

Math 235 Practice Midterm 1

Solutions

Q1: (a) $x_1 + x_2 + x_4 = 1$
 $x_3 + 3x_4 = 3$
 $x_4 = 1$
 $2x_1 + 2x_2 + 5x_4 = 5$

(b)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 0 & 5 & 5 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & 3 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

reduced echelon form

(c)

$$\left[\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

pivot positions are circled.

(d) The system has infinitely many solutions.

This is because :

- 1) It is consistent, as the last column is non-pivot column,
- 2) There is a free variable, as the 2nd column is a non-pivot column.

Q2. The vector \vec{v} is a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ if and only if the following equation has a solution:

$$x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 = \vec{v}$$

This equation is equivalent to:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ 0 & -1 & -1 & 2 \end{array} \right].$$

Transform it to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ 0 & -1 & -1 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{array} \right].$$

Since the last column is a pivot column, the system has no solutions. Hence \vec{v} is NOT a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$.

Q3.

$$\begin{bmatrix} 2 & b & -9 & -4 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 1 & 4 & -2 & 1 & 0 \end{bmatrix}$$

← the augmented matrix.

Transform it to reduced echelon form:

$$\begin{bmatrix} 2 & b & -9 & -4 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 1 & 4 & -2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -2 & 1 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 2 & b & -9 & -4 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 4 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & -2 & -5 & -6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 4 & 0 & -3 & 0 \\ 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -35 & 0 \\ 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

The system of equations:

$$x_1 - 35x_4 = 0$$

$$x_2 + 8x_4 = 0$$

$$x_3 - 2x_4 = 0$$

The solution in parametric form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 35 \\ -8 \\ 2 \\ 1 \end{bmatrix} s, \quad s \in \mathbb{R}.$$

$$\underline{\text{Q4. (1) (a)}} \quad A\vec{b} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 5 & -5 \\ -2 & 4 & -8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \times (-2) + 3 \times 1 + (-1) \times 1 \\ 0 \times (-2) + 5 \times 1 + (-5) \times 1 \\ (-2) \times (-2) + 4 \times 1 + (-8) \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence \vec{b} is a solution to $A\vec{x} = \vec{0}$

$$(b) \quad A\vec{b} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 5 & -5 \\ -2 & 4 & -8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ -5 \\ -8 \end{bmatrix}$$

So $A\vec{b} = \vec{0}$ implies

$$(-2) \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ -5 \\ -8 \end{bmatrix} = \vec{0}.$$

This shows that the columns of A , i.e., $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ -8 \end{bmatrix}$,

are linearly dependent, as the above equation provides an explicit linear dependence relation.

(2) The columns of A are not linearly independent, this is because, if $A\vec{x} = \vec{0}$ has free variables, then $A\vec{x}$ must have a nontrivial solution, which gives rise to an explicit linear dependence relation to the columns of A .

Q5. (a) The first statement (i) $\vec{v} = \vec{0}$ is not true, because of the following counterexample:

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation defined by

$$T\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}, \text{ where } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Consider vector $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, such that $\vec{v} \neq \vec{0}$.

One can check

$$T(2\vec{v}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad T(3\vec{v}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The 2nd statement (ii) $T(\vec{v}) = \vec{0}$ is true, This is because

$$T(\vec{v}) = T(3\vec{v} - 2\vec{v}) = T(3\vec{v}) - T(2\vec{v}) = \vec{0} - \vec{0} = \vec{0}.$$

(b) (1) Reflection over y-axis is given by matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ as it sends \vec{e}_1 to $-\vec{e}_1$, and \vec{e}_2 to \vec{e}_2 .

(2) Contraction in y-direction by a factor of $\frac{1}{3}$ and expansion in the x-direction by a factor 2, is given by matrix $\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$, as it sends \vec{e}_1 to $2\vec{e}_1$ and sends \vec{e}_2 to $\frac{1}{3}\vec{e}_2$. So the composition is given by the product matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$\text{Q6. (a)} \quad T(\vec{e}_1) = T(1, 0) = (1 \ -1 \ 3)$$

$$T(\vec{e}_2) = T(0, 1) = (-2 \ 3 \ -2)$$

So the standard matrix for T is

$$\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}$$

$$(b) \quad \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since the last row contains no pivot position,

T is not onto.

(c) T is one to one, because in the echelon form $\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, every column has a pivot position.

Q7. (a) compute AB and BA , and check if $AB=BA$.

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \times 3 + 2 \times 4 + 4 \times (-1) & 20 \\ 4 & 2 \times 3 + (-1) \times 4 + 3 \times (-1) & 15 \\ 14 & 7 \times 3 + 2 \times 4 + 1 \times (-1) & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 7 & 20 \\ 4 & -1 & 15 \\ 14 & 28 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ 7 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 1 + 3 \times 2 & 2 \times 2 + 3 \times (-1) & 2 \times 4 + 3 \times 3 \\ 8 & -4 & 12 \\ (-1) \times 2 + 5 \times 7 & (-1) \times (-1) + 5 \times 2 & (-1) \times 3 + 5 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 1 & 17 \\ 8 & -4 & 12 \\ 33 & 11 & 2 \end{bmatrix}$$

Hence $AB \neq BA$, i.e., A does not commute with B.

(b) The standard matrix associated to the composite

$$T_B \circ T_A \quad \text{(circled)} \quad \text{is } BA = \begin{bmatrix} 8 & 1 & 17 \\ 8 & -4 & 12 \\ 33 & 11 & 2 \end{bmatrix}$$

(c) The standard matrix associated to $T_A \circ T_B$ is

$$AB = \begin{bmatrix} 2 & 7 & 20 \\ 4 & -1 & 15 \\ 14 & 28 & 5 \end{bmatrix}$$

Since $AB \neq BA$, $T_A \circ T_B \neq T_B \circ T_A$.