Hurwitz-type Theorems for Automorphisms of Smooth 4-manifolds

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Theorem (Hurwitz, 1893) Let $C$ be a complex curve of genus $g \geq 2$. Then the automorphism group $\text{Aut}(C)$ satisfies

$$|\text{Aut}(C)| \leq 84(g - 1) = 42 \deg K_C.$$ 

Furthermore, the bound $84(g - 1)$ is optimal.

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Some Examples

The Hurwitz curve of the smallest genus is given by the Klein quartic $C_{Klein}$, which has genus 3. Furthermore,

$$\text{Aut}(C_{Klein}) \cong PSL(2,7),$$

which is the unique simple group of order 168.

Note: $168 = 84(3 - 1)$.

A projective model of $C_{Klein}$: $xy^3 + yz^3 + zx^3 = 0$. (Klein, 1878)

The next Hurwitz curve has genus 7, which is called the Macbeath curve, with automorphism group $PSL(2,8)$ of order 504. The next smallest Hurwitz group has order 1092, with three distinct Hurwitz curves of genus 14 (called the first Hurwitz triplet). There are infinitely many Hurwitz groups (e.g., the alternating groups $A_n$ are Hurwitz groups for large $n$).
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Generalization of Hurwitz’s theorem to complex surfaces

Theorem (G. Xiao, 1994, 1995) Let $X$ be a minimal complex surface of general type. Then

$$|Aut(X)| \leq (42)^2 c_1^2(K_X).$$

Remarks 1. The bound is optimal: let $C$ be a Hurwitz curve, then $X = C \times C$ attains the bound because

$$|Aut(C \times C)| = 2 \cdot |Aut(C)|^2, \quad c_1^2(K_{C \times C}) = 2 \cdot (\deg K_C)^2.$$

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Topological version of Hurwitz’s theorem

Theorem: Let $G$ be a finite group acting on an orientable surface of genus $g \geq 2$ by orientation-preserving homeomorphisms. Then

$$|G| \leq 84(g - 1).$$

Topological version of Xiao’s theorem?

1. In Xiao’s theorem, the bound $(42)^2 c_1^2(K_X)$ is topological:

$$c_1^2(K_X) = 2\chi(X) + 3 \text{Sign}(X) \leq 5(1 + b_1(X) + b_2(X)).$$

2. Two distinct categories and four distinct classes of 4-manifolds:

$$\{\text{Topological}\} > \{\text{Smooth}\} > \{\text{Symplectic}\} > \{\text{Kähler}\}.$$
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**Our Goal:** Generalizations of Xiao’s theorem to symplectic 4-manifolds, or even more generally, to smooth 4-manifolds.

A smooth, oriented 4-manifold $X$ is called symplectic if it admits a symplectic structure $\omega$, i.e., a closed, non-degenerate 2-form, such that $\omega^2$ orients the manifold. Kähler surfaces are natural examples of symplectic 4-manifolds, with the Kähler form being the symplectic structure. Symplectic 4-manifolds are prominent in the study of differential topology in dimension 4.

**Reformulation of Xiao’s theorem:** two issues

1. The automorphism group $\text{Aut}(X)$ needs to be replaced by a finite group $G$ acting on $X$.

2. The "general type" condition – an appropriate substitute needed.
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Kodaira dimension of symplectic 4-manifolds: Let \((X, \omega)\) be a symplectic 4-manifold and let \((X_{\text{min}}, \omega_{\text{min}})\) be a symplectic minimal model of \((X, \omega)\). Then the Kodaira dimension \(\kappa(X, \omega)\) is defined as follows according to the following four possibilities:

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\kappa(X, \omega) = \begin{cases} 
-\infty & \text{if } K_{X_{\text{min}}} \cdot [\omega_{\text{min}}] < 0 \text{ or } K_{X_{\text{min}}}^2 < 0 \\
0 & \text{if } K_{X_{\text{min}}} \cdot [\omega_{\text{min}}] = 0 \text{ and } K_{X_{\text{min}}}^2 = 0 \\
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Theorem (Consequence of Taubes’ "SW=Gr"): \(\kappa(X, \omega)\) is well-defined, coincides with the complex version when \((X, \omega)\) is Kähler, and depends only on the diffeomorphism type of \(X\).

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\[ |G| \leq c \cdot c_1^2(K_X)? \]

Some evidence

Fact: Let \((X, \omega)\) be a symplectic 4-manifold, and suppose the circle group \(S^1\) embeds in the symplectomorphism group \(\text{Symp}(X, \omega)\). Then

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Question 2 (smooth Hurwitz-Xiao): Let $X$ be a smooth 4-manifold which supports no smooth $S^1$-actions. Is there a constant $C > 0$ such that there are no smooth $\mathbb{Z}_p$-actions of prime order on $X$ for $p > C$?

Remarks: 1. We will call such a constant $C$ a Hurwitz-type bound for smooth $\mathbb{Z}_p$-actions. Besides its existence, it is also an interesting question as what structures of $X$, e.g., homology, homotopy, smooth structure, etc., a Hurwitz-type bound may depend on.

2. No Hurwitz-type bound exists for locally linear, topological $\mathbb{Z}_p$-actions on 4-manifolds (this follows from work of R. Fintushel and A. Edmonds).

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Theorem (Chen, 2011) Let $X$ be a compact complex surface with $b_2^+(X) > 0$ which does not admit any smooth $S^1$-actions. Then for any holomorphic $Z_p$-actions of prime order on $X$, the order $p$ obeys

$$p \leq C := c \cdot (1 + b_1(X) + b_2(X) + |\text{Tor } H_2(X)|),$$

where $c > 0$ is a universal constant.

Theorem (Chen, 2011) For any prime number $p > 3$, there is a symplectic 4-manifold $X_p$ with the following properties:
- $X_p$ is homeomorphic to $CP^2 \# 9(-CP^2)$;
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Remarks: There exists a symplectic structure $\omega_p$ on $X_p$ with $[\omega_p] \in H^2_{dR}(X_p)$ integral, such that $K_{X_p} \cdot [\omega_p] \geq 2p - 1$. 
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**Remarks:**

1. The above bound is weaker than the one in the symplectic Hurwitz-Xiao (i.e., Question 1), but it is valid under a weaker assumption.

2. The symplectic Hurwitz-Xiao (i.e. Question 1) is still open: the examples of symplectic 4-manifolds $(X_p, \omega_p)$ mentioned earlier do not give counterexamples because $\kappa(X_p, \omega_p) = 1$, and furthermore, these examples show that the condition $\kappa(X, \omega) = 2$ is necessary.
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Theorem (Chen, 2014) For each integer $n > 1$, there is a smooth 4-manifold $X_n$ which has the following properties.

- $X_n$ has the same integral homology, intersection form and Seiberg-Witten invariant as the Kodaira-Thurston manifold;
- $X_n$ supports no smooth $S^1$-actions;
- $X_n$ admits a smooth $Z_n$-action.

Remarks: 1. The Kodaira-Thurston manifold is the smooth 4-manifold $S^1 \times M^3$, where $M^3 = [0, 1] \times T^2 / \sim$ with $(0, x, y) \sim (1, x + y, y)$. It is a complex non-Kähler surface, and furthermore, it is a symplectic 4-manifold. In particular, it has nonzero Seiberg-Witten invariant.

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**Remarks**: By a theorem of Atiyah-Hirzebruch, a simply connected 4-manifold with even intersection form and non-zero signature does not support any smoothable $S^1$-actions.

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Thank You!

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