}$ and $k = 3$. With this understood, note that one can modify $\tilde{h}_1$ by a suitable power of h_1 to arrange so that $b = -2$. With this choice, we then have $s = -b = 2$ and $v = 1$, where s, v appear in the relations (see the proof of Proposition 3.7)

$$g^2 t_1 g^{-2} = t_2^k t_1^{-s}, \quad g^2 t_2 g^{-2} = t_2^{s-1} t_1^{-v}.$$

Moreover, $b = -2$ implies $u = -1$, hence the presentation of $\langle \Gamma, g^2, \tau \rangle$:

$$g^2 t_1 g^{-2} = t_2^3 t_1^{-2}, \quad g^2 t_2 g^{-2} = t_2 t_1^{-1}, \quad \tau t_1 \tau = t_1, \quad \tau t_2 \tau = t_2^{-1} t_1$$

With this presentation, the subgroup $\langle \Gamma, g^2, \tau \rangle$ can be identified with $\tilde{G}_{n,3,2}$ by identifying

$$g^2 = [z_2, z_0, z_1], \quad \tau = [z_0, z_2, z_1], \quad t_1 = [\mu_{n/3} z_0, z_1, z_2], \quad t_2 = [\mu_n^2 z_0, \mu_n z_1, z_2].$$

<div align="right">q.e.d.</div>

It is clear that G is a semi-direct product of the imprimitive subgroup of $PGL(3)$ and $\mathbb{Z}_2$. The proof of Theorem 1.3 is completed.

The rest of this section is occupied by the proof of Theorem 1.8, where we assume (X, ω) admits a minimal symplectic G-conic bundle structure $\pi : X \to \mathbb{S}^2$. Furthermore, we assume $N \geq 4$. Recall that Q is the subgroup of G which leaves each fiber of π invariant, and G_0 is the subgroup of Q which acts trivially on $H^2(X)$.

3.3. Proof of Theorem 1.8.
We begin with some useful observations about the rotation numbers of an element of Q at a fixed point.

Rotation numbers of fixed points.

Suppose $q \in X$ is fixed by a nontrivial element $g \in Q$ and q is not the singular point of a singular fiber. Then g must fix the symplectic orthogonal direction of the fiber at q, because g induces a trivial action on the base. It follows that the rotation numbers of g at q are $(a, 0)$ for some $a \neq 0$.

Furthermore, if q lies in a singular fiber, then g must leave the (-1)-sphere containing q invariant, and the rotation numbers of g at the other fixed point, which is the singular point of the singular fiber, must be $(a, -a)$. (See Section 3.2.2)

Structure of fixed point sets.

Note that the fixed-point set of a nontrivial element $g \in Q$ consists of embedded J-holomorphic curves and isolated points. Since each regular fiber contains two fixed points of g, it follows that the fixed-point set of g consists of a bisection and isolated points which are the singular points of those singular fibers of which g leaves each component invariant.

Clearly, one of the following must be true:

(i) g leaves components of a singular fiber invariant.

   Then there is a fixed point q as described above, and the singularity p on the fiber has rotation number $(a, -a)$ hence is an isolated fixed point (and there is one more fixed point on either component of the fiber).

(ii) g switches the 2 components of the singular fiber.

   Then

   a) the singularity p on the fiber is contained in the fixed bisection,

b) p is a branched point of the double branched covering from the bisection to the base,

c) g must be an involution, because if $g^2 \neq 1$, then p must be an isolated fixed point of g^2 from (i), a contradiction.

With the preceding understood, note that the subgroup G_0 can be identified with the subgroup which leaves each (-1)-sphere in a singular fiber invariant. Since there are $N - 1 \geq 3$ singular fibers, the induced action of G_0 on the base $\mathbb{S}^2$ has at least 3 fixed points. It follows that the action of G_0 on the base must be trivial, and G_0 is a subgroup of Q. It is clear that either G_0 is trivial, or it is finite cyclic.

Furthermore, the fixed-point set of each nontrivial element of G_0 consists of $N - 1$ isolated points with rotation numbers $(a, -a)$ for some $a \neq 0$ (these are the singular points of the singular fibers) and two disjoint fixed sections from the two other fixed points other than the singular points on each fiber. The two fixed sections must have the same self-intersection number because there exists a $g \in G$ which switches the two sections. Applying Lemma 2.1 to the fixed sections, we see that each has self-intersection $(N - 1)/2$; in particular, N must be odd if G_0 is nontrivial. Finally, we note that each element of $Q \setminus G_0$ must be an involution (the square fixes 4 points in a general fiber: intersections with the bisection, and the intersections of the two disjoint sections as above).

Lemma 3.11. *The group Q contains an involution; in particular, it has an even order.*

Proof. Let F be the singular fiber which contains the exceptional sphere representing E_N. The minimality assumption implies that there is a $g \in G$ which switches the two (-1)-spheres in F. Clearly, g has an even order, say $2m$. If $g \in Q$, then as we have shown earlier, g must be an involution, and we are done. Suppose $m > 1$. We let $h = g^m$, which is an involution. We claim that $h \in Q$.

Suppose h is not contained in Q. Then h induces a rotation on the base, so that the fixed-point set of h must be contained in the two fibers which h leaves invariant. With this understood, we claim that h must fix one of the (-1)-spheres in F. Denote Σ as the two dimensional component of the fixed point set of h, which could be empty If h does not fix either of the (-1)-spheres in F, we must have $E_N \cdot \Sigma = 0$. On the other hand, by Proposition 5.1 in [**7**], we have

$$(h \cdot E_N) \cdot E_N = E_N \cdot \Sigma \pmod 2.$$

This is a contradiction because $h \cdot E_N = E_N$ or $H - E_1 - E_N$, so that $(h \cdot E_N) \cdot E_N = \pm 1$. Hence h fixes one of the (-1)-spheres in F. Now since g commutes with h and g switches the two (-1)-spheres in F, h must also fix the other (-1)-sphere. But this clearly contradicts the fact that h is nontrivial. Hence $h \in Q$ and the lemma is proved.

<div style="text-align: right">q.e.d.</div>

Proposition 3.12. *If G_0 is nontrivial, say, of finite order $m > 1$, and $Q \neq G_0$, then Q is the dihedral group D_{2m}. Moreover, any involution $\tau \in Q \setminus G_0$ switches the two fixed sections of G_0, hence the two (-1)-spheres in each singular fiber.*

Proof. For any $g \in Q$, since G_0 is normal in Q, g leaves the two fixed sections of G_0 invariant. Note that if g leaves each of the section invariant, then it must fix both of them because the induced action of g on the base is trivial. Consequently, Q/G_0 is either trivial or $\mathbb{Z}_2$, depending on whether there is a $g \in Q$ switching the two fixed sections of G_0. It follows easily that every element in $Q \setminus G_0$ switches the two (-1)-spheres in each singular fiber. Finally, if $Q \neq G_0$, then Q must be the corresponding dihedral group because each element in $Q \setminus G_0$ is an involution. (g^2 preserves all exceptional spheres hence in G_0) This finishes the proof of the proposition.

<div style="text-align: right">q.e.d.</div>

Next we consider the case where G_0 is trivial. Let Σ be the set of singular fibers.

Proposition 3.13. *Suppose G_0 is trivial. Then $Q = \mathbb{Z}_2$ or $(\mathbb{Z}_2)^2$. In the latter case, let τ_1, τ_2, τ_3 be the distinct involutions in Q. Then Σ is partitioned into subsets Σ_1, Σ_2, Σ_3, where Σ_i parametrizes the set of singular fibers of which τ_i leaves each (-1)-sphere invariant, and $\#\Sigma_i \equiv N - 1 \pmod 2$, for $i = 1, 2, 3$.*

Proof. First of all, Q consists of involutions as G_0 is trivial. Suppose $Q \neq \mathbb{Z}_2$, and let $\tau, \tau' \in Q$ be two distinct nontrivial elements. We claim that there is no singular fiber such that both τ, τ' leave both of the (-1)-spheres in this fiber invariant. This is because if there is such a singular fiber, then by examining the action of $\tau\tau'$ at the singular point of the fiber, we see $\tau\tau'$ must be trivial (the rotation numbers at this signular point is $(0, 0)$). But this contradicts the assumption that $\tau \neq \tau'$.

With the preceding understood, let $\tau_1, \cdots, \tau_n$, $n > 1$, be the distinct involutions in Q, and let Σ_i be the set of singular fibers of which τ_i leaves each (-1)-sphere invariant. Then the previous paragraph shows that $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$. On the other hand, suppose $\tau_k = \tau_i\tau_j$, then $\Sigma \setminus (\Sigma_i \cup \Sigma_j) \subset \Sigma_k$, implying

$$\Sigma \setminus (\Sigma_i \cup \Sigma_j) = \Sigma_k.$$

It follows easily that $n = 3$ and $Q = (\mathbb{Z}_2)^2$.

It remains to see that for each i, $\#\Sigma_i = N - 1 \pmod 2$. Consider the fixed-point set S_i of τ_i. Then S_i is a bisection and the projection of S_i onto the base is a double branched covering which ramifies exactly at the singular points of those singular fibers not parametrized by the set Σ_i.

Since the number of ramifications must be even, we have $\#\Sigma_i = N - 1$ (mod 2) as claimed.

<div align="right">q.e.d.</div>

The proof of Theorem 1.8 is completed.

4. Minimality and equivariant symplectic cones

Now let $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, $N \geq 2$, which is equipped with a smooth action of a finite group G. Suppose there is a G-invariant symplectic form ω_0 on X such that the corresponding symplectic rational G-surface (X, ω_0) is minimal. With this understood, we denote by $\Omega(X, G)$ the set of G-invariant symplectic forms on X.

The following is a crucial observation.

Lemma 4.1. *For any $\omega \in \Omega(X, G)$, the canonical class $K_\omega = K_{\omega_0}$ or $-K_{\omega_0}$.*

Proof. The lemma is obvious if $H^2(X)^G$ has rank 1 because $[\omega_0], [\omega]$ and K_ω are proportional.

Assume that $H^2(X)^G$ has rank 2. From Theorem 1.3, $N \neq 6$. Recall also from the proof of Theorem 1.3, there is a reduced basis $H, E_1, \cdots, E_N$ of (X, ω_0) such that $K_{\omega_0} = -3H + E_1 + E_2 + \cdots + E_N$. Now after changing ω by sign and with a further scaling if necessary, we can write $[\omega] = -K_{\omega_0} + bF = (3+b)H - (1+b)E_1 - \cdots - E_N$ for some $b \in \mathbb{R}$. We claim that $[\omega]$ is always a *reduced class* in the sense of [**10**] when $N \geq 9$. To see this, note that the condition $[\omega]^2 > 0$ implies $4b > N - 9 \geq 0$, which implies $b > 0$. This gives $3 + b > 1 + b \geq 1$ and $3 + b = 1 + b + 1 + 1$. Now the conclusion follows from [**10**, Lemma 3.4 (part 5)], which asserts that any symplectic class with reduced form has canonical class K_{ω_0}.

For the case of $N = 5, 7, 8$, we have $b > -1, -\frac{1}{2}$ and $-\frac{1}{4}$, respectively. A complete list of K_0-exceptional class in these cases are given by

- $N = 5$: E_i, $H - E_i - E_j$, $2H - E_1 - \cdots - E_5$;
- $N = 7$: E_i, $H - E_i - E_j$, $2H - E_{i_1} - \cdots - E_{i_5}$, $3H - 2E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4} - E_{i_5} - E_{i_6} - E_{i_7}$;
- $N = 8$: E_i, $H - E_i - E_j$, $2H - E_{i_1} - \cdots - E_{i_5}$,
  $3H - E_{i_1} - 2E_{i_2} - E_{i_3} - E_{i_4} - E_{i_5} - E_{i_6} - E_{i_7}$,
  $4H - 2E_{i_1} - 2E_{i_2} - 2E_{i_3} - E_{i_4} - E_{i_5} - E_{i_6} - E_{i_7} - E_{i_8}$,
  $5H - 2E_{i_1} - 2E_{i_2} - 2E_{i_3} - 2E_{i_4} - 2E_{i_5} - 2E_{i_6} - E_{i_7} - E_{i_8}$,
  $6H - 3E_{i_1} - 2E_{i_2} - 2E_{i_3} - 2E_{i_4} - 2E_{i_5} - 2E_{i_6} - 2E_{i_7} - 2E_{i_8}$.

It is not hard to check that in the respective range of b for each N, all such exceptional classes would have $\omega(E) = (-K_0 + bF) \cdot E > 0$ for any E picked from the above list. This implies that $K_\omega = K_0$ from [**12**, Theorem 3].

<div align="right">q.e.d.</div>

Recall that we have shown that a minimal symplectic rational G-surface where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ admits a minimal symplectic G-conic bundle only when $N \geq 5$ and $N \neq 6$. The following lemma deals with precisely these cases and gives the converse of this fact.

Lemma 4.2. *Let (X, ω) be a symplectic rational G-surface where $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ with $N \geq 5$ and $N \neq 6$. Suppose (X, ω) admits a minimal symplectic G-conic bundle structure. Then for any $\omega' \in \Omega(X, G)$ such that $K_{\omega'} = K_\omega$, the symplectic rational G-surface (X, ω') is minimal.*

Proof. By Lemma 2.2, $H^2(X)^G$ is of rank 2 spanned by K_ω and the fiber class F of the symplectic G-conic bundle. Now suppose to the contrary that (X, ω') is not minimal. Then there is a G-invariant, disjoint union of ω'-symplectic (-1)-spheres $C_1, C_2, \cdots, C_m$. Let $C = C_1 + \cdots + C_m$. Then since $K_{\omega'} = K_\omega$, we have

$$-m = C^2 = K_\omega \cdot C.$$

On the other hand, $C \in H^2(X)^G$, so that

(9) $$C = aK_\omega + bF, \quad a, b \in \mathbb{Z}.$$

The key ingredient for deriving a contradiction is the fact that $F \cdot C \geq 0$, which as a corollary implies that $a < 0$, and so that $F \cdot C > 0$. To see this, Note that F is represented by an embedded J-holomorphic sphere S where J is ω-compatible. On the other hand, since $K_{\omega'} = K_\omega$, ω and ω' have the same set of symplectic (-1)-classes. Hence since the class of each C_i is represented by a ω'-symplectic (-1)-sphere, it is also represented by a ω-symplectic (-1)-sphere. Consequently, the class of C_i can be represented by $\cup_j m_j D_j$ where D_j is a J-holomorphic curve and $m_j > 0$. Now since S is irreducible and $S^2 = 0$, we have $S \cdot D_j \geq 0$ for each j, which implies that $F \cdot C_i \geq 0$ for each i. Hence our claim that $F \cdot C \geq 0$.

With this understood, we consider the pairing of $C = aK_\omega + bF$ with K_ω and then square both sides, we have

$$aK_\omega^2 - 2b = -m$$
$$-m = a^2 K_\omega^2 - 4ab$$

which gives

(10) $$m = -\frac{a^2 K_\omega^2}{2a - 1}.$$

Notice that $m > 0$, hence $K_\omega^2 > 0$, which excludes all cases for $N \geq 9$. Moreover, a^2 and $2a-1$ are co-prime for all $a \leq -1$, therefore, $2a-1 | K_\omega^2$. But this is not possible because of the assumption $N \geq 5$ and $N \neq 6$.

<div style="text-align: right">q.e.d.</div>

Proof of Theorem 1.10

The claim regarding the canonical class is proved in Lemma 4.1. The minimality claims are trivial if $H^2(X)^G$ is of rank 1. For the case $\mathrm{rank}H^2(X)^G > 1$, they follow from Lemma 4.1, Lemma 4.2 for $N \geq 5$ and $N \neq 6$; and Lemma 2.7 for $N = 6$. Theorem 1.3 implies these are the only cases to consider. Combining these results with Theorem 3.8 of [**6**], we cover the case (2) of minimal complex rational G-surfaces.q.e.d.

In Theorem 2.8, the case of complex rational G-surface admits an alternative proof, which we sketch it here. Let X be a minimal complex rational G-surface which is $\mathbb{CP}^2$ blown up at 6 points, and assume X is a conic bundle. Then by Proposition 5.2 of [**6**], X must be Del Pezzo, and hence has a G-invariant monotone Kähler form ω. Part (1) of Lemma 4.3 below implies there are two distinct fiber classes in $H^2(X)^G$, but $N = 6$ contradicts part (2) of the lemma.

We now turn to the proof of Theorem 1.12. Fixing the canonical class K_{ω_0}, we recall that $F \in H^2(X)^G$ is called a **fiber class** if it is the class of the fibers of a symplectic G-conic bundle on X for some G-invariant symplectic form ω with $K_\omega = K_{\omega_0}$.

Lemma 4.3. *Suppose $H^2(X)^G$ has rank 2.*

(1) *If X admits a G-invariant monotone symplectic form, then there are at least two distinct fiber classes in $H^2(X)^G$.*
(2) *Suppose $F, F' \in H^2(X)^G$ are distinct fiber classes. Then $F + F' = -aK_{\omega_0}$ for some integer $a > 0$, and $N = 5, 7$ or 8 where $a = 1, 2, 4$ respectively. In particular, there are at most two distinct fiber classes in $H^2(X)^G$.*

Proof. To prove (1), let ω be a G-invariant monotone form on X, and let F be the fiber class of the G-conic bundle structure obtained from Theorem 1.3. Pick a G-invariant closed 2-form η representing F. Then for sufficiently small $\epsilon \neq 0$, the G-invariant symplectic form $\omega' := \omega + \epsilon\eta$ is non-monotone, and the symplectic G-manifold (X, ω') is minimal (as shown in the proof of Theorem 1.3). By Theorem 1.3, (X, ω') admits a symplectic G-conic bundle with small fiber. An easy check with the symplectic areas shows that for $\epsilon > 0$, the small fiber class (whose area is twice the minimal exceptional spheres) of the symplectic G-conic bundle equals F, but for $\epsilon < 0$, it is not F.

To prove (2), let ω, ω' be the G-invariant symplectic forms associated with the symplectic G-conic bundles whose fiber classes are F, F' respectively. For simplicity, we set $K = K_{\omega_0}$.

Write

(11)
$$F' = -aK + bF,$$

for some $a, b \in \mathbb{Z}$. Note that $K \cdot F = K \cdot F' = -2$, $F^2 = (F')^2 = 0$, and the assumption that $F \neq F'$ implies $a \neq 0$. Then by pairing (11) with K, F and F', respectively, one has $-aK^2 = -4$ and $b = -1$. Therefore, a and K^2 are both divisors of 4, and $F + F' = -aK$.

We claim $2a = F \cdot F' \geq 0$, which implies $a > 0$ and $N = 5, 7, 8$. The point is that F can be represented by an embedded J-holomorphic sphere V with $V^2 = 0$, where J is ω-compatible. Since $K_\omega = K_{\omega'}$, this fact implies the Gromov-Witten invariant of F' is nontrivial, hence F' can be represented by a stable J-holomorphic curves, from which our claim $F \cdot F' \geq 0$ follows easily.

<div align="right">q.e.d.</div>

Lemma 4.4. *Suppose G_0 is nontrivial. Then there is a unique fiber class.*

Proof. We note first that the 2-dimensional fixed point set of G_0 consists of two embedded 2-spheres S_1, S_2 , each with self-intersection $-(N-1)/2$ (in particular, N must be odd). Suppose to the contrary that there are two distinct fiber classes F, F'. Then S_1 (and S_2) is a J-holomorphic section of the corresponding symplectic G-conic bundles with fiber classes F, F', for an appropriate ω or ω'-compatible G-invariant J. This implies that $S_1 \cdot F = S_1 \cdot F' = 1$. On the other hand, by Lemma 4.3, $F + F' = -aK_{\omega_0}$ for some $a > 0$, implying that $K_{\omega_0} \cdot S_1 < 0$. This violates the adjunction formula because $S_1^2 = -(N-1)/2 \leq -2$, and $K_\omega = K_{\omega'} = K_{\omega_0}$.

<div align="right">q.e.d.</div>

Proof of Theorem 1.12

Part (1) and part (2) follow immediately from Lemma 4.3 and Lemma 4.4, and the proof of Theorem 1.3. In the case (2) when there is a unique fiber class F for the G-conic bundle, any G-invariant symplectic form has the form $(3H - E_1 - \cdots - E_m) + bF$. From Theorem 1.3, $\omega(E_1) \geq \omega(E_k)$, hence $b \geq 0$. This also shows $C(X, G, F) \cup C(X, G, F') = C(X, G)$.

To see that $C(X, G, F) \cap C(X, G, F')$ is either empty or consists of classes of G-invariant monotone symplectic forms, let $[\omega] \in C(X, G, F)$ and $[\omega'] \in C(X, G, F')$ such that $[\omega] = [\omega']$. Then $[\omega] = -aK_{\omega_0} + bF$, $[\omega'] = -a'K_{\omega_0} + b'F'$, where $a, a' > 0$ and $b, b' \geq 0$. If both b, b' are non-zero, then $[\omega] = [\omega']$, together with the fact that $F + F'$ is a multiple of K_{ω_0} (cf. Lemma 4.3), would imply that F, F' are linearly dependent, a contradiction.

To see (3), first we note that if $[\omega] \in \hat{C}(X, G, F)$, then for any $\delta > \delta_{\omega, F}$, there is an $\omega' \in \Omega(X, G)$ such that $[\omega'] \in \hat{C}(X, G, F)$ with $\delta_{\omega', F} = \delta$. We can simply take $\omega' := \omega + (\delta - \delta_{\omega, F})\pi^*\eta$, where π is a symplectic G-conic bundle on (X, ω) with fiber class F, and η is an area form on the base of π with total area 1. Secondly, if $\delta_{X, G, F} > 0$, then it

can not be attained. Suppose to the contrary that there is an ω such that $\delta_{\omega,F} = \delta_{X,G,F}$. Then take $0 < \epsilon < \delta_{X,G,F}$ sufficiently small, the G-invariant form $\omega' := \omega - \epsilon\eta$, where η is a G-invariant closed 2-form representing F, is a symplectic form. The condition $\epsilon < \delta_{X,G,F}$ implies that $[\omega'] \in \hat{C}(X,G,F)$, which contradicts the definition of $\delta_{X,G,F}$.

We end this section with a uniqueness result on the subgroup Q from Definition 1.7.

Proposition 4.5. *Suppose (X, G) has a unique fiber class F. Then the normal subgroup Q of G is uniquely determined, i.e., it is independent of $\omega \in \Omega(X, G)$, nor the symplectic G-conic bundle structure on (X, ω) involved in the definition of Q.*

Proof. Let $\omega, \omega' \in \Omega(X, G)$, and let π, π' be symplectic G-conic bundles with fiber class F, and let Q, Q' be the subgroups of G defined using π, π' respectively. Let $H, E_1, \cdots, E_N$ and $H', E_1', \cdots, E_N'$ be a reduced basis associated to π, π' respectively.

Let $Q_{(H,E_i)} = \{g \in G | g \cdot E_j = E_j \text{ or } H - E_1 - E_j\}$. We claim $Q = Q_{(H,E_i)}$. First, it is clear that $Q \subset Q_{(H,E_i)}$. Secondly, if $g \in Q_{(H,E_i)}$, then g leaves each singular fiber invariant. Since the number of singular fibers is $N - 1$ which is greater than 3, the induced action of g on the base $\mathbb{S}^2$ has at least 3 fixed points. This implies that the action of g on the base must be trivial, and $g \in Q$. Hence $Q = Q_{(H,E_i)}$. Similarly, $Q' = Q_{(H',E_i')}$.

If we normalize so that $\omega(F) = \omega'(F) = 2$, then for each $j = 2, \cdots, N$, $\omega(E_j') = \omega'(E_j) = 1$ also. In particular, $H', E_1', \cdots, E_N'$ is a reduced basis for (X, ω). By Lemma 2.6, and the fact that ω is not monotone, we have, for each $j > 1$, $E_j' = E_k$ or $H - E_1 - E_k$ for some $k > 1$. It follows that $H' - E_1' - E_j' = H - E_1 - E_k$ or E_k for the same k. From these relations we see immediately that $Q_{(H,E_i)} = Q_{(H',E_i')}$. Hence $Q = Q'$.

<div align="right">q.e.d.</div>

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WEIMIN CHEN, DEPARTMENT OF MATHEMATICS, UNVIERSITY OF MASSACHUSETTS AT AMHERST, AMHERST, MA 01003

E-mail address: `wchen@math.umass.edu`

TIAN-JUN LI, SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455

E-mail address: `tjli@math.umn.edu`

WEIWEI WU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GA 30606

E-mail address: `weiwei@math.uga.edu`