HOMEWORK 5

1. Let S be a surface of revolution, given with the following standard parametrization

$$F(\theta, z) = (r(z)\cos\theta, r(z)\sin\theta, z),$$

for some smooth function $r(t) > 0$, $t \in [a, b]$. We assume S is given with the first fundamental form as the Riemannian metric on S.

(1) Write down the geodesic equations in the coordinates (θ, z), and show that every longitude $\theta = constant$ satisfies the geodesic equations after appropriately parametrized.

(2) Compute the geodesic curvature of each latitude $z = constant$, and show that a latitude is a geodesic exactly when $r'(z) = 0$.

2. Let $S = \{(u, v) \in \mathbb{R}^2 | u^2 + v^2 < 1\}$ be the unit disc, given with the Riemannian metric g, where $g_{ij}(u, v) = \frac{4}{(1-(u^2+v^2))^2}\delta_{ij}$, and let $\tilde{S} = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ be the upper half plane, given with the Riemannian metric $\tilde{g}$, where $\tilde{g}_{ij}(x, y) = \frac{1}{y^2}\delta_{ij}$. (Here (S, g), $(\tilde{S}, \tilde{g})$ are called the Poincaré disc model and Poincaré half-plane model of the hyperbolic geometry.)

(1) Define a diffeomorphism $f : S \to \tilde{S}$, where in complex variables $w = u + iv$ and $z = x + iy$, f is given by the formula $w \mapsto z = (-i) \cdot \frac{w+i}{w-i}$. Show that f is an isometry between (S, g) and $(\tilde{S}, \tilde{g})$.

(2) We showed in class that the geodesics in the Poincaré half-plane model $(\tilde{S}, \tilde{g})$ are given by vertical lines $x = x_0$ and the semi-circles $(x - x_0)^2 + y^2 = R^2$. Use this information to give a description of the geodesics in the Poincaré disc model (S, g).

3. In a Riemann normal coordinates, express the Riemann curvature tensor R_{ijk}^l at $(0, 0)$ in terms of the 2nd partial derivatives of g_{ij} at $(0, 0)$. Furthermore, use this to do the following two parts:

(1) Show that the Riemann curvature tensor is completely determined by the Gauss curvature. More precisely, they are related by the following equation: for any vectors $v, w, z \in T_pS$,

$$R(v, w)z = K(p)(g_p(w, z)v - g_p(v, z)w).$$

(2) Show that for any vectors $v, w, x, y \in T_pS$,

$$g_p(R(v, w)x, y) = g_p(R(x, y)v, w).$$

4. Suppose in a geodesic polar coordinates centered at a point $p \in S$, the function $f(r, \phi) = f(r)$ is constant in the variable ϕ. For any $r_0 > 0$, let c be the circle defined by the equation $r = r_0$, which is oriented such that ϕ is increasing according to the orientation.

(1) Compute the geodesic curvature of c for any given $r_0 > 0$.

(2) Show that the holonomy around c is given by a counterclockwise rotation of an angle which equals $\int_{D(r_0)} K dA$, where $D(r_0)$ is the disc in S that is bounded by c, K is the Gauss curvature, and dA is the area element on S.