MATH 705: PART 3: ASPECTS OF CONTACT GEOMETRY

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Contents

1. Basic notions and examples

Definition 1.1. (1) Let N be a $(2n+1)$-dimensional manifold, $\xi \subset TN$ be a hyperplane distribution. We say ξ is a **contact structure** on N if for any $p \in N$, there is a local defining 1-form α near p (i.e., $\xi = \ker \alpha$) such that $d\alpha|_\xi$ is non-degenerate (equivalently, $\alpha \wedge (d\alpha)^n \neq 0$ near p). The pair (N, ξ) is called a **contact manifold**. A contact structure ξ is called **co-oriented** if the line bundle TN/ξ is trivial and is oriented. This condition is equivalent to the existence of a global defining 1-form α, which is unique up to multiplication by a positive function. The 1-form α is called a **contact form** associated to the contact structure ξ. A co-oriented contact structure ξ (with a contact form α) is also canonically oriented by $(d\alpha)^n$, and determines a canonical orientation on N by the volume form $\alpha \wedge (d\alpha)^n$.

(2) Let ξ be a co-oriented contact structure on N, and α be a contact form associated to ξ. The **Reeb vector field** is the vector field R_α on N uniquely determined by

$$i_{R_\alpha} d\alpha = 0, \;\; \alpha(R_\alpha) = 1.$$

Exercise: Let N be a $(2n+1)$-dimensional manifold where n is odd. Show that if N admits a contact structure, then N must be orientable.

Exercise: Show that after passing to a double cover of N if necessary, every contact structure on N can be made co-oriented.

Example 1.2. (Euclidean spaces). Let $x_1, \cdots, x_n$, $y_1, \cdots, y_n$, z be the coordinates on $\mathbb{R}^{2n+1}$. Then the hyperplane distribution $\xi_0 := \ker \alpha_0$, where

$$\alpha_0 = dz - \sum_{j=1}^{n} y_j dx_j$$

is a co-oriented contact structure on $\mathbb{R}^{2n+1}$, called the **standard contact structure**. Let's visualize ξ_0 for the case of $\mathbb{R}^3$. Let x, y, z be the coordinates on $\mathbb{R}^3$. Then $\alpha_0 = dz - ydx$. It is easily seen that

$$\xi_0 = \bigcup_{(x,y,z)\in\mathbb{R}^3} \{a\partial_y + b(\partial_x + y\partial_z) | a, b \in \mathbb{R}\}.$$

The associated Reeb vector field $R_{\alpha_0} = \frac{\partial}{\partial z}$.

Example 1.3. (1-jet bundles). Let L be a n-dimensional manifold, and let $N := T^*L \times \mathbb{R}$ be the 1-jet bundle over L. Then

$$\alpha = dz - \lambda$$

is a contact form on N, where z is the coordinate on the $\mathbb{R}$ factor and λ is the canonical 1-form on the T^*L factor. The associated Reeb vector field $R_\alpha = \frac{\partial}{\partial z}$.

Note that Example 1.2 is a special case of Example 1.3. The following exercise is the general construction behind these two examples.

Exercise: Let (W, ω) be a symplectic manifold where $\omega = d\lambda$. Let $N := \mathbb{R} \times W$. Show that $\alpha := dz + \lambda$, where z is the coordinate on the $\mathbb{R}$ factor, is a contact form on N. (Note a similar construction for $N := \mathbb{S}^1 \times W$.)

Example 1.4. (Principal $\mathbb{S}^1$-bundles). Let N be a circle bundle over Σ. Then for any connection 1-form $A \in \Omega^1(N; i\mathbb{R})$ of the bundle, $F_A = dA$ descents to a 2-form on Σ such that the first Chern class of the circle bundle $\pi : N \to \Sigma$ is represented by $iF_A/2\pi$. Now suppose that the first Chern class of the circle bundle $\pi : N \to \Sigma$ is represented by a 2-form ω which is a symplectic structure on Σ. Then one can choose a connection 1-form A such that $F_A = -2\pi i \pi^*\omega$. We set

$$\alpha := -iA.$$

Then α is a contact 1-form on N with $d\alpha = -2\pi\pi^*\omega$. Note that the associated Reeb vector field R_α is simply the vector field which generates the $\mathbb{S}^1$-action. One basic example of the above construction is given by the Hopf fibration $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$, where the first Chern class is represented by the Kähler form of the Fubini-Study metric on $\mathbb{CP}^n$.

Exercise: Let N be an oriented Seifert 3-manifold. Show that N admits a contact structure ξ which is transversal to the fibers, invariant under the $\mathbb{S}^1$-action, and defines a compatible orientation on N if and only if the Euler number of the Seifert fibration on N is negative.

Example 1.5. (Space of contact elements) Let B be a n-dimensional manifold. A contact element of B is a pair (b, H_b), where $b \in B$ and $H_b \subset T_bB$ is a hyperplane.

We let N be the space of contact elements of B. Then N is a naturally a $(2n-1)$-dimensional manifold, with a natural projection $\pi : N \to B$ sending (b, H_b) to b. The fiber of π is diffeomorphic to $\mathbb{RP}^{n-1}$.

There is a natural (tautological) contact structure ξ on N, which is defined as follows: for any $p = (b, H_b) \in N$, $\xi_p \subset T_pN$ is defined to be $\pi_{p,*}^{-1}(H_b)$, where $\pi_{p,*} : T_pN \to T_bB$ is the differential of $\pi : N \to B$ at p. To see ξ is a contact structure, we identify N naturally with $\mathbb{P}(T^*B)$, the projectization of T^*B, where $(b, H_b) \in N$ is identified with $(b, [\sigma_b]) \in \mathbb{P}(T^*B)$. Here $\sigma_b \in T_b^*B$ such that $H_b = \ker(\sigma_b)$, and $[\sigma_b] \in \mathbb{P}(T_b^*B)$ is the corresponding element. With this understood, we regard ξ as a hyperplane distribution of $\mathbb{P}(T^*B)$. Now for any $(b, [\sigma_b]) \in \mathbb{P}(T^*B)$, let $q_1, q_2, \cdots, q_n$ be local coordinates near b, let $p_1, p_2, \cdots, p_n$ be the corresponding coordinates on the cotangent spaces. Moreover, suppose $\sigma_b = p_1dq_1 + p_2dq_2 + \cdots + p_ndq_n$ where $p_1 \neq 0$. Then $(q_1, q_2, \cdots, q_n, p_2, \cdots, p_n)$ is a local coordinate system near $(b, [\sigma_b])$ where $[\sigma_b]$ is identified with $dq_1 + p_2dq_2 + \cdots + p_ndq_n$. Then near $(b, [\sigma_b])$,

$$\xi = \ker(dq_1 + p_2dq_2 + \cdots + p_ndq_n),$$

from which it follows easily that ξ is a contact structure.

Exercise: In the above example,
(1) Show that the contact structure ξ on N is not co-oriented.
(2) For $B = \mathbb{S}^2, \mathbb{T}^2$, determine $\mathbb{P}(T^*B)$ and describe the contact structure ξ on it.

Let (N, ξ) be a contact manifold. A diffeomorphism $\psi \in \mathrm{Diff}(N)$ is called a **contactomorphism** if ψ preserves the hyperplane distribution ξ, i.e., $\psi_*(\xi) = \xi$. If ξ is co-oriented, with a contact form α, then equivalently, ψ is a contactomorphism if there is a smooth function h such that $\psi^*\alpha = e^h\alpha$. A **contact isotopy** is a smooth family ψ_t of contactomorphisms, $t \in [0, 1]$, where $\psi_0 = id$, $\psi_1 = \psi$. In case ξ is co-oriented with a contact form α, $\psi_t^*\alpha = e^{h_t}\alpha$ for a smooth family of functions h_t.

Consider a time-dependent vector field X_t and suppose a smooth family of diffeomorphisms ψ_t is generated by X_t, i.e.,

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_0 = \mathrm{id}.$$

Lemma 1.6. *Suppose ξ is co-oriented with contact form α. Then ψ_t is a contact isotopy if and only if $L_{X_t}\alpha = g_t\alpha$ for some time-dependent smooth functions g_t.*

Proof. Suppose $L_{X_t}\alpha = g_t\alpha$ for some time-dependent smooth functions g_t. Then

$$\frac{d}{dt}\psi_t^*\alpha = \psi_t^* L_{X_t}\alpha = \psi_t^* g_t \cdot \psi_t^*\alpha,$$

so that $\psi_t^*\alpha = e^{h_t}\alpha$ where $h_t = \int_0^t \psi_s^* g_s ds$. Hence ψ_t is a contact isotopy. Conversely, if ψ_t is a contact isotopy with $\psi_t^*\alpha = e^{h_t}\alpha$, then $L_{X_t}\alpha = g_t\alpha$ where $g_t = (\psi_t^{-1})^* \frac{d}{dt}h_t$. ∎

A vector field X on a contact manifold (N, ξ) is called a **contact vector field** if $L_X\alpha = g\alpha$ for a contact form α associated to ξ, where g is a smooth function on N. By Lemma 1.6, the (local) flow generated by a contact vector field X is a (local) contact isotopy. Furthermore, the time-dependent vector field X_t generating any contact isotopy ψ_t is a contact vector field for each t. Note that the Reeb vector field R_α is a contact vector field as $L_{R_\alpha}\alpha = 0$.

Lemma 1.7. *Fix a contact form α. There is a 1-1 correspondence between contact vector fields and smooth functions: for any smooth function H, the contact vector field X_H corresponding to H is given by*

$$X_H = H \cdot R_\alpha + Y,$$

where $Y \in \xi$ is uniquely determined by $i_Y d\alpha = -dH + R_\alpha(H) \cdot \alpha$. Furthermore, $L_{X_H}\alpha = R_\alpha(H) \cdot \alpha$.

Note that Y is tangent to the level sets of H, i.e., $i_Y dH = 0$. Moreover, the Reeb vector field R_α corresponds to $H \equiv 1$. The function H is called the **Hamiltonian** associated to the contact vector field. Combining with Lemma 1.6, we note that any contact isotopy ψ_t is given by a time-dependent Hamiltonian H_t.

Proof. First, it is straightforward to check that for any smooth function H, $L_{X_H}\alpha = R_\alpha(H) \cdot \alpha$, which implies that X_H is a contact vector field. On the other hand, let X be any contact vector field. We may write $X = H \cdot R_\alpha + Y$ for some smooth function H and $Y \in \xi$. Then $L_X\alpha = g\alpha$ gives

$$d(H) + i_Y d\alpha = g\alpha.$$

Applying both sides to R_α, we obtain $g = R_\alpha(H)$. It follows also that Y is determined by $i_Y d\alpha = -dH + R_\alpha(H)\alpha$. Consequently, $X = X_H$. □

Exercise: Show that X_H is the Reeb vector field associated to some contact form if and only if $H > 0$ everywhere on N.

Let M be a complex manifold of complex dimension n, $N \subset M$ be a compact, real co-dimension 1 submanifold. Let $\rho : M \to \mathbb{R}$ be a smooth function such that N is a regular level surface of ρ. We say N is **strictly pseudoconvex** if for any $p \in N$, there are local holomorphic coordinates $z_1, z_2, \cdots, z_n$ near p such that the $n \times n$ Hermitian matrix $(\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k})$ is positive definite at p.

Lemma 1.8. *Let N be a strictly pseudoconvex hypersurface of a complex manifold M, and let $\xi := TN \cap J(TN)$ be the hyperplane distribution consisting of complex tangents. Then ξ is a co-oriented contact structure on N.*

Proof. Let $\alpha := -d\rho \circ J$ be the 1-form on M. Then $\alpha = -(\partial\rho + \bar{\partial}\rho) \circ J = -i(\partial\rho - \bar{\partial}\rho)$. In local holomorphic coordinates $z_1, z_2, \cdots, z_n$, $\alpha = -i(\sum_{j=1}^n \frac{\partial\rho}{\partial z_j} dz_j - \frac{\partial\rho}{\partial \bar{z}_j} d\bar{z}_j)$ so that

$$\omega := d\alpha = 2i \sum_{k,j=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

By the assumption that N is strictly pseudoconvex and the fact that N is compact, it follows easily that ω is a Kähler form in a regular neighborhood of N.

With this understood, note that $\xi = \ker(\alpha|_{TN})$. On the other hand, $d\alpha = \omega$ is non-degenerate on ξ because ξ is complex linear and ω is Kähler. This shows that ξ is a co-oriented contact structure on N.

□

Exercise: Show that the sphere of radius r in $\mathbb{C}^n$, for any $r > 0$, is a strictly pseudoconvex hypersurface.

Let (M, ω) be a symplectic manifold, $N \subset M$ be a compact hypersurface. We say N is of **contact type** if there is a contact form α on N such that $\omega|_{TN} = d\alpha$. As an example, let N be a strictly pseudoconvex hypersurface of a complex manifold M. Then N is of contact type with respect to the Kähler form $\omega := d(-d\rho \circ J)$ defined in a regular neighborhood of N (cf. the proof of Lemma 1.8).

Let (M, ω) be a symplectic manifold. A 1-form λ on M is called a **Liouville 1-form** if $\omega = d\lambda$. A vector field X is called a **Liouville vector field** if $L_X\omega = \omega$. The Liouville vector fields and Liouville 1-forms are in $1-1$ correspondence by the relation $i_X\omega = \lambda$.

Example 1.9. Let (N, ξ) be a contact manifold where ξ is co-oriented, with a contact form α. The **symplectization** of (N, α) is the symplectic manifold $(\mathbb{R} \times N, d(e^t\alpha))$, where t is the coordinate on $\mathbb{R}$. Note that for the Liouville 1-form $\lambda = e^t\alpha$, the corresponding Liouville vector field $X = \frac{\partial}{\partial t}$.

Exercise: Let (M, ω) be a symplectic manifold, $N \subset M$ be a compact hypersurface.

(1) Show that if X is a Liouville vector field defined in a regular neighborhood of N which is normal along N, then N is of contact type.

(2) Suppose N is of contact type. Show that there is a a Liouville vector field defined in a regular neighborhood of N which is normal along N.

(3) Suppose N is of contact type. Show that a regular neighborhood of N is symplectomorphic to a regular neighborhood of $\{0\} \times N$ in a symplectization of N.

Exercise: Let N be a strictly pseudoconvex hypersurface of a complex manifold M. Let ω, α be defined as in Lemma 1.8. Let X be the gradient vector field of the function ρ with respect to the Hermitian metric associated to the Kähler form ω. Show that X is the Liouville vector field corresponding to α. Moreover, the Reeb vector field associated to the contact form $\alpha|_{TN}$ is $JX/||JX||^2$.

Exercise: Let (M, ω) be a symplectic manifold, X a Liouville vector field on M. Suppose $N_1, N_2 \subset M$ are hypersurfaces transverse to X, such that there is a diffeomorphism $\psi : N_1 \to N_2$ defined by following the flow lines of X. Show that $\psi_*(\xi_1) = \xi_2$, where ξ_i, $i = 1, 2$, is the induced contact structure on N_i.

Exercise: Let (M, ω) be a symplectic manifold where $\omega = d\lambda$, and let X be the corresponding Liouville vector field, which we assume is complete. Let M^* be the complement of the zeroes of λ.

(1) Show that the flow of X generates a free, proper $\mathbb{R}$-action on M^*.

(2) Let $N := M^*/\mathbb{R}$ be the quotient manifold. Show that λ determines a natural, co-oriented contact structure ξ on N.

(3) Show that (M^*, ω) is symplectomorphic to a symplectization of (N, ξ).

Exercise: As an example of the above, consider $M = T^*B$, with $\omega = d\lambda$ where λ is the canonical 1-form on T^*B. Then M^* is the complement of the zero section.

(1) Show that the Liouville vector field X, where $i_X\omega = \lambda$, is complete.

(2) Prove that $M^*/\mathbb{R}$ is the associated sphere bundle $\mathbb{S}(T^*B)$. Show that the contact structure on $\mathbb{S}(T^*B)$ is the pull-back of the canonical contact structure on $\mathbb{P}(T^*B)$ in Example 1.5 under the canonical double covering $\pi : \mathbb{S}(T^*B) \to \mathbb{P}(T^*B)$.

Finally, we discuss submanifolds of contact manifolds.

Definition 1.10. Let (N, ξ) be a contact manifold, where $\dim N = 2n+1$. Let $L \subset N$ be a submanifold. We say L is a **contact submanifold** if $\xi|_L \cap TL$ is a contact structure on L. We say L is an **isotropic submanifold** if $TL \subset \xi$. An isotropic submanifold is called **Legendrian** if it has a dimension equaling to $n = (\dim N - 1)/2$.

Let α be an associated contact form (even locally defined). Then $L \subset N$ is a contact submanifold if and only if $\alpha|_{TL}$ is a contact form on L, for any α. On the other hand, if $L \subset N$ is isotropic, then $\alpha|_{TL} = 0$, which implies $d\alpha|_{TL} = 0$ as well. In particular, $TL \subset \xi$ is an isotropic sub-bundle with respect to the symplectic form $d\alpha$ on ξ. It follows that $\dim L \leq \frac{1}{2} \dim \xi = (\dim N - 1)/2$. So a Legendrian submanifold is an isotropic submanifold of the maximal possible dimension.

Example 1.11. Let (N, ξ) be a contact manifold, where $\dim N = 2n+1$. Let $L \subset N$ be a 1-dimensional submanifold. Then L is a contact submanifold if and only if $\xi \cap TL = \{0\}$, i.e., the tangent vectors of L are transverse to ξ. When N is a 3-manifold, such a L is called a **transverse knot or link** (when L is closed).

Example 1.12. Let L be a n-dimensional manifold, and let $N \equiv T^*L \times \mathbb{R}$ be the 1-jet bundle over L. Then

$$\alpha = dz - \lambda$$

is a contact form on N, where z is the coordinate on the $\mathbb{R}$ factor and λ is the canonical 1-form on the T^*L factor. Then for any smooth function $f : L \to \mathbb{R}$, the submanifold

$$L_f \equiv \{(x, df(x), f(x)) | x \in L\} \subset N$$

is Legendrian.

Exercise: Let (N, ξ) be a contact manifold, where ξ is co-oriented with contact form α. Let $(M, \omega) = (\mathbb{R} \times N, d(e^t\alpha))$ be the symplectization.

(1) Show that a submanifold $L \subset N$ is a contact submanifold if and only if $\mathbb{R} \times L$ is a symplectic submanifold of (M, ω).

(2) Show that a submanifold $L \subset N$ is an isotropic submanifold if and only if $\mathbb{R} \times L$ is an isotropic submanifold of (M, ω). In particular, L is Legendrian if and only if $\mathbb{R} \times L$ is Lagrangian.

2. Stability and neighborhood theorems

Definition 2.1. Let ξ_0, ξ_1 be two contact structures on N. We say ξ_0, ξ_1 are **isomorphic** if there is a diffeomorphism $\psi \in \mathrm{Diff}(N)$ such that $\psi_*(\xi_0) = \xi_1$. We say ξ_0, ξ_1 are **homotopic through contact structures** if there is a smooth family of contact structures ξ_t, $t \in [0, 1]$, such that $\xi_t = \xi_0$ for $t = 0$ and $\xi_t = \xi_1$ for $t = 1$. Finally, we say ξ_0, ξ_1 are **isotopic** if there is an isotopy ψ_t, $t \in [0, 1]$, such that $\psi_0 = id$, $(\psi_1)_*(\xi_0) = \xi_1$.

Clearly, isotopic contact structures are isomorphic, and are also homotopic through contact structures. Our first theorem says that contact structures which are homotopic through contact structures are isotopic.

Theorem 2.2. *(Gray's stability theorem). Let N be a compact, closed manifold and suppose that α_t is a smooth family of contact forms on N. Then there exist a smooth family of $\psi_t \in Diff(N)$ and a smooth family of nonvanishing fuctions f_t on N such that*

$$\psi_0 = id, \ \ \psi_t^* \alpha_t = f_t \alpha_0.$$

In particular, the smooth family of contact structures $\xi_t \equiv \ker \alpha_t$ are diffeomorphic under ψ_t.

Proof. We shall obtain ψ_t by finding the corresponding vector fields X_t such that

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \ \ \psi_0 = \mathrm{id}.$$

To this end, we differentiate both sides of $\psi_t^* \alpha_t = f_t \alpha_0$ and obtain

$$\psi_t^*(\frac{d}{dt}\alpha_t + L_{X_t}\alpha_t) = g_t \psi_t^* \alpha_t, \text{ where } g_t = f_t^{-1}\frac{d}{dt}f_t.$$

Hence we must find X_t and $h_t = g_t \circ \psi_t^{-1}$ such that

$$\frac{d}{dt}\alpha_t + L_{X_t}\alpha_t = h_t \alpha_t.$$

Let Y_t be the Reeb vecter field associated to α_t. Then the above equation has a unique solution

$$h_t = i_{Y_t}\frac{d}{dt}\alpha_t, \ \ i_{X_t} d\alpha_t = -(\frac{d}{dt}\alpha_t)|_{\ker \alpha_t}, \text{ and } X_t \in \ker \alpha_t.$$

Gray's stability theorem follows by integrating X_t to obtain ψ_t. Note that f_t is uniquely determined from h_t and ψ_t, with the initial value $f_0 = 1$.

□

Next we present several neighborhood theorems in contact geometry.

Theorem 2.3. *(Darboux theorem in contact geometry). Every contact structure is locally diffeomorphic to the standard contact structure on $\mathbb{R}^{2n+1}$, defined by the standard contact form*

$$\alpha_0 = dz - \sum_{j=1}^n y_j dx_j.$$

Proof. Let α be a contact form on N and $q \in N$ be any point. Let $\xi = \ker \alpha$ be the corresponding contact structure, and Y be the Reeb vector field associated to α. We fix a Riemannian metric near q, and let $u_1, u_2, \cdots, u_n$, $v_1, v_2, \cdots, v_n$ be a symplectic basis of $(\xi_q, d\alpha_q)$. Then we define a chart centered at q by

$$\phi : \mathbb{R}^{2n+1} \to N : (x_1, \cdots, x_n, y_1, \cdots, y_n, z) \mapsto \exp_q(\sum_{j=1}^n (x_j u_j + y_j v_j) + zY).$$

By the nature of construction, we see that $\ker(\phi^*\alpha)_0 = \mathbb{R}^{2n} \times \{0\}$, and the Reeb vector field associated to $\phi^*\alpha$ equals ∂z at $0 \in \mathbb{R}^{2n+1}$, so that $\phi^*\alpha = \alpha_0$ at the origin 0. Moreover, note that $d(\phi^*\alpha) = d\alpha_0$ also holds at the origin.

We set $\alpha_1 = \phi^*\alpha$, and consider the smooth family of 1-forms $\alpha_t \equiv \alpha_0 + t(\alpha_1 - \alpha_0)$ near the origin $0 \in \mathbb{R}^{2n+1}$. Since at the origin $\alpha_t = \alpha_0$, $d\alpha_t = d\alpha_0$ for all $t \in [0,1]$, α_t is a smooth family of contact forms in a neighborhhod of the origin. We apply Moser's argument as in the proof of previous theorem, and obtain vector fields X_t and functions h_t which are determined by conditions

$$h_t = i_{Y_t}\frac{d}{dt}\alpha_t, \;\; i_{X_t}d\alpha_t = -(\frac{d}{dt}\alpha_t)|_{\ker\alpha_t}, \text{ and } X_t \in \ker\alpha_t.$$

(Here Y_t is the Reeb vecter field associated to α_t.) Since $\frac{d}{dt}\alpha_t = 0$ at the origin, we have $h_t(0) = 0$ and $X_t(0) = 0$ for all $t \in [0,1]$. The condition $X_t(0) = 0$ for all t ensures that there exists a neighborhood of the origin such that the diffeomorphisms ψ_t are defined on that neighborhood for all $t \in [0,1]$. By construction $\psi_t^*\alpha_t = f_t\alpha_0$ for some nonvanishing f_t, where $f_t(0) = 1$ because $h_t(0) = 0$ and $\psi_t(0) = 0$. It follows immediately that the contact structure $\xi = \ker\alpha$ is diffeomorphic to $\xi_0 = \ker\alpha_0$ by the local diffeomorphism $\phi \circ \psi_1$.

$\square$

Let (N,ξ) be a contact manifold of dimension $2n+1$, where ξ is co-oriented, and let $L \subset N$ be a compact closed isotropic submanifold of dimension k, where $k \leq n$. We pick a contact form α, and denote by $TL^\perp \subset \xi|_L$ the symplectic orthogonal complement of $TL \subset \xi|_L$ with respect to $d\alpha$. We remark that $TL^\perp$ depends only on ξ, as if we change α to $e^f\alpha$ conformally, $d\alpha|_\xi$ is changed conformally to $e^f d\alpha|_\xi$. Since L is isotropic, $TL \subset TL^\perp$. We set

$$CSN_N(L) := TL^\perp/TL,$$

which is called the **conformal symplectic normal bundle** of L in (N,ξ); it has a conformal symplectic form depending only on ξ.

Let $\nu_L := TN|_L/TL$ be the normal bundle of L in N. Then ν_L is a direct sum

$$\nu_L = \mathbb{R}(R_\alpha) \oplus \xi_L/TL^\perp \oplus CSN_N(L),$$

where $\mathbb{R}(R_\alpha)$ is the trivial line bundle over L with a preferred section R_α (the Reeb vector field associated to the contact form α). There is a canonical identification (depending on $d\alpha$) $\gamma : \xi_L/TL^\perp \to T^*L$ given as follows: for any $q \in L$, $Y \in \xi_q$, $\gamma(Y) = i_Y d\alpha|_{T_qL} \in T_q^*L$. Hence

$$\nu_L = \mathbb{R}(R_\alpha) \oplus T^*L \oplus CSN_N(L).$$

Theorem 2.4. *Let (N_i,ξ_i), $i = 0,1$, be a $(2n+1)$-dimensional contact manifold, where ξ_i is co-oriented, and let $L_i \subset N_i$ be a compact closed isotropic submanifold of dimension $k \leq n$. Suppose there is an isomorphism $\Phi : CSN_{N_0}(L_0) \to CSN_{N_1}(L_1)$ of the conformal symplectic normal bundles, covering a diffeomorphism $\phi : L_0 \to L_1$. Then this diffeomorphism ϕ extends to a contactomorphism $\psi : \mathcal{N}(L_0) \to \mathcal{N}(L_1)$ of suitable neighborhoods $\mathcal{N}(L_i)$ of L_i such that the bundle maps $d\psi|_{CSN_{N_0}(L_0)}$ and Φ are bundle homotopic (as conformal symplectic normal bundle isomorphisms).*

Proof. We fix a contact form α_0 for (N_0, ξ_0). This gives us the Reeb vector field R_{α_0} and a symplectic form $d\alpha_0$ on ξ_0. Then we fix a $d\alpha_0$-compatible complex structure J_0 of ξ_0, also giving rise to a corresponding Hermitian metric on ξ_0. With this understood, we note that T^*L_0 can be identified with $J_0(TL_0) \subset \xi_0|_{L_0}$ (see Lemma 3.5 in §3 of Part 1), and $CSN_{N_0}(L_0)$ with $(TL_0 \oplus J_0(TL_0))^\perp \subset \xi_0|_{L_0}$, the symplectic orthogonal complement of $TL_0 \oplus J_0(TL_0)$ in $\xi_0|_{L_0}$. Similarly, we fix a contact form α for (N_1, ξ_1), a $d\alpha$-compatible complex structure J of ξ_1, such that $\Phi^*(d\alpha) = d\alpha_0$ on $CSN_{N_0}(L_0)$. Then there is a bundle isomorphism $\tilde{\Phi} : \nu_{L_0} \to \nu_{L_1}$ covering $\phi : L_0 \to L_1$, such that $\tilde{\Phi}(R_{\alpha_0}) = R_\alpha$, $\tilde{\Phi} = J \circ d\phi \circ J_0^{-1}$ on $J_0(TL_0)$, and $\tilde{\Phi} = \Phi$ on $(TL_0 \oplus J_0(TL_0))^\perp$. Then via appropriate exponential maps, we obtain a diffeomorphism ϕ' between neighborhoods of L_0 and L_1, covering $\phi : L_0 \to L_1$, such that $d\phi' = \tilde{\Phi} : \nu_{L_0} \to \nu_{L_1}$. With this understood, we set $\alpha_1 := (\phi')^*\alpha$. Then it follows easily that

$$\alpha_1|_{TN_0|_{L_0}} = \alpha_0|_{TN_0|_{L_0}}, \;\; d\alpha_1|_{TN_0|_{L_0}} = d\alpha_0|_{TN_0|_{L_0}}.$$

(For the second equation, see the proof of Theorem 3.4 in §3 of Part 1.) For $t \in [0,1]$, let $\alpha_t = \alpha_0 + t(\alpha_1 - \alpha_0)$. Then

$$\alpha_t|_{TN_0|_{L_0}} = \alpha_0|_{TN_0|_{L_0}}, \;\; d\alpha_t|_{TN_0|_{L_0}} = d\alpha_0|_{TN_0|_{L_0}}, \forall t \in [0,1].$$

In particular, since L_0 is compact, there is a neighborhood of L_0 such that α_t is a contact form for any $t \in [0,1]$. Then as in the proof of Darboux theorem, let Y_t be the Reeb vecter field associated to α_t. We determine functions h_t and vector fields X_t by the following equations

$$h_t = i_{Y_t} \frac{d}{dt}\alpha_t, \;\; i_{X_t} d\alpha_t = -(\frac{d}{dt}\alpha_t)|_{\ker \alpha_t}, \text{ and } X_t \in \ker \alpha_t.$$

Note that $\frac{d}{dt}\alpha_t = 0$ along L_0, so that we have $h_t = 0$ and $X_t = 0$ for all $t \in [0,1]$ along L_0. The fact that $X_t = 0$ for all $t \in [0,1]$ along L_0 (and the fact that L_0 is compact) ensures that there is a regular neighborhood of L_0 such that the diffeomorphisms ψ_t generated by X_t are defined on that neighborhood for all $t \in [0,1]$. Moreover, $\psi_t^*\alpha_t = f_t\alpha_0$ for some positive functions f_t, with $f_t \equiv 1$ on L_0. Then $\phi' \circ \psi_1$ is the desired contactomorphism between suitable neighborhoods of L_0 and L_1.

□

Example 2.5. If L is a closed Legendrian submanifold of (N, ξ), then $CSN_N(L) = \{0\}$. By Theorem 2.4, a neighborhood of L in N has a standard model of contact structures, depending only on the diffeomorphism class of L. More concretely, a neighborhood of L in N is contactomorphic to a neighborhood of the zero section of the 1-jet bundle of L, i.e., $T^*L \times \mathbb{R}$ with a contact structure defined by $dz - \lambda$, where z is the coordinate on $\mathbb{R}$ and λ is the canonical 1-form on T^*L.

More generally, let L be a closed isotropic submanifold of dimension k, where $k < n = \frac{1}{2}(\dim N - 1)$. Set $l := n - k$. Assume $CSN_N(L)$ is trivial. Then fixing any identification of $CSN_N(L)$ with $L \times \mathbb{R}^{2l}$, there is a contactomorphism from a neighborhood of L in N to a neighborhood of $L_0 \times \{0\}$ in $T^*L \times \mathbb{R} \times \mathbb{R}^{2l}$ (L_0 denotes the zero section of $T^*L \times \mathbb{R}$), where the contact structure on the latter is defined by $dz - \lambda + \sum_{j=1}^{l} x_j dy_j$. (Here $x_1, \cdots, x_l, y_1, \cdots, y_l$ are the standard coordinates of $\mathbb{R}^{2l}$.)

Now let $L \subset N$ be a contact submanifold of (N, ξ). Denote by $\eta := \xi|_L \cap TL$ the contact structure on L. Then η is a conformal symplectic sub-bundle of $\xi|_L$, so we can talk about its orthogonal complement $\eta^\perp \subset \xi|_L$, which is also a conformal symplectic sub-bundle of $\xi|_L$. Clearly, $\eta^\perp$ is isomorphic to the normal bundle of L in N. We let

$$CSN_N(L) := \eta^\perp,$$

and call it the **conformal symplectic normal bundle** of L in (N, ξ).

Theorem 2.6. *Let (N_i, ξ_i), $i = 0, 1$, be a $(2n+1)$-dimensional contact manifold, where ξ_i is co-oriented, and let $(L_i, \eta_i) \subset (N_i, \xi_i)$ be a compact closed contact submanifold of dimension $2k + 1$, $k < n$. Suppose there is an isomorphism $\Phi : CSN_{N_0}(L_0) \to CSN_{N_1}(L_1)$ of the conformal symplectic normal bundles, covering a contactomorphism $\phi : (L_0, \eta_0) \to (L_1, \eta_1)$. Then ϕ extends to a contactomorphism $\psi : \mathcal{N}(L_0) \to \mathcal{N}(L_1)$ of suitable neighborhoods $\mathcal{N}(L_i)$ of L_i such that the bundle maps $d\psi|_{CSN_{N_0}(L_0)}$ and Φ are bundle homotopic (as conformal symplectic normal bundle isomorphisms).*

Proof. We choose contact forms β_i on L_i such that $\eta_i = \ker \beta_i$, and $\beta_0 = \phi^*\beta_1$. We shall choose appropriate contact forms α_i on N_i accordingly next.

First, we pick a contact form α_i on N_i such that $\alpha_i(R_{\beta_i}) = 1$ along L_i, where R_{β_i} is the Reeb vector field of β_i on L_i. With this choice we note that $\alpha_i|_{TL_i} = \beta_i$ and $d\alpha_i|_{TL_i} = d\beta_i$. Next we choose a smooth function $f_i > 0$ in a neighborhood of L_i, where $f_i \equiv 1$ on L_i, such that $i_{R_{\beta_i}} d(f_i\alpha_i) \equiv 0$ on $TN_i|_{L_i}$. This is equivalent to

$$i_{R_{\beta_i}} df_i \cdot \alpha_i - df_i \cdot i_{R_{\beta_i}} \alpha_i + f_i \cdot i_{R_{\beta_i}} d\alpha_i = 0 \text{ on } TN_i|_{L_i}.$$

With $f_i \equiv 1$ on L_i and $\alpha_i|_{TL_i} = \beta_i$, we obtain $-df_i + i_{R_{\beta_i}} d\alpha_i = 0$ on $TN_i|_{L_i}$. To see such a smooth function f_i exists, we simply extend $f_i \equiv 1$ along L_i to a neighborhood of L_i, such that for any normal vector X of L_i, $df_i(X) = i_{R_{\beta_i}} d\alpha_i(X)$.

With the preceding understood, we replace α_i by $f_i\alpha_i$, which is still denoted by α_i for simplicity. Then $R_{\alpha_i} = R_{\beta_i}$ along L_i. Finally, note that after scaling Φ if necessary, we may assume $\Phi^*(d\alpha_1) = d\alpha_0$. With these preparations, it follows that there is a diffeomorphism ϕ' between neighborhoods of L_0 and L_1, covering $\phi : L_0 \to L_1$, such that $(\phi')^*\alpha_1|_{TN_0|_{L_0}} = \alpha_0|_{TN_0|_{L_0}}$, $(\phi')^* d\alpha_1|_{TN_0|_{L_0}} = d\alpha_0|_{TN_0|_{L_0}}$. Then

$$\gamma_t := \alpha_0 + t((\phi')^*\alpha_1 - \alpha_0), \ t \in [0, 1]$$

is a smooth family of contact forms in a neighborhood of L_0, such that $\gamma_t|_{TN_0|_{L_0}} = \alpha_0|_{TN_0|_{L_0}}$, $d\gamma_t|_{TN_0|_{L_0}} = d\alpha_0|_{TN_0|_{L_0}}$. As we argued before, this gives rise to a smooth family of diffeomorphisms ψ_t defined in a neighborhood of L_0, $\psi_t = id$ on L_0, such that $\psi_t^*\gamma_t = f_t\gamma_0$. Since $\gamma_0 = \alpha_0$, $\gamma_1 = (\phi')^*\alpha_1$, $\phi' \circ \psi_1$ is the desired contactomorphism between suitable neighborhoods of L_0 and L_1.

□

Example 2.7. Suppose L is a compact closed, connected contact submanifold of dimension 1. Then the contact structure on L is trivially unique, and $CSN_N(L)$ is a trivial bundle. With this understood, fixing any trivialization of $CSN_N(L)$, there is a contactomorphism from a neighborhood of L to a neighborhood of $\mathbb{S}^1 \times \{0\}$ in

$\mathbb{S}^1 \times \mathbb{R}^{2n}$, where $\mathbb{S}^1 \times \{0\}$ in $\mathbb{S}^1 \times \mathbb{R}^{2n}$ is given the contact structure defined by

$$dz + \sum_{j=1}^{n} x_j dy_j.$$

Here z is the coordinate on $\mathbb{S}^1$ and $x_1, \cdots, x_n, y_1, \cdots, y_n$ are the coordinates on $\mathbb{R}^{2n}$.

Exercise: Let L be a compact closed submanifold of contact manifold (N, ξ) (here ξ is assumed to be co-oriented), where L is an isotropic (resp. contact) submanifold. Let $j_t : L \to N$, for $t \in [0,1]$, be a smooth family of smooth embeddings with $j_0 = Id$, such that each $j_t(L)$ is an isotropic (resp. contact) submanifold. Show that there is a contact isotopy $\phi_t : (N, \xi) \to (N, \xi)$, $t \in [0,1]$, with $\phi_0 = Id$, such that $\phi_t|_L = j_t$ for each t, and ϕ_t is supported in a neighborhood of $\cup_t j_t(L)$ in N. (*Hint: by the neighborhood theorems, Theorems 2.4 and 2.6, the submanifolds $j_t(L)$ have contactmorphic regular neighborhoods. This fact makes it possible for us to first extend the isotopy of embeddings j_t to an isotopy of contact embeddings of a regular neighborhood of L, which corresponds to a time-dependent Hamiltonian H_t defined on a neighborhood of $\cup_t j_t(L)$ in N. Simply extend H_t to N in a compactly supported way, cf. Lemmas 1.6 and 1.7.*)

3. Pseudoholomorphic curves in symplectizations

Let (M, ω) be a symplectic manifold, $N := H^{-1}(c)$ be a regular level surface of a smooth function $H : M \to \mathbb{R}$. Recall that as a hypersurface, N is co-isotropic, i.e., $TN^\omega \subset TN$, which generates a 1-dimensional foliation on N, called the **characteristic foliation** on N. Note that the corresponding Hamiltonian vector field X_H is tangent to the characteristic foliation on N. A fundamental question in Hamiltonian dynamics asks whether there is always a closed orbit of a Hamiltonian vector field; in particular, whether the characteristic foliation on N always contains a closed orbit.

In 1978, Weinstein conjectured that if N is compact closed and is of contact type, then the characteristic foliation on N must contain a closed orbit. There is an equivalent formulation: let α be a contact form on N such that $\omega|_{TN} = d\alpha$, then it is easy to see that the Reeb vector field R_α is tangent to the characteristic foliation. Thus Weinstein's conjecture asserts that on a compact closed contact manifold, any Reeb vector field admits a closed orbit. The pseudoholomorphic curve theory for symplectizations of contact manifolds, introduced by Hofer, was designed to attack the Weinstein conjecture on closed Reeb orbits.

For the rest of this section, let (M, ξ) be a compact closed $(2n+1)$-dimensional co-oriented contact manifold, let λ be an associated contact form. We shall denote the Reeb vector field of λ by X. Then $TM = \mathbb{R}X \oplus \xi$. We let $\pi : TM \to \xi$ denote the projection. Finally, we equip ξ with the symplectic form $d\lambda$, making it into a symplectic vector bundle.

Fixing any $d\alpha$-compatible complex structure $J : \xi \to \xi$, we can canonically extend it to any almost complex structure $\tilde{J}$ on $W := \mathbb{R} \times M$ by $\tilde{J}|_\xi = J$ and

$$\tilde{J}(\frac{\partial}{\partial t}) = X, \;\; \tilde{J}(X) = -\frac{\partial}{\partial t},$$

where t is the coordinate on the $\mathbb{R}$-factor. Now let (S, j) be a compact closed Riemann surface, $\Gamma \subset S$ be a subset of finitely many points, called the *punctures*, and denote by $\dot{S} = S \setminus \Gamma$ the corresponding punctured Riemann surface. We shall consider $\tilde{J}$-holomorphic maps $\tilde{u} = (a, u) : \dot{S} \to W = \mathbb{R} \times M$, which obey

$$\tilde{J} \circ d\tilde{u} = d\tilde{u} \circ j.$$

If $z = s + it$ is a local holomorphic coordinate on $\dot{S}$, the above equation becomes

$$\pi u_s + J(u)\pi u_t = 0, \;\; a_s = \lambda(u_t), \;\; a_t = -\lambda(u_s).$$

As shown in the following example, $\tilde{J}$-holomorphic maps are closely related to integral curves of the Reeb vector field X.

Example 3.1. (1) Let $x(t)$ be an integral curve of X which is periodic (i.e., a closed orbit), and let $T > 0$ such that $x(T) = x(0)$. Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Then the map $\tilde{u} = (a, u) : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R} \times M$, where $a(s, t) = sT$, $u(s, t) = x(tT)$, is $\tilde{J}$-holomorphic.

(2) Let $x(t)$ be an integral curve of X, $t \in \mathbb{R}$, which is not periodic. Let $z = s+it \in \mathbb{C}$ be the holomorphic coordinate on $\mathbb{C}$. Then the map $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times M$, where $a(s, t) = s$, $u(s, t) = x(t)$, is $\tilde{J}$-holomorphic.

The type of $\tilde{J}$-holomorphic maps in Example 3.1(2) are obviously not desirable; in particular, it is not even a proper map. In order to exclude these ill-behaved cases, one needs to introduce a finite energy condition.

Let Σ be the set of smooth maps $\phi : \mathbb{R} \to [0, 1]$ such that $\phi' \geq 0$. For each $\phi \in \Sigma$, we set $\lambda_\phi := \phi\lambda$, where ϕ is considered as a function on $\mathbb{R} \times M$ and λ as the pull-back 1-form on $\mathbb{R} \times M$. With this understood, for any $\tilde{J}$-holomorphic map $\tilde{u} = (a, u) : \dot{S} \to \mathbb{R} \times M$, we define the **energy of** $\tilde{u}$ to be

$$E(\tilde{u}) = \sup_{\phi \in \Sigma} \int_{\dot{S}} \tilde{u}^* d\lambda_\phi.$$

Lemma 3.2. *Let $\tilde{u} = (a, u)$ be a $\tilde{J}$-holomorphic map and $z = s + it$ be a local holomorphic coordinate on the domain $\dot{S}$. Then for any $\phi \in \Sigma$,*

$$\tilde{u}^* d\lambda_\phi = \frac{1}{2}(\phi'(a)(|a_s|^2 + |a_t|^2 + |\lambda(u_s)|^2 + |\lambda(u_t)|^2) + \phi(a)(|\pi u_s|_J^2 + |\pi u_t|_J^2))ds \wedge dt,$$

where the norm $|v|_J := d\lambda(v, Jv)^{1/2}$, $\forall v \in \xi$. Consequently, $E(\tilde{u}) \geq 0$ and $\tilde{u}$ is non-constant if and only if $E(\tilde{u}) > 0$.

Exercise: (1) Prove Lemma 3.2 and compute $E(\tilde{u})$ for the maps $\tilde{u}$ in Example 3.1.
(2) Show that $0 \leq \int_{\dot{S}} u^* d\lambda \leq E(\tilde{u})$.

Definition 3.3. A $\tilde{J}$-holomorphic map $\tilde{u} : \dot{S} \to \mathbb{R} \times M$ is called a (non-constant) **finite energy surface** if $0 < E(\tilde{u}) < \infty$. If there is a $z_0 \in \dot{S}$ such that $d\tilde{u}(z_0) \neq 0$ and $\tilde{u}^{-1}(\tilde{u}(z_0)) = \{z_0\}$, then $\tilde{u}$ is called **somewhere injective**. The image $C := \tilde{u}(\dot{S})$ of a finite energy surface is called an **unparametrized finite energy surface**, and $\tilde{u}$ is called a **parametrization** of C if $\tilde{u}$ is somewhere injective.

The following is a generalization of Corollary 5.4 in §5 of Part 2.

Proposition 3.4. *Let $\tilde{u} : \dot{S} \to \mathbb{R} \times M$ be any finite energy surface. There exist a compact closed Riemann surface S' with a finite set of punctures $\Gamma' \subset S'$ and a holomorphic map $\psi : S \to S'$, with $\Gamma' = \psi(\Gamma)$, and a somewhere injective finite energy surface $\tilde{v} : \dot{S}' \to \mathbb{R} \times M$ such that $\tilde{u} = \tilde{v} \circ \psi$.*

All the local properties discussed in §5 of Part 2 continue to hold in the current setting, in particular, the removal of singularities (Theorem 5.6 in Part 2), which plays a role in the classification of punctures of a finite energy surface.

Let $z \in \Gamma$ be a given puncture. In a disc neighborhood of z in S, we introduce coordinates $(s,t) \in [c, +\infty) \times \mathbb{R}/\mathbb{Z}$, and let $\tilde{v}(s,t) := \tilde{u}(e^{-2\pi(s+it)})$, and write $\tilde{v} = (b, v) \in \mathbb{R} \times M$. Then $\tilde{v}$ is $\tilde{J}$-holomorphic, which obeys

$$\pi v_s + J(v)\pi v_t = 0, \quad b_s = \lambda(v_t), \quad b_t = -\lambda(v_s).$$

Lemma 3.5. *The limit $\lim_{s\to+\infty} \int_0^1 v^*\lambda(s,\cdot)$ exists.*

Proof. We have $\int_0^1 v^*\lambda(s,\cdot) - \int_0^1 v^*\lambda(c,\cdot) = \int_{[c,s]\times\mathbb{R}/\mathbb{Z}} v^*d\lambda$ by Stokes' theorem. On the other hand, $v^*d\lambda = |\pi v_s|_J^2 ds \wedge dt$ and $\int_{[c,s]\times\mathbb{R}/\mathbb{Z}} v^*d\lambda \leq E(\tilde{u})$. It follows easily that $\int_0^1 v^*\lambda(s,\cdot)$ is monotone and is bounded from above. The lemma follows immediately. $\square$

We denote the limit $\lim_{s\to+\infty} \int_0^1 v^*\lambda(s,\cdot)$ by $m(\tilde{u}, z)$, which is called the **charge** of $\tilde{u}$ at the puncture $z \in \Gamma$. We have the following basic result.

Theorem 3.6. *Let $\tilde{u} : \dot{S} \to \mathbb{R} \times M$ be a finite energy surface and $z \in \Gamma$ a given puncture, and $\tilde{v}(s,t) = \tilde{u}(e^{-2\pi(s+it)}) = (b, v) \in \mathbb{R} \times M$. Then the following hold true.*

- *If $m(\tilde{u}, z) = 0$, then the map $\tilde{u}$ can be extended over z, i.e., z is a removable puncture.*
- *If $m(\tilde{u}, z) \neq 0$, then for any sequence $\{s_k\}$ converging to $+\infty$, there is a subsequence, continuing to be denoted by $\{s_k\}$, and an integral curve $x(t)$ of the Reeb vector field X, such that $v(s_k, t) \to x(mt)$ in C^∞-topology as $k \to +\infty$, where $m = m(\tilde{u}, z)$ and $t \in \mathbb{R}/\mathbb{Z}$.*
- $\lim_{s\to+\infty} \frac{b(s,t)}{s} = m(\tilde{u}, z)$, *$\forall t \in \mathbb{R}/\mathbb{Z}$.*

We remark that when $m(\tilde{u}, z) \neq 0$, the limit of $v(s_k, t)$ is a closed orbit of the Reeb vector field, which is periodic of period $T := |m(\tilde{u}, z)| > 0$ (not necessarily the minimal period). Moreover, if we write $\tilde{u} = (a, u) \in \mathbb{R} \times M$, then $a(w) \to +\infty$ as $w \to z$ if $m(\tilde{u}, z) > 0$, and $a(w) \to -\infty$ as $w \to z$ if $m(\tilde{u}, z) < 0$. Without loss of generality, we assume that there are no removable punctures. Then $\Gamma = \Gamma^+ \sqcup \Gamma^-$, where Γ^+ (resp. Γ^-) is the set of punctures z with positive (resp. negative) charges $m(\tilde{u}, z)$.

Moreover, assuming $m := m(\tilde{u}, z) \neq 0$. If the limiting periodic orbit $x(t)$ (for some sequence $\{s_k\}$) is non-degenerate in the sense of Definition 3.8 below, then $v(s,t) \to x(mt)$ in C^∞-topology as $s \to +\infty$. In fact, the finite energy surface $\tilde{u}$ converges to the $\tilde{J}$-holomorphic cylinder $\tilde{u}_0$ associated to the periodic orbit $x(t)$, i.e., $\tilde{u}_0 = (a, u) : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R} \times M$, where $a(s,t) = ms + c$, $u(s,t) = x(mt)$, exponentially fast as it approaches to the puncture $z \in \Gamma$. This improved exponential decay convergence under the non-degeneracy condition of the periodic orbits at the punctures z is important in analyzing the structure of the moduli space of the finite energy surface $\tilde{u}$.

Exercise: Show that $\Gamma^+ \neq \emptyset$. (*Hint:* note that $\int_{\dot{S}} u^* d\lambda \geq 0$ and then apply Theorem 3.6 and Stokes' theorem.)

Corollary 3.7. *(Hofer) For any compact closed contact manifold (M, λ), the Weinstein conjecture holds, i.e., there exists a closed Reeb orbit, if and only if there is a finite energy surface in the symplectization of (M, λ).*

In particular, in dimension three Hofer was able to establish the existence of finite energy surfaces under the assumption that the contact structure ξ is overtwisted (see §4 for definition) or the contact 3-manifold M has $\pi_2(M) \neq 0$, thus proving the Weinstein conjecture in these cases.

The remaining part of this section is devoted to the structure of moduli spaces of finite energy surfaces.

Let ψ_t be the family of maps generated by the Reeb vector field X, i.e., $t \in \mathbb{R} \mapsto \psi_t \in \mathrm{Diff}(M)$, be the corresponding $\mathbb{R}$-action on M. Since $L_X \lambda = 0$, it follows that $(\psi_t)_* \xi = \xi$ and $\psi_t^* d\lambda = d\lambda$. Let $x(t)$ be an integral curve of X, i.e., $x(t) = \psi_t(x(0))$, $\forall t \in \mathbb{R}$. Recall that $x(t)$ is called periodic if there is a $T > 0$ such that $x(t+T) = x(t)$ for any $t \in \mathbb{R}$. The (parametrized) closed curve $\{x(t) | 0 \leq t \leq T\}$ is called a periodic orbit of X, where T is called its period. We remark that here T needs not to be minimal; there is a minimal period $T_0 > 0$ and $T = kT_0$ for some $k \in \mathbb{Z}$. Moreover, the periodic orbit of period T is a k-fold cover of the periodic orbit of period T_0.

Definition 3.8. A periodic orbit $x(t)$ of period $T > 0$ is said to be **non-degenerate** if the map $(\psi_T)_* : \xi|_{x(0)} \to \xi|_{x(T)=x(0)}$ does not have 1 contained in its spectrum. The contact form λ is called **non-degenerate** if every periodic orbit of its Reeb vector field is non-degenerate.

Next we recall the notion of **Conley-Zehnder index**. For each integer $n \geq 1$, we let $\Sigma(n)$ denote the set of continuous path $\Phi = \Phi(t) \in \mathrm{Sp}(2n)$, $t \in [0,1]$, such that $\Phi(0) = Id$ and $\Phi(1)$ does not have 1 contained in its spectrum. Moreover, let $G(n)$ denote the set of continuous loops $g = g(t) \in \mathrm{Sp}(2n)$, $t \in [0,1]$, where $g(0) = g(1) = Id$. For each $g = g(t) \in G(n)$, we denote the Maslov index of g by $\mu_M(g)$. (Recall that $\mu_M : \pi_1(\mathrm{Sp}(2n)) \to \mathbb{Z}$ is an isomorphism.) The Conley-Zehnder index is characterized in the following theorem.

Theorem 3.9. *There exists a unique family of maps $\mu = \mu^n : \Sigma(n) \to \mathbb{Z}$, where $n \geq 1$ an integer, having the following properties:*

- *Homotopic paths in $\Sigma(n)$ have the same index (i.e., the value under μ).*
- *For any $g \in G(n)$, $\Phi \in \Sigma(n)$, $\mu(g\Phi) = \mu(\Phi) + 2\mu_M(g)$.*
- *$\mu(\Phi^{-1}) = -\mu(\Phi)$ for any $\Phi \in \Sigma(n)$, where $\Phi^{-1} = \Phi(t)^{-1}$.*
- *$\mu(\Phi_0) = 1$ where $\Phi_0 = e^{\pi i t} Id \in \Sigma(1)$.*
- *$\mu^{n+m}(\Phi \oplus \Psi) = \mu^n(\Phi) + \mu^m(\Psi)$, where $\Phi \in \Sigma(n)$, $\Psi \in \Sigma(m)$.*

Example 3.10. In this example, we explain how to determine the Coney-Zehnder index of a $\Phi \in \Sigma(1)$. Note that $\Phi = \Phi(t) \in \mathrm{Sp}(2)$, where $\mathrm{Sp}(2)$ consists of 2×2 matrices of determinant 1. Thus for $\Phi(1)$, there are two possibilities for its spectrum: (1) a pair of complex numbers $\sigma, \bar{\sigma}$ where $|\sigma| = 1$, (2) a pair of real numbers β, β^{-1}, where $\beta \neq 1$. In case (1), $\Phi \in \Sigma(1)$ is called **elliptic**, and in case (2), Φ is called **positive hyperbolic** (resp. **negative hyperbolic**) when $\beta > 0$ (resp. $\beta < 0$).

We shall change Φ through a homotopy of paths in $\Sigma(1)$, such that $\Phi(1) = -Id$ when it is elliptic, and $\Phi(1) = diag(\beta, \beta^{-1})$ when it is hyperbolic. With this understood, recall that $\Phi(t) = P(t)Q(t)$ where $P(t) = (\Phi(t)\Phi(t)^T)^{1/2}$ and $Q(t) \in U(1)$. It follows easily that when Φ is elliptic, $P(1) = Id$ and $Q(1) = \Phi(1) = -Id$, and when Φ is hyperbolic, $P(1) = diag(|\beta|, |\beta|^{-1})$, so that $Q(1) = -Id$ when $\beta < 0$ and $Q(1) = Id$ when $\beta > 0$. With this understood, the Conley-Zehnder index $\mu(\Phi)$ equals the degree of the loop $\det Q(t)^2$ in $\mathbb{S}^1$. We observe that $\mu(\Phi)$ is an odd number when Φ is elliptic or negative hyperbolic, and $\mu(\Phi)$ is an even number when Φ is positive hyperbolic.

Now let $x(t)$ be a periodic orbit of period $T > 0$ which is non-degenerate. For any trivialization of $(\xi, d\lambda)|_{x(t)}$ as a symplectic vector bundle, we can associate $x(t)$ with a Conley-Zehnder index as follows. Observe that the trivialization of $(\xi, d\lambda)|_{x(t)}$ is given by a smooth family of isomorphisms of symplectic vector spaces $\Psi(t) : (\xi, d\lambda)|_{x(tT)} \to (\mathbb{R}^{2n}, \omega_0)$, $t \in [0,1]$, where $\Psi(0) = \Psi(1)$. With this understood, we consider the linearized Reeb flow around the periodic orbit $\Phi \in \Sigma(n)$, where $\Phi(t) = \Psi(t) \circ (\psi_{tT})_* \circ \Psi(0)^{-1}$, $t \in [0,1]$. Note that $\Phi(1) = \Psi(0) \circ (\psi_T)_* \circ \Psi(0)^{-1}$ which is conjugate to $(\psi_T)_*$, so that it does not have 1 contained in its spectrum. The Conley-Zehnder index of $x(t)$ with the trivialization Ψ is defined to be

$$\mu(x(t), \Psi) := \mu(\Phi).$$

Exercise: Let Ψ, Ψ' be two different trivializations of $(\xi, d\lambda)|_{x(t)}$. Show that $\mu(x(t), \Psi') = \mu(x(t), \Psi) + 2\mu_M(\Psi'(t)\Psi(t)^{-1})$.

Accordingly, a periodic orbit $x(t)$ in a contact 3-manifold is called **elliptic (resp. negative hyperbolic, positive hyperbolic)** if the linearized Reeb flow around $x(t)$ is elliptic (resp. negative hyperbolic, positive hyperbolic) as defined in Example 3.10.

Example 3.11. Consider $\mathbb{C}^2$ equipped with the standard symplectic structure ω_0. Let (r_i, θ_i), $i = 1, 2$, be the polar coordinates on $\mathbb{C}^2$. Then

$$\omega_0 = r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2.$$

Let $Y := \frac{1}{2}(r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2})$, which is easily seen a Liouville vector field, with the corresponding Liouville 1-form

$$\lambda := i_Y \omega_0 = \frac{1}{2}(r_1^2 d\theta_1 + r_2^2 d\theta_2).$$

For any fixed real numbers $a_1 > 0$, $a_2 > 0$, we consider smooth function

$$H(z_1, z_2) = \frac{1}{2}(a_1 r_1^2 + a_2 r_2^2).$$

The corresponding Hamiltonian vector field $X = a_1 \frac{\partial}{\partial \theta_1} + a_2 \frac{\partial}{\partial \theta_2}$, where $i_X \omega_0 = -dH$. Since Y is transversal to the regular level surfaces of H, λ restricts to a contact form the level surfaces. Note that

$$\lambda(X) = \frac{1}{2}(a_1 r_1^2 + a_2 r_2^2) = H,$$

so if we consider the level surface $M := \{H \equiv 1\}$, X is the Reeb vector field on M associated to the contact form λ.

Next we examine the integral curves of X. For any $c_1 > 0, c_2 > 0$ such that $\frac{1}{2}(a_1 c_1^2 + a_2 c_2^2) = 1$, we consider the torus $T_{c_1,c_2} := \{r_1 \equiv c_1, r_2 \equiv c_2\}$ in M. Clearly, each T_{c_1,c_2} is invariant under the flow of X. For any $x_0 \in T_{c_1,c_2}$, the integral curve $x(t)$ with $x(0) = x_0$ is given by, in the coordinates $(r_1, \theta_1, r_2, \theta_2)$,

$$x(t) = x_0 + (0, a_1, 0, a_2)t.$$

It follows easily that $x(t)$ is periodic with period $T > 0$ if and only if there are $m, n \in \mathbb{Z}$, $m > 0$, $n > 0$, such that $a_1 T = 2\pi m$, $a_2 T = 2\pi n$. Moreover, T is the minimal period if m, n are relatively prime. In conclusion, we have

- If $a_1/a_2 \in \mathbb{Q}$, then every integral curve $x(t)$ on T_{c_1,c_2}, for any c_1, c_2, is periodic with the same minimal period independent of c_1, c_2.
- If a_1/a_2 is irrational, then the torus T_{c_1,c_2}, for any c_1, c_2, does not contain any periodic Reeb orbit.

We point out that when $a_1/a_2 \in \mathbb{Q}$, each periodic orbit on the tori T_{c_1,c_2} fails to be non-degenerate. To see this note that the vector field $Z := a_2 r_2 \frac{\partial}{\partial r_1} - a_1 r_1 \frac{\partial}{\partial r_2}$ along T_{c_1,c_2} lies in the contact structure $\xi|_{T_{c_1,c_2}}$. Moreover, $L_X Z = [X, Z] = 0$, so that $(\psi_t)_* Z = Z$ for any $t \in \mathbb{R}$, where ψ_t is the flow generated by X. It follows easily that the map $(\psi_T)_* : \xi|_{x(0)} \to \xi|_{x(T)=x(0)}$ has 1 contained in its spectrum, with $Z|_{x(0)}$ being the corresponding eigenvector.

Finally, there are two integral curves of X, given by $r_1 = 0$ and $r_2 = 0$ respectively, which are both closed Reeb orbits. It is easy to see that the minimal periods are $2\pi/a_2$, $2\pi/a_1$ respectively. Let $x(t)$ be the periodic orbit given by $r_1 = 0$ with period $T = 2\pi k/a_2$ for some $k \in \mathbb{Z}$. Note that the contact structure $\xi|_{x(t)}$ can be identified with $\mathbb{C} \times \{x(t)\} \subset \mathbb{C}^2$, where (r_1, θ_1) is the polar coordinate on $\mathbb{C}$. With this understood, it is easy to see that $(\psi_T)_* : \mathbb{C} \to \mathbb{C}$ is given by a rotation of angle $2\pi \cdot \frac{a_1 k}{a_2}$. Thus when $\frac{a_1 k}{a_2}$ is not an integer, $x(t)$ is non-degenerate. Similar discussions apply to the periodic orbits given by $r_2 = 0$. In particular, when a_1/a_2 is irrational, the Reeb vector field X has only two isolated closed orbits, with all the associated periodic orbits (i.e., the orbits and their iterates) being non-degenerate. (In this case, the contact form λ is non-degenerate.) Furthermore, the periodic orbits are all elliptic in the sense of Example 3.10. We remark that when $a_1/a_2 \in \mathbb{Q}$, the Reeb flow actually generates a $\mathbb{S}^1$-action on M. We also observe that it is possible that a multiple cover (i.e., an iterate) of a non-degenerate periodic Reeb orbit can become degenerate.

Exercise: For any $a \in \mathbb{R}$, we denote by $[a] \in \mathbb{Z}$ the unique integer such that $a - 1 < [a] \leq a$. Consider the contact form λ in Example 3.11, where we assume that a_1/a_2 is irrational. Let $x(t)$ be a periodic orbit given by $r_1 = 0$, whose period equals k times of the minimal period for some $k > 0$. With the natural trivialization of $(\xi, d\lambda)|_{x(t)}$ mentioned in Example 3.11, show that the Conley-Zehnder index of $x(t)$ equals $2[\frac{a_1 k}{a_2}] + 1$.

Let $C := \tilde{u}(\dot{S})$ be an unparametrized finite energy surface, $\tilde{u} = (a, u) : \dot{S} \to \mathbb{R} \times M$ be a parametrization of C. Let $\Gamma = \{z_i\}$ be the set of punctures, and for each i, let

$x_i = x_i(t)$ be the periodic orbit of period $T_i > 0$ which is the limit of u at the puncture z_i. We assume all of x_i are non-degenerate. We shall define the Conley-Zehnder index of C, denoted by $\mu(C)$, as follows.

First, we denote by $\overline{\dot{S}}$ a compatification of $\dot{S}$ obtained by adding a circle at each puncture. Clearly, the map $u : \dot{S} \to M$ can be extended to a map $\bar{u} : \overline{\dot{S}} \to M$. With this understood, we fix a trivialization Ψ of the symplectic vector bundle $\bar{u}^*\xi \to \overline{\dot{S}}$. It gives rise to a trivialization Ψ_i of ξ over each periodic orbit $x_i(t)$. We define

$$\mu(C) := \sum_{z_i \in \Gamma^+} \mu(x_i(t), \Psi_i) - \sum_{z_i \in \Gamma^-} \mu(x_i(t), \Psi_i).$$

Exercise: Show that $\mu(C)$ is well-defined, i.e., it is independent of the choices of the parametrization $\tilde{u}$ and the trivialization Ψ.

With the preceding understood, one can introduce an appropriate Banach space $\mathcal{J}$ of almost complex structures J (and hence the almost complex structures $\tilde{J}$). We let $\mathcal{M}$ be the set of pairs $(C, \tilde{J})$ where C is an unparametrized finite energy surface which is $\tilde{J}$-holomorphic, $\tilde{J} \in \mathcal{J}$, and whose asymptotic limits at the punctures are non-degenerate periodic orbits.

Theorem 3.12. *The set $\mathcal{M}$ carries a structure of a separable Banach manifold such that the map $\eta : \mathcal{M} \to \mathcal{J}$ sending $(C, \tilde{J})$ to $\tilde{J}$ is a Fredholm map. Moreover, at each $(C, \tilde{J}) \in \mathcal{M}$, the Fredholm index of η is given by*

$$\textit{Ind}\ (C) := \mu(C) + (n-2)(\chi(S) - \#\Gamma),$$

where $n = \frac{1}{2}(\dim M - 1)$, and C is parametrized by a $\tilde{J}$-holomorphic map $\tilde{u} : S \setminus \Gamma \to \mathbb{R} \times M$. ($\chi(S)$ is the Euler characteristic of S and $\#\Gamma$ the number of punctures.)

Corollary 3.13. *For a generic $\tilde{J} \in \mathcal{J}$, $\eta^{-1}(\tilde{J})$ is a union of finiate dimensional smooth manifolds. Moreover, for any $C \in \eta^{-1}(\tilde{J})$ which is not a $\tilde{J}$-holomorphic cylinder associated to a periodic orbit of the Reeb vector field, the following holds:*

$$\textit{Ind}\ (C) := \mu(C) + (n-2)(\chi(S) - \#\Gamma) \geq 1.$$

Note that the translations in the $\mathbb{R}$-factor of $\mathbb{R} \times M$ defines a $\mathbb{R}$-action on $\mathcal{M}$, whose fixed points are precisely the $\tilde{J}$-holomorphic cylinders associated to a periodic orbit of the Reeb vector field. The inequality follows from this fact.

4. Contact topology in dimension 3

4.1. **Existence.** It is a nontrivial question as if every closed, orientable 3-manifold admits a contact structure.

Theorem 4.1. *(Martinet) Every closed, orientable 3-manifold admits a co-orientable contact structure.*

Lemma 4.2. *Let ξ_0 be a co-orientable contact structure on a 3-manifold M_0, let K be a transverse knot in (M_0, ξ_0). Suppose M is a 3-manifold obtained from a Dehn surgery along K. Then there is a co-orientable contact structure ξ on M, which coincides with ξ_0 off a neighborhood of K in M_0.*

Proof. Pick a contact form α_0 such that $\xi_0 = \ker \alpha_0$. Then by Theorem 2.6, there is an identification of a regular neighborhood of K in M_0 with $\mathbb{S}^1 \times D(\delta)$ for some $\delta > 0$, where $D(\delta) \subset \mathbb{R}^2$ is the open disc of radius δ, such that over $\mathbb{S}^1 \times D(\delta)$,

$$e^h \alpha_0 = d\gamma + r^2 d\theta$$

for some smooth function h, where $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ is the coordinate on the $\mathbb{S}^1$-factor, and (r, θ) is the polar coordinate on $D(\delta)$. We extend h to the whole M_0 and replace α_0 by the contact form $e^h\alpha_0$, which is still denoted by α_0 for simplicity. Then $\alpha_0 = d\gamma + r^2 d\theta$ in the neighborhood $\mathbb{S}^1 \times D(\delta)$ of K.

The Dehn surgery amounts to removing the regular neighborhood $\mathbb{S}^1 \times D(\delta)$ of K in M_0, and then gluing it back along the boundary $\mathbb{T}^2 =: \mathbb{S}^1 \times \partial D(\delta)$ via some diffeomorphism $\phi : \mathbb{T}^2 \to \mathbb{T}^2$, where

$$\phi : (\gamma, \theta) \mapsto (q\theta + n\gamma, p\theta + m\gamma)$$

for some $p, q, m, n \in \mathbb{Z}$ with $pn - mq = 1$. (Note that the meridian $\{*\} \times \partial D(\delta)$ goes to p times of the meridian $\{*\} \times \partial D(\delta)$ and q times of the longitude $\mathbb{S}^1 \times \{*\}$, so the resulting 3-manifold M depends only on the integers p, q.)

We will consider contact 1-form α on $\mathbb{S}^1 \times D(\delta)$ of the form

$$\alpha := f(r)d\gamma + g(r)d\theta,$$

such that $\alpha = \phi^*\alpha_0 = (n + mr^2)d\gamma + (q + pr^2)d\theta$ near $r = \delta$, and $\alpha = c(d\gamma + r^2 d\theta)$ for $0 \leq r \leq \frac{1}{2}\delta$ for some small constant $c > 0$. Note that $\alpha \wedge d\alpha > 0$ is equivalent to $fg' - gf' > 0$, which is equivalent to the ratio g/f being strictly increasing as r increases. In particular, one can obtain such a 1-form α by connecting the point $(n+m\delta^2, q+p\delta^2) \in \mathbb{R}^2$ to the point $(c, \frac{1}{2}c\delta^2) \in \mathbb{R}^2$ by a smooth curve $(f(r), g(r))$, which runs clockwise around the origin as r decreases from δ to $\frac{1}{2}\delta$. With this understood, we obtain a contact form on M which equals α_0 on the M_0 part and equals α on $\mathbb{S}^1 \times D(\delta)$. The contact structure $\xi = \ker \alpha$.

□

Recall that every closed, orientable 3-manifold can be obtained from $\mathbb{S}^3$ via a sequence of Dehn surgeries, and note that $\mathbb{S}^3$ admits a contact structure, Theorem 4.1 follows immediately if one can show that every knot is isotopic to a transverse knot.

Lemma 4.3. *Every knot is isotopic to a transverse knot.*

Proof. There are two steps for the proof. For Step 1, we show every knot admits a C^0-approximation by a Legendrian knot, which is clearly isotopic to the original knot.

Let K be the knot. We cover K by finitely many Darboux charts (cf. Theorem 2.3), over which the contact structure can be identified with the standard contact structure on $\mathbb{R}^3$, defined by the contact form $\alpha_0 = dz + xdy$. The C^0-approximation of K is then obtained by constructing a C^0-approximation over each Darboux chart by a Legendrian curve.

With this understood, note that a parametrized curve $\gamma(s)$ in $\mathbb{R}^3$, where $\gamma(s) = (x(s), y(s), z(s))$, is Legendrian if and only if $z'(s) + x(s)y'(s) = 0$, or if $y'(s) \neq 0$,

$$x(s) = -\frac{z'(s)}{y'(s)} = -\frac{dz}{dy}.$$

The projection of $\gamma(s)$ under $(x,y,z)\mapsto(y,z)$ is called the **front projection**. Then it follows easily that any smooth curve in the yz-plane, which is immersed with only transverse self-intersections and no vertical tangent lines, except at finitely many cusp points, where at each cusp point, the curve can be parametrized by

$$(*) \quad (y(s),z(s))=(\lambda s^2+b,-\lambda(\frac{2}{3}s^3+as^2)+c), \;\; \lambda\neq 0, \; -\epsilon<s<\epsilon,$$

is the front projection of a smoothly embedded Legendrian curve in $\mathbb{R}^3$. Note that in the model of a cusp point in $(*)$, the point above the cusp point, i.e., $(x(0),y(0),z(0))$, can be chosen to be any point (a,b,c) in $\mathbb{R}^3$. Moreover, the x-coordinate of the Legendrian curve is given by $x(s)=s+a$. Note also that the cusp points occur at exactly the images of the Legendrian curve where the y-coordinate has a local maximum or a local minimum under the front projection.

Now given any smooth curve $\gamma(s)=(x(s),y(s),z(s))$, we pick a C^0-approximation of $(y(s),z(s))$ by an immersed curve $(\tilde{y}(s),\tilde{z}(s))$, with only transverse self-intersections and no vertical tangent lines, with finitely many cusp points modeled as in $(*)$ above, such that the lifting x-coordinate under the front projection, i.e., $-\frac{\tilde{z}'(s)}{\tilde{y}'(s)}$, is C^0-close to $x(s)$. Then the corresponding Legendrian curve is a C^0-approximation of $\gamma(s)$.

For Step 2, we show that any Legendrian knot K admits a C^∞-approximation by a transverse knot. To this end, note that by Theorem 2.4, a neighborhood of K can be modeled by $(\mathbb{S}^1\times D(\delta),\xi)$, with $\xi=\ker\alpha$, where, with $\theta\in\mathbb{S}^1$, $(x,y)\in D(\delta)$,

$$\alpha=\cos\theta dx-\sin\theta dy,$$

with K being identified with $\mathbb{S}^1\times\{0\}$. Then for any $0<r<\delta$, the knots

$$\gamma_\pm(\theta)=(\theta,\pm r\sin\theta,\pm r\cos\theta), \;\; \theta\in\mathbb{S}^1,$$

are transverse; note that $\alpha(\gamma_\pm'(\theta))=\pm r$. In particular, γ_+ is a positive transverse knot and γ_- is a negative transverse knot.

□

4.2. Overtwisted contact structures and homotopy classes of 2-plane fields.

Definition 4.4. Let M be a closed, orientable 3-manifold, ξ be an co-orientable contact structure on M. Let $\Delta\subset M$ be a smoothly embedded disc. We say Δ is an **overtwisted disc** if for any $p\in\partial\Delta$, $\xi_p=T_p\Delta$. In particular, $\partial\Delta$ is a Legendrian knot in M. The contact structure ξ is called **overtwisted** if there exists an overtwisted disc in M. Otherwise, ξ is called **tight**.

Example 4.5. Consider the contact structure ξ on $\mathbb{S}^1\times D(\delta)$ in the proof of Lemma 4.2, which is defined by a contact form α of the following form:

$$\alpha=f(r)d\gamma+g(r)d\theta.$$

Note that the vector field $\frac{\partial}{\partial r}\in\xi$. Thus in order to visualize the contact structure ξ, we consider its intersection with a family of tori T_r parametrized by r, where $T_r:=\mathbb{S}^1\times\partial D(r)$. The intersection $\xi\cap TT_r$ generates a foliation of T_r by Legendrian curves, and it is easy to see that in the coordinate (θ,γ), the Legendrian curves on T_r has a constant slope $-g(r)/f(r)$. It follows that as we move towards the core of the solid torus $\mathbb{S}^1 \times D(\delta)$, i.e., as r decreases to 0, the Legendrian curves on T_r will twist by turning around counter-clockwise. Moreover, note that the Legendrian curves on T_r will become meridians, i.e., of slope 0, if and only if $g(r) = 0$. In particular, if there is a $r_0 > 0$ such that $g(r_0) = 0$, then the intersection $(\{*\} \times D(\delta)) \cap T_{r_0}$ bounds an overtwisted disc $\Delta := \{*\} \times D(r_0)$ in the contact manifold $(\mathbb{S}^1 \times D(\delta), \xi)$.

One can change the contact structure ξ by adding one or more full twists to the Legendrian curves on T_r as r decreases to 0. (Such a twist is called a **Lutz twist**.) It is clear that the resulting contact structure, denoted by $\tilde{\xi}$, is always overtwisted. More concretely, $\tilde{\xi}$ is a contact structure defined by a contact form $\tilde{\alpha} = \tilde{f}(r)d\gamma + \tilde{g}(r)d\theta$, where the functions $\tilde{f}(r)$, $\tilde{g}(r)$ are chosen such that the curve $(\tilde{f}(r), \tilde{g}(r))$ runs one or more full rounds around the origin clockwise than the curve $(f(r), g(r))$ as $r \downarrow 0$.

Theorem 4.6. *(Eliashberg) The set of co-oriented overtwisted contact structures up to contact isotopy is in one to one correspondence with the set of homotopy classes of oriented 2-plane fields, where the correspondence sends each contact structure to the underlying 2-plane field.*

The set of homotopy classes of oriented 2-plane fields on a closed, oriented 3-manifold M can be easily described. The basic fact is that M has trivial tangent bundle. After fixing an identification $TM = M \times \mathbb{R}^3$, where $\mathbb{R}^3$ is oriented, each oriented 2-plane field on M determines a map from M to $\mathbb{S}^2$, sending $x \in M$ to the oriented unit vector in $\mathbb{R}^3$ perpendicular to the 2-plane at x. Consequently, the set of homotopy classes of oriented 2-plane fields on M is identified with the set of homotopy classes of maps from M to $\mathbb{S}^2$. For example, the set of homotopy classes of oriented 2-plane fields on $\mathbb{S}^3$ is identified with $\pi_3(\mathbb{S}^2) = \mathbb{Z}$.

Example 4.7. We consider the standard contact structure ξ_0 on $\mathbb{S}^3$, where $\mathbb{S}^3 \subset \mathbb{C}^2$ and $\xi_0 := T\mathbb{S}^3 \cap J(T\mathbb{S}^3)$ is the field of complex tangency. We shall determine the map $F : \mathbb{S}^3 \to \mathbb{S}^2$ which corresponds to ξ_0.

To this end, we fix a trivialization of $T\mathbb{S}^3$ as follows. We identify $\mathbb{S}^3$ with the Lie group of unit quaternions under $\mathbb{C}^2 = \mathbb{H}$ with $(z_1, z_2) = z_1 + z_2 j$. Then the left-invariant vector fields X_1, X_2, X_3, where for any $p = (z_1, z_2) \in \mathbb{S}^3$, $X_1(p) = (iz_1, -iz_2)$, $X_2(p) = (-z_2, z_1)$, and $X_3(p) = (iz_2, iz_1)$, defines a trivialization of $T\mathbb{S}^3$, and X_1, X_2, X_3 is a positively oriented global frame with respect to the orientation on $\mathbb{S}^3$.

Now let $a, b, c \in \mathbb{R}$ such that $aX_1(p) + bX_2(p) + cX_3(p) \in \xi_0|_p$. This happens if and only if $J(aX_1(p) + bX_2(p) + cX_3(p)) \in T_p\mathbb{S}^3$, which means $aX_1(p) + bX_2(p) + cX_3(p)$ is orthogonal to the vector $p \in \mathbb{S}^3$. This condition can be expressed as

$$(|z_2|^2 - |z_1|^2)a + (iz_1\bar{z}_2 - iz_2\bar{z}_1)b - (z_1\bar{z}_2 + z_2\bar{z}_1)c = 0.$$

Equivalently, $2Re(z_1\bar{z}_2)c + 2Im(z_1\bar{z}_2)b + (|z_1|^2 - |z_2|^2)a = 0$. Now observe that, with $|z_1|^2 + |z_2|^2 = 1$, the vector $(2Re(z_1\bar{z}_2), 2Im(z_1\bar{z}_2), |z_1|^2 - |z_2|^2)$ is a unit vector. Finally, $(2Re(z_1\bar{z}_2), 2Im(z_1\bar{z}_2), |z_1|^2 - |z_2|^2)$ is orthogonal to $\xi_0|_p$, thus we can define the map $F : \mathbb{S}^3 \to \mathbb{S}^2$ by

$$F(z_1, z_2) = (2Re(z_1\bar{z}_2), 2Im(z_1\bar{z}_2), |z_1|^2 - |z_2|^2).$$

In order to understand the map $F : \mathbb{S}^3 \to \mathbb{S}^2$, we observe $F(\lambda z_1, \lambda z_2) = F(z_1, z_2)$ for any $\lambda \in \mathbb{S}^1$, which implies that F factors through the Hopf fibration $H : \mathbb{S}^3 \to \mathbb{CP}^1$

to a map $\bar{F}:\mathbb{CP}^1\to\mathbb{S}^2$. With this understood, note that $\bar{F}([1:0])=(0,0,1)$, so we shall consider next the case $\bar{F}([z_1:z_2])$ where $z_2\neq 0$. For any $0<r\leq 1$, let $(z_1,z_2)=(\sqrt{1-r^2}e^{i\theta},r)$, and identify $\mathbb{CP}^1\setminus\{[1:0]\}$ with $\mathbb{C}$ by sending $[z_1:z_2]$ to $z=\rho e^{i\theta}$, where $\rho=\frac{\sqrt{1-r^2}}{r}$. Then under this identification,

$$\bar{F}(\rho e^{i\theta})=(\frac{2\rho\cos\theta}{1+\rho^2},\frac{2\rho\sin\theta}{1+\rho^2},\frac{\rho^2-1}{1+\rho^2}),$$

which is the inverse of the stereographic projection. Hence in conclusion, the map $F:\mathbb{S}^3\to\mathbb{S}^2$ is given by the Hopf fibration after identifying $\mathbb{CP}^1$ with $\mathbb{S}^2$ canonically. (Note that the Hopf fibration defines a generator of $\pi_3(\mathbb{S}^2)=\mathbb{Z}$.)

In contrast to overtwisted contact structures, the existence and classification of tight contact structures are much more subtle. The standard contact structure ξ_0 on $\mathbb{S}^3$ is tight, and it is known that ξ_0 is the only tight contact structure on $\mathbb{S}^3$.

4.3. **Symplectic fillings.** A fundamental fact is that fillable contact structures on a closed 3-manifold are always tight.

Definition 4.8. Let ξ be a co-oriented contact structure on a closed 3-manifold M.

(1) Let W be a compact complex surface with boundary. We say that W is a **holomorphic filling** of (M,ξ) if $M=\partial W$ is a strictly pseudoconvex boundary of W (i.e. there exists a real smooth function ρ defined near M such that M is a regular level surface, with ρ being increasing along the outward normals of M, and the Levi form of ρ is positive definite), and $\xi=TM\cap J(TM)$. In this case, (M,ξ) is called **holomorphically fillable**. Furthermore, W is called a **Stein filling** if W is a Stein domain (i.e., the function ρ is a globally defined, strictly plurisubharmonic function on W), and in this case, (M,ξ) is called **Stein fillable**.

(2) Let (W,ω) be a compact symplectic 4-manifold with $\partial W=M$. We say that (W,ω) is a **strong symplectic filling** of (M,ξ) if there is a Liouville vector field X defined near M pointing outward, such that the 1-form $\alpha:=i_X\omega|_M$ is a contact form defining ξ. In this case, (M,ξ) is called **strongly (symplectically) fillable**. More generally, if $\omega|_\xi>0$, then (W,ω) is called a **weak symplectic filling** of (M,ξ), and (M,ξ) is called **weakly (symplectically) fillable**.

Remark 4.9. It is known that (M,ξ) is holomorphically fillable if and only if it is Stain fillable (possibly by different compact complex surfaces with boundary). On the other hand, (M,ξ) is Stain fillable if and only if it is the (contact) boundary of a Weinstein domain (cf. Definition 4.4 in §4 of Part 2).

Exercise: Let (W,ω) be a strong symplectic filling of both (M,ξ_1) and (M,ξ_2). Show that ξ_1,ξ_2 are contact isotopic.

The following inclusions are easy to see:

$$\{Stein\ fillable\}\subset\{strongly\ fillable\}\subset\{weakly\ fillable\}\subset\{tight\}.$$

It is also known that each of the above inclusions is a proper subset. However, when M is a rational homology sphere, strongly fillable and weakly fillable are equivalent (see Prop. 4.11 below). It is also known that there are closed, orientable 3-manifolds which do not admit any tight contact structures.

Example 4.10. (1) The standard contact structure ξ_0 on $\mathbb{S}^3$ is both holomorphically fillable and strongly (symplectically) fillable, by the unit 4-ball.

(2) Let Σ be a compact Riemann surface, M be the $\mathbb{S}^1$-bundle associated to $\pi : T^*\Sigma \to \Sigma$, naturally embedded in $T^*\Sigma$ as a compact hypersurface. Note that if $\Sigma = \mathbb{S}^2$, then $M = \mathbb{RP}^3$, and for $\Sigma = \mathbb{T}^2$, $M = \mathbb{T}^3$.

Let $\omega = -d\lambda$ be the standard symplectic structure on $T^*\Sigma$, where λ is the canonical 1-form on $T^*\Sigma$. Consider the Liouville vector field X associated to $-\lambda$, i.e., $i_X\omega = -\lambda$. Then X is transversal to M. To see this, note that in a local coordinate system x_1, x_2 on Σ, $\lambda = y_1 dx_1 + y_2 dx_2$, and $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. With this understood, $X = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$, which shows that X is transversal to M. Let $\xi := \ker(i_X\omega|_M)$ be the contact structure on M, and let $W \subset T^*\Sigma$ be the compact domain bounded by M. Then (W, ω) is a strong symplectic filling of (M, ξ).

In fact, (W, ω) is actually a Weinstein domain, so (M, ξ) is even Stein fillable. To see this, we fix a Riemannian metric $\langle , \rangle$ on Σ, and pick a Morse function $f : \Sigma \to \mathbb{R}$. Let ∇f be its gradient vector. Let $\tilde{X}$ be the Liouville vector field associated to the 1-form $-\lambda + \epsilon dF$, where $F : T^*\Sigma \to \mathbb{R}$ is defined by $F = \lambda(\nabla f)$, and $\epsilon > 0$ is chosen sufficiently small. Then $\tilde{X}$ is transversal to M, and is gradient-like with respect to the Morse function $\phi : T^*\Sigma \to \mathbb{R}$, where for any $v \in T^*\Sigma$, $\phi(v) := ||v||^2 + \rho(v) \cdot (\epsilon f(\pi(v)))$ for some cut-off function ρ defined on a small neighborhood of the zero-section in $T^*\Sigma$.

Proposition 4.11. *(Eliashberg) Let (W, ω) be a weak symplectic filling of (M, ξ). Suppose the de Rham cohomology class $[\omega|_M] = 0$. Then there is a symplectic form ω' on W and a contact structure ξ' on M, such that (1) (W, ω') is a strong symplectic filling of (M, ξ'), (2) ξ' is contact isotopic to ξ.*

Proof. We fix a parametrization of a small neighborhood of M in W by $(1-\epsilon, 1] \times M$ as follows: let $\xi^\perp \subset TW|_M$ be the symplectic complement of ξ with respect to ω. Since $\omega|_M > 0$ on ξ, we have $TW|_M = \xi \oplus \xi^\perp$; in particular, we can pick an outward normal vector field $v \in \xi^\perp$. We use v to identify a small neighborhood of M in W with $(1-\epsilon, 1] \times M$ such that $v = \frac{\partial}{\partial t}$ on M.

Next we fix a contact form α of ξ. Since (W, ω) is a weak symplectic filling of (M, ξ), by choosing $\epsilon > 0$ small enough, $\omega|_{\{t\} \times M} > 0$ on ξ for any $t \in (1-\epsilon, 1]$. Consequently, there is a smooth family of smooth functions $f_t > 0$ on M such that $\omega|_{\{t\} \times M} = f_t d\alpha$ when restricted to ξ. Set $\tilde{\alpha} = f_t \alpha$, which is regarded as a 1-form on $(1-\epsilon, 1] \times M$. Then on $(1-\epsilon, 1] \times M$

$$\omega = d_M\tilde{\alpha} + \tilde{\alpha} \wedge \beta_1(t) + g dt \wedge \tilde{\alpha} + dt \wedge \beta_2(t),$$

for some $\beta_1(t), \beta_2(t) \in \Omega^1(M)$ and a smooth function g, where d_M is the exterior differential on M. Note that $\beta_2(t) = 0$ at $t = 1$ as $\frac{\partial}{\partial t} = v \in \xi^\perp$. Consequently,

$$\omega \wedge \omega|_M = 2g|_M \cdot (d_M\tilde{\alpha} \wedge dt \wedge \tilde{\alpha})|_M > 0,$$

which implies that $g > 0$ as long as we choose $\epsilon > 0$ small.

Now pick a smooth function $\rho : [1-\epsilon, 1] \to \mathbb{R}$ such that $\rho(t) \geq 0$, $\rho'(t) \geq 0$, $\rho(t) = 0$ near $t = 1-\epsilon$ and $\rho(t) = 1$ near $t = 1$. For any constant $C > 0$, we let

$$\omega' := \omega + Cd(\rho\tilde{\alpha}).$$

Then it is easy to check that $\omega' \wedge \omega' > 0$ (here we use the fact that $g > 0$ and $\beta_2(t)$ is small near $t = 1$, so that $\omega \wedge d(\rho\tilde{\alpha}) \geq 0$).

Since $[\omega|_M] = 0$, there is a 1-form γ on M such that $\omega|_M = d\gamma$. Then notice that $\omega'|_M = d(\gamma + C\tilde{\alpha}|_M)$, and for large $C > 0$, the 1-form $\gamma + C\tilde{\alpha}|_M$ is a contact form. We set $\xi' := \ker(\gamma + C\tilde{\alpha}|_M)$. Then (W, ω') is a strong symplectic filling of (M, ξ'). Furthermore, $\lambda_s := s\gamma + C\tilde{\alpha}|_M$, $s \in [0, 1]$, is a smooth family of contact forms, which gives rise to a contact isotopy between ξ and ξ'. This finishes the proof.

□

4.4. **Open book decompositions.** Let M be a closed, oriented 3-manifold. An **open book decomposition** of M, denoted by (B, π), consists of the following data: $B = \sqcup_i B_i$ is an oriented link in M (called the **binding**), and $\pi : M \setminus B \to \mathbb{S}^1$ is a fibration with the following property, i.e., for each i, if we fix a regular neighborhood N_i of B_i in M, then for each $\theta \in \mathbb{S}^1$, the fiber $\pi^{-1}(\theta)$ (called a **page**) intersects with each torus ∂N_i in a longitude λ_i. The longitude λ_i determines an identification of N_i with $\mathbb{S}^1 \times D^2$. The restriction of π to $M \setminus \sqcup_i N_i$ defines it as a fiber bundle over $\mathbb{S}^1$ whose fibers are diffeomorphic to a compact Riemann surface with boundary Σ. There is a $1 : 1$ correspondence between the components of $\partial\Sigma$ and the components of $B = \sqcup_i B_i$. We assume that the boundary orientation of $\partial\Sigma$ matches with the orientation of B under this correspondence. With the orientation of M, Σ determines a co-orientation, which orients the base $\mathbb{S}^1$ of the fibration π. This allows us to identify $\mathbb{S}^1$ with $[0, 1]/0 \sim 1$. Finally, the identification of N_i with $\mathbb{S}^1 \times D^2$ induces an identification of ∂N_i with $\mathbb{S}^1 \times \mathbb{S}^1$, which determines a trivialization of the fiber bundle $\pi : M \setminus \sqcup_i N_i \to \mathbb{S}^1$ on the boundary of $M \setminus \sqcup_i N_i$. It follows that

$$M \setminus \sqcup_i N_i = \Sigma \times [0, 1]/(x, 1) \sim (h(x), 0)$$

for some diffeomorphism $h : \Sigma \to \Sigma$ such that $h = Id$ on $\partial\Sigma$. Moreover, h is uniquely determined up to isotopy rel $\partial\Sigma$. It is easy to see that one can completely recover the 3-manifold M and the open book decomposition (B, π) from the data (Σ, h).

Example 4.12. (1) Consider $M = \mathbb{S}^3 \subset \mathbb{C}^2$, oriented as the boundary of the unit ball, and the function $f(z_1, z_2) = z_1 z_2$. Let $B = \mathbb{S}^3 \cap f^{-1}(0)$, which consists of two components $B_i = \{z_i = 0\}$, where $i = 1, 2$. The map $\pi : M \setminus B \to \mathbb{S}^1$ defined by $\pi(z_1, z_2) = f(z_1, z_2)/|f(z_1, z_2)|$ is an open book decomposition of M.

For each $e^{i\theta} \in \mathbb{S}^1$, the page $\pi^{-1}(e^{i\theta})$ is an annulus. An explicit parametrization (as oriented surfaces) is given by $\phi_\theta : (0, 1) \times \mathbb{S}^1 \to \pi^{-1}(e^{i\theta})$, sending (r, e^{it}) to $(re^{it}, \sqrt{1 - r^2} e^{i(\theta - t)})$. Note that from this parametrization, it is easily seen that the orientation of B is given by the natural orientation as unit circles $B_i \subset \mathbb{C}$.

In order to see the monodromy map, we consider the $\mathbb{S}^1$-action on $\mathbb{S}^3$, defined by $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2)$. Note that $\pi(\lambda \cdot (z_1, z_2)) = \lambda^2 \pi(z_1, z_2)$. It follows easily that the flow of the $\mathbb{S}^1$-action is transverse to the pages $\pi^{-1}(e^{i\theta})$, and the return map is given by the multiplication by $\lambda = -1$. Since the flow of the $\mathbb{S}^1$-action moves in the positive direction of the binding B, the monodromy h is given by a right-handed Dehn twist of the annulus $[\epsilon, 1 - \epsilon] \times \mathbb{S}^1$.

Similarly, one can consider the function $g(z_1, z_2) = z_1 \bar{z}_2$ instead. The binding $B = \mathbb{S}^3 \cap g^{-1}(0)$ still consists of two components $B_i = \{z_i = 0\}$, $i = 1, 2$. In this case, an oriented parametrization of the pages is given by $(r, e^{it}) \mapsto (re^{i(\theta-t)}, \sqrt{1-r^2}e^{-it})$. Note that with this parametrization, the orientation of B_1 is given by the complex orientation as the unit circle in $\mathbb{C}$, but B_2 is given with the opposite orientation. The flow of the $\mathbb{S}^1$-action $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2)$ is transverse to the pages, and the return map is given again by the multiplication by $\lambda = -1$. However, since the flow of the $\mathbb{S}^1$-action is in the opposite direction of the binding components, the monodromy h is given by a left-handed Dehn twist of the annulus $[\epsilon, 1-\epsilon] \times \mathbb{S}^1$ this time.

(2) Let $M = \mathbb{S}^3 \subset \mathbb{C}^2$, oriented as the boundary of the unit ball, and let $f(z_1, z_2) = z_1^2 - z_2^3$. The binding $B = \mathbb{S}^3 \cap f^{-1}(0)$ is a $(2,3)$-torus knot, parametrized by $(r_1 e^{3it}, r_2 e^{2it})$, where $r_1^2 = r_2^3$ and $r_1^2 + r_2^2 = 1$. Consider the $\mathbb{S}^1$-action on $M = \mathbb{S}^3$ given by $\lambda \cdot (z_1, z_2) = (\lambda^3 z_1, \lambda^2 z_2)$. Then $f(\lambda \cdot (z_1, z_2)) = \lambda^6 f(z_1, z_2)$. It follows easily that the flow of the $\mathbb{S}^1$-action is transverse to the pages of the open book decomposition $\pi = f/|f| : M \setminus B \to \mathbb{S}^1$, with the return map given by the multiplication of $\lambda = \exp(2\pi i/6)$. Note that the return map acts on B as the translation $t \mapsto t + \pi/3$. It follows that the page Σ of $\pi : M \setminus N \to \mathbb{S}^1$ is $\mathbb{T}^2$ with a disc removed, and the monodromy $h : \Sigma \to \Sigma$ is a periodic diffeomorphism of order 6, composed with a twisting near its boundary undoing the translation $t \mapsto t + \pi/3$ (so $h = Id$ near $\partial\Sigma$).

Definition 4.13. Let ξ be a co-oriented contact structure on M, and let (B, π) be an open book decomposition on M. We say that ξ is **supported** by (B, π) if there is a contact form α associated to ξ such that $d\alpha > 0$ on the pages $\pi^{-1}(\theta)$ and $\alpha > 0$ on the binding B.

Exercise: Show that the standard contact structure ξ_0 on $\mathbb{S}^3$ is supported by the first open book decomposition in Example 4.12(1) and it is also supported by the open book decomposition in Example 4.12(2).

Theorem 4.14. *(1) Let (B, π) be an open book decomposition of M. Then there is a contact structure ξ on M such that ξ is supported by (B, π). Moreover, any two contact structures supported by (B, ξ) are contact isotopic.*

(2) (Giroux) Let ξ be a co-oriented contact structure on M. Then there is an open book decomposition (B, π) which supports ξ.

Part (1) of Theorem 4.14 gives an alternative proof for the existence of contact structures on a closed orientable 3-manifold, as it is known that every closed oriented 3-manifold admits an open book decomposition. On the other hand, it is possible to characterize contact structures in terms of an open book decomposition that supports the contact structure. For example, it is known that a contact structure is Stein fillable if and only if it is supported by an open book decomposition whose monodromy map is the product of finitely many right-handed Dehn twists.

Example 4.15. Let M be the link of an isolated normal surface singularity (X, x). Then the plane field of complex tangency on M defines a contact structure, which is known to be Stein fillable. The diffeomorphism type of M depends only on the topological type of the singularity, but there could be different analytical structures for the singularity. It turns out that the contact structures defined by different analytic structures are all contactomorphic; this contact structure is called the **Milnor fillable contact structure** on M.

The idea of proof is to consider holomorphic functions f on X which vanishes at x and is nonsingular on $X \setminus \{x\}$. For any such f, the map $\pi := f/|f| : M \setminus f^{-1}(0) \to \mathbb{S}^1$ defines an open book decomposition which supports the contact structure on M (called a **Milnor open book**). With this understood, one can show that for any analytic structure of the singularity, there is a Milnor open book which is isomorphic to a canonical open book on M. Then the claim follows from Theorem 4.14(1).

In the case where M is a Seifert manifold such that the Euler number of the Seifert fibration $e(M) < 0$, then M can be realized as the link of a weighted homogeneous singularity (X, x). There is a holomorphic $\mathbb{C}^*$-action on (X, x), inducing the $\mathbb{S}^1$-action on M. The Milnor fillable contact structure on M coincides with the $\mathbb{S}^1$-invariant contact structure on M (cf. Example 1.4 and the exercise following it).

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