HOMEWORK 2

1. Let S be the sphere of radius R in $\mathbb{R}^3$ which is centered at the origin. For any point $p \in S$, determine the Gauss map $N : S \to \mathbb{S}^2$ and then the Weingarten map $W_p; = -dN_p : T_pS \to T_pS$. Use it to compute the Gauss curvature and the mean curvature at p. Here S is oriented by the standard unit normal vector field N (which is outward-pointing).

2. Let $S \subset \mathbb{R}^3$ be the graph of the function $f(x, y) = 1+xy$. We give S the standard parametrization $F(x, y) = (x, y, f(x, y))$, and we orient S by the standard unit normal vector field N associated to the parametrization. Use the definition of the Weingarten map to compute, for any $u = (x, y)$, $p = F(u) \in S$, the 2×2 matrix which represents the Weingarten map $W_p : T_pS \to T_pS$ with respect to the basis $D_uF(e_1), D_uF(e_2)$ associated to the parametrization. Use this to compute the Gauss curvature and the mean curvature of S.

3. Let $S = \{(x, y, z) \in \mathbb{R}^3 | (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\}$.

(1) Show that S is a regular surface in $\mathbb{R}^3$.
(2) Describe and sketch the surface S in $\mathbb{R}^3$.
(3) Use your geometric intuition (e.g. the meaning of the Weingarten map) to guess which points on S are elliptic, hyperbolic, or parabolic (i.e., where the Gauss curvature is positive, negative or zero). Pick some examples of such points and confirm your assertion by explicitly calculating the Gauss curvature at these points.

4. Let $S \subset \mathbb{R}^3$ be the graph of the function $f(x, y) = 1 - x^2 - y^2$. We consider the parametrization $F(r, \theta) = (r \cos\theta, r \sin\theta, 1 - r^2)$, where $r > 0$, of S, and we orient S by the standard unit normal vector field N associated to the parametrization.

(1) Compute the first and second fundamental forms of S with respect to the given parametrization (i.e., the corresponding 2×2 matrices which represent the first and second fundamental forms).
(2) Use (1) to compute the 2×2 matrix which represents the Weingarten map at each $p \in S$ where $r > 0$.
(3) Compute the principal curvatures of S at each $p \in S$.
(4) Describe the curvature lines on S going through each point $p \in S$ where $r > 0$. Furthermore, verify that if $c(t)$ is a curvature line on S, then the tangent vector of the smooth curve $N_{c(t)} \subset \mathbb{S}^2$ satisfies the following equation

$$\frac{d}{dt} N_{c(t)} = -\kappa(t)c'(t),$$

where $\kappa(t)$ is the principal curvature at $c(t)$ in the direction given by the tangent vector of $c(t)$.