MATH 563H MIDTERM EXAM SPRING 2018

Name:

Total points: 60. Show all your work !!!

1 (10 pts). Let c be the space curve defined by the parametrization $c(t) = (\sin 2t, \cos 2t, t)$, where $t \in \mathbb{R}$.

(1) Find a reparametrization of c which is by arc-length.
(2) Find the curvature $k(t)$ and the torsion $\tau(t)$ of the space curve c.

2 (20 pts). Let $c(t) = \rho(t)(\cos t, \sin t)$ be a parametrized plane curve, for $t \in \mathbb{R}$, where $\rho(t) = r + \cos t$ for some real number r.

(1) Show that for any r, $c(t)$ is periodic. Moreover, find the period of $c(t)$.
(2) For what values of r, $c(t)$ is not a regular parametrized curve? Explain your reasoning.
(3) Explain why for each of the values of r where $r > 2$, $c(t)$ has at least four vertices?
(4) Suppose $r = 1.5$. Determine whether $c(t)$ is convex. Moreover, what is the winding number of $c(t)$. Explain your reasoning.

3 (10 pts). Let S be the surface (called helicoid) which is parametrized by the map $F : \mathbb{R}^2 \to \mathbb{R}^3$, where

$$F(x, y) = (x \sin y, -x \cos y, y).$$

Show that S is a minimal surface.

4 (20 pts). Let S be the surface defined by the equation $x^2 + y^2 = 1 + z^2$. For any fixed real number θ_0, consider the space curve $c(t) = (\cos\theta_0 - t\sin\theta_0, \sin\theta_0 + t\cos\theta_0, t)$.

(1) Show that for any $t \in \mathbb{R}$, $c(t) \in S$, i.e., $c(t)$ is a smooth curve on the surface S.
(2) Find the normal curvature of $c(t)$.
(3) Determine whether $c(t)$ is a curvature line on S. Explain your reasoning.
(4) Prove, without computing the principal curvatures or the Gauss curvature, that both principal curvatures of S must be nonzero and they must have opposite sign, at every point of the surface.

Math 563H Midterm Solutions

#1. (1) $c'(t) = (2\cos 2t, -2\sin 2t, 1)$

$$\|c'(t)\| = \sqrt{(2\cos 2t)^2 + (-2\sin 2t)^2 + 1^2}$$
$$= \sqrt{4\cos^2 2t + 4\sin^2 2t + 1} = \sqrt{5}$$

So the arc-length parameter

$$\tilde{t} = \int_0^t \sqrt{5}\, ds = \sqrt{5}\, t.$$

$\Rightarrow t = \frac{1}{\sqrt{5}}\tilde{t}$. Hence the reparametrization by arc-length is $\tilde{c}(\tilde{t}) = c\left(\frac{1}{\sqrt{5}}\tilde{t}\right) = \left(\sin\frac{2}{\sqrt{5}}\tilde{t}, \cos\frac{2}{\sqrt{5}}\tilde{t}, \frac{\tilde{t}}{\sqrt{5}}\right)$.

(2) <u>Formulae</u>: $\kappa(t) = \dfrac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}$

$$\tau(t) = \frac{\det(c'(t), c''(t), c'''(t))}{\|c'(t) \times c''(t)\|^2}$$

we compute: $c'(t) = (2\cos 2t, -2\sin 2t, 1)$
$c''(t) = (-4\sin 2t, -4\cos 2t, 0)$
$c'''(t) = (-8\cos 2t, 8\sin 2t, 0)$

$$c'(t) \times c''(t) = (4\cos 2t, -4\sin 2t, -8)$$

$$\det(c'(t), c''(t), c'''(t)) = \begin{vmatrix} 2\cos 2t & -4\sin 2t & -8\cos 2t \\ -2\sin 2t & -4\cos 2t & 8\sin 2t \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -4\sin 2t & -8\cos 2t \\ -4\cos 2t & 8\sin 2t \end{vmatrix} = -32$$

Hence $\kappa(t) = \dfrac{\sqrt{(4\cos 2t)^2 + (-4\sin 2t)^2 + (-8)^2}}{(\sqrt{5})^3} = \dfrac{\sqrt{16+64}}{5\sqrt{5}} = \dfrac{4}{5}$

$$\tau(t) = \frac{-32}{(4\cos 2t)^2 + (4\sin 2t)^2 + (-8)^2} = \frac{-32}{80} = -\frac{2}{5}$$

#2. (1) Since $\cos t$, $\sin t$, and $\rho(t) = r + \cos t$ are all periodic with period 2π, $c(t)$ is periodic with period 2π.

(2) $c'(t) = \rho'(t)\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + \rho(t)\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

Hence $\|c'(t)\| = \sqrt{(\rho'(t))^2 + (\rho(t))^2}$

$$= \sqrt{(-\sin t)^2 + (r+\cos t)^2}$$

$$= \sqrt{\sin^2 t + r^2 + 2r\cos t + \cos^2 t}$$

$$= \sqrt{r^2 + 2r\cos t + 1}$$

Note: $r^2 + 2r\cos t + 1 = (r+\cos t)^2 + 1 - \cos^2 t > 0$ for any t whenever $r \neq \pm 1$. But if $r = 1$, when $t = \pi$, $r^2 + 2r\cos t + 1 = 1^2 + 2 \times 1 \times (-1) + 1 = 0$, and if $r = -1$, when $t = 0$, $r^2 + 2r\cos t + 1 = (-1)^2 + 2(-1) \times 1 + 1 = 0$.

Hence for $r = \pm 1$, $c(t)$ is not regular parametrized.

(3) <u>Formula</u>: the curvature $k(t) = \dfrac{\det(c'(t), c''(t))}{\|c'(t)\|^3}$

$$c'(t) = \rho'(t)\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} + \rho(t)\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}$$

$$c''(t) = \rho''(t)\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} + \rho'(t)\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix} + \rho'(t)\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}$$
$$+ \rho(t)\begin{bmatrix}-\cos t\\ -\sin t\end{bmatrix}$$
$$= \left[\rho''(t) - \rho(t)\right]\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} + 2\rho'(t)\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}$$

So $\det(c'(t), c''(t)) = -\rho(t)\cdot(\rho''(t) - \rho(t)) + 2(\rho'(t))^2$

$$= -(r+\cos t)(-\cos t - r - \cos t) + 2(-\sin t)^2$$
$$= (r+\cos t)(r+2\cos t) + 2\sin^2 t = r^2 + 2 + 3r\cos t$$

Hence $k(t) = \dfrac{r^2+2+3r\cos t}{\left(\sqrt{r^2+2r\cos t+1}\right)^3}$

when $r>2$, $k(t) > 0$ for all t.

On the other hand, when $r>2$, $c(t)$ is simple closed curve. Hence by the Four-Vertex Theorem, $c(t)$ has at least 4 vertices when $r>2$.

(4) Suppose $r=1.5$. In this case,

$$K(0)=\frac{(1.5)^2+2+3\times 1.5\times 1}{\left(\sqrt{1.5^2+2\times 1.5\times 1+1}\right)^3}>0$$

$$K(\pi)=\frac{(1.5)^2+2+3\times 1.5\times(-1)}{\sqrt{(1.5)^2+2\times 1.5\times(-1)+1}}<0$$

So $c(t)$ is not convex.

On the other hand, $c(t)$ is simple closed curve when $r=1.5$, so by Hopf's theorem, the winding number of $c(t)$ is ± 1. Since the parametrization is counter-clockwise, it is $+1$.

#3. $\frac{\partial F}{\partial x}=\begin{bmatrix}\sin y\\ -\cos y\\ 0\end{bmatrix}$, $\frac{\partial F}{\partial y}=\begin{bmatrix}x\cos y\\ x\sin y\\ 1\end{bmatrix}$, so the 1st fundamental form is given by

$$(g_{ij})=\begin{bmatrix}\left\|\frac{\partial F}{\partial x}\right\|^2 & \left\langle\frac{\partial F}{\partial x},\frac{\partial F}{\partial y}\right\rangle\\ \left\langle\frac{\partial F}{\partial x},\frac{\partial F}{\partial y}\right\rangle & \left\|\frac{\partial F}{\partial y}\right\|^2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1+x^2\end{bmatrix}\cancel{=(g_{ij})}$$

For the 2nd fundamental form (h_{ij}):

$$\frac{\partial^2 F}{\partial x^2}=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad \frac{\partial^2 F}{\partial y^2}=\begin{bmatrix}-x\sin y\\ x\cos y\\ 0\end{bmatrix},\quad \frac{\partial^2 F}{\partial x\partial y}=\begin{bmatrix}\cos y\\ \sin y\\ 0\end{bmatrix}$$

The ~~norm~~ unit normal vector:

$$N_p = \frac{\frac{\partial F}{\partial x} \times \frac{\partial F}{\partial y}}{\left\| \frac{\partial F}{\partial x} \times \frac{\partial F}{\partial y} \right\|} = \frac{1}{\sqrt{1+x^2}} \begin{bmatrix} -\cos y \\ -\sin y \\ x \end{bmatrix},$$

$$\frac{\partial F}{\partial x} \times \frac{\partial F}{\partial y} = \begin{vmatrix} i & \sin y & x\cos y \\ j & -\cos y & x\sin y \\ k & 0 & 1 \end{vmatrix} = \begin{bmatrix} -\cos y \\ -\sin y \\ x \end{bmatrix}$$

$$\text{So } (h_{ij}) = \begin{bmatrix} \langle \frac{\partial^2 F}{\partial x^2}, N \rangle & \langle \frac{\partial^2 F}{\partial x \partial y}, N \rangle \\ \langle \frac{\partial^2 F}{\partial x \partial y}, N \rangle & \langle \frac{\partial^2 F}{\partial y^2}, N \rangle \end{bmatrix} = \begin{pmatrix} 0 & \frac{-1}{\sqrt{1+x^2}} \\ \frac{-1}{\sqrt{1+x^2}} & 0 \end{pmatrix}$$

Hence the Weingarten map

$$(w_i^j) = (h_{ij})(g_{ij})^{-1} = \begin{pmatrix} 0 & \frac{-1}{\sqrt{1+x^2}} \\ \frac{-1}{\sqrt{1+x^2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+x^2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{-1}{(\sqrt{1+x^2})^3} \\ \frac{-1}{(\sqrt{1+x^2})^3} & 0 \end{pmatrix}$$

The mean curvature $H(p) = \frac{1}{2} \operatorname{trace} W_p = \frac{1}{2}(0+0) \equiv 0$, for any point $p \in S$, Hence S is minimal surface.

#4. (1) for any t, the coordinates of $c(t)$ are $x = \cos\theta_0 - t\sin\theta_0$, $y = \sin\theta_0 + t\cos\theta_0$, $z = t$. Hence

$$x^2 + y^2 = (\cos\theta_0 - t\sin\theta_0)^2 + (\sin\theta_0 + t\cos\theta_0)^2$$
$$= \cos^2\theta_0 - 2\cos\theta_0\sin\theta_0 t + t^2\sin^2\theta_0 + \sin^2\theta_0 + 2t\cos\theta_0\sin\theta_0 + t^2\cos^2\theta_0$$
$$= 1 + t^2 = 1 + z^2. \quad \text{Hence } c(t) \in S, \ \forall t.$$

(2) $c'(t) = (-\sin\theta_0, \cos\theta_0, 1)$

$c''(t) = 0, \quad \forall t.$

Hence ~~the~~ curvature $\kappa(t) = \dfrac{\| c'(t) \times c''(t) \|}{\| c'(t) \|^3} = \dfrac{\| c'(t) \times 0 \|}{\| c'(t) \|^3} = 0$

Hence $\kappa_{nor}(t) = 0, \quad \forall t.$

(3) Let $F(x,y,z) = x^2+y^2-z^2-1$. Then $S = \{(x,y,z) \mid F(x,y,z) = 0\}$

Hence the unit normal vector N is given by

$$N = \begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{bmatrix} \Bigg/ \left\| \begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{bmatrix} \right\| = \begin{bmatrix} 2x \\ 2y \\ -2z \end{bmatrix} \Bigg/ \sqrt{(2x)^2+(2y)^2+(-2z)^2}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \begin{bmatrix} x \\ y \\ -z \end{bmatrix} = \frac{1}{\sqrt{1+2z^2}} \begin{bmatrix} x \\ y \\ -z \end{bmatrix}$$

Along the curve $c(t)$,

$$N_{c(t)} = \begin{bmatrix} \cos\theta_0 - t\sin\theta_0 \\ \sin\theta_0 + t\cos\theta_0 \\ -t \end{bmatrix} \cdot \frac{1}{\sqrt{1+2t^2}}$$

$$\frac{d}{dt}\left(N_{c(t)}\right) = -\frac{1}{2} \cdot \frac{4t}{(\sqrt{1+2t^2})^3} \begin{bmatrix} \cos\theta_0 - t\sin\theta_0 \\ \sin\theta_0 + t\cos\theta_0 \\ -t \end{bmatrix} + \frac{1}{\sqrt{1+2t^2}} \begin{bmatrix} -\sin\theta_0 \\ \cos\theta_0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{(\sqrt{1+2t^2})^3} \left(2t \begin{bmatrix} \cos\theta_0 - t\sin\theta_0 \\ \sin\theta_0 + t\cos\theta_0 \\ -t \end{bmatrix} + (1+2t^2) \begin{bmatrix} -\sin\theta_0 \\ \cos\theta_0 \\ -1 \end{bmatrix} \right)$$

$$= \frac{1}{\left(\sqrt{1+2t^2}\right)^3} \begin{bmatrix} \cancel{-\sin\theta_0} + 2t\cos\theta_0 - (4t^2 \cancel{+1})\sin\theta_0 \\ 2t\sin\theta_0 + (4t^2+1)\cos\theta_0 \\ -(4t^2+1) \end{bmatrix}$$

$$= \frac{4t^2+1}{\left(\sqrt{1+2t^2}\right)^3} \begin{bmatrix} -\sin\theta_0 + \frac{2t}{4t^2+1}\cos\theta_0 \\ \cos\theta_0 + \frac{2t}{4t^2+1}\sin\theta_0 \\ -1 \end{bmatrix}$$

Since for any t, $\frac{d}{dt} N_{c(t)}$ is not a multiple of $c'(t)$, $c(t)$ is not a curvature line.

(4) For any $p \in S$, one can choose a θ_0 such that p is contained in the curve $c(t)$. Denote by $X \in T_pS$ which is tangent to $c(t)$ at p, ~~Then~~ such that $\|X\| = 1$. (e.g, choose $X = \frac{c'(t)}{\|c'(t)\|}$.) Then $K_{nor}(t) \equiv 0$ implies that $\Pi_p(X,X) = 0$. Now let k_1, k_2 be the principal curvatures at p, let φ be the angle between X and the principle curvature direction X_1. Then

$$\Pi_p(X,X) = k_1\cos^2\varphi + k_2\sin^2\varphi = 0.$$

If both $k_1, k_2 > 0$, or both $k_1, k_2 < 0$, this is not possible. If one of k_1, k_2 is zero, say $k_1 = 0$, then $k_2\sin^2\varphi = 0$ implies $\varphi = 0$, so that $X = X_1$ is a principal curvature direction. But this contradicts the calculation in (3). Hence k_1, k_2 are both nonzero, with opposite sign.