MATH 705: PART 2: FUNDAMENTAL CONSTRUCTIONS AND METHODS

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Contents

1. Symplectic blowing up and symplectic cutting

In complex geometry the blowing-up operation amounts to replace a point in a space by the space of complex tangent lines through that point. It is a local operation which can be explicitly written down as follows.

Consider the blow-up of $\mathbb{C}^n$ at the origin. This is the complex submanifold of $\mathbb{C}^n \times \mathbb{CP}^{n-1}$:

$$\widetilde{\mathbb{C}}^n \equiv \{((z_1, z_2, \cdots, z_n), [w_1, w_2, \cdots, w_n]) | (z_1, z_2, \cdots, z_n) \in [w_1, w_2, \cdots, w_n]\}.$$

Note that as a set $\widetilde{\mathbb{C}}^n = \mathbb{C}^n \setminus \{0\} \sqcup \mathbb{CP}^{n-1}$, with $\mathbb{CP}^{n-1}$ being the space of complex tangent lines through the origin. $\mathbb{CP}^{n-1} \subset \widetilde{\mathbb{C}}^n$ is called the **exceptional divisor**. The projection onto the first factor $\mathbb{C}^n$, which is called blowing-down, induces a biholomorphism between $\widetilde{\mathbb{C}}^n \setminus \mathbb{CP}^{n-1}$ and $\mathbb{C}^n \setminus \{0\}$, collapsing the exceptional divisor $\mathbb{CP}^{n-1}$ onto the origin $0 \in \mathbb{C}^n$. On the other hand, the projection onto the second factor $\mathbb{CP}^{n-1}$ defines $\widetilde{\mathbb{C}}^n$ as a holomorphic line bundle, i.e., the so-called tautological line bundle over $\mathbb{CP}^{n-1}$.

Blowing up is often used in resolving singularities of complex subvarieties. We give some examples next to illustrate this.

Example 1.1. (1) Consider the complex curve $C \equiv \{(z_1, z_2) \in \mathbb{C}^2 | z_1 z_2 = 0\}$. It is the union of two lines $C_1 = \{z_1 = 0\}$ and $C_2 = \{z_2 = 0\}$ which intersect transversely at $0 \in \mathbb{C}^2$. Let us consider the pre-image of the portion of C_1, C_2 in $\mathbb{C}^2 \setminus \{0\}$ in the blowup $\widetilde{\mathbb{C}}^2$ of $\mathbb{C}^2$ at 0. It can be compactified into a complex curve $\widetilde{C}$ in $\widetilde{\mathbb{C}}^2$, which is a disjoint union of two smooth complex curves $\widetilde{C}_1$ and $\widetilde{C}_2$. The point here is that since C_1, C_2 intersect transversely at $0 \in \mathbb{C}^2$, the compactification $\widetilde{C}_1$ and $\widetilde{C}_2$ in $\widetilde{\mathbb{C}}^2$ are obtained by adding two distinct points in the exceptional divisor $\mathbb{CP}^1 \subset \widetilde{\mathbb{C}}^2$, which parametrizes the complex lines through $0 \in \mathbb{C}^2$. More concretely, note that

$$\widetilde{C}_1 = \{(0,z),[0,1]) \in \mathbb{C}^2 \times \mathbb{CP}^1 | z \in \mathbb{C}\},$$

and

$$\widetilde{C}_2 = \{(z,0),[1,0]) \in \mathbb{C}^2 \times \mathbb{CP}^1 | z \in \mathbb{C}\}.$$

We call $\widetilde{C}_1$, $\widetilde{C}_2$ the **proper transform** of C_1, C_2 under the blowing up $\widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$.

(2) Consider the complex curve $C = \{(z_1, z_2) \in \mathbb{C}^2 | z_1^2 = z_2^3\}$. It is called a cusp curve and is singular at $0 \in \mathbb{C}^2$. The proper transform $\widetilde{C}$ of C in the blowing up $\widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ is identified with

$$\widetilde{C} = \{(z^3, z^2), [z,1]) \in \mathbb{C}^2 \times \mathbb{CP}^1 | z \in \mathbb{C}\}.$$

Note that $\widetilde{C}$ is a smooth curve in $\widetilde{\mathbb{C}}^2$ because the projection of $\widetilde{C}$ onto the exceptional curve $\mathbb{CP}^1 \subset \widetilde{\mathbb{C}}^2$ is non-singular at $z = 0$. Note that $\widetilde{C}$ intersects the exceptional divisor $\mathbb{CP}^1$ at $z = 0$ with a tangency of order 2.

Exercise: Let $p := [0, 0, \cdots, 0, 1] \in \mathbb{CP}^n$. We let $\pi : \mathbb{CP}^n \setminus \{p\} \to \mathbb{CP}^{n-1}$ be the map which sends $[z_0, z_1, \cdots, z_n]$ to $[z_0, z_1, \cdots, z_{n-1}]$, where $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ via $[z_0, z_1, \cdots, z_{n-1}] \mapsto [z_0, z_1, \cdots, z_{n-1}, 0]$. Show that $\pi : \mathbb{CP}^n \setminus \{p\} \to \mathbb{CP}^{n-1}$ can be made into a holomorphic line bundle over $\mathbb{CP}^{n-1}$, which is isomorphic to the dual of the tautological line bundle.

As a consequence, note that $\widetilde{\mathbb{C}}^n$ is diffeomorphic to the connected sum $\mathbb{C}^n \# \overline{\mathbb{CP}^n}$. More generally, we have

Theorem 1.2. *Let X be an n-dimensional complex manifold. Then the blow up of X at one point is an n-dimensional complex manifold $\widetilde{X}$ which is diffeomorphic to $X \# \overline{\mathbb{CP}^n}$.*

Let (M, ω) be a symplectic manifold of dimension $2n$, and let $p \in M$ be a point. We would like to define a symplectic analog of blowing up of M at p. To this end, note that topologically $M \# \overline{\mathbb{CP}^n}$ can be obtained by removing a ball centered at p and then collapsing the boundary $\mathbb{S}^{2n-1}$ along the fibers of the Hopf fibration $\mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1}$. We will show that there is a canonical symplectic structure on the resulting manifold (depending on the "symplectic size" of the ball removed); this is a special case of the so-called **symplectic cutting** due to E. Lerman, which we shall describe next.

Let (M, ω) be a symplectic manifold equipped with a Hamiltonian $\mathbb{S}^1$-action, and let $h : M \to \mathbb{R}$ be a moment map of the $\mathbb{S}^1$-action and let ϵ be a regular value of h. For simplicity we assume that the $\mathbb{S}^1$-action on $h^{-1}(\epsilon)$ is free; this condition is unnecessary if one works with orbifolds. We introduce the following notations: we denote by $M_{h>\epsilon}$, $M_{h\geq\epsilon}$ the pre-images of (ϵ, ∞) and $[\epsilon, \infty)$ under $h : M \to \mathbb{R}$, and denote by $\overline{M_{h\geq\epsilon}}$ the manifold which is obtained by collapsing the boundary $h^{-1}(\epsilon)$ of $M_{h\geq\epsilon}$ along the orbits of the $\mathbb{S}^1$-action.

Theorem 1.3. *(E. Lerman). There is a natural symplectic structure ω_ϵ on $\overline{M_{h\geq\epsilon}}$ such that the restriction of ω_ϵ to $M_{h>\epsilon}\subset\overline{M_{h\geq\epsilon}}$ equals ω.*

Proof. Consider the symplectic product $(M\times\mathbb{C},\omega\oplus\omega_0)$ and the Hamiltonian $\mathbb{S}^1$-action on it given by

$$t\cdot(m,z)=(t\cdot m,e^{it}z),\ \ m\in M,z\in\mathbb{C}.$$

The moment map is $H(m,z)=h(m)-\frac{1}{2}|z|^2$.

Observe the following identification

$$\begin{aligned} H^{-1}(\epsilon) &= \{(m,z)|h(m)>\epsilon,|z|=\sqrt{2(h(m)-\epsilon)}\}\sqcup\{(m,0)|h(m)=\epsilon\}\\ &= M_{h>\epsilon}\times\mathbb{S}^1\sqcup h^{-1}(\epsilon). \end{aligned}$$

The theorem follows immediately from $H^{-1}(\epsilon)/\mathbb{S}^1=M_{h>\epsilon}\sqcup h^{-1}(\epsilon)/\mathbb{S}^1=\overline{M_{h\geq\epsilon}}$, and that the symplectic structure ω_ϵ on $H^{-1}(\epsilon)/\mathbb{S}^1$ equals ω when restricted to the open submanifold $M_{h>\epsilon}$.

□

Let $\delta>0$ and let $M=\{(z_1,z_2,\cdots,z_n)\in\mathbb{C}^n|\sum_{j=1}^n|z_j|^2<\delta\}$ be the ball of radius $\sqrt{\delta}$ in $\mathbb{C}^n$ centered at the origin. Given with the standard symplectic structure ω_0, M admits a Hamiltonian $\mathbb{S}^1$-action

$$t\cdot(z_1,z_2,\cdots,z_n)=(e^{-it}z_1,e^{-it}z_2,\cdots,e^{-it}z_n),$$

which has a moment map $h(z_1,z_2,\cdots,z_n)=\frac{1}{2}\sum_{j=1}^n|z_j|^2$. For any $0<\epsilon<\frac{\delta}{2}$, ϵ is a regular value of h such that the $\mathbb{S}^1$-action on $h^{-1}(\epsilon)$ is free. The symplectic manifold $\overline{M_{h\geq\epsilon}}$ is called the **symplectic blowing up** of M at $0\in M$ (of size 2ϵ). Note that $h^{-1}(\epsilon)/\mathbb{S}^1=\mathbb{CP}^{n-1}$, which is embedded in $\overline{M_{h\geq\epsilon}}$ as a symplectic submanifold; it is the symplectic analog of the exceptional divisor in the complex blowing up.

Exercise: Show that the normal bundle of the symplectic exceptional divisor $\mathbb{CP}^{n-1}$ in $\overline{M_{h\geq\epsilon}}$ is isomorphic to the tautological line bundle over $\mathbb{CP}^{n-1}$. Consequently, $\overline{M_{h\geq\epsilon}}=M\#\overline{\mathbb{CP}^n}$.

Symplectic blowing up of a general symplectic manifold can be defined by grafting the above construction into the manifold. More precisely, let (M,ω) be a symplectic manifold and let $p\in M$ be a point. By Darboux theorem, there exists a $\delta>0$ such that a neighborhood of p in M is symplectomorphic to the standard symplectic ball of radius $\sqrt{\delta}$. By Theorem 1.3, since the symplectic structure ω_ϵ on the symplectic blow-up of the standard symplectic ball of radius $\sqrt{\delta}$ equals the standard symplectic structure ω_0 when restricted to the shell region $\{(z_1,z_2,\cdots,z_n)\in\mathbb{C}^n|2\epsilon<\sum_{j=1}^n|z_j|^2<\delta\}$, one can graft it into the symplectic manifold (M,ω).

We remark that there are several other approaches to symplectic blowing up, each with different advantages.

Example 1.4. Let M be the 4-dimensional symplectic ball of radius $\sqrt{\delta}$ with standard symplectic structure ω_0. Let C_1,C_2 be two embedded symplectic surfaces in M which intersect at $0\in M$ such that in an open neighborhood of the closed ball of radius 2ϵ, C_1,C_2 are given by complex lines. Under this assumption notice that in the symplectic blowing up construction (i.e., the symplectic cutting), $h^{-1}(\epsilon)$ intersects each of C_1,C_2

at an orbit of the $\mathbb{S}^1$-action on $h^{-1}(\epsilon)$, which collapses to a point in $\overline{M_{h\geq\epsilon}}$. Thus the proper transform $\widetilde{C}_1$, $\widetilde{C}_2$ of C_1, C_2 in the symplectic blowing up $\overline{M_{h\geq\epsilon}}$ are simply the embedded symplectic surfaces obtained by removing a disc from C_1, C_2 and then collapsing the boundary to a point lying in the symplectic exceptional divisor. Clearly, the intersection of C_1, C_2 is resolved under the symplectic blowing up.

For the cusp curve $C = \{(z_1, z_2)|z_1^2 = z_2^3\} \subset M$, its proper transform $\widetilde{C}$ in $\overline{M_{h\geq\epsilon}}$ can be constructed, at least when $\epsilon > 0$ is sufficiently small. It is again an embedded symplectic disc intersecting the symplectic exceptional divisor at one point with a tangency of order 2. The construction of $\widetilde{C}$, however, is much more involved hence the details are left out here.

The reversing operation, called **symplectic blowing down**, can be defined, at least in low dimensions, i.e., 4 dimensions or 6 dimensions. More concretely, let M be a 4-dimensional symplectic manifold, $C \subset M$ be an embedded symplectic $\mathbb{S}^2$ with $C^2 = -1$. Then by the symplectic neighborhood theorem, a neighborhood of C in M is symplectomorphic to a neighborhood of the symplectic exceptional divisor in a symplectic blowing up. This allows us to remove C from M and then graft back a symplectic 4-ball of appropriate size (with the standard symplectic structure ω_0) into M. (Topologically, this has the effect of collapsing C to a point.) Similarly, let M be a 6-dimensional symplectic manifold, where $\mathbb{CP}^2$ is embedded in M as a symplectic submanifold whose normal bundle is isomorphic to the tautological line bundle of $\mathbb{CP}^2$. Then, by a highly nontrivial result that any symplectic structure on $\mathbb{CP}^2$ is equivalent to a constant multiple of the Kähler form of the Fubini-Study metric, a neighborhood of $\mathbb{CP}^2$ in M is again symplectomorphic to a neighborhood of the symplectic exceptional divisor in a symplectic blowing up. We can then remove the $\mathbb{CP}^2$ from M and put back a symplectic 6-ball of appropriate size.

We next discuss symplectic blowing up in the presence of a Hamiltonian torus action. Let (M, ω) be a symplectic manifold with a Hamiltonian $\mathbb{T}^n$-action, and let $\mu : M \rightarrow (\mathbf{t}^n)^*$ be the moment map. Suppose $\xi_0 \in \mathbf{t}^n$ generates a circle $\mathbb{T}^1 \subset \mathbb{T}^n$. Then the induced $\mathbb{S}^1$-action on M has the moment map $h = \langle \mu, \xi_0 \rangle$. Because the induced $\mathbb{S}^1$-action commutes with the $\mathbb{T}^n$-action, and both $M_{h>\epsilon}$ and $h^{-1}(\epsilon)$ are $\mathbb{T}^n$-invariant, there is an induced $\mathbb{T}^n$-action on $M_{h>\epsilon} \sqcup h^{-1}(\epsilon)/\mathbb{S}^1 = \overline{M_{h\geq\epsilon}}$, which is also Hamiltonian. The corresponding moment map $\mu_\epsilon : \overline{M_{h\geq\epsilon}} \rightarrow (\mathbf{t}^n)^*$ is induced from the restriction of μ on $M_{h\geq\epsilon}$. Hence the image of μ_ϵ in $(\mathbf{t}^n)^*$ is

$$\mu_\epsilon(\overline{M_{h\geq\epsilon}}) = \mu(M) \cap \{\xi^* \in (\mathbf{t}^n)^* | \langle \xi^*, \xi_0 \rangle \geq \epsilon\}.$$

Example 1.5. (Equivariant blowing-up at a fixed point). Consider the standard Hamiltonian $\mathbb{T}^2$-action on $\mathbb{CP}^2$ given by

$$(t_1, t_2) \cdot [z_0, z_1, z_2] = [z_0, e^{-it_1} z_1, e^{-it_2} z_2]$$

which is considered in Example 4.15(1) of Section 4 of Part 1. Here we double the symplectic form used in Example 4.15(1), and consequently the moment map becomes

$$\mu([z_0, z_1, z_2]) = (\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}),$$

and the image of μ is $\{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 \leq 1, x_1, x_2 \geq 0\}$.

Now consider the $\mathbb{S}^1$-action

$$t \cdot [z_0, z_1, z_2] = [z_0, z_1, e^{it} z_2],$$

which has moment map $h([z_0, z_1, z_2]) = -\frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2}$. The symplectic cut $\overline{\mathbb{CP}^2_{h\geq -\frac{1}{2}}}$, which is a symplectic blowing-up of $\mathbb{CP}^2$ at the fixed point $[0,0,1]$, has an induced Hamiltonian $\mathbb{T}^2$-action. The image of the corresponding moment map is

$$\{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 \leq 1, x_1 \geq 0, 0 \leq x_2 \leq \frac{1}{2}\}.$$

Note that this is the same as the image of the moment map in Example 4.15(4) in Section 4 of Part 1 (a $\mathbb{T}^2$-action on a Hirzebruch surface). By Delzant's classification theorem, these two $\mathbb{T}^2$-actions are equivalent.

Example 1.6. Here we describe another application of symplectic cutting. Let $\pi : Y \to \Sigma$ be a principal $\mathbb{S}^1$-bundle over a compact Riemann surface, and denote by E the associated complex line bundle over Σ. We fix any 1-form α on Y such that $i\alpha$ is a connection 1-form of the principal $\mathbb{S}^1$-bundle $\pi : Y \to \Sigma$. We denote by κ the 2-form on Σ such that $\pi^*\kappa = -d\alpha$. Note that the de Rham cohomology class of κ equals $2\pi c_1(E)$. Finally, we fix an area form σ on Σ.

Let $M := Y \times (-\epsilon, \epsilon)$, and consider $\omega := \pi^*(\sigma - t\kappa) + dt \wedge \alpha$, where t is the coordinate on the interval $(-\epsilon, \epsilon)$. It is easy to check that $d\omega = 0$, and moreover, ω is non-degenerate when $\epsilon > 0$ is sufficiently small.

Observe that the natural $\mathbb{S}^1$-action on M preserves ω. Furthermore, it is a Hamiltonian $\mathbb{S}^1$-action with a moment map $h(y,t) = t$, $\forall y \in Y$. With this understood, we consider the symplectic cutting $\overline{M_{h\geq 0}}$. It is easy to see that topologically, $\overline{M_{h\geq 0}}$ is diffeomorphic to a disc bundle of E. Moreover, Σ, embedded in $\overline{M_{h\geq 0}}$ as the zero-section, is a symplectic surface with normal bundle E. Finally, the symplectic form on $\overline{M_{h\geq 0}}$ restricts to the area form σ on Σ. With this understood, we remark that, by the symplectic neighborhood theorem, $\overline{M_{h\geq 0}}$ can serve as a model for the symplectic structure of a neighborhood of any symplectic embedding of Σ into a symplectic 4-manifold with normal bundle E whose total area equals $\int_\Sigma \sigma$.

Exercise: (Symplectic branched covering) Let M be a smooth 4-manifold, equipped with a smooth $\mathbb{Z}_n$-action, such that the fixed-point set of the action consists of a disjoint union of embedded surfaces $B = \cup_i B_i$, and the action is free in the complement of B. Let $\pi : M \to M'$ be the quotient of the action. Clearly, M' is naturally a smooth 4-manifold. We denote by B' the image of B in M'. The map $\pi : M \to M'$ is called a branched covering, with branch locus B'. When M is a complex surface and the $\mathbb{Z}_n$-action is holomorphic, M' is naturally a (non-singular) complex surface, with B' being a holomorphic curve in M', and vice versa.

Prove the following symplectic analog using the symplectic model in Example 1.6:

(1) Suppose ω is a symplectic form on M which is preserved by the $\mathbb{Z}_n$-action. Then ω descends to a symplectic form ω' on $M' \setminus B'$. Show that one can extend ω' across B' to a symplectic form on M' such that B' is a symplectic surface.

(2) Suppose ω' is a symplectic form on M' such that B' is a symplectic surface. Then ω' can be lifted to a symplectic form ω on $M \setminus B$, which is naturally invariant

under the $\mathbb{Z}_n$-action. Show that ω can be extended across B to a symplectic form on M, with respect to which the $\mathbb{Z}_n$-action is symplectic.

2. Symplectic fiber bundles

One of the early approaches for constructing compact closed symplectic manifolds which go beyond the natural examples is through symplectic fiber bundles. This construction is due to Thurston.

Let M, B be compact closed, connected smooth manifolds, $\pi : M \to B$ be a smooth, surjective submersion. Then π is a locally trivial fibration, with each fiber $F_b := \pi^{-1}(b)$, $b \in B$, diffeomorphic to a fixed manifold F, and M is called a fiber bundle over B with fiber F. Moreover, there is an open cover $\{U_\alpha\}$ of B and a collection of diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that $\pi = pr_1 \circ \phi_\alpha$, where $pr_1 : U_\alpha \times F \to U_\alpha$ is the projection. The maps ϕ_α are called local trivializations. We denote by $\phi_\alpha(b) : F_b \to F$ the restriction of ϕ_α to F_b followed by the projection onto F. Then the maps $\phi_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ defined by $\phi_{\beta\alpha}(b) = \phi_\beta(b) \circ \phi_\alpha(b)^{-1}$ are called the transition functions.

If F is an oriented manifold and each $\phi_{\beta\alpha}(b) \in \mathrm{Diff}^+(F)$, $\forall b \in U_\alpha \cap U_\beta$, then M is called an oriented fiber bundle. If F is a symplectic manifold with a symplectic form σ, and for each pair α, β, $\phi_{\beta\alpha}(b) \in \mathrm{Symp}(F, \sigma)$, $\forall b \in U_\alpha \cap U_\beta$, then M is called a **symplectic fiber bundle** (or **symplectic fibration**). In this case, each fiber F_b has an induced symplectic form $\sigma_b = \phi_\alpha(b)^*(\sigma)$, which is clearly independent of the choice of α as $\phi_{\beta\alpha}(b) \in \mathrm{Symp}(F, \sigma)$, $\forall b \in U_\alpha \cap U_\beta$. Note that if we give F the canonical orientation by the volume form σ^k, where $2k = \dim F$, then $\mathrm{Symp}(F, \sigma) \subset \mathrm{Diff}^+(F)$, i.e., a symplectic fiber bundle is necessarily an oriented fiber bundle.

Lemma 2.1. *Let $\pi : M \to B$ be a fiber bundle with fiber F. Suppose there is a closed 2-form τ on M such that the restriction of τ to each fiber F_b is a symplectic form on F_b, and that the base B is a symplectic manifold with a symplectic form β. Then for large enough $N > 0$, the 2-form $\omega_N \equiv \tau + N\pi^*\beta$ is a symplectic structure on M.*

Proof. Let $2m = \dim M$ and $2n = \dim B$. Then $\dim F = 2m - 2n$. Note that $(\pi^*\beta)^k = 0$ for any $k > n$. Hence

$$\omega_N^m = N^n\left(\frac{m!}{(m-n)!n!}\tau^{m-n} \wedge (\pi^*\beta)^n + \sum_{l=0}^{n-1} N^{l-n}\frac{m!}{(m-l)!l!}\tau^{m-l} \wedge (\pi^*\beta)^l\right).$$

Since M is compact, the lemma follows easily from the fact that $\tau^{m-n} \wedge (\pi^*\beta)^n$ is a volume form on M. To see that $\tau^{m-n} \wedge (\pi^*\beta)^n$ is a volume form, let $p \in M$ be any point and let $b = \pi(p)$. Pick a basis $v_1, \cdots, v_{2m-2n}$ of T_pF_b and pick a basis $u_1, \cdots, u_{2n}$ of T_bB. Let $u_i' \in T_pM$ be a lift of u_i, i.e., $\pi_*(u_i') = u_i$, $i = 1, \cdots, 2n$. Then $v_1, \cdots, v_{2m-2n}, u_1', \cdots, u_{2n}'$ form a basis of T_pM. With this understood, we have

$$\begin{aligned} & \tau^{m-n} \wedge (\pi^*\beta)^n(v_1, \cdots, v_{2m-2n}, u_1', \cdots, u_{2n}') \\ = & \pm\tau^{m-n}(v_1, \cdots, v_{2m-2n}) \cdot (\pi^*\beta)^n(u_1', \cdots, u_{2n}') \\ = & \pm\tau^{m-n}(v_1, \cdots, v_{2m-2n}) \cdot \beta^n(u_1, \cdots, u_{2n}) \end{aligned}$$

because $\pi^*\beta(v_j,\cdot) = \beta(\pi_*(v_j),\cdot) = 0$ for all j. Since the restriction of τ to F_b is a symplectic form, $\tau^{m-n}(v_1,\cdots,v_{2m-2n}) \neq 0$, and since β is a symplectic form on B, $\beta^n(u_1,\cdots,u_{2n}) \neq 0$ also. Hence

$$\tau^{m-n} \wedge (\pi^*\beta)^n(v_1,\cdots,v_{2m-2n},u_1',\cdots,u_{2n}') \neq 0$$

and $\tau^{m-n} \wedge (\pi^*\beta)^n$ is a volume form on M.

□

Exercise: Let $\pi : M \to B$ be a fiber bundle with fiber F. Suppose there is a closed 2-form τ on M such that the restriction of τ to each fiber F_b is a symplectic form on F_b. Show that $\pi : M \to B$ is a symplectic fiber bundle.

For a symplectic fiber bundle, the existence of a closed 2-form τ as in Lemma 2.1 is equivalent to a cohomological condition as shown by the next lemma.

Lemma 2.2. *Suppose $\pi : M \to B$ is a symplectic fiber bundle with fiber (F,σ). If there exists a cohomology class $a \in H^2(M,\mathbb{R})$ such that $i_b^*a = [\sigma_b]$, where $i_b : F_b \to M$ is the inclusion, then there exists a closed 2-form $\tau \in \Omega^2(M)$ such that $i_b^*\tau = \sigma_b$, $\forall b \in B$, and $[\tau] = a \in H^2(M,\mathbb{R})$.*

Proof. Pick a closed 2-form τ_0 on M which represents a. Let $\{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F\}$ be a set of local trivializations where $\{U_\alpha\}$ is an open cover of B by balls. Let $\sigma_\alpha \in \Omega^2(U_\alpha \times F)$ be the pull-back of $\sigma \in \Omega^2(F)$ by the projection to F. Then note that for any $b \in B$, $i_b^*(\phi_\alpha^*\sigma_\alpha) = \sigma_b$. Since $i_b^*a = [\sigma_b]$, $[\tau_0] = a$, and each U_α is contractible, we see that τ_0 and $\phi_\alpha^*\sigma_\alpha$ are cohomologous on $\pi^{-1}(U_\alpha)$, therefore, there exists a 1-form λ_α such that

$$\phi_\alpha^*\sigma_\alpha - \tau_0 = d\lambda_\alpha, \quad \forall \alpha.$$

Pick a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$, i.e., $\sum_\alpha \rho_\alpha = 1$ and $\mathrm{supp}\rho_\alpha \subset U_\alpha$. We define

$$\tau = \tau_0 + \sum_\alpha d((\pi \circ \rho_\alpha)\lambda_\alpha).$$

Note that $d((\pi \circ \rho_\alpha)\lambda_\alpha) = d(\pi \circ \rho_\alpha) \wedge \lambda_\alpha + (\pi \circ \rho_\alpha)d\lambda_\alpha$ and $i_b^*(d(\pi \circ \rho_\alpha)) = 0$. Hence

$$\begin{aligned} i_b^*\tau &= i_b^*\tau_0 + \sum_\alpha \rho_\alpha(b) i_b^*(d\lambda_\alpha) \\ &= \sum_\alpha \rho_\alpha(b) i_b^*(\tau_0 + d\lambda_\alpha) \\ &= \sum_\alpha \rho_\alpha(b) i_b^*\phi_\alpha^*\sigma_\alpha \\ &= \sum_\alpha \rho_\alpha(b)\sigma_b \\ &= \sigma_b. \end{aligned}$$

Finally, $[\tau] = [\tau_0] = a \in H^2(M,\mathbb{R})$, and the lemma is proved.

□

In the following exercise, we give some conditions which ensure the existence of a cohomological class $a \in H^2(M, \mathbb{R})$ in Lemma 2.2.

Exercise: Let $\pi : M \to B$ be a symplectic fiber bundle with fiber (F, σ). Let E be the sub-bundle of TM defined as follows: for any $p \in M$, the fiber $E_p = \ker d\pi : T_pM \to T_{\pi(p)}B$. Prove that

(1) E is a naturally symplectic vector bundle.

(2) If $c_1(TF)$ is a non-zero multiple of $[\sigma]$, then there exists a cohomology class $a \in H^2(M, \mathbb{R})$ such that $i_b^* a = [\sigma_b]$, where $i_b : F_b \to M$ is the inclusion.

According to the general theory of fiber bundles, given an oriented fiber bundle $\pi : M \to B$ with fiber F, where F is a symplectic manifold with an orientation-compatible symplectic form σ, the question as whether $\pi : M \to B$ is a symplectic fiber bundle with fiber (F, σ) boils down to the understanding of the homotopy type of the space $\mathrm{Diff}^+(F)/\mathrm{Symp}(F, \sigma)$. In this regard, the situation when F is 2-dimensional is particularly simple (and nice).

Lemma 2.3. *Let F be a compact, closed, oriented surface. Then $\mathit{Diff}^+(F)/\mathit{Symp}(F, \sigma)$ is contractible. Consequently, any oriented surface bundle over a smooth manifold is a symplectic fiber bundle.*

Proof. Let T be the space of orientation-compatible symplectic structures on F which has the same total area of σ. Then T is contractible, with $(\sigma', t) \mapsto (1 - t)\sigma' + t\sigma$, where $\sigma' \in T$ and $0 \le t \le 1$, being the contraction of T to the point $\sigma \in T$.

We will show that $\mathrm{Diff}^+(F)/\mathrm{Symp}(F, \sigma)$ is homeomorphic to T, from which the lemma follows. To see this, we consider the action of $\mathrm{Diff}^+(F)$ on T by

$$g \cdot \sigma' = g^*(\sigma'), \ \ \forall g \in \mathrm{Diff}^+(F), \sigma' \in T.$$

The action is obviously continuous with respect to appropriate topology on the two spaces. We claim it is transitive. To see this, let $\alpha_0, \alpha_1 \in T$ be any two elements. Then $\alpha_t = (1 - t)\alpha_0 + t\alpha_1 \in T$ for $0 \le t \le 1$, and $[\alpha_t] = [\alpha_0]$ for all t. By Moser's stability theorem, there exists a smooth family of $g_t \in \mathrm{Diff}^+(F)$ with $g_0 = id$, such that $g_t^* \alpha_t = \alpha_0$. Particular, $g_1^* \alpha_1 = \alpha_0$, so that the action of $\mathrm{Diff}^+(F)$ on T is transitive. This implies that $\mathrm{Diff}^+(F)/\mathrm{Symp}(F, \sigma)$ is homeomorphic to T as $\mathrm{Symp}(F, \sigma)$ is the isotropy subgroup at $\sigma \in T$.

□

In the case when F is 2-dimensional, the existence of a cohomological class $a \in H^2(M, \mathbb{R})$ in Lemma 2.2 has a simple criterion.

Lemma 2.4. *Let $\pi : M \to B$ be an oriented surface bundle, which is a symplectic fiber bundle with fiber (F, σ) by Lemma 2.3. There exists a cohomological class $a \in H^2(M, \mathbb{R})$ such that $i_b^* a = [\sigma_b]$ iff the fiber class $[F_b] \in H_2(M, \mathbb{R})$ is nonzero.*

Proof. Suppose a cohomological class $a \in H^2(M, \mathbb{R})$ such that $i_b^* a = [\sigma_b]$ exists. Then $a([F_b]) = i_b^* a([F_b]) = \sigma_b(F_b) \neq 0$, which implies that $[F_b] \in H_2(M, \mathbb{R})$ is nonzero. On the other hand, since $H^2(M; \mathbb{R}) = Hom(H_2(M, \mathbb{R}), \mathbb{R})$, if $[F_b] \in H_2(M, \mathbb{R})$ is nonzero, there must exist an $a \in H^2(M, \mathbb{R})$ such that $a([F_b]) \neq 0$. With $H^2(F, \mathbb{R}) = \mathbb{R}$, this implies that $i_b^* a = \lambda[\sigma_b]$ for some $\lambda \neq 0$. It is clear that λ is constant in b, and we simply replace a by $\frac{1}{\lambda}a$.

□

Exercise: Let $\pi : M \to B$ be an oriented surface bundle such that the fiber F has genus $\neq 1$. Show that the fiber class $[F_b] \in H_2(M, \mathbb{R})$ is nonzero.

We obtain the following corollary of Lemma 2.4.

Corollary 2.5. *Let $\pi : M \to B$ be an oriented surface bundle over a symplectic manifold, where the fiber class $[F_b] \in H_2(M, \mathbb{R})$ is nonzero. Then M admits a symplectic structure such that each fiber is a symplectic submanifold. In particular, if the fiber F has genus $\neq 1$, M admits a symplectic structure.*

We remark that the above result is not necessarily true if the fiber F has genus 1, as shown by the following example.

Example 2.6. Let $H : \mathbb{S}^3 \to \mathbb{CP}^1$ be the Hopf fibration. Then $\pi : \mathbb{S}^3 \times \mathbb{S}^1 \to \mathbb{CP}^1$ defined by $(x, t) \mapsto H(x)$ is a $\mathbb{T}^2$-bundle over $\mathbb{CP}^1$. Note that

$$H^2(\mathbb{S}^3 \times \mathbb{S}^1) = H^2(\mathbb{S}^3) \otimes H^0(\mathbb{S}^1) \oplus H^1(\mathbb{S}^3) \otimes H^1(\mathbb{S}^1) = 0$$

by the Kunneth formula, so that $\mathbb{S}^3 \times \mathbb{S}^1$ can not be symplectic. In particular, the fiber class is zero.

Example 2.7. (The Kodaira-Thurston Manifold, cf. Example 1.8, §1 of Part 1). Consider the 4-manifold $M = \mathbb{S}^1 \times N$ where N is the nontrivial $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ defined by $N = [0, 1] \times \mathbb{T}^2/\sim$, where $(0, x, y) \sim (1, x + y, y)$. Naturally M is a $\mathbb{T}^2$-bundle over $\mathbb{T}^2$. We claim that the fiber class is nonzero in $H_2(M, \mathbb{R})$. This is equivalent to say that the fiber class is nonzero in $H_2(N, \mathbb{R})$. But this follows from the fact that $N \to \mathbb{S}^1$ has a section $[(t, 0, 0)]$, $t \in [0, 1]$, which has a nonzero intersection product with the fiber.

By Corollary 2.5, M is a symplectic manifold. M can not be Kähler, because $H_1(M) = \mathbb{R}^3$, which has an odd dimension. This is the first example of symplectic, non-Kähler manifold discovered by Thurston.

Remark 2.8. Let $\pi : M \to B$ be an oriented surface bundle over a symplectic manifold. It is possible that the fiber class $[F_b] \in H_2(M, \mathbb{R})$ is zero and M still admits a symplectic structure. Of course in this case, the fibers can not be symplectic submanifolds of M.

A particularly interesting case is when M is an oriented $\mathbb{T}^2$-bundle over $\mathbb{T}^2$. Such bundles are completely classified topologically. It is known that all such bundles admit a symplectic structure, even though some of them have a zero fiber class. (In fact in this case, the fibers are actually Lagrangian tori in M.) It is also known that such bundles M all have $c_1(TM) = 0$. A symplectic 4-manifold M with $c_1(TM) = 0$ is called **symplectic Calabi-Yau**. There is a short list of known examples of symplectic Calabi-Yau 4-manifolds, with oriented $\mathbb{T}^2$-bundles over $\mathbb{T}^2$ included. It is a challenging problem to construct new examples of symplectic Calabi-Yau 4-manifolds, as well as understanding the smooth classification of them.

3. Symplectic normal connected sum

Recall the symplectic neighborhood theorem that the symplectic structure on a regular neighborhood of a compact symplectic submanifold is determined by the induced symplectic structure on the symplectic submanifold and the isomorphism class of the normal bundle as a symplectic vector bundle, or equivalently as a complex vector bundle (cf. Theorem 3.3, §3 of Part 1). This theorem is the basis of a connected sum construction in symplectic category, called the **symplectic normal connected sum**. We note that such a connected sum construction is not available in the holomorphic category. In fact, the symplectic normal connected sum construction is the major technique of investigating the difference between the category of symplectic manifolds and that of Kähler manifolds. The following theorem, due to R. Gompf, is a simple, but an important, example.

Theorem 3.1. *(Gompf, 1995) Every finitely presentable group is the fundamental group of a compact symplectic* 4*-manifold.*

It is known that every finitely presentable group is the fundamental group of a compact 4-manifold. On the other hand, it was proved that there exist no algorithms which can be used to classify all the finitely presentable groups. As a consequence, we obtain the following complexity result about compact 4-manifolds: there exist no algorithms which can be used to classify all the compact 4-manifolds (topological or smooth). The above theorem of Gompf shows that the same holds for symplectic 4-manifolds. On the other hand, it is known that there are severe constraints on the fundamental group of a Kähler surface. Gompf's theorem shows that the set of symplectic 4-manifolds is significantly larger than that of Kähler surfaces.

We shall focus our attention on a special case of symplectic normal connected sum, which is called **symplectic fiber sum**. For $j = 1, 2$ let (M_j, ω_j) be a symplectic manifold of dimension $2n$ and let $Q_j \subset M_j$ be a compact symplectic submanifold of dimension $2n-2$ which has a trivial normal bundle. (For example, if M_j is a symplectic fiber bundle over a compact Riemann surface, we could take Q_j to be a fiber of M_j, hence the name symplectic fiber sum.)

By the symplectic neighborhood theorem, a regular neighborhood of Q_j in M_j is symplectomorphic to

$$(Q_j \times B^2(r_0), \omega_j \oplus dx \wedge dy)$$

for some $r_0 > 0$, where $B^2(r_0) = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < r_0^2\}$.

Now suppose there exists a symplectomorphism $\phi : (Q_1, \omega_1) \to (Q_2, \omega_2)$. Then for any $0 < r_1 < r_0$, there is a symplectomorphism

$$\Phi : (Q_1 \times A(r_1, r_0), \omega_1 \oplus dx \wedge dy) \to (Q_2 \times A(r_1, r_0), \omega_2 \oplus dx \wedge dy)$$

lifting ϕ, where $A(r_1, r_0) = \{(x, y) \in \mathbb{R}^2 | r_1^2 < x^2 + y^2 < r_0^2\}$, which interchanges the inner boundary and the outer boundary of $A(r_1, r_0)$. To define Φ, we let r, θ be the polar coordinates on $B^2(r_0)$. Then $dx \wedge dy = rdr \wedge d\theta = du \wedge d\theta$ where $u = \frac{1}{2}r^2$. With this understood, we define

$$\Phi : (q, u, \theta) \mapsto (\phi(q), \frac{1}{2}(r_0^2 + r_1^2) - u, -\theta).$$

We can construct a new symplectic manifold by taking out a regular neighborhood of Q_1, Q_2 and gluing the complements via Φ:

$$(M_1 \setminus Q_1 \times \overline{B^2(r_1)}) \sqcup (M_2 \setminus Q_2 \times \overline{B^2(r_1)})/\sim$$

where $(q,u,\theta) \sim \Phi(q,u,\theta)$, $(q,u,\theta) \in Q_1 \times A(r_1,r_0)$. We denote it by $M_1 \#_{Q_1=Q_2} M_2$.

Remark 3.2. (1) The most useful case of this construction is when $\dim M_j = 4$. In this case, Q_j is an embedded symplectic surface with self-intersection $Q_j^2 = 0$. Note that by Moser's argument, there exists a symplectomorphism $\phi : (Q_1,\omega_1) \to (Q_2,\omega_2)$ if and only if Q_1, Q_2 have the same genus, and the total areas $\int_{Q_1} \omega_1 = \int_{Q_2} \omega_2$. However, the second condition is not essential because it can always be arranged by replacing of one of ω_1, ω_2 with an appropriate multiple.

(2) The diffeomorphism type of the resulting manifold $M_1 \#_{Q_1=Q_2} M_2$ depends on a number of things. First, it depends on the identification of a regular neighborhood of Q_j to $Q_j \times B^2(r_0)$. Such identifications are parametrized by the so-called "framings", i.e., the set of trivializations of the trivial bundle $Q_j \times \mathbb{R}^2$ over Q_j, which may be identified with $H^1(Q_j;\mathbb{Z})$ (i.e., the set of homotopy classes of maps from Q_j to $\mathbb{S}^1$). Second, it depends on the isotopy class of $\phi : Q_1 \to Q_2$. For example, when $Q_j = \mathbb{T}^2$ is a torus, the gluing data may be summarized into a 3×3 matrix

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ m & n & -1 \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ parametrizes the isotopy classes of symplectomorphisms $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ and $(m,n) \in H^1(\mathbb{T}^2;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ are the "framings".

(3) Note that the symplectic fibre sum construction requires the existence of a symplectomorphism $A(r_1,r_0) \to A(r_1,r_0)$ of the annulus which interchanges the inner and outer boundaries. There exist no higher dimensional analogs because such a symplectomorphism could be used to glue two higher dimensional balls to obtain a symplectic $\mathbb{S}^{2k}$ for some $k > 1$, which we know does not exist. Thus the symplectic fiber sum construction can only be performed along a symplectic submanifold of co-dimension 2.

(4) Notice in the gluing region $Q_1 \times A(r_1,r_0)$ in a symplectic fiber sum $M_1 \#_{Q_1=Q_2} M_2$, there is a free Hamiltonian $\mathbb{S}^1$-action which acts trivially on the Q_1 factor and acts as complex multiplication on the $A(r_1,r_0)$ factor. Lerman's symplectic cutting, when applied in this setting, allows one to undo the symplectic fiber sum, i.e., producing M_1, M_2 from $M_1 \#_{Q_1=Q_2} M_2$. This is where the name "symplectic cutting" was coming from, i.e., it gives an inverse of symplectic gluing.

In the remaining of this section we shall explain the basic ideas of the proof of Gompf's theorem (i.e. Theorem 3.1) with a simple example. A key ingredient is the following fact.

Lemma 3.3. *There exists a compact symplectic 4-manifold V with an embedded symplectic torus T of self-intersection $T^2 = 0$ such that the complement of a regular neighborhood of T in V is simply-connected.*

Proof. The manifold V will be the Kähler surface which is $\mathbb{CP}^2$ blown up at 9 points, and T will be the proper transform of a smooth cubic curve in $\mathbb{CP}^2$.

More precisely, take two generic cubic polynomials P_1, P_2 such that the zeroes $\{P_1 = 0\}, \{P_2 = 0\} \subset \mathbb{CP}^2$ are smooth curves which intersect transversely at 9 distinct points. For any $\lambda = [a, b] \in \mathbb{CP}^1$, the cubic curve $\{P_\lambda \equiv aP_1 + bP_2 = 0\} \subset \mathbb{CP}^2$ contains all of the 9 points where P_1, P_2 intersect. Since for distinct λ, λ' the cubic curves $\{P_\lambda = 0\}$, $\{P_{\lambda'} = 0\}$ are distinct, and since their intersection product is 9, it follows that $\{P_\lambda = 0\}$ and $\{P_{\lambda'} = 0\}$ intersect only at these 9 points, and furthermore, the intersection is transversal. It follows that $\mathbb{CP}^2$ is the union of these cubic curves, and the complex surface obtained by blowing up at these 9 points is a disjoint union of the proper transform of these cubic curves in $\mathbb{CP}^2$, which is parametrized by $\mathbb{CP}^1$. A generic member is a smoothly embedded torus of self-intersection 0. Since the blow up of an algebraic surface is still an algebraic surface, we see in particular that V is Kähler.

To see that the complement of a regular neighborhood of T in V is simply-connected, we use the Van-Kampen theorem. To this end, we denote by $\nu(T)$ a regular neighborhood of T in V. Then V is the union of the complement $V \setminus \nu(T)$ and $\overline{\nu(T)}$ along a 3-torus $\partial\overline{\nu(T)}$. We pick a base point $x_0 \in \partial\overline{\nu(T)}$. First, we observe that the class of the meridian of T in $\pi_1(V \setminus \nu(T), x_0)$ is zero because the meridian bounds an embedded disc in $V \setminus \nu(T)$. To see this, recall that V is $\mathbb{CP}^2$ blown up at 9 points where the family of cubic curves intersect transversely, and that T is the proper transform of a fixed smooth cubic. In particular, the exceptional curve at any of the blown up point intersects T transversely, so that the part of the exceptional curve in $V \setminus \nu(T)$ is an embedded disc bounded by the meridian of T. With this understood, now observe that there is a homomorphism $\pi_1(\overline{\nu(T)}, x_0) \to \pi_1(V \setminus \nu(T), x_0)$ such that the natural homomorphism induced by inclusion $\pi_1(\partial\overline{\nu(T)}, x_0) \to \pi_1(\overline{\nu(T)}, x_0)$ followed by this homomorphism equals the natural homomorphism induced by inclusion $\pi_1(\partial\overline{\nu(T)}, x_0) \to \pi_1(V \setminus \nu(T), x_0)$. It follows from the Van-Kampen theorem that there exists a homomorphism $\pi_1(V, x_0) \to \pi_1(V \setminus \nu(T), x_0)$, such that the natural homomorphism induced by inclusion $\pi_1(V \setminus \nu(T), x_0) \to \pi_1(V, x_0)$ followed by this homomorphism equals the identity on $\pi_1(V \setminus \nu(T), x_0)$. This implies that $V \setminus \nu(T)$ is simply-connected because V is simply-connected.

□

Now suppose X is a symplectic 4-manifold with an embedded symplectic torus T' of self-intersection 0. Let $Y \equiv V\#_{T=T'}X$ be the symplectic fiber sum. Then by Van-Kampen theorem, $\pi_1(Y, x_0)$ is obtained from $\pi_1(X, x_0)$ by setting the free loops contained in T' null-homotopic, for any $x_0 \in X$.

Example 3.4. In this example we will illustrate how to construct a compact symplectic 4-manifold with fundamental group $\mathbb{Z}$, using the symplectic fiber sum construction. To this end, we consider the symplectic 4-manifold X, where $X = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. The symplectic structure on X is $\omega = d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4 + d\theta_2 \wedge d\theta_3$, where θ_j, $j = 1, 2, 3, 4$, is the angular coordinate on the j-th copy of $\mathbb{S}^1$ in X. There are two disjoint, embedded symplectic tori in X:

$$T_1 = \{1\} \times \{1\} \times \mathbb{S}^1 \times \mathbb{S}^1,$$

and

$$T_2 = \{-1\} \times \mathbb{S}^1 \times \mathbb{S}^1 \times \{1\}.$$

The symplectic fiber sum $Y = V\#_{T\equiv T_1} X\#_{T_2=T} V$ has a fundamental group which is obtained from $\pi_1(X)$ by setting the j-th copy of $\mathbb{S}^1$ in X null-homotopic, for $j = 2, 3, 4$. Clearly $\pi_1(Y) = \mathbb{Z}$. We remark that Y can not be Kähler because the first Betti number $b_1(Y) = 1$ which is odd.

The general version of symplectic normal connected sum is given in the following

Theorem 3.5. *For $j = 1, 2$ let (M_j, ω_j) be a symplectic manifold of dimension $2n$ and let $Q_j \subset M_j$ be a compact symplectic submanifold of dimension $2n-2$ such that there exists a symplectomorphism $\phi : (Q_1, \omega_1) \to (Q_2, \omega_2)$. Moreover, for the normal bundles, ν_{Q_1} is isomorphic to $\phi^*\nu_{Q_2}^{-1}$ as complex line bundles. Then for any choice of isomorphism $\nu_{Q_1} \cong \phi^*\nu_{Q_2}^{-1}$, there is a regular neighborhood N_j of Q_j, $j = 1, 2$, and a symplectomorphism $\Phi : (N_1 \setminus Q_1, \omega_1) \to (N_2 \setminus Q_2, \omega_2)$, such that we can symplectically glue $(M_1 \setminus Q_1, \omega_1)$ and $(M_2 \setminus Q_2, \omega_2)$ via Φ to form a new symplectic manifold $M_1\#_{Q_1=Q_2} M_2$.*

Exercise: Prove Theorem 3.5.
Hint: Use a model for (N_j, ω_j), $j = 1, 2$, as in Example 1.6, §1.

4. Symplectic handlebodies and Weinstein manifolds

For the relevant material in contact geometry, see §1,§2 of Part 3.

Definition 4.1. Let $(N_\pm, \xi_\pm)$ be compact closed contact manifolds of dimension $2n-1$ with co-oriented contact structures, which induce the orientation of the respective manifold. A **symplectic cobordism** from (N_-, ξ_-) to (N_+, ξ_+) is a compact $2n$-dimensional symplectic manifold (M, ω), canonically oriented by ω^n, such that

- $\partial M = N_+ \sqcup \overline{N_-}$, where $\overline{N_-}$ stands for N_- with reversed orientation.
- In a neighborhood of ∂M, there is a Liouville vector field X for ω, which is transverse to ∂M, pointing inwards along N_- and outwards along N_+.
- The 1-form $\lambda := i_X\omega$ restricts to $TN_\pm$ as a contact form for $\xi_\pm$.

We call (N_-, ξ_-) the **concave boundary** and (N_+, ξ_+) the **convex boundary** of the symplectic cobordism. The cobordism (M, ω) is called a **Liouville cobordism** if the Liouville vector field X is defined everywhere on M.

Exercise: Let N be a compact closed hypersurface in a symplectic manifold (M, ω). Suppose X_i, $i = 1, 2$, are two Liouville vector fields defined in a neighborhood of N which are both transverse to N. Show that the induced contact structures on N, i.e., $\xi_i := \ker(i_{X_i}\omega|_{TN})$, are isotopic.

The above exercise shows that the specific choice of the Liouville vector field X in Definition 4.1 is not relevant up to contact isotopy of the boundary.

Example 4.2. Let (N, ξ) be a compact closed contact manifold, α a contact form for ξ. For any smooth functions h, k on N, consider the following subset in $\mathbb{R} \times N$,

$$M := \{(t, x) \in \mathbb{R} \times N | h(x) \le t \le k(x)\}.$$

Then $(M, d(e^t\alpha))$ is a symplectic (in fact, Liouville) cobordism from (N, ξ) to itself.

The symplectic (resp. Liouville) cobordism relation is reflexive as shown above. It is also transitive as show below.

Proposition 4.3. *Let (M_-, ω_-) be a symplectic (resp. Liouville) cobordism from (N_-, ξ_-) to (N, ξ), and (M_+, ω_+) be a symplectic (resp. Liouville) cobordism from (N, ξ) to (N_+, ξ_+). Then there is a symplectic (resp. Liouville) cobordism from (N_-, ξ_-) to (N_+, ξ_+), which topologically is obtained by gluing M_- to M_+ along the boundary components N.*

Proof. Let $X_\pm$ be a Liouville vector field defined in a neighborhood of $\partial M_\pm$ (resp. everywhere in $M_\pm$). Let $\alpha_\pm := i_{X_\pm}\omega_\pm|_{TN}$ be the induced contact forms for the contact structure ξ on N. Then a neighborhood of N in M_- is symplectomorphic to $((-\epsilon, 0] \times N, d(e^t\alpha_-))$, and a neighborhood of N in M_+ is symplectomorphic to $([0, \epsilon) \times N, d(e^t\alpha_+))$, with $X_\pm$ identified with $\frac{\partial}{\partial t}$ in the corresponding symplectization. On the other hand, there is a smooth function h on N such that $\alpha_+ = e^h\alpha_-$. We pick a constant C such that $h(x) > C$ for any $x \in N$. Then we scale the symplectic form ω_- to $e^C\omega_-$. With the new symplectic structure on M_-, a neighborhood of N is symplectomorphic to $((-\epsilon + C, C] \times N, d(e^t\alpha_-))$. Let

$$W := \{(t, x) \in \mathbb{R} \times N | C \leq t \leq h(x)\}.$$

Then we obtain a symplectic (resp. Liouville) cobordism (M, ω) by gluing $(M_-, e^C\omega_-)$, $(W, d(e^t\alpha_-))$, and (M_+, ω_+) along the corresponding boundaries.

□

Exercise: Let α_1, α_2 be contact forms on N such that $\alpha_2 = e^h\alpha_1$ for some smooth function h on N. Show that for any $a < b$, the map $\phi : ((a, b) \times N, d(e^t\alpha_2)) \to (\mathbb{R} \times N, d(e^t\alpha_1))$ where $\phi(t, x) = (t + h(x), x)$ is a symplectomorphism onto its image.

By Morse theory, each smooth cobordism can be decomposed into a sequence of elementary cobordisms, where an elementary cobordism is one obtained by attaching a handle to a trivial cobordism. It turns out that the handle attaching operation can be extended to the symplectic category, which is due to Weinstein.

We first recall the topological version of handle attaching. Let N be a compact closed $(n-1)$-dimensional smooth manifold. Let the $(k-1)$-dimensional sphere $\mathbb{S}^{k-1}$, where $k \geq 1$, be smoothly embedded in N with a trivial normal bundle. Then fixing any trivialization of the normal bundle, which means fixing any specific identification of a neighborhood of $\mathbb{S}^{k-1} \subset N$ with $\mathbb{S}^{k-1} \times D^{n-k}$, we glue a n-dimensional k-handle, i.e., $D^k \times D^{n-k}$, to $[-1, 0] \times N$ by identifying $\mathbb{S}^{k-1} \times D^{n-k} \subset \partial(D^k \times D^{n-k}) = \mathbb{S}^{k-1} \times D^{n-k} \cup D^k \times \mathbb{S}^{n-k-1}$ with the neighborhood $\mathbb{S}^{k-1} \times D^{n-k}$ of $\mathbb{S}^{k-1}$ in $\{0\} \times N$. After smoothing off the corners, the resulting n-dimensional manifold M has boundary $\{-1\} \times N \sqcup N'$, and is called an elementary cobordism from N to N'. Note that topologically, N' is obtained from N by surgery, i.e., removing a neighborhood $\mathbb{S}^{k-1} \times D^{n-k}$ of $\mathbb{S}^{k-1}$ in N and then gluing back $D^k \times \mathbb{S}^{n-k-1}$ along the boundary $\mathbb{S}^{k-1} \times \mathbb{S}^{n-k-1}$. The k-dimensional disc $D^k \times \{0\}$ is called the **core** of the handle, the $(n-k)$-dimensional disc $\{0\} \times D^{n-k}$ is called the **co-core**, and the $(k-1)$-dimensional sphere $\mathbb{S}^{k-1} \subset N$ is called the **attaching sphere**, and the $(n-k-1)$-dimensional sphere $\{0\} \times \mathbb{S}^{n-k-1}$

is called the **belt sphere** of the handle. We remark that this operation requires that the attaching sphere $\mathbb{S}^{k-1} \subset N$ has a trivial normal bundle, and it depends and is determined by the trivialization of the normal bundle.

In order to extend the construction to the symplectic category, we first describe the relevant symplectic geometry for a $2n$-dimensional k-handle. To this end, consider $\mathbb{R}^{2n}$ which is given with the standard symplectic structure

$$\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$

Consider the Morse function

$$f(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) = \sum_{j=1}^{k}(-\frac{1}{2}y_j^2 + x_j^2) + \frac{1}{4}\sum_{j=k+1}^{n}(y_j^2 + x_j^2).$$

With respect to the standard Euclidean metric, the gradient vector of f is given by

$$X := \sum_{j=1}^{k}(-y_j\frac{\partial}{\partial y_j} + 2x_j\frac{\partial}{\partial x_j}) + \frac{1}{2}\sum_{j=k+1}^{n}(y_j\frac{\partial}{\partial y_j} + x_j\frac{\partial}{\partial x_j}).$$

It is easy to check that $L_X\omega_0 = \omega_0$, i.e., X is also a Liouville vector field. We will consider level surfaces of f: $H_- := f^{-1}(-1)$ and $H_+ := f^{-1}(1)$. Since X is transverse to H_- and H_+, each of H_-, H_+ has an induced, co-oriented contact structure, to be denoted by ξ_- and ξ_+ respectively. Let α_-, α_+ be the corresponding contact forms.

The stable manifold E_-^k is the k-dimensional subspace

$$E_-^k := \{x_1 = x_2 = \cdots = x_n = y_{k+1} = \cdots = y_n = 0\}.$$

Note that E_-^k is an isotropic submanifold and the Liouville vector field X is tangent to E_-^k. Consequently, the ascending sphere $\mathbb{S}^{k-1} = E_-^k \cap H_-$ is isotropic in the contact manifold (H_-, ξ_-). Recall from §2 of Part 3 that the normal bundle of an isotropic submanifold L of a contact manifold N has a canonical decomposition

$$\nu_L = \mathbb{R}(R_\alpha) \oplus T^*L \oplus CSN_N(L),$$

where R_α is the Reeb vector field and $CSN_N(L)$ is the conformal symplectic normal bundle of L in N. In the present case, note that $\mathbb{R}(R_\alpha) \oplus T^*L$ is spanned by $\{\frac{\partial}{\partial x_j} | j = 1, 2, \cdots, k\}$, and $CSN_N(L)$ is spanned by $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} | j = k+1, \cdots, n\}$.

To describe the symplectic handle, we begin by fixing an arbitrarily small regular neighborhood U_- of the isotropic $(k-1)$-sphere $\mathbb{S}^{k-1} = E_-^k \cap H_-$ in H_-. The boundary $\partial U_- = \mathbb{S}^{k-1} \times \mathbb{S}^{2n-k-1}$. Let $\Sigma = \mathbb{R} \times \partial U_-$ be the $(2n-1)$-dimensional submanifold of $\mathbb{R}^{2n}$ which is obtained by moving ∂U_- along the flow of the Liouville vector field X. Then the symplectic handle is the region bounded by H_-, H_+ and Σ.

Now let (N, ξ) be a $(2n-1)$-dimensional contact manifold, $L \subset N$ an isotropic $(k-1)$-dimensional sphere with a trivial conformal symplectic normal bundle $CSN_N(L)$. As L is a sphere, $\mathbb{R}(R_\alpha) \oplus T^*L$ has a natural trivialization, so any trivialization of $CSN_N(L)$ determines a trivialization of the normal bundle ν_L in N, which is part of the data to be fixed in the (topological) handle attaching or surgery operation as we recalled earlier. Let α be a contact form associated to ξ, which is given as part of the data in the setup. We shall describe how to attach a symplectic $2n$-dimensional

k-handle to the trivial symplectic cobordism $([-1,0]\times N, d(e^t\alpha))$ with $L\subset\{0\}\times N$ being the attaching sphere.

By Theorem 2.4 of §2, Part 3, for any trivialization of $CSN_N(L)$, there is a regular neighborhood U_- of the ascending sphere $\mathbb{S}^{k-1}=E^k_-\cap H_-$ in H_- and a regular neighborhood U of L in $\{0\}\times N$, and a contactomorphism $\phi: U\rightarrow U_-$ sending L diffeomorphic to $\mathbb{S}^{k-1}$. Then there is a smooth function h on U such that $\phi^*\alpha_-=e^h\alpha$. We extend h to the entire manifold N, which is still denoted by h for simplicity. By scaling α by an appropriate positive constant as we did in the proof of Proposition 4.3, we may assume $h(x)>0$, $\forall x\in N$.

Now we fix a smaller regular neighborhood U' of L such that its closure $\overline{U'}\subset U$, and we set $A:=U\setminus\overline{U'}$. Let $A_-:=\phi(A)\subset U_-$. There is a smooth map $\psi: A_-\rightarrow H_+$ which is defined by moving the points of A_- along the flow lines of the Liouville vector field X until hitting H_+. Then $\psi^*\alpha_+=e^{k_0}\alpha_-$ for some smooth function $k_0>0$ on A_-. We let $k=k_0\circ\phi$ be the pull-back of k_0 to A. We extend k to a positive smooth function on the entire N and still denote it by k for simplicity.

We let M be the following subset of the symplectization $(\mathbb{R}\times N, d(e^t\alpha))$:

$$M:=\{(t,x)|-1\leq t\leq h(x)+k(x) \text{ if } x\in N\setminus\overline{U'}, \text{ and } -1\leq t\leq h(x) \text{ if } x\in\overline{U'}\}.$$

Let $V:=\{(t,x)|h(x)\leq t\leq h(x)+k(x), x\in A\}\subset M$ and let W be the region in the symplectic handle which consists of the flow lines of the Liouville vector field X starting from points in A_-. Then $\phi: A\rightarrow A_-$ induces a symplectomorphism $\Phi: V\rightarrow W$ by mapping the flow lines of $\frac{\partial}{\partial t}$ in V to the corresponding flow lines of the Liouville vector field X in W. With this understood, the symplectic handle attaching operation is done by gluing M and the symplectic handle via Φ. The result is a Liouville cobordism (P,ω) from (N,ξ) to a contact manifold (N',ξ'), where N' is a manifold diffeomorphic to the surgery of N along L. We call (N',ξ') the corresponding **contact surgery** of (N,ξ).

In fact, the Liouville cobordism (P,ω) above is an example of the so-called Weinstein cobordism, as it possesses an additional structure of a Morse function.

Definition 4.4. A **Weinstein cobordism** (M,ω,X,ϕ) is a Liouville cobordism (M,ω,X) with a Morse function $\phi: M\rightarrow\mathbb{R}$ which is constant on ∂M, such that the Liouville vector field X is gradient-like for ϕ, i.e.,

$$i_X d\phi\geq\delta(|X|^2+|d\phi|^2)$$

holds for some $\delta>0$. (Here $|X|$ is the norm of X with respect to some Riemannian metric on M and $|d\phi|$ is the dual norm of $d\phi$.) A **Weinstein domain** is a Weinstein cobordism with empty concave boundary.

A **Weinstein manifold** (W,ω,X,ϕ) is a symplectic manifold (W,ω) with a complete Liouville vector field X and an exhausting Morse function ϕ, such that X is gradient-like for ϕ. A Weinstein manifold is called **finite type** if the Morse function has only finitely many critical points.

For either a Weinstein cobordism or a Weinstein manifold, the triple (ω,X,ϕ) is called a **Weinstein structure**.

For further discussions about Weinstein manifolds, see [2].

5. Gromov's theory of J-holomorphic curves

5.1. **Basic elements of J-holomorphic curve theory.** Let (M, ω) be a *compact closed* symplectic manifold of dimension $2n$, and let $J \in \mathcal{J}(M, \omega)$ be an ω-compatible almost complex structure. Let $g_J(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$ be the corresponding hermitian metric (i.e. J-invariant Riemannian metric) on M.

Let (Σ, j) be a Riemann surface (not necessarily compact) with complex structure j. A smooth map $u : \Sigma \to M$ is called a (J, j)**-holomorphic map** (or simply a J**-holomorphic map**) if $du \circ j = J \circ du$, or equivalently,

$$\bar{\partial}_J(u) \equiv \frac{1}{2}(du + J \circ du \circ j) = 0.$$

The equation $\bar{\partial}_J(u) = 0$ is a first order, non-linear equation of Cauchy-Riemann type. We give a description of it in a local coordinate system. Let $z_0 \in \Sigma$ be any point and let $p = u(z_0) \in M$ be its image in M under u. Supppose $s + it$ is a local holomorphic coordinate centered at z_0 and $\phi : U \to \mathbb{R}^{2n}$ is a local chart centered at $p \in M$. Set $\phi \circ u = (u^1, \cdots, u^{2n})^T$. Then

$$\bar{\partial}_J(u) = \frac{1}{2}((\partial_s u^j) + J(u^1, \cdots, u^{2n})(\partial_t u^j))ds + \frac{1}{2}((\partial_t u^j) - J(u^1, \cdots, u^{2n})(\partial_s u^j))dt,$$

and $\bar{\partial}_J(u) = 0$ is equivalent to

$$(\partial_s u^j) + J(u^1, \cdots, u^{2n})(\partial_t u^j) = 0.$$

If J is integrable and $(u^1, \cdots, u^{2n})$ is coming from a local holomorphic coordinate system $(z^1, \cdots, z^n)$ with $z^j = u^j + iu^{j+n}$, $j = 1, \cdots, n$, then $J(u^1, \cdots, u^{2n})$ is constant in $u^1, \cdots, u^{2n}$ and equals the matrix

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

where I denotes the $n \times n$ identity matrix. In this case, $\bar{\partial}_J(u) = 0$ becomes the Cauchy-Riemann equations

$$\partial_s u^j - \partial_t u^{j+n} = 0, \quad \partial_s u^{j+n} + \partial_t u^j = 0, \quad j = 1, \cdots, n.$$

Hence when J is integrable, J-holomorphic maps are simply the usual holomorphic maps. On the other hand, it is easy to see that for a general J, the linearization of the non-linear equation $\bar{\partial}_J(u) = 0$ is a zero-th order perturbation of the Cauchy-Riemann equations.

Local properties. We shall next list several relevant local analytical properties of J-holomorphic maps.

Let $u, v : \Sigma \to M$ be two smooth maps and let $z_0 \in \Sigma$ be a point. We say that u, v agree to the infinite order at z_0 if $u(z_0) = v(z_0) = p_0$, and there is a local chart centered at p_0, $\phi : U \to \mathbb{R}^{2n}$, such that all partial derivatives of the $\mathbb{R}^{2n}$-valued function $\phi \circ u - \phi \circ v$ vanish at z_0.

Proposition 5.1. *(Unique continuation). If $u, v : \Sigma \to M$ are two J-holomorphic maps which agree to the infinite order at a point $z_0 \in \Sigma$, then $u \equiv v$ in the connetced component of Σ which contains z_0.*

Let $u : \Sigma \to M$ be a J-holomorphic map. A point $z \in \Sigma$ is called a **critical point** if $du(z) = 0$. Correspondingly the image $u(z) \in M$ is called a **critical value**. We remark that u is locally an embedding at any point which is not a critical point. To see this, we suppose $du(z) \neq 0$ for some $z \in \Sigma$. Let $u(z) = p$ and let $s + it$ be a local holomorphic coordinate centered at z. Then $du(z) \neq 0$ means that either $\partial_s u(z) \in T_pM$ or $\partial_t u(z) \in T_pM$ is non-zero. But u is J-holomorphic so that $\partial_s u + J(u)\partial_t u = 0$, which implies that both $\partial_s u(z), \partial_t u(z) \in T_pM$ are non-zero. Hence u is locally an embedding near z.

Lemma 5.2. *A critical point of a non-constant J-holomorphic map is isolated. In particular, a non-constant J-holomorphic map from a compact Riemann surface has only finitely many critical points.*

Lemma 5.3. *Let $\Omega \subset \mathbb{C}$ be an open neighborhood of $0 \in \mathbb{C}$ and let $u, v : \Omega \to M$ be J-holomorphic maps such that*

$$u(0) = v(0), \;\; du(0) \neq 0.$$

Moreover, assume that there exist sequences $z_n, w_n \in \Omega$ such that

$$u(z_n) = v(w_n), \;\; \lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = 0, \;\; w_n \neq 0.$$

Then there exists a holomorphic function $\phi : B_\epsilon(0) \to \Omega$ defined in some neighborhood of $0 \in \mathbb{C}$ such that $\phi(0) = 0$ and

$$v = u \circ \phi.$$

Lemmas 5.2 and 5.3 have the following consequence.

Corollary 5.4. *Let $u : \Sigma \to M$ be a non-constant J-holomorphic map from a compact Riemann surface. Then there exists a compact Riemann surface Σ' and a non-constant J-holomorphic map $v : \Sigma' \to M$ such that in the complement of finitely many points, v is an embedding onto its image. Moreover, there exists a biholomorphism or branched covering map $\phi : \Sigma \to \Sigma'$ such that*

$$u = v \circ \phi.$$

The map v in the above corollary is called **simple** and the map u is called **multiply covered** if $\deg(\phi) > 1$. The image $C \equiv \mathrm{Im}\; v$ is called a ***J*-holomorphic curve** in M, and the map $v : \Sigma' \to M$ is called a **parametrization** of C. We call C a **rational** J-holomorphic curve if $\Sigma' = \mathbb{S}^2$.

Exercise: (Carleman Similarity Principle) A key analytic fact in local properties of J-holomorphic curves is the following: Let $D \subset \mathbb{C}$ be the unit disc $|z| \leq 1$. Then the $\bar\partial$-operator $\bar\partial : L^{1,p}(D, \mathbb{C}) \to L^p(D, \mathbb{C})$ is surjective, where $1 < p < \infty$. Here $\bar\partial u = u_s + iu_t$, where $z = s + it$. In fact, for any $g \in L^p(D, \mathbb{C})$, the Cauchy Integral Formula defines

$$f := \frac{1}{2\pi i}\int_D \frac{g(w)}{w - z} dw \wedge d\bar w,$$

which obeys $f \in L^{1,p}(D, \mathbb{C})$ and $\bar\partial f = g$.

Now suppose $w \in L^{1,p}(D, \mathbb{C})$, $p > 2$, obeys the equation

$$\bar\partial w - A(z)w = 0,$$

where in $A(z)w$, w is regarded as a vector in $\mathbb{R}^2$ under $\mathbb{C} = \mathbb{R}^2$, and $A(z)$ is a 2×2 matrix whose entires are L^∞ functions on D. Show that there is a $u \in \bigcap_{p>1} L^{1,p}(D,\mathbb{C})$ (note in particular, $u \in C^{0,\alpha}(D,\mathbb{C})$ for any $0<\alpha<1$), such that

$$w = e^u f$$

for some holomorphic function f on D.

Hint: consider the function g on D, where $g(z) := \frac{A(z)w(z)}{w(z)}$ if $w(z)\neq 0$ and $g(z)=0$ if $w(z)=0$. Then $g \in L^\infty(D,\mathbb{C})$. In particular, $g\in L^p(D,\mathbb{C})$ for any $p>1$. On the other hand, note that $g(z)w = A(z)w$.

Let $u : \Sigma \to M$ be a smooth map, where Σ is given a complex structure j, M is given a $J \in \mathcal{J}(M,\omega)$. We denote by g_J the associated hermitian metric on M. In order to define the **energy** of the map u, we fix a Kähler metric h on Σ, and with h and g_J, the norm $|du|$ is well-defined. We define the energy of u to be

$$E(u) \equiv \int_\Sigma |du|^2 dvol_\Sigma.$$

An important fact about $E(u)$ is that even though the energy density $|du|^2$ may depend on the choice of the Kähler metric h on Σ, the energy $E(u)$ depends only on the complex structure j, i.e., $E(u)$ is invariant under comformal transformations on the domain of u.

The following energy identity can be easily derived

$$E(u) = \int_\Sigma |\bar\partial_J(u)|^2 dvol_\Sigma + \int_\Sigma u^*\omega,$$

which has the following important consequence. (This is where the closedness of ω plays a real role.)

Proposition 5.5. *J-holomorphic maps are the absolute minima of the energy functional $E(u)$ amongst the smooth maps u which carry a fixed homology class in M. In particular, J-holomorphic maps are harmonic maps, and the energy of a J-holomorphic map depends only on the homology class it carries, and a J-holomorphic map must be constant if it carries a trivial homology class.*

Finally, we mention the following important local analytical property of J-holomorphic maps.

Theorem 5.6. *(Removal of singularities) Let $D\subset\mathbb{C}$ be the unit disc containing 0 and let $u : D\setminus\{0\}\to M$ be a J-holomorphic map such that $E(u)<\infty$. Then u may be extended to a J-holomorphic map $\hat u : D\to M$ with $\hat u|_{D\setminus\{0\}} = u$.*

Next we consider the moduli space of J-holomorphic maps. For simplicity, we shall assume $\Sigma = \mathbb{S}^2$. In this case, the complex structure j is unique, and the group of biholomorphisms of Σ is the group of Möbius trnsformations $G = \mathrm{PSL}(2,\mathbb{C})$:

$$z \mapsto \frac{az+b}{cz+d},\;\; a,b,c,d\in\mathbb{C},\;\; ad-bc=1.$$

Fix a non-zero homology class $0 \neq A \in H_2(M;\mathbb{Z})$. We consider the space of J-holomorphic maps

$$\mathcal{M}(A,J) = \{u : \mathbb{S}^2 \rightarrow M | u \text{ is } J\text{-holomorphic and } u_*[\mathbb{S}^2] = A\},$$

and the subspace of $\mathcal{M}(A,J)$ consisting of simple J-holomorphic maps

$$\mathcal{M}^*(A,J) = \{u : \mathbb{S}^2 \rightarrow M | u \text{ is } J\text{-holomorphic and simple, and } u_*[\mathbb{S}^2] = A\}.$$

Note that the group $G = \mathrm{PSL}(2,\mathbb{C})$ acts on $\mathcal{M}(A,J)$ via reparametrization

$$\phi \cdot u = u \circ \phi^{-1}, \;\; \forall \phi \in G, u \in \mathcal{M}(A,J),$$

which is free when restricted on the subspace $\mathcal{M}^*(A,J)$. We denote the quotient space by $\widetilde{\mathcal{M}}(A,J)$ and $\widetilde{\mathcal{M}}^*(A,J)$ respectively. Note that $\widetilde{\mathcal{M}}^*(A,J)$ is exactly the space of J-holomorphic curves C such that the homology class of C is A. We remark that when A is a primitive class, i.e., A is not an integral multiple of another integral class, $\mathcal{M}(A,J) = \mathcal{M}^*(A,J)$.

Compactness. One of the fundamental issues concerning the moduli spaces is compactness. Note that the group $G = \mathrm{PSL}(2,\mathbb{C})$ acts freely on $\mathcal{M}^*(A,J)$ and G is not a compact group. Hence the moduli space of J-holomorphic maps $\mathcal{M}(A,J)$ and $\mathcal{M}^*(A,J)$ can not be compact, and one could best hope that the quotient spaces $\widetilde{\mathcal{M}}(A,J)$ and $\widetilde{\mathcal{M}}^*(A,J)$ are compact. However, this is also not true in general, as illustrated in the following example.

Example 5.7. Consider a family of holomorphic curves of degree 2 in $\mathbb{CP}^2$ parametrized by $0 \neq \lambda \in \mathbb{C}$

$$C_\lambda = \{[z_0,z_1,z_2] | \lambda z_0^2 = z_1 z_2\} \in \widetilde{\mathcal{M}}^*(2[\mathbb{CP}^1], J_0).$$

Here $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2;\mathbb{Z})$ is the class of a line, and J_0 is the complex structure of $\mathbb{CP}^2$. As $\lambda \rightarrow 0$, C_λ converges to a union of two lines

$$C_0 = \{[z_0,z_1,z_2] | z_1 z_2 = 0\} = \{[z_0,0,z_2]\} \cup \{[z_0,z_1,0]\}$$

which intersect transversely at $[1,0,0]$. It is known that C_0 can not be the image of a holomorphic map $u : \mathbb{S}^2 \rightarrow \mathbb{CP}^2$, hence C_0 does not lie in $\widetilde{\mathcal{M}}(2[\mathbb{CP}^1], J_0)$. This shows that both $\widetilde{\mathcal{M}}(2[\mathbb{CP}^1], J_0)$ and $\widetilde{\mathcal{M}}^*(2[\mathbb{CP}^1], J_0)$ are non-compact.

The phenomenon illustrated in the above example is called **bubbling**, i.e., during the limiting process as $\lambda \rightarrow 0$, the holomorphic curves C_λ split off a (non-constant) J-holomorphic 2-sphere which carries strictly less energy than the original curves. The bubbling phenomenon is the primary cause of non-compactness of moduli space of J-holomorphic curves, and when $\Sigma = \mathbb{S}^2$ as what we currently consider, it is the only cause. In other words, if there is no bubbling, the space $\widetilde{\mathcal{M}}(A,J)$ is compact.

Next we mention a simple criterion which ensures compactness. Recall that a homology class $B \in H_2(M;\mathbb{Z})$ is called **spherical** if it may be represented by a map from $\mathbb{S}^2$ into M. Suppose the symplectic manifold (M,ω) contains no spherical classes B such that

$$0 < \omega(B) < \omega(A).$$

Such a condition has two consequences: (1) every element $u \in \mathcal{M}(A,J)$ is simple because otherwise the image of u represents a spherical class B satisfying $0 <$

$\omega(B) < \omega(A)$, this gives $\mathcal{M}^*(A, J) = \mathcal{M}(A, J)$, (2) there is no bubbling for elements in $\widetilde{\mathcal{M}}(A, J)$ because a split-off J-holomorphic 2-sphere would represent a spherical class B satisfying $0 < \omega(B) < \omega(A)$. This gives rise to the following simple version of the Gromov Compactness Theorem.

Theorem 5.8. *(Gromov). Suppose there are no spherical classes B such that*

$$0 < \omega(B) < \omega(A).$$

Then for any compact subset $W \in \mathcal{J}(M, \omega)$ (given with C^∞-topology), $\cup_{J \in W} \widetilde{\mathcal{M}}(A, J)$ is compact with respect to the C^∞-topology.

The full version of the Gromov Compactness Theorem states that the moduli space of J-holomorphic curves carrying a fixed homology class can be suitably compactified. This is where the closedness of ω plays a real role, cf. Proposition 5.5.

Fredholm theory. Finally, we discuss the Fredholm theory of J-holomorphic maps, which allows us to analyze the topological structure of the moduli spaces.

Fix a sufficiently large integer $l > 0$, we consider the Banach manifold

$$B \equiv \{u : \mathbb{S}^2 \to M | u \text{ is a } C^l\text{-map and } u_*[\mathbb{S}^2] = A\}$$

and the Banach bundle $E \to B$, where the fiber over $u \in B$ is

$$E_u \equiv \{v | v \text{ is a } C^{l-1}\text{-section of } Hom(T\mathbb{S}^2, u^*TM) \to \mathbb{S}^2 \text{ such that } v \circ j = -J \circ v\}.$$

The Banach bundle $E \to B$ has a natural smooth section $s : B \to E$ defined by

$$s : u \mapsto (u, \bar{\partial}_J(u)).$$

By the elliptic regularity of the equation $\bar{\partial}_J(u) = 0$, any C^l-solution is automatically a smooth solution, so that the moduli space of J-holomorphic maps $\mathcal{M}(A, J)$ is simply the zero loci of s, i.e.,

$$s^{-1}(\text{zero-section}) = \mathcal{M}(A, J).$$

A crucial fact is that $s : B \to E$ is a Fredholm section, which means that the linearization of $\bar{\partial}_J(u)$ for each $u \in B$, $D_u : T_uB \to E_u$, is a Fredholm operator between the Banach spaces. This has the following implication on the topological structure of the moduli space $\mathcal{M}(A, J)$.

- For any open subset $U \subset \mathcal{M}(A, J)$, if $D_u : T_uB \to E_u$ is onto for any $u \in U$, then U is a canonically oriented, finite dimensional smooth manifold whose dimension is given by the index of D_u, which can be computed via the Atiyah-Singer index theorem in the following formula

  $$\text{Index } D_u = 2n(1 - g_{\mathbb{S}^2}) + 2c_1(TM) \cdot A.$$

  Here $2n = \dim M$ and $g_{\mathbb{S}^2} = 0$ is the genus of $\mathbb{S}^2$. Such a J is called **regular** (with respect to U). (We remark that the same holds true if one allows J to vary in an oriented finite dimensional space.)

When J is integrable, the operator $D_u : T_uB \to E_u$ is simply the $\bar{\partial}$-operator $\bar{\partial} : \Omega^0(\mathbb{CP}^1, V) \to \Omega^{0,1}(\mathbb{CP}^1, V)$ where $V = u^*TM$ is a holomorphic vector bundle over $\mathbb{CP}^1$. The cokernel of D_u is simply the Dolbeault cohomology group $H^{0,1}_{\bar{\partial}}(\mathbb{CP}^1, V)$, which by Kodaira-Serre duality is isomorphic to the space of holomorphic sections of

$V^* \otimes K$. Here V^* is the dual of V and K is the canonical bundle of $\mathbb{CP}^1$. The following lemma follows immediately from vanishing theorems of holomorphic vector bundles.

Lemma 5.9. *Suppose J is integrable and $V = u^*TM \rightarrow \mathbb{CP}^1$ is a holomorphic vector bundle of non-negative curvature tensor. Then D_u is onto.*

In general, using the Sard-Smale theorem one has

Theorem 5.10. *There exists an open, dense subset $\mathcal{J}_{reg}(A) \subset \mathcal{J}(M, \omega)$ of second Bair category such that for any $J \in \mathcal{J}_{reg}(A)$, J is regular with respect to $\mathcal{M}^*(A, J)$, so that $\mathcal{M}^*(A, J)$ is a smooth manifold of dimension*

$$\dim M + 2c_1(TM) \cdot A.$$

Moreover, for any $J_1, J_2 \in \mathcal{J}_{reg}(A)$, there exists a path $J_t \in \mathcal{J}(M, \omega)$ connecting J_1, J_2 such that

$$\cup_t \mathcal{M}^*(A, J_t)$$

is an oriented smooth manifold with boundary which is the disjoint union of $\mathcal{M}^*(A, J_1)$ and $\mathcal{M}^*(A, J_2)$.

5.2. **The non-squeezing theorem and Gromov invariant.** As one of the first applications of J-holomorphic curve theory, we describe the proof of the following non-squeezing theorem, where $B^{2n}(R)$ denotes the closed ball of radius R in $\mathbb{R}^{2n}$ which is equipped with the standard symplectic structure ω_0.

Theorem 5.11. *(Gromov, 1985). There exist no symplectic embeddings $B^{2n}(1) \rightarrow B^2(r) \times \mathbb{R}^{2n-2}$ if $r < 1$.*

Proof. Suppose to the contrary, there exists an symplectic embedding $\psi : B^{2n}(1) \rightarrow B^2(r) \times \mathbb{R}^{2n-2}$ for some $r < 1$. Fix any $\epsilon > 0$, we consider $B^2(r)$ as a subset of $\mathbb{S}^2$ which is given a symplectic form σ with total area $\pi r^2 + \epsilon$. On the other hand, since $\psi(B^{2n}(1))$ is compact, its projection into the $\mathbb{R}^{2n-2}$ factor is contained in an open ball of radius λ centered at the origin. Let T^{2n-2} be the torus which is $\mathbb{R}^{2n-2}$ modulo the lattice $\{(x_1, \cdots, x_{2n}) \cdot \lambda | x_j \in \mathbb{Z}\}$, which inherits a natural symplectic form ω_0. We set $M = \mathbb{S}^2 \times T^{2n-2}$, which is given with the product symplectic structure $\omega = \sigma \oplus \omega_0$. With this understood, note that there is a symplectic embedding $\psi : (B^{2n}(1), \omega_0) \rightarrow (M, \omega)$. We set $p_0 = \psi(0)$ where $0 \in B^{2n}(1)$ is the origin.

Lemma 5.12. *For any $J \in \mathcal{J}(M, \omega)$, there exists a ratinal J-holomorphic curve C which contains p_0 and carries the homology class $[\mathbb{S}^2 \times \{pt\}]$.*

Assuming Lemma 5.12 momentarily, the proof of Theorem 5.11 goes as follows. Note that there is a $J \in \mathcal{J}(M, \omega)$ such that the pull-back almost complex structure ψ^*J is the standard complex structure J_0 on $B^{2n}(1)$. Let C be the rational J-holomorphic curve which contains p_0 and carries the homology class $[\mathbb{S}^2 \times \{pt\}]$. We set $C' \equiv \psi^{-1}(C) \subset B^{2n}(1)$. Then C' is a holomorphic curve in $B^{2n}(1)$ containing the origin. Particularly, C' is a minimal surface, and by the theory of minimal surfaces, the area of C' is at least the area of the flat plane contained in $B^{2n}(1)$, which equals π. This gives rise to the following inequalities

$$\pi \leq \text{Area}(C') = \int_{C'} \omega_0 = \int_{\psi(C')} \omega \leq \int_C \omega = \int_{\mathbb{S}^2} \sigma = \pi r^2 + \epsilon.$$

Let $\epsilon \to 0$, we obtain $\pi \leq \pi r^2$, which contradicts the assumption $r < 1$. This proves the non-squeezing theorem.

$\square$

The basic idea behind the proof of Lemma 5.12 is the so-called **Gromov invariant**, which is the "number" of rational J-holomorphic curves (counted with signs) for a given J, that carries a given homology class and satisfies a certain topological constraint. (Such a count of J-holomorphic curves is supposed to be independent of the choice of J.) Lemma 5.12 basically says that the Gromov invariant which counts the number of rational J-holomorphic curves carrying a homology class $[\mathbb{S}^2 \times \{pt\}]$ and passing through a given point in M is non-zero.

We shall next explain how to define such a Gromov invarint in the current context, and explain why the Gromov invariant is non-zero.

To this end, we set $A = [\mathbb{S}^2 \times \{pt\}] \in H_2(M;\mathbb{Z})$. Since $\omega = \sigma \oplus \omega_0$ is a product symplectic structure, $c_1(TM) = c_1(T\mathbb{S}^2) + c_1(TT^{2n-2})$, so that

$$c_1(TM) \cdot A = c_1(T\mathbb{S}^2) \cdot A = 2.$$

By Theorem 5.10, there is an open, dense subset of second Bair category $\mathcal{J}_{reg}(A) \subset \mathcal{J}(M,\omega)$, such that for any $J \in \mathcal{J}_{reg}(A)$, the space $\mathcal{M}^*(A,J)$ is an oriented smooth manifold of dimension

$$\dim M + 2c_1(TM) \cdot A = 2n + 4.$$

In the present case, since A is a generator of $H_2(M,\mathbb{Z}) = \mathbb{Z}$, there are no spherical classes B such that $0 < \omega(B) < \omega(A)$, so that by Theorem 5.8, $\mathcal{M}(A,J) = \mathcal{M}^*(A,J)$, and the quotient space $\widetilde{\mathcal{M}}(A,J)$ is compact, which is an oriented smooth manifold of dimension

$$\dim \widetilde{\mathcal{M}}(A,J) = \dim \mathcal{M}^*(A,J) - \dim \mathrm{PSL}(2,\mathbb{C}) = 2n + 4 - 6 = 2n - 2.$$

Denote $\mathrm{PSL}(2,\mathbb{C})$ by G, and set $\mathcal{M}(A,J) \times_G \mathbb{S}^2 \equiv (\mathcal{M}(A,J) \times \mathbb{S}^2)/G$ where G acts on $\mathcal{M}(A,J) \times \mathbb{S}^2$ via $\phi \cdot (u,z) = (u \circ \phi^{-1}, \phi(z))$. Then $\mathcal{M}(A,J) \times_G \mathbb{S}^2$ is a compact, oriented smooth manifold of dimension $2n$, which is a $\mathbb{S}^2$-bundle over $\widetilde{\mathcal{M}}(A,J)$. The evaluation map

$$ev : \mathcal{M}(A,J) \times_G \mathbb{S}^2 \to M, \quad [(u,z)] \mapsto u(z)$$

is a smooth map between two compact, oriented smooth manifolds of the same dimension. The degree of ev, which is the image of the fundamental class of $\mathcal{M}(A,J) \times_G \mathbb{S}^2$ under $ev_* : H_{2n}(\mathcal{M}(A,J) \times_G \mathbb{S}^2;\mathbb{Z}) \to H_{2n}(M;\mathbb{Z}) = \mathbb{Z}$, can be geometrically interpreted as a count with signs of the points in the pre-image $ev^{-1}(p)$ for any generic point $p \in M$. On the other hand, $\mathcal{M}(A,J) \times_G \mathbb{S}^2$ as a $\mathbb{S}^2$-bundle over $\widetilde{\mathcal{M}}(A,J)$ may be regarded as the space of rational J-holomorphic curves $C \in \widetilde{\mathcal{M}}(A,J)$ with a marked point $z \in \mathbb{S}^2$ in the de-singularization of C. Thus the degree of ev is a count with signs of the number of rational J-holomorphic curves with a marked point, which carry the homology class A and pass through a given generic point $p \in M$ at the marked point. The Gromov invariant involved in the current problem is defined to be the degree of the evaluation map $ev : \mathcal{M}(A,J) \times_G \mathbb{S}^2 \to M$. Note that the Gromov invariant is independent of the choice of $J \in \mathcal{J}_{reg}(A)$. This is because by Theorem 5.10, for any $J_1, J_2 \in \mathcal{J}_{reg}(A)$, there exists a path $J_t \in \mathcal{J}(M,\omega)$ connecting J_1, J_2 such that $\cup_t \mathcal{M}(A, J_t)$ is an oriented smooth manifold with boundary which is the disjoint union of $\mathcal{M}(A, J_1)$ and $\mathcal{M}(A, J_2)$. It follows that $\cup_t \mathcal{M}(A, J_t) \times_G \mathbb{S}^2$ is a cobordism between $\mathcal{M}(A, J_1) \times_G \mathbb{S}^2$ and $\mathcal{M}(A, J_2) \times_G \mathbb{S}^2$, hence the degree of ev is the same for J_1, J_2. This shows that the Gromov invariant is independent of the choice of $J \in \mathcal{J}_{reg}(A)$.

In order to show that the Gromov invariant is non-zero, we consider a special $J \in \mathcal{J}_{reg}(A)$. Let j, J_0 be the complex structure on $\mathbb{S}^2$ and T^{2n-2} respectively, and let $J = j \times J_0$ be the product which lies in $\mathcal{J}(M,\omega)$.

For any $u \in \mathcal{M}(A, J)$, since $J = j \times J_0$, the map $pr \circ u : \mathbb{S}^2 \to T^{2n-2}$, where $pr : M \to T^{2n-2}$ is the projection, is J_0-holomorphic. But $pr \circ u$ carries a trivial homology class, hence by Proposition 5.5, $pr \circ u$ is a constant map. This shows that any $u \in \mathcal{M}(A, J)$ has the form $u : z \mapsto (\phi(z), x)$ for some $\phi \in G = \mathrm{PSL}(2, \mathbb{C})$ and $x \in T^{2n-2}$.

There are two consequences of this fact: (1) For any $u \in \mathcal{M}(A, J)$, u^*TM is isomorphic as a holomorphic vector bundle to $T\mathbb{S}^2 \oplus E$ where E is a trivial bundle of rank $n-1$. By Lemma 5.9, D_u is onto for any $u \in \mathcal{M}(A, J)$, so that $J \in \mathcal{J}_{reg}(A)$. (2) The correspondence $u \mapsto (\phi, x)$ gives an identification of $\mathcal{M}(A, J)$ with $G \times T^{2n-2}$, and hence $\widetilde{\mathcal{M}}(A, J)$ with T^{2n-2} and $\mathcal{M}(A, J) \times_G \mathbb{S}^2$ with $\mathbb{S}^2 \times T^{2n-2} = M$. It follows that the evaluation map $ev : \mathcal{M}(A, J) \times_G \mathbb{S}^2 \to M$ is a diffeomorphism, and the degree of ev is ± 1. This proves that the Gromov invariant is non-zero.

Proof of Lemma 5.12.

Note that the non-vanishing of Gromov invariant only implies immediately that for any $J \in \mathcal{J}_{reg}(A)$, and for any generic point $p \in M$, there exists a J-holomorphic curve $C \in \widetilde{\mathcal{M}}(A, J)$ such that $p \in C$. This is different from the claim in Lemma 5.12 that in fact such a J-holomorphic curve exists for any $J \in \mathcal{J}(M,\omega)$ and any point $p \in M$ (in particular, $p_0 \in M$).

To get around of this, we use the Gromov Compactness Theorem, Theorem 5.8. We pick a sequence of $J_n \in \mathcal{J}_{reg}(A)$, since $\mathcal{J}_{reg}(A)$ is dense in $\mathcal{J}(M,\omega)$, which converges to $J \in \mathcal{J}(M,\omega)$ in C^∞-topology, and we pick a sequence of generic points p_n converging to $p_0 \in M$, such that for each n, there exists a J_n-holomorphic curve C_n such that $p_n \in C_n$. By Theorem 5.8, a subsequence of $\{C_n\}$ converges to a $C \in \widetilde{\mathcal{M}}(A, J)$ such that $p_0 = \lim_{n\to\infty} p_n \in C$. This proves Lemma 5.12.

∎

5.3. **J-holomorphic curves in dimension 4.** The J-holomorphic curve theory in dimension 4 is particularly more powerful because there are additional tools which allow one to analyse the singularities of a J-holomorphic curve. On the other hand, the existence of certain types of J-holomorphic curves actually can be derived from the underlying differential topology of the symplectic 4-manifold, due to the deep analytical work of Cliff Taubes.

Let (M, J) be an almost complex 4-manifold, and let $C \subset M$ be a J-holomorphic curve parametrized by a simple J-holomorphic map $u : \Sigma \to M$. The following theorem gives a criterion, amongst other things, for the embeddedness of C.

Theorem 5.13. *(Adjunction Inequality). Let g_Σ be the genus of Σ. Then the inequality*

$$\frac{1}{2}(C^2 - c_1(TM)\cdot C) + 1 \geq g_\Sigma$$

holds with equality if and only if C is embedded.

In particular, a rational J-holomorphic curve must be embedded if it is homologous to an embedded rational J-holomorphic curve. This explains why the singular curve C_0 in Example 5.7 can not be the image of a holomorphic map $u : \mathbb{S}^2 \to \mathbb{CP}^2$.

Example 5.14. (Algebraic curves in $\mathbb{CP}^2$). For notations we denote by $[\mathbb{CP}^1]$ the generator of $H_2(\mathbb{CP}^2;\mathbb{Z})$ which is the class of a line. Using Poincaré duality, we identify $H^2(\mathbb{CP}^2;\mathbb{Z})$ with $H_2(\mathbb{CP}^2;\mathbb{Z})$.

We first consider non-singular algebraic curves in $\mathbb{CP}^2$. Let C be a line in $\mathbb{CP}^2$. Then $C^2 = 1$, and

$$\frac{1}{2}(C^2 - c_1(T\mathbb{CP}^2)\cdot C) + 1 = 0,$$

which implies that $c_1(T\mathbb{CP}^2) = 3\cdot[\mathbb{CP}^1]$.

Now let C_d be any non-singular algebraic curve of degree d. Then $C_d^2 = d^2$ and $c_1(T\mathbb{CP}^2)\cdot C_d = 3d$. This gives rise to the following genus formula for C_d:

$$\text{genus }(C_d) = \frac{1}{2}(C_d^2 - c_1(T\mathbb{CP}^2)\cdot C_d) + 1 = \frac{1}{2}(d-1)(d-2).$$

Next we consider a singular algebraic curve, the cusp curve

$$C_0 = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 | z_1^3 = z_0 z_2^2\}.$$

C_0 is of degree 3 and has a cusp singularity at $[1, 0, 0]$. The left-hand side of the adjunction inequality for C_0 is

$$\frac{1}{2}(C_0^2 - c_1(T\mathbb{CP}^2)\cdot C_0) + 1 = \frac{1}{2}(3^2 - 3\cdot 3) + 1 = 1.$$

Since C_0 is singular, C_0 can only be parametrized by a holomorphic map from a genus zero Riemann surface, i.e., $\mathbb{S}^2$, so C_0 belongs to $\widetilde{\mathcal{M}}^*(3[\mathbb{CP}^1], J_0)$. On the other hand, C_0 is the limit of a family of non-singular cubic curves ($\lambda \neq 0$)

$$C_\lambda = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 | z_1^3 = z_0 z_2^2 + \lambda z_0^3\}$$

as $\lambda \to 0$. We remark that this also represents a certain kind of non-compactness phenomenon in the Gromov Compactness Theorem.

As an application of Theorem 5.13, we prove the following non-existence result.

Proposition 5.15. *For a generic almost complex structure J, there exist no rational J-holomorphic curves C such that $C^2 \leq -2$, and there exist at most embedded rational J-holomorphic curves C with $C^2 = -1$.*

Proof. Suppose C is a rational J-holomorphic curve such that $C^2 \leq -2$. Then the adjunction inequality implies that

$$c_1(TM)\cdot C \leq C^2 + 2 \leq -2 + 2 = 0.$$

On the other hand, for a generic almost complex structure J (cf. Theorem 5.10), the space $\mathcal{M}^*([C], J)$ is a smooth manifold of dimension $4 + 2c_1(TM) \cdot C \leq 4$. Since $G = \mathrm{PSL}(2, \mathbb{C})$ is 6-dimensional and acts on $\mathcal{M}^*([C], J)$ freely if it is non-empty, we see that $\mathcal{M}^*([C], J)$ must be at least 6-dimensional. This proves that for a generic almost complex structure, there exist no rational curves C with $C^2 \leq -2$. The proof for the case of $C^2 = -1$ is similar and we leave the details to the reader.

□

The above proposition shows that in a symplectic 4-manifold, the only interesting rational J-holomorphic curves are those with non-negative self-intersection. Because if J is taken generic, the only rational J-holomorphic curves with negative self-intersection are the embedded ones with self-intersection -1. By the symplectic neighborhood theorem, the symplectic 4-manifold can be symplectically blown down along these (-1)-curves, and the resulting symplectic 4-manifold does not have any rational J-holomorphic curves with negative self-intersection for a generic J. A symplectic 4-manifold is called **minimal** if it contains no embedded symplectic 2-spheres with self-intersection -1 (i.e., it can not be symplectically blown down).

The following theorem is useful in analysing the intersection of two distinct J-holomorphic curves.

Theorem 5.16. *(Positivity of Intersection) Let C, C' be two distinct J-holomorphic curves in a compact almost complex 4-manifold. Then the intersection of C, C' consists of at most finitely many points. Moreover, the intersection product*

$$C \cdot C' = \sum_{p \in C \cap C'} k_p$$

where $k_p \in \mathbb{Z}^+$, and $k_p = 1$ if and only if both C, C' are embedded near p and the intersection at p is transverse.

In particular, $C \cdot C' \geq 0$, and if $C \cdot C' = 0$, then C, C' are disjoint. If $C \cdot C' = 1$, then C, C' intersect at exactly one point and the intersection must be transverse.

Suppose C is an embedded rational J-holomorphic curve with $C^2 = 0$, and suppose C' is another rational J-holomorphic curve which is homologous to C. Then on the one hand, the adjunction inequality implies C' must also be embedded, and on the other hand, the positivity of intersection implies that C, C' must be disjoint. Thus if the moduli space of such rational curves has a positive dimension, they may be used to fill up the whole manifold. In order to do this, we need the following regularity criterion for J.

Lemma 5.17. *Suppose C is an immersed rational J-holomorphic curve in an almost complex 4-manifold (M, J) such that $c_1(TM) \cdot C > 0$. Then for any simple J-holomorphic map $u : \mathbb{S}^2 \to M$ parametrizing C, the linearization D_u of $\bar{\partial}_J(u) = 0$ is onto.*

We combine these tools to give a proof of the following structural theorem of symplectic 4-manifolds which contain an embedded rational curve of self-intersection 0.

Theorem 5.18. *Let (M, ω) be a symplectic 4-manifold which contains an embedded symplectic 2-sphere C with $C^2 = 0$. Suppose that M contains no spherical classes B*

such that $0 < \omega(B) < \omega(C)$. *Then* M *must be diffeomorphic to a* $\mathbb{S}^2$*-bundle over a surface.*

Proof. Pick a $J \in \mathcal{J}(M,\omega)$ such that C is J-holomorphic (cf. Proposition 2.16, §2 of Part 1). Then by the adjunction inequality every element in $\widetilde{\mathcal{M}}^*([C],J)$ is embedded, and since

$$c_1(TM)\cdot C = C^2 + 2 = 2 > 0,$$

by Lemma 5.17, the space $\mathcal{M}^*([C],J)$ is an oriented smooth manifold of dimension

$$\dim M + 2c_1(TM)\cdot C = 4 + 2\cdot 2 = 8.$$

Consequently, $\widetilde{\mathcal{M}}^*([C],J)$ is an oriented surface, which is compact by Theorem 5.8. With this understood, M is diffeomorphic to the $\mathbb{S}^2$-bundle over $\widetilde{\mathcal{M}}^*([C],J)$ via

$$ev : \mathcal{M}^*([C],J)\times_G \mathbb{S}^2 \to M, \;\; [(u,z)] \mapsto u(z),$$

where $G = \mathrm{PSL}(2,\mathbb{C})$.

□

Suppose in the above theorem, M contains another embedded symplectic 2-sphere C' such that $C'\cdot C' = 0$, which intersects with C transversely and positively at a single point. Then one can arrange $J \in \mathcal{J}(M,\omega)$ such that both C, C' are J-holomorphic. Suppose furthermore that there are also no spherical classes B such that $0 < \omega(B) < \omega(C')$, then $\widetilde{\mathcal{M}}^*([C'],J)$ is precisely the space of J-holomorphic sections of $\mathcal{M}^*([C],J)\times_G \mathbb{S}^2$ under ev. It is easily seen that in this case there is a diffeomorphism $\psi : \mathbb{S}^2\times\mathbb{S}^2 \to M$ such that the 2-spheres $\psi(\mathbb{S}^2\times\{pt\})$ and $\psi(\{pt\}\times\mathbb{S}^2)$ are symplectic in M. This fact was exploited in the proof of the following theorem.

Theorem 5.19. *(Gromov-Taubes). For any symplectic structure* ω *on* $\mathbb{CP}^2$*, there is a diffeomorphism* $\psi : \mathbb{CP}^2 \to \mathbb{CP}^2$ *such that* $\psi^*\omega$ *is a multiple of the standard Fubini-Study form* ω_0.

The proof of Theorem 5.19 consists of two steps. Step 1, which is due to Taubes and uses his deep work on Seiberg-Witten theory of symplectic 4-manifolds, asserts that there exists an embedded symplectic 2-sphere C with $C^2 = 1$. The complement $\mathbb{CP}^2 \setminus \nu(C)$ of a regular neighborhood of C is a homotopy 4-ball W with $\partial W = \mathbb{S}^3$ (Van-Kampen plus Mayer-Vietoris). By the symplectic neighborhood theorem, the symplectic form ω equals the standard symplectic form on $\mathbb{R}^4$ in a regular neighborhood of ∂W, which is identified with $\{x \in \mathbb{R}^4 | \delta_0 - \epsilon < ||x||^2 \leq \delta_0\}$ for some δ_0 and $\epsilon > 0$.

Step 2: (Gromov). There exists a symplectomorphism $B^4(\delta_0) \to W$ *which is identity near the boundaries.*

The proof of Step 2 goes as follows. Pick a large enough polydisc $D^2\times D^2 \subset \mathbb{R}^4$ which contains $B^4(\delta_0)$, and embedded $D^2\times D^2$ into $\mathbb{S}^2\times\mathbb{S}^2$ via some embedding $D^2 \subset \mathbb{S}^2$. Then one removes $B^4(\delta_0)$ from $\mathbb{S}^2\times\mathbb{S}^2$ and then glues back W. Call the resulting symplectic 4-manifold M. Apply the remarks following Theorem 5.18 to M (for details see [4], pages 314-319).

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