MATH 705: PART 1: FUNDATIONAL MATERIALS

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Contents

1. Basic notions and examples

1.1. Symplectic manifolds.

Definition 1.1. A symplectic structure on a smooth manifold M is a 2-form $\omega \in \Omega^2(M)$, which is (1) nondegenerate, and (2) closed (i.e. $d\omega = 0$). The pair (M, ω) is called a **symplectic manifold**.

Recall that a 2-form $\omega \in \Omega^2(M)$ is said to be nondegenerate if for every point $p \in M$, $\omega(u, v) = 0$ for all $u \in T_pM$ implies $v \in T_pM$ equals 0. The nondegeneracy condition on ω is equivalent to the condition that M has an even dimension $2n$ and the top wedge product

$$\omega^n \equiv \omega \wedge \omega \cdots \wedge \omega$$

is nowhere vanishing on M, i.e., ω^n is a volume form. In particular, M must be orientable, and is canonically oriented by ω^n. The nondegeneracy condition is also equivalent to the condition that M is almost complex, i.e., there exists an endomorphism J of TM such that $J^2 = -Id$. The latter is homotopy theoretic in nature as it means that the tangent bundle TM as a $SO(2n)$-bundle can be lifted to a $U(n)$-bundle under the natural homomorphism $U(n) \to SO(2n)$. (More systematic discussion in Section 2.) Note that this has nothing to do with the closedness of ω.

The closedness of ω is a very important, nontrivial geometric or analytical condition. For now, we simply observe that ω defines a deRham cohomology class $[\omega] \in H^2(M)$, which must be nonzero when M is *closed*. In fact in this case,

$$[\omega]^n \equiv [\omega] \cup [\omega] \cdots \cup [\omega] \in H^{2n}(M)$$

must be nonzero, since $[\omega]^n([M]) = \int_M \omega^n \neq 0$. Now we give some natural examples of symplectic manifolds.

Example 1.2. (Euclidean Spaces). The most basic examples are the Euclidean spaces $\mathbb{R}^{2n}$ equipped with the standard symplectic structure

$$\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$

Note that with $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ under $z_j = x_j + iy_j$, $j = 1, 2 \cdots, n$,

$$\omega_0 = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

Example 1.3. (Cotangent Bundles). Let L be a smooth manifold of dimension n and $M \equiv T^*L$ be the cotangent bundle. There is a canonical symplectic structure on M which has the form $\omega = -d\lambda$.

The 1-form λ is defined as follows. Let $\pi : M \equiv T^*L \to L$ be the natural projection. Then for any $v \in M \equiv T^*L$, we have $\pi^* : T^*_{\pi(v)}L \to T^*_v M$. With this understood, λ is defined by setting its value at v to be $\lambda(v) \equiv \pi^*(v)$. (Note that $v \in T^*_{\pi(v)}L$, so that $\lambda(v) \equiv \pi^*(v) \in T^*_v M$.)

Let $q_1, \cdots, q_n$ be local coordinates on L, and $p_1, \cdots, p_n$ be the corresponding coordinates on the fibers, i.e, if a cotangent vector $v = \sum_{j=1}^n p_j dq_j$, then v has coordinates $p_1, \cdots, p_n$ on the fiber. Together $q_1, \cdots, q_n$ and $p_1, \cdots, p_n$ form a system of local coordinates on $M \equiv T^*L$. In the above local coordinates, we claim $\lambda = \sum_{j=1}^n p_j dq_j$. Accepting it momentarily, we see immediately that $\omega \equiv -d\lambda$ is a symplectic structure, because $\omega = \sum_{j=1}^n dq_j \wedge dp_j$ in these coordinates.

To see the claim $\lambda = \sum_{j=1}^n p_j dq_j$, recall that the value of λ at $v = \sum_{j=1}^n p_j dq_j$ equals $\pi^*(v) = \sum_{j=1}^n p_j \pi^*(dq_j) = \sum_{j=1}^n p_j d(\pi \circ q_j)$. Since $q_1, \cdots, q_n$ are regarded as local coordinates on $M \equiv T^*L$, $q_j = \pi \circ q_j$, and with this understood, $v = \sum_{j=1}^n p_j dq_j$ has coordinates $q_1, \cdots, q_n, p_1, \cdots, p_n$. This shows that $\lambda = \sum_{j=1}^n p_j dq_j$. It is interesting to check that λ is characterized by the following property: for any 1-form σ on L, which can be regarded as a section of the cotangent bundle $\pi : M \equiv T^*L \to L$, one has $\sigma^* \lambda = \sigma$.

We remark that cotangent bundles form a fundamental class of symplectic manifolds. These are the phase spaces in classical mechanics, with coordinates q and p corresponding to position and momentum.

Example 1.4. (Orientable Surfaces). Let M be a 2-dimensional orientable manifold, and let ω be any volume form on M. Then (M, ω) is a symplectic manifold because in this case ω is automatically nondegenerate and closed.

Example 1.5. (Kähler Manifolds). Let M be a complex manifold of dimension n, and let h be a Hermitian metric on M. Then in a local complex coordinates $z_1, z_2, \cdots, z_n$,

h may be written as $h = \sum_{j,k=1}^n h_{j\bar{k}} dz_j \otimes d\bar{z}_k$, where $(h_{j\bar{k}})$ is a $n \times n$ Hermitian matrix, i.e., $h_{j\bar{k}} = \bar{h}_{k\bar{j}}$. The associated 2-form to h is $\omega = i \sum_{j,k=1}^n h_{j\bar{k}} dz_j \wedge d\bar{z}_k$. With this understood, h is called a *Kähler metric* if ω is closed. ω is clearly nondegenerate, hence the underlying real manifold, also denoted by M, is a $2n$-dimensional symplectic manifold with a symplectic structure ω. Note that the nondegeneracy condition on ω follows from the identity

$$\omega^n = i^n \det(h_{j\bar{k}})\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

The complex projective spaces $\mathbb{CP}^n$ and its complex submanifolds form a fundamental class of Kähler manifolds. There is a canonical Kähler metric on $\mathbb{CP}^n$, called the *Fubini-Study metric*. In terms of the homogeneous coordinates $z_0, z_1, \cdots, z_n$ of $\mathbb{CP}^n$, the associated Kähler form of the Fubini-Study metric is

$$\omega_0 = \frac{i}{2\pi \cdot (\sum_{\kappa=0}^n z_\kappa \bar{z}_\kappa)^2} \sum_{k=0}^n \sum_{j \neq k} (\bar{z}_j z_j dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k).$$

(We point out that ω_0 is normalized such that $\int_{\mathbb{CP}^n} \omega_0^n = 1$.) Note that a complex submanifold of $\mathbb{CP}^n$ is naturally a Kähler manifold with the induced metric.

Another important class of Kähler manifolds is given by properly embedded complex submanifolds of $\mathbb{C}^N$, which are called *Stein manifolds*.

Example 1.6. (Product of Symplectic Manifolds). Given two symplectic manifolds (M_1, ω_1), (M_2, ω_2), the product $M_1 \times M_2$ is also symplectic with symplectic structures $\pi_1^* \omega_1 \oplus \pi_2^*(\pm\omega_2)$. Here $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, and we canonically identity $T^*(M_1 \times M_2)$ with $\pi_1^*(T^* M_1) \oplus \pi_2^*(T^* M_2)$.

Example 1.7. Consider $\mathbb{R}^{2n}$ with the standard symplectic structure ω_0, and $G = \mathbb{Z}^{2n}$ the integral lattice, which acts freely on $\mathbb{R}^{2n}$ by translations. The quotient is the $(2n)$-dimensional torus $\mathbb{T}^{2n}$, with a standard symplectic structure. (Note that this example can be also obtained by taking the n-fold product of the 2-torus $\mathbb{T}^2$.)

The following example is also of this sort, which gives the first example of a compact closed, symplectic but non-Kähler manifold. (The example was known to Kodaira in the 1950s and was rediscovered in the 1970s by Thurston.)

Example 1.8. (A Non-Kähler Manifold). Consider the group $G = \mathbb{Z}^2 \times \mathbb{Z}^2$ with the noncommutative group operation

$$(j', k') \circ (j, k) = (j + j', A_{j'} k + k'), \quad A_j = \begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix},$$

where $j = (j_1, j_2) \in \mathbb{Z}^2$, and similarly for k. This group acts on $\mathbb{R}^4$ via

$$G \to \mathrm{Diff}(\mathbb{R}^4) : (j, k) \mapsto \rho_{jk},$$

where $\rho_{jk}(x, y) = (x + j, A_j y + k)$. One can verify easily that the action is free and preserves the symplectic structure

$$\omega = dx_1 \wedge dx_2 + dy_1 \wedge dy_2.$$

Hence the quotient $M = \mathbb{R}^4/G$ is a symplectic manifold which is easily seen to be compact closed.

We will show that $H_1(M;\mathbb{Z})=\mathbb{Z}^3$ so that M is not a Kähler manifold. (Recall that by the Hodge theory the odd dimensional Betti numbers of a closed Kähler manifold must be even.) To see this, one first verifies that the commutator subgroup $[G,G]$, which consists of elements of the form $aba^{-1}b^{-1}$, $a,b\in G$, equals $0\oplus 0\oplus\mathbb{Z}\oplus 0$. Then note that $\pi_1(M)=G$ and $H_1(M;\mathbb{Z})=\pi_1(M)/[\pi_1(M),\pi_1(M)]$. Hence $H_1(M;\mathbb{Z})=G/[G,G]=\mathbb{Z}^3$. This implies that the first Betti number of M equals 3, and that M is not Kähler.

Topologically, M is a nontrivial $\mathbb{T}^2$ bundle over $\mathbb{T}^2$, or more precisely, $M=\mathbb{S}^1\times N$ where N is the nontrivial $\mathbb{T}^2$ bundle over $\mathbb{S}^1$ defined as follows. Let x,y be the standard coordinates on $\mathbb{T}^2$. Then $N=[0,1]\times\mathbb{T}^2/\sim$, where $(0,x,y)\sim(1,x+y,y)$.

A **symplectomorphism** of a symplectic manifold (M,ω) is a diffeomorphism $\psi\in\text{Diff}(M)$ which preserves the symplectic structure

$$\omega=\psi^*\omega.$$

Note that a symplectomorphism is particularly a volume-preserving diffeomorphism (hence is necessarily an orientation-preserving diffeomorphism), as one has $\psi^*(\omega^n)=(\psi^*\omega)^n=\omega^n$. There is an abundance of symplectomorphisms on a symplectic manifold. In fact, the group of symplectomorphisms, denoted by $\text{Symp}(M,\omega)$ or simply by $\text{Symp}(M)$, is an infinite dimensional Lie group, in contrast with the finite dimensionality of the isometry group of a Riemannian metric.

The nondegeneracy condition of a symplectic structure ω gives rise to the following canonical isomorphism

$$TM\to T^*M : X\mapsto i_X\omega=\omega(X,\cdot).$$

A vector field X is called **symplectic** if $i_X\omega$ is closed. The next result is one consequence of the closedness of a symplectic structure, which shows that when M is closed, the set of symplectic vector fields form the Lie algebra of the group $\text{Symp}(M)$.

Proposition 1.9. *Let (M,ω) be any symplectic manifold. If $t\mapsto\psi_t\in\mathit{Diff}(M)$ is a smooth family of diffeomorphisms generated by a family of vector fields X_t via*

$$\frac{d}{dt}\psi_t=X_t\circ\psi_t,\quad \psi_0=id,$$

then $\psi_t\in\mathit{Symp}(M)$ for every t iff X_t is a symplectic vector field for every t.

Proof. Recall Cartan's formula for the Lie derivative

$$L_X\omega=i_Xd\omega+d(i_X\omega).$$

Now the closedness of ω, i.e., $d\omega=0$, implies that

$$\frac{d}{dt}\psi_t^*\omega=\psi_t^*(L_{X_t}\omega)=\psi_t^*(d(i_{X_t}\omega)),$$

which vanishes if and only if $i_{X_t}\omega$ is closed. This proves that $\psi_t\in\text{Symp}(M)$ for every t iff X_t is a symplectic vector field for every t.

□

There is a simple way to obtain symplectic vector fields, and therefore to obtain symplectomorphisms, which shows their abundance. Let H be a smooth function (which has a compact support if M is not closed). There is a vector field X_H canonically associated to H by the following equation

$$i_{X_H}\omega = dH.$$

The flow ψ_H^t on M generated by X_H, i.e.,

$$\frac{d}{dt}\psi_H^t = X_H \circ \psi_H^t,$$

is called a **Hamiltonian flow**. Note that X_H is a symplectic vector field as $i_{X_H}\omega$, being exact, is closed. Thus ψ_H^t is a symplectomorphism for each t. The function H is called the **Hamiltonian function** and the vector field X_H is called the **Hamiltonian vector field**.

More generally, a symplectomorphism $\psi \in \mathrm{Symp}(M)$ is called **Hamiltonian** if there exists a smooth family of $\psi_t \in \mathrm{Symp}(M)$, $t \in [0,1]$, with $\psi_0 = \mathrm{id}$, $\psi_1 = \psi$, such that the corresponding (time-dependent) symplectic vector field X_t generating ψ_t is **Hamiltonian**, i.e., the 1-form $i_{X_t}\omega$ is exact for each t and has the form $i_{X_t}\omega = dH_t$ for a time-dependent smooth function H_t on M. The function H_t is called a **time-dependent Hamiltonian function**, and ψ_t is called an **Hamiltonian isotopy**.

Exercise: Consider the symplectic manifold (M,ω) where

$$M = \{(x_1,x_2,x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\},$$

and ω is the area form induced from the embedding $M \subset \mathbb{R}^3$. Show that the Hamiltonian flow on M associated to the "height" function

$$H(x_1,x_2,x_3) \equiv x_3$$

is the $\mathbb{S}^1$-action on M defined by the rotations about the x_3-axis.

Further discussions about Hamiltonian $\mathbb{S}^1$-actions (and more generally, Hamiltonian $\mathbb{T}^n$-actions) will be given in Section 4.

1.2. **Submanifolds of symplectic manifolds.** Let (M,ω) be a symplectic manifold, and let $Q \subset M$ be a submanifold of M. Note that the tangent bundle TQ is a sub-bundle of $TM|_Q$. We set

$$TQ^\omega \equiv \bigcup_{q\in Q} \{u \in TM_q | \omega(u,v) = 0 \quad \forall v \in TQ_q\},$$

which is also a sub-bundle of $TM|_Q$.

Definition 1.10. We say Q is **isotropic** if $TQ \subset TQ^\omega$, **coisotropic** if $TQ^\omega \subset TQ$, **symplectic** if $TQ \cap TQ^\omega = \{0\}$, and **Lagrangian** if $TQ = TQ^\omega$.

We remark that Q is symplectic iff the pull-back of ω to Q is a symplectic structure. Moreover, Q is Lagrangian iff Q is isotropic and half-dimensional. (More discussion in Section 2.) Amongst the four types of submanifolds of a symplectic manifold, Lagrangian submanifolds form the most important class.

Example 1.11. (Totally real submanifolds in a Kähler manifold). Let M be a Kähler manifold of complex dimension n. A real n-dimensional submanifold Q is called *totally real* if for every point $q \in Q$, $T_qQ \cap J(T_qQ) = \{0\}$. A totally real submanifold Q in a Kähler manifold is Lagrangian with respect to the Kähler form iff for each $q \in Q$, T_qQ and $J(T_qQ)$ are orthogonal with respect to the Kähler metric. When $n = 1$, every symplectic 2-dimensional manifold is Kähler, and every real 1-dimensional submanifold is Lagrangian and totally real. For higher dimensional examples, suppose M is a complex submanifold of $\mathbb{CP}^N$, which is Kähler under the induced metric. Then the fixed-point set of the anti-holomorphic involution $z_i \mapsto \bar{z}_i$ in M (assuming M is invariant under the involution), if nonempty, is a totally real, Lagrangian submanifold of M.

Example 1.12. (Lagrangian submanifolds in $(\mathbb{R}^{2n}, \omega_0)$). The n-dimensional spaces defined by $x_j =$ constant, $j = 1, \cdots, n$, or $y_j =$ constant, $j = 1, \cdots, n$, are Lagrangian submanifolds in $(\mathbb{R}^{2n}, \omega_0)$. For a different type of examples, consider the n-torus $T^n \subset \mathbb{R}^{2n}$, where we think $(\mathbb{R}^{2n}, \omega_0)$ as a product of symplectic manifold $(\mathbb{R}^2, \omega_0)$ and

$$T^n \equiv S^1 \times \cdots \times S^1,$$

where the j-th copy of S^1 is the unit circle $\{x_j^2 + y_j^2 = 1\}$ in the j-th copy of $\mathbb{R}^2$. The torus T^n is Lagrangian because (1) any 1-dimensional manifold in a 2-dimensional symplectic manifold is Lagrangian, and (2) the product of Lagrangian submanifolds in a product symplectic manifold is again Lagrangian.

Example 1.13. (Lagrangian submanifolds and symplectomorphisms). Let (M, ω) be any symplectic manifold. Then in the product symplectic manifold $(M \times M, \omega \times (-\omega))$, the diagonal $\Delta \equiv \{(x, x) \in M \times M | x \in M\}$ is Lagrangian. More generally, for any $\psi \in \mathrm{Symp}(M, \omega)$, the gragh of ψ in $M \times M$,

$$\mathrm{gragh}(\psi) \equiv \{(x, \psi(x)) \in M \times M | x \in M\},$$

is a Lagrangian submanifold.

Example 1.14. (Lagrangian submanifolds in cotangent bundles). Let $M \equiv T^*L$ be the cotangent bundle of L equipped with the canonical symplectic structure $\omega = -d\lambda$. (In local coordinates $\lambda = \sum_j p_j dq_j$ and $\omega = \sum_j dq_j \wedge dp_j$.) Clearly, the fibers of T^*L (defined by $q_j =$ constant$, \forall j$) and the zero section $L \subset T^*L$ (defined by $p_j = 0, \forall j$) are Lagrangian submanifolds.

Next we consider submanifolds Q_σ in M which is the graph of a 1-form σ on L, regarded as a smooth section of T^*L. Since Q_σ is of half dimension of M, it follows that Q_σ is a Lagrangian submanifold if and only if the pull-back of ω to Q_σ equals 0, which is equivalent to the condition that the pull-back $\sigma^*\omega = 0$. But

$$\sigma^*\omega = \sigma^*(-d\lambda) = -d(\sigma^*\lambda) = -d\sigma,$$

which implies that Q_σ is Lagrangian iff σ is closed.

Exercise: Suppose σ is a closed 1-form on L and Q_σ is the corresponding Lagrangian submanifold in T^*L. Let $\psi_t \in \mathrm{Symp}(T^*L)$, $t \in [0, 1]$, be an isotopy of symplectomorphisms such that $\psi_0 = \mathrm{id}$ and $\psi_1(L) = Q_\sigma$. Show that σ is an exact 1-form if ψ_t is an Hamiltonian isotopy.

Exercise: Show that every 1-dimensional submanifold is isotropic and every codimension 1 submanifold is coisotropic.

Exercise: Let Q be a coisotropic submanifold. Show that $TQ^\omega \subset TQ$ is an integrable distribution on Q.

2. Linear symplectic geometry

2.1. **Symplectic vector spaces.**

Definition 2.1. A **symplectic vector space** is a pair (V, ω) where V is a finite dimensional real vector space and ω is a bilinear form which satisfies

- Skew-symmetry: for any $u, v \in V$,
$$\omega(u, v) = -\omega(v, u).$$
- Nondegeneracy: for any $u \in V$,
$$\omega(u, v) = 0 \ \ \forall v \in V \text{ implies } u = 0.$$

A **linear symplectomorphism** of a symplectic vector space (V, ω) is a vector space isomorphism $\psi : V \to V$ such that
$$\omega(\psi u, \psi v) = \omega(u, v) \quad \forall u, v \in V.$$
The group of linear symplectomorphisms of (V, ω) is denoted by $\mathrm{Sp}(V, \omega)$.

Example 2.2. The Euclidean space $\mathbb{R}^{2n}$ carries a standard skew-symmetric, nondegenerate bilinear form ω_0 defined as follows. For $u = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T$, $u' = (x_1', x_2', \cdots, x_n', y_1', y_2', \cdots, y_n')^T$,
$$\omega_0(u, u') = \sum_{i=1}^{n} (x_i y_i' - x_i' y_i) = -u^T J_0 u',$$
where $J_0 = \begin{pmatrix} 0, -I \\ I, 0 \end{pmatrix}$. (Here I is the $n \times n$ identity matrix.)

The group of linear symplectomorphisms of $(\mathbb{R}^{2n}, \omega_0)$, which is denoted by $\mathrm{Sp}(2n)$, can be identified with the group of $2n \times 2n$ symplectic matrices. Recall a symplectic matrix Ψ is one which satisfies $\Psi^T J_0 \Psi = J_0$. For the case of $n = 1$, a symplectic matrix is simply a matrix Ψ with $\det \Psi = 1$.

Let (V, ω) be any symplectic vector space, and let $W \subset V$ be any linear subspace. The **symplectic complement** of W in V is defined and denoted by
$$W^\omega = \{v \in V | \omega(v, w) = 0, \ \ \forall w \in W\}.$$
A subspace W is called **isotropic** if $W \subset W^\omega$, **coisotropic** if $W^\omega \subset W$, **symplectic** if $W \cap W^\omega = \{0\}$, and **Lagrangian** if $W = W^\omega$.

Lemma 2.3. *(1)* $\dim W + \dim W^\omega = \dim V$, *(2)* $(W^\omega)^\omega = W$.

Proof. Define $\iota_\omega : V \to V^*$ by $\iota_\omega(v) : w \mapsto \omega(v,w)$, $\forall v, w \in V$, where V^* is the dual space of V. Since ω is nondegenerate, ι_ω is an isomorphism. Now observe that $\iota_\omega(W^\omega) = W^\perp$ where $W^\perp \subset V^*$ is the annihilator of W, i.e.,

$$W^\perp \equiv \{v^* \in V^* | v^*(w) = 0 \ \ \forall w \in W\}.$$

Part (1) follows immediately from the fact that

$$\dim W + \dim W^\perp = \dim V.$$

Part (2) follows easily from $W \subset (W^\omega)^\omega$ and the equations

$$\dim W = \dim V - \dim W^\omega = \dim (W^\omega)^\omega$$

which are derived from (1). □

Exercise: Show that if $W \subset V$ is isotropic, then ω induces a symplectic structure on the quotient space W^ω/W. Similarly, if $W \subset V$ is coisotropic, then ω induces a symplectic structure on W/W^ω.

Lemma 2.4. *For any symplectic vector space (V, ω), there exists a basis of V, denoted by $u_1, u_2, \cdots, u_n$, $v_1, v_2, \cdots, v_n$, such that*

$$\omega(u_j, u_k) = \omega(v_j, v_k) = 0, \ \ \omega(u_j, v_k) = \delta_{jk}.$$

(Such a basis is called a **symplectic basis***.) In particular,* $\dim V = 2n$ *is even.*

Proof. We prove by induction on $\dim V$. Note that $\dim V \geq 2$.

When $\dim V = 2$, the nondegeneracy condition of ω implies that there exist $u, v \in V$ such that $\omega(u, v) \neq 0$. Clearly u, v are linearly independent so that they form a basis of V since $\dim V = 2$. We can replace v by an appropriate nonzero multiple so that the condition $\omega(u, v) = 1$ is satisfied. Hence the theorem is true for the case of $\dim V = 2$.

Now suppose the theorem is true when $\dim V \leq m - 1$. We shall prove that it is also true when $\dim V = m$. Again the nondegeneracy condition of ω implies that there exist $u_1, v_1 \in V$ such that u_1, v_1 are linearly independent and $\omega(u_1, v_1) = 1$. Set $W \equiv \mathrm{span}(u_1, v_1)$. Then we claim that $(W^\omega, \omega|_{W^\omega})$ is a symplectic vector space. It suffices to show that $\omega|_{W^\omega}$ is nondegenerate. To see this, suppose $w \in W^\omega$ such that $\omega(w, z) = 0$ for all $z \in W^\omega$. We need to show that w must be zero. To this end, note that $W \cap W^\omega = \{0\}$, so that $V = W \oplus W^\omega$ by (1) of the previous lemma. Now for any $z \in V$, write $z = z_1 + z_2$ where $z_1 \in W$ and $z_2 \in W^\omega$. Then $\omega(w, z_1) = 0$ because $w \in W^\omega$ and $\omega(w, z_2) = 0$ by $z_2 \in W^\omega$ and the assumption on w. Hence $\omega(w, z) = 0$, and therefore $w = 0$ by the nondegeneracy condition of ω on V.

Note that $\dim W^\omega = \dim V - 2 \leq m - 1$, so that by the induction hypothesis, there is a symplectic basis $u_2, \cdots, u_n, v_2, \cdots, v_n$ of $(W^\omega, \omega|_{W^\omega})$. It is clear that $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ is a symplectic basis of (V, ω), and the lemma is proved. □

The following two immediate corollaries are left as exercises.

Exercise: Let ω be any skew-symmetric bilinear form on V. Show that ω is nondegenerate if and only if $\dim V = 2n$ is even and

$$\omega^n = \omega \wedge \cdots \wedge \omega \neq 0.$$

Exercise: Let (V,ω) be any symplectic vector space. Show that there exists an $n>0$ and a vector space isomorphism $\phi:\mathbb{R}^{2n}\to V$ such that

$$\omega_0(z,z')=\omega(\phi z,\phi z'),\ \ \forall z,z'\in\mathbb{R}^{2n}.$$

Consequently, $\mathrm{Sp}(V,\omega)$ is isomorphic to $\mathrm{Sp}(2n)$.

Exercise: (Space of Lagrangians) Let $\mathcal{L}(n)$ denote the set of Lagrangian subspaces of $(\mathbb{R}^{2n},\omega_0)$. Let $\Lambda\subset\mathbb{R}^{2n}$ be any n-dimensional subspace.

(1) Show that $\Lambda\in\mathcal{L}(n)$ iff $J_0(\Lambda)$ is orthogonal to Λ.

(2) Let $u_1,u_2,\cdots,u_n$ be an orthonormal basis of Λ, and write $(u_1,\cdots,u_n)=\begin{pmatrix} X\\ Y\end{pmatrix}$ where X,Y are $n\times n$ matrices. Show that $X+iY\in U(n)$ iff $\Lambda\in\mathcal{L}(n)$.

(3) Show that $\mathcal{L}(n)$ can be identified with $U(n)/O(n)$, induced by $\Lambda\mapsto X+iY$.

(4) Let $\Lambda_t\in\mathcal{L}(n)$ be any loop of Lagrangian subspaces, and let $U_t\in U(n)$ be any path which is a lifting of Λ_t. Show that $\det(U_t^2)$ is a loop in $\mathbb{S}^1$, which depends only on the loop Λ_t. The degree of $\det(U_t^2)$ is called the **Maslov index** of Λ_t.

(5) Show that the Maslov index defines an isomorphism of $\pi_1(\mathcal{L}(n))$ to $\mathbb{Z}$.

A **complex structure** on a (finite dimensional) real vector space V is an automorphism $J:V\to V$ such that $J^2=-\mathrm{id}$. A **Hermitian structure** on (V,J) is an inner product g on V which is J-invariant, i.e., $g(Jv,Jw)=g(v,w)$, for all $v,w\in V$. Let J be a complex structure on V. Then V becomes a complex vector space by defining the complex multiplication by

$$\mathbb{C}\times V\to V:(x+iy,v)\mapsto xv+yJv.$$

Definition 2.5. Let (V,ω) be a symplectic vector space. A complex structure J on V is called **ω-compatible** if

- $\omega(Jv,Jw)=\omega(v,w)$ for all $v,w\in V$,
- $\omega(v,Jv)>0$ for any $0\neq v\in V$.

We denote the set of ω-compatible complex structures by $\mathcal{J}(V,\omega)$. Note that $\mathcal{J}(V,\omega)$ is nonempty: let $u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n$ be a symplectic basis of (V,ω), then $J:V\to V$ defined by $Ju_i=v_i,Jv_i=-u_i$ is a ω-compatible complex structure. Finally, for any $J\in\mathcal{J}(V,\omega)$, $g_J(v,w)\equiv\omega(v,Jw)$, $\forall v,w\in V$, is a (canonically associated) Hermitian structure on (V,J).

Lemma 2.6. *Suppose* $\dim V=2n$. *Then for any* $J\in\mathcal{J}(V,\omega)$, *there is a vector space isomorphism* $\phi_J:\mathbb{R}^{2n}\to V$ *such that*

$$\phi_J^*\omega=\omega_0,\quad \phi_J^*J\equiv\phi_J^{-1}\circ J\circ\phi_J=J_0.$$

Moreover, $\phi_J^*:J'\mapsto\phi_J^{-1}\circ J'\circ\phi_J$ *identifies* $\mathcal{J}(V,\omega)$ *with* $\mathcal{J}(\mathbb{R}^{2n},\omega_0)$.

Proof. Consider the canonical Hermitian structure $g_J(v,w)\equiv\omega(v,Jw)$, $\forall v,w\in V$, and extend g_J to $V\times\mathbb{C}$ complex linearly. Recall that $V\times\mathbb{C}=V_{1,0}\oplus V_{0,1}$ where $V_{1,0}$, $V_{0,1}$ are the $(+i)$-eigenspace and $(-i)$-eigenspace of J. The projection of (V,J) to $V_{1,0}$ is an isomorphism of complex vector spaces.

Let $(e_1,e_2,\cdots,e_n,f_1,f_2,\cdots,f_n)$ be any basis of V where $Je_k=f_k$, $Jf_k=-e_k$ for each k, and let $z_k:=\frac{1}{2}(e_k-if_k)$, $\bar{z}_k:=\frac{1}{2}(e_k+if_k)$ be the projections of e_k to $V_{1,0}$,

$V_{0,1}$ respectively. Then

$$g_J(z_k, \bar{z}_l) = \frac{1}{2}(\omega(e_k, f_l) - i\omega(e_k, e_l)).$$

Set $u_k := \frac{1}{\sqrt{2}} e_k$ for each k. It follows easily that if $z_1, z_2, \cdots, z_n$ is a unitary basis of $(V_{1,0}, g_J)$, then $u_1, u_2, \cdots, u_n, Ju_1, Ju_2, \cdots, Ju_n$ is a symplectic basis of (V, ω). Furthermore, if we define $\phi_J : \mathbb{R}^{2n} \to V$, where

$$\phi_J : (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \mapsto \sum_{i=1}^{n} (x_i u_i + y_i J u_i),$$

then it is obvious that $\phi_J^* \omega = \omega_0$ and $\phi_J^* J = J_0$. Moreover, the verification that $\phi_J^* : J' \mapsto \phi_J^{-1} \circ J' \circ \phi_J$ identifies $\mathcal{J}(V, \omega)$ with $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ is straightforward. $\square$

Note that under the natural inclusion of $GL(n, \mathbb{C})$ in $GL(2n, \mathbb{R})$, $U(n)$ lies in $\mathrm{Sp}(2n)$.

Theorem 2.7. *The space $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ is naturally identified with the homogeneous space $Sp(2n)/U(n)$. Moreover, there is a canonically defined smooth map*

$$H : Sp(2n)/U(n) \times [0, 1] \to Sp(2n)/U(n)$$

such that $H(x, 0) = x$, $H(x, 1) = *$ for any $x \in Sp(2n)/U(n)$, $H(*, t) = *$ for any $t \in [0, 1]$, where $*$ denote the orbit of the identity matrix $I \in Sp(2n)$ in $Sp(2n)/U(n)$. In particular, $Sp(2n)/U(n)$ (and $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ as well) is contractible.

Proof. There is a canonical left action of $\mathrm{Sp}(2n)$ on $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$: for any $\psi \in \mathrm{Sp}(2n)$, $J \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$, we define $\psi \cdot J := \psi \circ J \circ \psi^{-1}$. Then Lemma 2.6 implies that the action of $\mathrm{Sp}(2n)$ on $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ is transitive. The isotropy subgroup at J_0 is clearly $U(n)$, which implies the identification of $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ with $\mathrm{Sp}(2n)/U(n)$.

It remains to construct the map H. First of all, for any $\psi \in \mathrm{Sp}(2n)$, ψ^T is also in $\mathrm{Sp}(2n)$, so that $\psi\psi^T$ is a symmetric, positive definite symplectic matrix. We will show that $(\psi\psi^T)^\alpha$ is also a symplectic matrix for any real number $\alpha > 0$.

To this end, we decompose $\mathbb{R}^{2n} = \oplus_\lambda V_\lambda$ where V_λ is the λ-eigenspace of $\psi\psi^T$, and $\lambda > 0$. Then note that for any $z \in V_\lambda$, $z' \in V_{\lambda'}$, $\omega_0(z, z') = 0$ if $\lambda\lambda' \neq 1$. Our claim that $(\psi\psi^T)^\alpha$ is a symplectic matrix for any real number $\alpha > 0$ follows easily from this observation.

Now for any $\psi \in \mathrm{Sp}(2n)$, we decompose $\psi = PQ$ where $P = (\psi\psi^T)^{1/2}$ is symmetric and $Q \in O(2n)$. Note that $Q = \psi P^{-1} \in \mathrm{Sp}(2n) \cap O(2n) = U(n)$, which shows that ψ and $P = (\psi\psi^T)^{1/2}$ are in the same orbit in $\mathrm{Sp}(2n)/U(n)$. With this understood, we define

$$H : (\psi \cdot U(n), t) \mapsto (\psi\psi^T)^{(1-t)/2} \cdot U(n), \quad \psi \in \mathrm{Sp}(2n), \ t \in [0, 1].$$

$\square$

Recall that for any $J \in \mathcal{J}(V, \omega)$, there is a canonically associated Hermitian structure (i.e. a J-invariant inner product) $g_J(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$. The next theorem shows that one can construct ω-compatible complex structures from inner products on V. Let $\mathrm{Met}(V)$ denote the space of inner products on V.

Theorem 2.8. *There exists a canonically defined map $r : Met(V) \to \mathcal{J}(V,\omega)$ such that*

$$r(g_J) = J, \;\; r(\psi^* g) = \psi^* r(g)$$

for all $J \in \mathcal{J}(V,\omega)$, $g \in Met(V)$, and $\psi \in Sp(V,\omega)$.

Proof. For any given $g \in \mathrm{Met}(V)$, we define $A : V \to V$ by

$$\omega(v,w) = g(Av,w), \quad \forall v,w \in V.$$

Then the skew-symmetry of ω implies that A is g-skew-adjoint. It follows that $P \equiv -A^2$ is g-self-adjoint and g-positive definite. Set $Q \equiv P^{1/2}$, which is also g-self-adjoint and g-positive definite.

We define the map r by $g \mapsto J_g \equiv Q^{-1}A$. Then $J_g^2 = Q^{-1}AQ^{-1}A = Q^{-2}A^2 = -I$ is a complex structure. To check that J_g is ω-compatible, note that

$$\omega(Q^{-1}Av, Q^{-1}Aw) = g(AQ^{-1}Av, Q^{-1}Aw) = -g(v, Aw) = \omega(v,w), \;\; \forall v,w \in V,$$

$$\omega(v, Q^{-1}Av) = g(Av, Q^{-1}Av) > 0 \;\; \forall 0 \neq v \in V$$

because Q^{-1} is g-self-adjoint and g-positive definite.

Finally, for any $\psi \in \mathrm{Sp}(V,\omega)$, replacing g with $\psi^* g$ changes A to $\psi^{-1}A\psi$, and therefore changes Q to $\psi^{-1}Q\psi$. This implies $r(\psi^* g) = \psi^* r(g)$. If $g = g_J$, then $A = J$ and $Q = I$, so that $r(g_J) = J$.

□

Exercise: For any $\psi \in \mathrm{Sp}(2n)$, the decomposition $\psi = PQ$, where $P = (\psi\psi^T)^{1/2}$ and $Q \in U(n)$, allows us to define the so-called **Maslov index** of a loop in $\mathrm{Sp}(2n)$. More precisely, for any loop $\psi(t) \in \mathrm{Sp}(2n)$, we decompose $\psi(t) = P(t)Q(t)$, and define the Maslov index of $\psi(t)$ to be the degree of the loop $\det Q(t)$ in $\mathbb{S}^1$. Show that two loops in $\mathrm{Sp}(2n)$ are homotopic if and only if they have the same Maslov index.

2.2. **Symplectic vector bundles.**

Definition 2.9. A **symplectic vector bundle** over a smooth manifold M is a pair (E,ω), where $E \to M$ is a real vector bundle and ω is a smooth section of $E^* \wedge E^*$ such that for each $p \in M$, (E_p, ω_p) is a symplectic vector space. (Here E^* is the dual of E.) The section ω is called a **symplectic bilinear form** on E. Two symplectic vector bundles (E_1,ω_1), (E_2,ω_2) are said to be **isomorphic** if there exists an isomorphism $\phi : E_1 \to E_2$ (which is identity over M) such that $\phi^*\omega_2 = \omega_1$.

The standard constructions in bundle theory carry over to the case of symplectic vector bundles. For example, for any smooth map $f : N \to M$ and symplectic vector bundle (E,ω) over M, the pull-back $(f^*E, f^*\omega)$ is a symplectic vector bundle over N. In particular, for any submanifold $Q \subset M$, the restriction $(E|_Q, \omega|_Q)$ is a symplectic vector bundle over Q. Let F be a sub-bundle of E such that for each $p \in M$, $(F_p, \omega_p|_{F_p})$ is a symplectic vector space. Then $(F, \omega|_F)$ is naturally a symplectic vector bundle. We call F (or $(F,\omega|_F)$) a **symplectic sub-bundle** of (E,ω). The **symplectic complement** of F is the sub-bundle

$$F^\omega \equiv \cup_{p\in M} F_p^{\omega_p} = \cup_{p\in M}\{v \in E_p | \omega_p(v,w) = 0, \; \forall w \in F_p\},$$

which is naturally a symplectic sub-bundle of (E,ω). Note that as a real vector bundle, F^ω is isomorphic to the quotient bundle E/F.

Given any symplectic vector bundles (E_1,ω_1), (E_2,ω_2), the symplectic direct sum $(E_1\oplus E_2,\omega_1\oplus\omega_2)$ is naturally a symplectic vector bundle. With this understood, note that for any symplectic sub-bundle F of (E,ω), one has

$$(E,\omega)=(F,\omega|_F)\oplus(F^\omega,\omega|_{F^\omega}).$$

Example 2.10. Let (M,ω) be a symplectic manifold. Note that ω as a 2-form on M is a smooth section of $\Omega^2(M)\equiv T^*M\wedge T^*M$. The nondegeneracy condition on ω implies that (TM,ω) is a symplectic vector bundle. Note that the closedness of ω is irrelevant here.

Suppose Q is a symplectic submanifold of (M,ω). Then TQ is a symplectic sub-bundle of $(TM|_Q,\omega|_Q)$. The normal bundle $\nu_Q\equiv TM|_Q/TQ$ of Q in M is also naturally a symplectic sub-bundle of $(TM|_Q,\omega|_Q)$ by identifying ν_Q with the symplectic complement TQ^ω of TQ. Notice the symplectic direct sum

$$TM|_Q=TQ\oplus\nu_Q.$$

Definition 2.11. Let (E,ω) be a symplectic vector bundle over M. A **complex structure** J of E, i.e., a smooth section J of $\mathrm{Aut}(E)\to M$ such that $J^2=-I$, is said to be ω-**compatible** if for each $p\in M$, J_p is ω_p-compatible, i.e., $J_p\in\mathcal{J}(E_p,\omega_p)$. The space of all ω-compatible complex structures of E is denoted by $\mathcal{J}(E,\omega)$.

Example 2.12. Let (M,ω) be a symplectic manifold. Then a complex structure of TM is simply what we call an almost complex structure on M. An almost complex structure J on M is said to be ω-**compatible** if $J\in\mathcal{J}(TM,\omega)$. In this context, we denote $\mathcal{J}(TM,\omega)$, the set of ω-compatible almost complex structures on M, by $\mathcal{J}(M,\omega)$. Notice that the closedness of ω is irrelevant here.

In what follows, we will address the issue of classification of symplectic vector bundles up to isomorphisms, and determine the topology of the space $\mathcal{J}(E,\omega)$.

Lemma 2.13. *Let (E,ω) be a symplectic vector bundle over M of rank $2n$.*

(1) There exists an open cover $\{U_i\}$ of M such that for each i, there is a symplectic trivialization $\phi_i:(E|_{U_i},\omega|_{U_i})\to(U_i\times\mathbb{R}^{2n},\omega_0)$. In particular, the transition functions $\phi_{ji}(p)\equiv\phi_j\circ\phi_i^{-1}(p)\in Sp(2n)$ for each $p\in U_i\cap U_j$, and E becomes a $Sp(2n)$-vector bundle. Conversely, any $Sp(2n)$-vector bundle is a symplectic vector bundle, and their classification up to isomorphisms is identical.

(2) Any $Sp(2n)$-vector bundle over a smooth manifold admits a lifting to a $U(n)$-vector bundle, which is unique up to isomorphisms (as $U(n)$-vector bundles). Consequently, for any $J_1,J_2\in\mathcal{J}(E,\omega)$, the complex vector bundles (E,J_1), (E,J_2) are isomorphic. (In other words, every symplectic vector bundle has a underlying complex vector bundle structure unique up to isomorphisms.)

Proof. For any $p\in M$, one can prove by induction as in Lemma 2.4 (with a parametric version) that there exists a small neighborhood U_p of p and smooth sections $u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n$ of E over U_p such that for each $q\in U_p$,

$$u_1(q),u_2(q),\cdots,u_n(q),v_1(q),v_2(q),\cdots,v_n(q)$$

form a symplectic basis of (E_q, ω_q). Part (1) follows immediately from this by defining $\phi_p : (E|_{U_p}, \omega|_{U_p}) \to (U_p \times \mathbb{R}^{2n}, \omega_0)$ to be the inverse of

$$(q, (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T) \mapsto \sum_{i=1}^{n} (x_i u_i(q) + y_i v_i(q)).$$

For part (2), it follows from Theorem 2.7, i.e., $\mathrm{Sp}(2n)/U(n)$ is contractible.

□

Theorem 2.14. *Let (E_1, ω_1), (E_2, ω_2) be two symplectic vector bundles. Then they are isomorphic as symplectic vector bundles iff they are isomorphic as complex vector bundles.*

Proof. Pick $J_1 \in \mathcal{J}(E_1, \omega_1)$, $J_2 \in \mathcal{J}(E_2, \omega_2)$. Then by the previous lemma (E_1, ω_1), (E_2, ω_2) are isomorphic as symplectic vector bundles iff (E_1, J_1, ω_1), (E_2, J_2, ω_2) are isomorphic as $U(n)$-vector bundles. But the classification of $U(n)$-vector bundles up to isomorphisms is the same as classification of the underlying complex vector bundles because $GL(n, \mathbb{C})/U(n)$ is contractible. The theorem follows immediately.

□

Theorem 2.15. *For any symplectic vector bundle (E, ω), the space of ω-compatible complex structures $\mathcal{J}(E, \omega)$ is nonempty and contractible. In particular, for any symplectic manifold (M, ω), the space of ω-compatible almost complex structures on M is nonempty and contractible.*

Proof. There are actually two proofs:

Proof 1: The space $\mathcal{J}(E, \omega)$ is the space of smooth sections of a fiber bundle over M with fiber $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$. The claim that $\mathcal{J}(E, \omega)$ is nonempty and contractible follows immediately from Theorem 2.7 that $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ is contractible.

Proof 2: A parametric version of Theorem 2.8 gives rise to a similar map $r : \mathrm{Met}(E) \to \mathcal{J}(E, \omega)$. Contractibility of $\mathcal{J}(E, \omega)$ follows from convexity of $\mathrm{Met}(E)$.

□

Proof 2 of Theorem 2.15, which uses Theorem 2.8, is less conceptual than Proof 1 but more useful in various concrete constructions. As an example of illustration, we prove the following

Proposition 2.16. *Let Q be a symplectic submanifold of (M, ω). Then for any $J \in \mathcal{J}(Q, \omega|_Q)$, there exists a $\hat{J} \in \mathcal{J}(M, \omega)$ such that $\hat{J}|_{TQ} = J$. In particular, every symplectic submanifold of (M, ω) is a pseudo-holomorphic submanifold for some ω-compatible almost complex structure on M.*

Proof. Recall the symplectic direct sum decomposition

$$(TM|_Q, \omega|_Q) = (TQ, \omega|_{TQ}) \oplus (\nu_Q, \omega|_{\nu_Q}),$$

where ν_Q is the normal bundle of Q in M. For any $J \in \mathcal{J}(Q, \omega|_Q)$, we can extend it to $J' = (J, J^\nu)$ by choosing a $J^\nu \in \mathcal{J}(\nu_Q, \omega|_{\nu_Q})$. We then extend the corresponding metric $\omega(\cdot, J'\cdot)$ on $TM|_Q$ over the whole M to a metric g on TM. Let $r : \mathrm{Met}(M) \to \mathcal{J}(M, \omega)$ be the parametric version of the map in Theorem 2.8. Then $\hat{J} \equiv r(g)$ satisfies

$\hat{J}|_{TQ} = J$, and in particular, Q is a pseudo-holomorphic submanifold with respect to the ω-compatible almost complex structure $\hat{J}$ on M.

□

Exercise: Let (M, ω) be a symplectic manifold, G be a compact Lie group acting smoothly on M which preserves the symplectic structure ω, i.e., $g^*\omega = \omega$, for any $g \in G$. Show that there exists a $J \in \mathcal{J}(M, \omega)$ such that $g^*J = J$ for any $g \in G$, i.e., G acts pseudo-holomorphically on M with respect to J.

3. Moser's argument

Moser's argument is one of the fundamental techniques in symplectic geometry. We shall sketch its main ideas first. Various applications are given in the individual subsections below.

Let M be a smooth manifold (not necessarily closed), and let $\omega_t \in \Omega^2(M)$, $t \in [0,1]$, be a smooth family of 2-forms on M each of which is closed and nondegenerate. We further assume that there exists a smooth family of 1-forms $\sigma_t \in \Omega^1(M)$ such that

$$\frac{d}{dt}\omega_t = d\sigma_t, \quad \forall t \in [0,1].$$

The goal of Moser's argument is to construct a smooth family of $\psi_t \in \mathrm{Diff}(M)$ such that

$$\psi_t^*\omega_t = \omega_0, \quad \forall t \in [0,1].$$

Suppose ψ_t is generated by a smooth family of (time-dependent) vector fields X_t. Let's find what conditions X_t has to satisfy.

Note that $\psi_t^*\omega_t = \omega_0, \ \forall t \in [0,1]$, is equivalent to

$$0 = \frac{d}{dt}(\psi_t^*\omega_t) = \psi_t^*(\frac{d}{dt}\omega_t + L_{X_t}\omega_t) = \psi_t^*(d\sigma_t + d(i_{X_t}\omega_t))$$

since ω_t is closed. The above equation is satisfied if we define X_t by the following equation:

$$i_{X_t}\omega_t + \sigma_t = 0.$$

With X_t determined as such, we can integrate X_t to obtain the diffeomorphisms ψ_t by solving the following initial value problem of ODE

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_0 = \mathrm{id}.$$

In general, the above problem has a (unique) solution for small t. When M is compact, ψ_t exists for all $t \in [0,1]$. But for the case where M is open, one needs some additional assumptions on ω_t to ensure that ψ_t exists for all $t \in [0,1]$.

3.1. **Various neighborhood theorems.** We first prove a basic lemma.

Lemma 3.1. *Let M be a $2n$-dimensional smooth manifold and $Q \subset M$ be a compact closed submanifold. Suppose that ω_0, ω_1 are closed and nondegenerate 2-forms in a neighborhood of Q, such that $\omega_0 = \omega_1$ on T_qM for each $q \in Q$. Then there exists open neighborhoods N_0, N_1 of Q in M and a diffeomorphism $\phi : N_0 \to N_1$ such that*

$$\phi^*\omega_1 = \omega_0, \ \text{and } \phi|_Q = id.$$

Proof. First of all, we will show that there exists a neighborhood N of Q and a 1-form $\sigma \in \Omega^1(N)$ such that

$$\omega_1 - \omega_0 = d\sigma, \text{ and } \sigma|_{T_qM} = 0, \ \forall q \in Q.$$

To this end, we fix a Riemannian metric on M and identify the normal bundle ν_Q of Q with the orthogonal complement $TQ^{\perp}$. Then the exponential map $\exp : \nu_Q \to M$ is a diffeomorphism on $U_\epsilon \equiv \{(q,v) \in \nu_Q || v| < \epsilon\}$ for a sufficiently small $\epsilon > 0$. We set $N \equiv \exp(U_\epsilon)$. (Note that such an $\epsilon > 0$ exists because Q is compact.)

Now for $t \in (-\infty, 0]$ we define $\phi_t : N \to N$ by $\phi_t(\exp(q,v)) = \exp(q, e^t v)$. Then for any t, ϕ_t is a diffeomorphism from N onto its image in N. Moreover, we have $\phi_0 = \text{id}$, $\lim_{t\to-\infty} \phi_t(N) = Q$ and $\phi_t|_Q = \text{id}_Q$ for any t. It is easy to see that $\phi_{s+t} = \phi_s \circ \phi_t$, so that $Y \equiv (\frac{d}{dt}\phi_t) \circ \phi_t^{-1} = \frac{d}{dt}\phi_t|_{t=0}$ is a time-independent vector field on N. Moreover, note that $Y = 0$ on Q.

We set $\tau \equiv \omega_1 - \omega_0$. Then $\tau = 0$ on T_qM for any $q \in Q$. It follows easily that $\lim_{t\to-\infty} \phi_t^*\tau = 0$ as $\lim_{t\to-\infty} \phi_t(N) = Q$. With this understood, we observe

$$\begin{aligned}
\tau = \phi_0^*\tau &= \int_{-\infty}^0 \frac{d}{dt}\phi_t^*\tau \, dt = \int_{-\infty}^0 \phi_t^*(L_Y\tau) \, dt \\
&= \int_{-\infty}^0 \phi_t^*(i_Y d\tau + d(i_Y\tau)) \, dt \\
&= d(\int_{-\infty}^0 \phi_t^*(i_Y\tau) \, dt) = d\sigma,
\end{aligned}$$

where $\sigma \equiv \int_{-\infty}^0 \phi_t^*(i_Y\tau) \, dt$. Note that $\sigma = 0$ on T_qM for any $q \in Q$, because for any t, $\phi_t^*(i_Y\tau) = i_Y\tau = 0$ on T_qM for any $q \in Q$, as $\phi_t|_Q = \text{id}_Q$ for any t.

Given the 1-form σ as above, we shall consider the family of closed 2-forms $\omega_t \equiv \omega_0 + t(\omega_1 - \omega_0)$, $t \in [0,1]$. Since nondegeneracy is an open condition, ω_t is a smooth family of symplectic forms in a perhaps smaller neighborhood of Q, which, for simplicity, is still denoted by N. Note that $\frac{d}{dt}\omega_t = \tau = d\sigma$.

In order to run Moser's argument on ω_t, we define time-dependent vector fields X_t by the equation

$$i_{X_t}\omega_t + \sigma = 0, \ \forall t \in [0,1].$$

Observe that $X_t = 0$ on Q because of the condition $\sigma|_{T_qM} = 0, \ \forall q \in Q$.

It remains to show that there exists a neighborhood $N_0 \subset N$ of Q such that the family of maps ψ_t defined by solving the following initial value problem of ODE

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_0 = \text{id}$$

is defined on N_0 for all $t \in [0,1]$. Suppose to the contrary, no such a neighborhood exists. Then there exists a sequence of points $p_n \in N$ and a sequence of $t_n \in [0,1)$ such that $\psi_t(p_n)$ is defined for $t \in [0,t_n]$ and $\psi_{t_n}(p_n) \in \partial N$, and as $n \to \infty$, the distance between p_n and Q converges to 0. Since Q is compact, a subsequence of p_n converges to $q_0 \in Q$, and for that subsequence of p_n, $t_n \to t_0 \in [0,1]$. Now the integral curves of X_t $\psi_t(q_n)$, $t \in [0,t_n]$, converges to the integral curve $\psi_t(q_0)$, $t \in [0,t_0]$. Since

$\psi_{t_n}(p_n) \in \partial N$, and ∂N is compact, we see that $\psi_{t_0}(q_0) \in \partial N$ also. But this is a contradiction, because $X_t = 0$ on Q so that $\psi_t(q_0) = q_0 \in Q$ for any $t \in [0,1]$.

Set $N_1 \equiv \psi_1(N_0) \subset N$. Then $\phi \equiv \psi_1 : N_0 \to N_1$ is the desired diffeomorphism: $\phi^*\omega_1 = \omega_0$ and $\phi|_Q = \mathrm{id}$ (because $X_t = 0$ on Q).

□

Next we use this lemma to prove several standard neighborhood theorems in symplectic geometry.

Theorem 3.2. *(Darboux Theorem). Every symplectic form is locally diffeomorphic to the standard symplectic form ω_0 on $\mathbb{R}^{2n}$.*

Proof. Suppose ω is a symplectic form defined near $q \in M$. In order to apply the previous lemma, we fix a Riemannian metric near q, and fix a choice of symplectic basis $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ of the symplectic vector space $(T_qM, \omega(p))$. With this given, we define $\phi : \mathbb{R}^{2n} \to M$ by

$$\phi : (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T \mapsto \exp_q(\sum_{j=1}^{n}(x_j u_j + y_j v_j)).$$

The map ϕ is a diffeomorphism from a neighborhood U of $0 \in \mathbb{R}^{2n}$ onto a neighborhood of $q \in M$, hence define a chart centered at q. Since the differential of $\exp_q$ at the origin is identity, and $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ is a symplectic basis, it follows easily that $\phi^*\omega$ and ω_0 as symplectic forms on U are equal on T_qU. The theorem follows immediately by applying Lemma 3.1 with $Q = \{q\}$.

□

Exercise: Formulate and prove an equivariant version of Darboux Theorem.

A similar argument gives the following

Theorem 3.3. *(Symplectic Neighborhood Theorem). For $j = 0, 1$, let (M_j, ω_j) be a symplectic manifold with a compact symplectic submanifold Q_j. Suppose that there exists an isomorphism $\Phi : (\nu_{Q_0}, \omega_0) \to (\nu_{Q_1}, \omega_1)$ between the normal bundles which covers a symplectomorphism $\phi : (Q_0, \omega_0) \to (Q_1, \omega_1)$. Then there are neighborhoods N_0 and N_1 of Q_0 and Q_1 respectively, such that ϕ extends to a symplectomorphism $\Psi : N_0 \to N_1$ with $d\Psi = \Phi$ on ν_{Q_0}.*

Proof. For each $j = 0, 1$ we fix a $J_j \in \mathcal{J}(M_j, \omega_j)$ such that Q_j is pseudo-holomorphic submanifold with respect to J_j (cf. Proposition 2.16 in Section 2). Then with respect to the metric $g_{J_j}(\cdot,\cdot) \equiv \omega_j(\cdot, J_j\cdot)$, the symplectic complement $TQ_j^{\omega_j} = \nu_{Q_j}$ is identified with the orthogonal complement $TQ_j^{\perp}$. With this understood, let $\psi_j : \nu_{Q_j} \to M_j$ be defined by $(q, v) \mapsto \exp_q(v)$, where $q \in Q_j$, then $\Psi' \equiv \psi_1 \circ \Phi \circ \psi_0^{-1}$ is a diffeomorphism between a neighborhood of Q_0 onto a neighborhood of Q_1, such that $(\Psi')^*\omega_1 = \omega_0$ on T_qM_0 for each $q \in Q_0$. The theorem follows by applying Lemma 3.1.

□

Theorem 3.4. *(Lagrangian Neighborhood Theorem). Let (M,ω) be a symplectic manifold and $L \subset M$ be a compact Lagrangian submanifold. Then there exist a neighborhood U of the zero-section in T^*L and a neighborhood V of L in M, and a diffeomorphism $\phi : U \to V$ such that*

$$\phi^*\omega = -d\lambda, \;\; \phi|_L = id,$$

where λ is the canonical 1-form on T^*L (cf. Example 1.3 in Section 1).

Proof. Note that since L is Lagrangian in (M,ω), i.e. $\omega(u,v) = 0$ for any $u, v \in T_qL$, $q \in L$, the homomorphism $\beta : T_qM \to T_q^*L$ defined by $\beta(u)(v) = \omega(u,v)$ factors through T_qL to give rise to an isomorphism of bundles $\nu_L \to T^*L$. This at least shows that a neighborhood of L in M is diffeomorphic to a neighborhood of the zero-section in T^*L. The key issue is to find an appropriate diffeomorphism under which the two symplectic forms ω and $-d\lambda$ agree on TM along L.

Lemma 3.5. *Let (V,ω) be a symplectic vector space, and let $L \subset V$ be a Lagrangian subspace, i.e., $L^\omega = L$. Then for any $J \in \mathcal{J}(V,\omega)$, $JL \subset V$ is also a Lagrangian subspace, and moreover, with respect to the Hermitian structure $g_J(\cdot,\cdot) \equiv \omega(\cdot, J\cdot)$, L and JL are orthogonal to each other.*

Proof. First, $JL \subset V$ is also a Lagrangian subspace follows from the fact that for any $J \in \mathcal{J}(V,\omega)$, $\omega(\cdot,\cdot) = \omega(J\cdot, J\cdot)$, and the fact that $\dim JL = \dim L = \frac{1}{2}\dim V$. Second, L and JL are orthogonal to each other with respect to the Hermitian structure g_J because $g_J(u, Jv) = -\omega(u,v)$ for any $u, v \in V$.

∎

Now let's go back to the proof of Theorem 3.4. We fix a $J \in \mathcal{J}(M,\omega)$, and by the previous lemma we may identify JTL with the normal bundle ν_L. This gives rise to an isomorphism $\beta : JTL \to T^*L$ by $\beta(u)(v) = \omega(u,v)$. We fix the Riemannian metric $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$ on M, and define a map

$$\psi : T^*L \to M : (q,v) \mapsto \exp_q(-\beta^{-1}v).$$

We fix the decomposition $T_{(q,0)}T^*L = T_qL \oplus T_q^*L$, $q \in L$, and write any $v \in T_{(q,0)}T^*L$ as $v = (v_0, v_1^*)$. Then observe that $d\psi_{(q,0)}(v) = v_0 - \beta^{-1}v_1^*$, and therefore

$$\begin{aligned} \psi^*\omega_{(q,0)}(v,w) &= \omega_q(v_0 - \beta^{-1}v_1^*, w_0 - \beta^{-1}w_1^*) \\ &= w_1^*(v_0) - v_1^*(w_0), \end{aligned}$$

because TL is Lagrangian by assumption and JTL is Lagrangian by Lemma 3.5, and $\beta^{-1}v_1^*$, $\beta^{-1}w_1^* \in JT_qL$.

On the other hand, recall that (cf. Example 1.3 in Section 1) in the standard local coordinates $q_1, \cdots, q_n, p_1, \cdots, p_n$ on T^*L, $-d\lambda = \sum_{j=1}^n dq_j \wedge dp_j$. For each point $q \in L$, $\partial q_1, \cdots, \partial q_n$ and $dq_1, \cdots, dq_n$ form a basis of T_qL and T_q^*L respectively. If we write

$$v = \sum_{j=1}^n (x_j \partial q_j + y_j dq_j), \;\; w = \sum_{j=1}^n (s_j \partial q_j + t_j dq_j),$$

then $v_0=\sum_{j=1}^n x_j\partial q_j$, $v_1^\ast=\sum_{j=1}^n y_j dq_j$, and $w_0=\sum_{j=1}^n s_j\partial q_j$, $w_1^\ast=\sum_{j=1}^n t_j dq_j$. With these understood,

$$-d\lambda_{(q,0)}(v,w)=\sum_{j=1}^n (x_jt_j-s_jy_j)=w_1^\ast(v_0)-v_1^\ast(w_0).$$

Hence $\psi^\ast\omega=-d\lambda$ on $T_{(q,0)}T^\ast L$ for any $q\in L$. Theorem 3.4 follows from Lemma 3.1 immediately. $\square$

Exercise: Let L be a compact Lagrangian submanifold of (M,ω), and $L'=\psi_1(L)$ where ψ_1 is the time-1 map of some Hamiltonian isotopy ψ_t, $t\in[0,1]$. Show that there exists a regular neighborhood N of L in M such that if $L'\subset N$, then L and L' must intersect nontrivially, at more than one point.

3.2. **Stability theorem.** We apply Moser's argument to the case of compact, closed manifolds, and prove Moser's stability theorem for symplectic structures.

Theorem 3.6. *(Moser's stability theorem). Let M be a compact, closed manifold and suppose that ω_t, $t\in[0,1]$, is a smooth family of cohomologous symplectic forms on M, i.e., the deRham cohomology class $[\omega_t]$ is constant in t. Then there exists a smooth family of $\psi_t\in Diff(M)$ such that*

$$\psi_0=id,\ \ \psi_t^\ast\omega_t=\omega_0.$$

Proof. Since M is a compact closed manifold, the diffeomorphisms ψ_t in Moser's argument are defined over M for all $t\in[0,1]$. The key issue here is to show that there exists a smooth family of 1-forms σ_t such that $\frac{d}{dt}\omega_t=d\sigma_t$. This will follow from the Hodge theory.

Note that for each t, $\frac{d}{dt}\omega_t$ is an exact 2-form because ω_t are cohomologous. This implies that $\frac{d}{dt}\omega_t$ is in the L^2 orthogonal complement of the space of harmonic 2-forms, therefore lies in the image of the Laplacian Δ by the Hodge decomposition theorem. Consequently, we obtain a smooth family of 2-forms $\tau_t:=\Delta^{-1}(\frac{d}{dt}\omega_t)$. We will see below that $\frac{d}{dt}\omega_t=d\sigma_t$ where $\sigma_t=d^\ast\tau_t$.

By definition $\frac{d}{dt}\omega_t=\Delta\tau_t=dd^\ast\tau_t+d^\ast d\tau_t$, from which we see that our claim above follows if $d^\ast d\tau_t=0$. This is true because

$$\langle d^\ast d\tau_t,d^\ast d\tau_t\rangle=\langle(\frac{d}{dt}\omega_t-dd^\ast\tau_t),d^\ast d\tau_t\rangle=\langle d(\frac{d}{dt}\omega_t-dd^\ast\tau_t),d\tau_t\rangle=0.$$

(Note that in the last equation above, $d(\frac{d}{dt}\omega_t)=\frac{d}{dt}(d\omega_t)=0$.) Thus we have established the existence of σ_t, and Moser's stability theorem follows.

$\square$

Exercise: Let (M,ω) be a symplectic manifold with $\dim M=4$. Let Σ_1,Σ_2 be two embedded symplectic surfaces in M which have the same genus, self-intersection, and symplectic area (i.e., $\int_{\Sigma_1}\omega=\int_{\Sigma_2}\omega$). Show that there are regular neighborhoods N_1,N_2 of Σ_1,Σ_2 respectively, and a diffeomorphism $\psi:N_1\to N_2$ such that $\psi^\ast\omega=\omega$.

4. Symplectic group actions

4.1. **Symplectic circle actions.** We set $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ throughout.

Let (M, ω) be a symplectic manifold. A symplectic $\mathbb{S}^1$-action on (M, ω) is a smooth family $\psi_t \in \text{Symp}(M, \omega)$, $t \in \mathbb{S}^1$, such that $\psi_{t+s} = \psi_t \circ \psi_s$ for any $t, s \in \mathbb{S}^1$. One can easily check that the corresponding vector fields $X_t \equiv \frac{d}{dt}\psi_t \circ \psi_t^{-1}$ is time-independent, i.e., $X_t = X$ is constant in t. We call X the associated vector field of the given symplectic $\mathbb{S}^1$-action, which is a symplectic vector field, i.e., $i_X\omega$ is a closed 1-form.

When $i_X\omega = dH$ is an exact 1-form, the corresponding symplectic $\mathbb{S}^1$-action is called a **Hamiltonian $\mathbb{S}^1$-action**, and the function $H : M \to \mathbb{R}$ is called a **moment map**. Note that H is uniquely determined up to a constant. We point out that a symplectic $\mathbb{S}^1$-action on (M, ω) is automatically Hamiltonian if $H^1(M; \mathbb{R}) = 0$.

Exercise: If $i_X\omega = dH$ where $H : M \to \mathbb{R}/\mathbb{Z}$ is a circle-valued function, the function H is called a **generalized moment map**. Show that for any symplectic $\mathbb{S}^1$-action, there is always a generalized moment map after suitably changing the symplectic structure on the manifold.

Let H be a moment map or generalized moment map of a given symplectic $\mathbb{S}^1$-action. We shall make the following two observations.

(1) Each level surface $H^{-1}(\lambda)$ is invariant under the $\mathbb{S}^1$-action.
(2) A point $p \in M$ is a fixed point of the $\mathbb{S}^1$-action if and only if p is a critical point of H, i.e., $dH = 0$ at p.

Now consider a level surface $H^{-1}(\lambda)$ where λ is a regular value of H. Then $H^{-1}(\lambda)$ is a hypersurface in M which does not contain any fixed points of the $\mathbb{S}^1$-action. When the $\mathbb{S}^1$-action is free on $H^{-1}(\lambda)$, the quotient $B_\lambda \equiv H^{-1}(\lambda)/\mathbb{S}^1$ is a smooth manifold. In general, the $\mathbb{S}^1$-action may have finite isotropy on $H^{-1}(\lambda)$, and the quotient $B_\lambda \equiv H^{-1}(\lambda)/\mathbb{S}^1$ is a smooth orbifold. In any case, one observes that $\dim B_\lambda = \dim M - 2$.

The next theorem shows that there exists a natural symplectic structure ω_λ on B_λ. The symplectic manifold (or orbifold) $(B_\lambda, \omega_\lambda)$ is called the **symplectic quotient** or the **reduced space** at λ. The process of going from (M, ω) to $(B_\lambda, \omega_\lambda)$ is called **symplectic reduction**.

Theorem 4.1. *There exists a canonically defined symplectic structure ω_λ on B_λ such that $\pi^*\omega_\lambda = \omega|_{H^{-1}(\lambda)}$, where $\pi : H^{-1}(\lambda) \to H^{-1}(\lambda)/\mathbb{S}^1 \equiv B_\lambda$ is the projection.*

Proof. For simplicity, we assume the $\mathbb{S}^1$-action on $H^{-1}(\lambda)$ is free, and consequently B_λ is a smooth manifold. To simplify the notation, we set $Q = H^{-1}(\lambda)$.

Let's recall a basic result about differentiable Lie group actions on manifolds — the existence of local slice. In the present situation, the result amounts to say that for any point $q \in Q$, there exists a submanifold $\mathcal{O}_q$ of codimension 1 containing q, such that $\mathbb{S}^1 \times \mathcal{O}_q$ embedds into Q $\mathbb{S}^1$-equivariantly. $\mathcal{O}_q$ is called a local slice at q, and the set $\{\mathcal{O}_q | q \in Q\}$ forms an atlas of charts for the differentiable structure on the quotient $Q/\mathbb{S}^1$. If $q' \in Q$ lies in $\mathbb{S}^1 \times \mathcal{O}_q$, then there is a local diffeomorphism $\phi_{qq'}$ from a neighborhood U of q' in the slice $\mathcal{O}_{q'}$ into $\mathcal{O}_q$ and a function on U into $\mathbb{S}^1$, $f_{qq'}$, such that $U \subset \mathcal{O}_{q'}$ may be identified with the graph of $f_{qq'}$ over the image of $\phi_{qq'}$ in $\mathbb{S}^1 \times \mathcal{O}_q$. Note that with the differentiable structure on the quotient $Q/\mathbb{S}^1 = B_\lambda$

described above, the projection $\pi : Q \to B_\lambda$ becomes a principal $\mathbb{S}^1$-bundle over B_λ, with local trivializations of the bundle given by projections $\mathbb{S}^1 \times \mathcal{O}_q \to \mathcal{O}_q$, $q \in Q$.

The symplectic structure ω_λ is defined by pulling-back ω to each local slice $\mathcal{O}_q$. This definition immediately gives the closedness of ω_λ as well as the equation $\pi^*\omega_\lambda = \omega|_Q$. To see that ω_λ is well-defined, i.e., the pull-back of ω to each local slice can be patched up, we note that the local slices are graphs over each other locally, and that the tangent direction of $\mathbb{S}^1$ in $\mathbb{S}^1 \times \mathcal{O}_q$ lies in TQ^ω at each point. The nondegeneracy of ω_λ follows from the fact that $\dim T_qQ^\omega = \dim T_qM - \dim T_qQ = 1$, so that T_qQ^ω is actually generated by the tangent direction of $\mathbb{S}^1$.

□

Example 4.2. (Product of $\mathbb{S}^1$-actions). For $j = 1, 2$, let (M_j, ω_j) be a symplectic manifold with a symplectic $\mathbb{S}^1$-action $t \mapsto \psi_t^j$, $t \in \mathbb{S}^1$. Then for any $m_1, m_2 \in \mathbb{Z}$ such that $gcd(m_1, m_2) = 1$, there is a canonical $\mathbb{S}^1$-action on the product $(M_1 \times M_2, \omega_1 \times \omega_2)$, $t \mapsto \psi_t$, $t \in \mathbb{S}^1$, where $\psi_t = \psi^1_{m_1t} \times \psi^2_{m_2t}$. Moreover, if H_1, H_2 are moment maps of the $\mathbb{S}^1$-actions ψ_t^1, ψ_t^2 respectively, then $H = m_1H_1 + m_2H_2$ is a moment map of the product ψ_t. To see this, note that if X_j, $j = 1, 2$, is the vector field on M_j which generates the $\mathbb{S}^1$-action ψ_t^j, then $X = \langle m_1X_1, m_2X_2\rangle$ is the vector field on $M_1 \times M_2$ which generates the $\mathbb{S}^1$-action ψ_t. The claim about the moment maps follows immediately from $i_X(\omega_1 \times \omega_2) = m_1 i_{X_1}\omega_1 + m_2 i_{X_2}\omega_2$.

Example 4.3. (Holomorphic $\mathbb{S}^1$-actions on Kähler manifolds). For any holomorphic $\mathbb{S}^1$-action on a Kähler manifold, one can choose an invariant Kähler metric, so that the $\mathbb{S}^1$-action becomes a symplectic $\mathbb{S}^1$-action with respect to the invariant Kähler form.

Example 4.4. Consider $(\mathbb{R}^2, \omega_0)$ with symplectic $\mathbb{S}^1$-action given by the complex multiplication $z \mapsto e^{it}z$, $t \in \mathbb{S}^1$. Here we identify $\mathbb{R}^2 = \mathbb{C}$. To determine the moment map, we note that the $\mathbb{S}^1$-action is generated by the vector field $X = -y\partial x + x\partial y$. With this we see the moment map is given by $H(z) = -\frac{1}{2}|z|^2$, because $i_X\omega_0 = -ydy - xdx$.

Now for any $\mathbf{m} = (m_0, m_1, \cdots, m_n)$ where $m_i \in \mathbb{Z}$ and $gcd(m_0, m_1, \cdots, m_n) = 1$, consider more generally the symplectic $\mathbb{S}^1$-action on $(\mathbb{R}^{2n+2}, \omega_0)$, which is defined by

$$(z_0, z_1, \cdots, z_n) \mapsto (e^{im_0t}z_0, e^{im_1t}z_1, \cdots, e^{im_nt}z_n), \quad t \in \mathbb{S}^1.$$

By Example 4.2, the moment map of the $\mathbb{S}^1$-action is

$$H(z_0, z_1, \cdots, z_n) = -\frac{1}{2}(m_0|z_0|^2 + m_1|z_1|^2 + \cdots + m_n|z_n|^2).$$

For the special case where $\mathbf{m} = (1, 1, \cdots, 1)$, the level surface $H^{-1}(\lambda)$, $\lambda < 0$, is the $(2n + 1)$-dimensional sphere of radius -2λ, and the $\mathbb{S}^1$-action on $H^{-1}(\lambda)$ is given by the Hopf fibration. The reduced space $(B_\lambda, \omega_\lambda)$ at $\lambda = -\frac{1}{2}$ is $\mathbb{CP}^n$ with ω_λ being π times the Fubini-Study form ω_0 on $\mathbb{CP}^n$, see Example 1.5 in Section 1.

Example 4.5. Note that for any $\mathbf{m} = (m_0, m_1, \cdots, m_n)$, the corresponding symplectic $\mathbb{S}^1$-action on $(\mathbb{R}^{2n+2}, \omega_0)$ with weights $\mathbf{m}$ preserves the unit sphere $\mathbb{S}^{2n+1}$ and commutes with the Hopf fibration. Hence there is an induced $\mathbb{S}^1$-action on $\mathbb{CP}^n$, which must be symplectic with respect to the Fubini-Study form and has the moment map

$$H(z_0, z_1, \cdots, z_n) = -\frac{1}{2}(m_0|z_0|^2 + m_1|z_1|^2 + \cdots + m_n|z_n|^2)$$

where $|z_0|^2+|z_1|^2+\cdots+|z_n|^2=1$, due to the relation $\pi^*\omega_\lambda=\omega|_{H^{-1}(\lambda)}$ between the symplectic forms in the symplectic reduction given in Theorem 4.1, and the fact that the vector field on $\mathbb{CP}^n$ which generates the induced $\mathbb{S}^1$-action is the push-down of the corresponding vector filed on $\mathbb{S}^{2n+1}$ under the Hopf fibration $\pi:\mathbb{S}^{2n+1}\rightarrow\mathbb{CP}^n$.

In terms of the homogeneous coordinates $z_0,z_1,\cdots,z_n$ on $\mathbb{CP}^n$, the $\mathbb{S}^1$-action is given by

$$[z_0,z_1,\cdots,z_n]\mapsto[e^{im_0t}z_0,e^{im_1t}z_1,\cdots,e^{im_nt}z_n],\ \ t\in\mathbb{S}^1.$$

The corresponding moment map is given by

$$H([z_0,z_1,\cdots,z_n])=-\frac{1}{2\sum_{j=0}^n|z_j|^2}(m_0|z_0|^2+m_1|z_1|^2+\cdots+m_n|z_n|^2).$$

Note that in order for the $\mathbb{S}^1$-action on $\mathbb{CP}^n$ to be effective, one needs to impose additional conditions

$$gcd(m_0-m_j,\cdots,m_n-m_j)=1,\ \ \forall j=0,\cdots,n.$$

On the other hand, for any $m\in\mathbb{Z}$, the wights $\mathbf{m}-m\equiv(m_0-m,m_1-m,\cdots,m_n-m)$ defines the same $\mathbb{S}^1$-action on $\mathbb{CP}^n$ as the weights $\mathbf{m}=(m_0,m_1,\cdots,m_n)$. Note that the corresponding moment maps change by a constant $\frac{m}{2}$.

Consider the case where $n=1$. The above construction gives rise to symplectic $\mathbb{S}^1$-actions on $\mathbb{CP}^1=\mathbb{S}^2$, where different choices of weights $\mathbf{m}$ yield the same $\mathbb{S}^1$-action. If we let $\mathbf{m}=(0,-1)$ and identify $\mathbb{CP}^1=\mathbb{C}\cup\{\infty\}$, the $\mathbb{S}^1$-action is simply given by the complex multiplication on $\mathbb{C}$, which has moment map

$$H(z)=\frac{1}{2}\cdot\frac{1}{1+|z|^2}.$$

The fixed points are $\{0,\infty\}$, and the corresponding critical values of the moment map are $H(0)=\frac{1}{2}$, $H(\infty)=0$. The only difference between this example of a Hamiltonian $\mathbb{S}^1$-action on $\mathbb{S}^2$ and the one given in the exercise at the end of §1.1 is that the former has area π over $\mathbb{S}^2$ and the latter has area 4π.

Definition 4.6. A smooth function f on a manifold M is called **Morse-Bott** if for any critical point $p\in M$ of f, there is a chart $\phi:\mathbb{R}^n\rightarrow M$ centered at p such that

$$\phi^*f(x_1,\cdots,x_n)=a_1x_1^2+\cdots+a_nx_n^2+f(p),$$

where each a_j is $-1,0$ or 1. The number of a_j's where $a_j<0$ is called the **index** of f at p. If none of the a_j's is zero for every critical point p, the function f is called a **Morse function**. Note that a Morse function has only isolated critical points, and for a Morse-Bott function in general, the set of critical points consists of a disjoint union of submanifolds of various co-dimensions, called the **critical submanifolds**. In the above standard chart, the critical submanifold is given by the equations $x_j=0$ for all j with $a_j\neq 0$.

Proposition 4.7. *A moment map (or generalized moment map) of a symplectic $\mathbb{S}^1$-action is Morse-Bott with the following additional properties: (1) the critical submanifolds are symplectic submanifolds, (2) the index at each critical point is always an even number.*

Proof. The proposition follows from an equivariant version of the Darboux theorem, which also gives a model for the moment map near a critical point. Since the problem is local, there is no difference for the case of generalized moment map.

Let (M,ω) be a symplectic manifold with a symplectic $\mathbb{S}^1$-action ψ_t, $t\in\mathbb{S}^1$. The equivariant version of Darboux theorem states as follows: for any fixed point $p\in M$, there is a chart $\phi:\mathbb{R}^{2n}\rightarrow M$ centered at p, such that $\phi^*\omega=\omega_0$ the standard symplectic form on $\mathbb{R}^{2n}$, and the pull-back $\mathbb{S}^1$-action $\phi^*\psi_t\equiv\phi^{-1}\circ\psi_t\circ\phi$ on $\mathbb{R}^{2n}$ is linear and is given under the standard identification $\mathbb{R}^{2n}=\mathbb{C}^n$ by

$$(z_1,z_2,\cdots,z_n)\mapsto(e^{im_1t}z_1,e^{im_2t}z_2,\cdots,e^{im_nt}z_n),\ \ t\in\mathbb{S}^1,$$

for some $m_1,m_2,\cdots,m_n\in\mathbb{Z}$ with $gcd(m_1,m_2,\cdots,m_n)=1$.

Let H be a moment map of the symplectic $\mathbb{S}^1$-action on M. Then the equivariant Darboux theorem together with Example 4.4 implies that there exists a chart $\phi:\mathbb{R}^{2n}\rightarrow M$ centered at p such that

$$\phi^*H(x_1,\cdots,x_n,y_1,\cdots,y_n)=-\frac{1}{2}[m_1(x_1^2+y_1^2)+\cdots+m_n(x_n^2+y_n^2)]+H(p)$$

for some $m_1,m_2,\cdots,m_n\in\mathbb{Z}$ with $gcd(m_1,m_2,\cdots,m_n)=1$. With a further change of coordinates it follows immediately that H is a Morse-Bott function. The index at p is twice of the number of positive m_j's, hence is always an even number. The critical submanifold is locally defined near p by the equations $x_j=y_j=0$ for all j with $m_j\neq 0$, hence is symplectic.

□

The fact that a moment map of a symplectic $\mathbb{S}^1$-action is Morse-Bott with even index at each critical point has the following corollary by a standard argument in Morse theory.

Corollary 4.8. *Let $H:M\rightarrow\mathbb{R}$ be a moment map of a Hamiltonian $\mathbb{S}^1$-action on a compact, connected manifold. Then each level surface $H^{-1}(\lambda)$ is connected. In particular, the critical submanifolds at the maximal and minimal values of H are connected.*

The standard model for a moment map H near a critical point as we obtained in the proof of Proposition 4.7 allows one to explicitly analyzing the change of the topology of the reduced spaces passing a critical value of the moment map in terms of the weights m_j at each critical point in the pre-image of that critical value. On the other hand, for any interval I of regular values, Morse theory allows one to identify $H^{-1}(I)$ diffeomorphically with the product $H^{-1}(\lambda_0)\times I$ for any $\lambda_0\in I$ (e.g. using the gradient flow of H), which can be made $\mathbb{S}^1$-equivariantly. The next proposition describes the relation between the symplectic form ω on $H^{-1}(I)$ and the reduced spaces $(B_\lambda,\omega_\lambda)$, $\lambda\in I$, and the first Chern class of the $\mathbb{S}^1$-principal bunble $\pi:H^{-1}(\lambda_0)\rightarrow B_{\lambda_0}$. For simplicity we assume that the $\mathbb{S}^1$-action on $H^{-1}(I)$ is free, so that each reduced space B_λ, $\lambda\in I$, is a smooth manifold. Note that all $H^{-1}(\lambda)$, B_λ, $\lambda\in I$, are diffeomorphic; we denote the underlying manifolds by P, B respectively. (We warn that the $\mathbb{S}^1$-action on each $H^{-1}(\lambda)$, $\lambda\in I$, is assumed to be on the left, so when $H^{-1}(\lambda)$ is regarded as a $\mathbb{S}^1$-principal bundle, where the action is always assumed to be on the right, we mean the conjugate $\mathbb{S}^1$-action. For example, in this way the first Chern class of the Hopf fibration $\mathbb{S}^3 \to \mathbb{CP}^1$ evaluates positively on the fundamental class of $\mathbb{CP}^1$.)

Proposition 4.9. *(1) Let $c \in H^2(B;\mathbb{Z})$ be the first Chern class of the $\mathbb{S}^1$-principal bundle $\pi : P \to B$ and let $\{\omega_\lambda | \lambda \in I\}$ be a smooth family of symplectic forms on B such that their deRham cohomology classes satisfy*

$$[\omega_\lambda] = [\omega_\mu] - 2\pi(\lambda - \mu) \cdot c.$$

There is an $\mathbb{S}^1$-invariant symplectic form ω on $P \times I$ with a moment map H equal to the projection $P \times I \to I$ and with reduced spaces (B, ω_λ), $\lambda \in I$.

(2) Conversely, every $\mathbb{S}^1$-invariant symplectic form ω arises in the above way. Moreover, up to $\mathbb{S}^1$-equivariant symplectomorphisms such a $\mathbb{S}^1$-invariant symplectic form on $P \times I$ is uniquely determined by the family of symplectic forms $\{\omega_\lambda | \lambda \in I\}$ on B.

Proof. (1) Since the deRham cohomology class of $\frac{d}{d\lambda}\omega_\lambda$ represents $-2\pi c$, there exists a smooth family of imaginary valued 1-forms A_λ on P, i.e., the connection 1-forms, such that $\frac{i}{2\pi} dA_\lambda = -\frac{1}{2\pi}\pi^* \frac{d}{d\lambda}\omega_\lambda$. Let X be the vector field which generates the $\mathbb{S}^1$-action. Then $A_\lambda(X) = -i$ because as we remarked before P is regarded as an $\mathbb{S}^1$-principal bundle on B with the conjugate action. Set $\alpha_\lambda = iA_\lambda$. Then $\alpha_\lambda(X) = 1$, and $\pi^* \frac{d}{d\lambda}\omega_\lambda + d\alpha_\lambda = 0$. With these understood,

$$\omega = \pi^*\omega_\lambda + \alpha_\lambda \wedge d\lambda$$

is an $\mathbb{S}^1$-invariant symplectic form on $P \times I$ with a moment map H equal to the projection $P \times I \to I$ and with reduced spaces (B, ω_λ), $\lambda \in I$.

(2) Note that as a 2-form on $P \times I$, ω may be written as

$$\omega = \beta_\lambda + \alpha_\lambda \wedge d\lambda$$

for some $\alpha_\lambda \in \Omega^1(P)$ and $\beta_\lambda \in \Omega^2(P)$. Let X be the vector field which generates the $\mathbb{S}^1$-action. Since the moment map of the $\mathbb{S}^1$-action is the projection $P \times I \to I$, we see that $i_X\beta_\lambda = 0$ and $\alpha_\lambda(X) = 1$. The former implies that β_λ descents to a smooth family of 2-forms ω_λ on B, so that $\pi^*\omega_\lambda = \beta_\lambda$. The nondegeneracy of ω implies that each ω_λ is nondegenerate, and the closedness of ω implies

$$d\omega_\lambda = 0, \;\; \frac{d}{d\lambda}\beta_\lambda + d\alpha_\lambda = 0.$$

Note that $-i\alpha_\lambda$ are connection 1-forms on P, so that the first Chern class c is represented by $\frac{i}{2\pi} d(-i\alpha_\lambda) = \frac{1}{2\pi} d\alpha_\lambda$. This gives the relation

$$[\omega_\lambda] = [\omega_\mu] - 2\pi(\lambda - \mu) \cdot c.$$

For the uniqueness, consider two different such symplectic forms

$$\omega = \pi^*\omega_\lambda + \alpha_\lambda \wedge d\lambda, \;\; \omega' = \pi^*\omega_\lambda + \alpha'_\lambda \wedge d\lambda.$$

One can form a smooth family of such symplectic forms

$$\omega_t = \pi^*\omega_\lambda + ((1-t)\alpha_\lambda + t\alpha'_\lambda) \wedge d\lambda, \;\; t \in [0, 1].$$

The uniqueness follows from an equivariant version of Moser's stability theorem applied to ω_t. We leave it as an exercise.

□

Example 4.10. (1) Consider the $\mathbb{S}^1$-action on $\mathbb{CP}^1$ at the end of Example 4.5. Since the reduced spaces are a single point, $\omega_\lambda = 0$, and therefore the symplectic form

$$\omega = \alpha_\lambda \wedge d\lambda.$$

In the polar coordinates (r, θ) on $\mathbb{C} \subset \mathbb{CP}^1$, $\alpha_\lambda = d\theta$, so that

$$\omega = d\theta \wedge d(\frac{1}{2(1+r^2)}) = \frac{rdr \wedge d\theta}{(1+r^2)^2} = \frac{dx \wedge dy}{(1+x^2+y^2)^2}.$$

Direct calculation shows

$$\int_{\mathbb{CP}^1} \omega = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dxdy}{(1+x^2+y^2)^2} = \pi$$

as we claimed.

(2) Consider the $\mathbb{S}^1$-action on $(\mathbb{R}^{2n+2}, \omega_0)$ in Example 4.4 with weights $\mathbf{m} = (1, 1, \cdots, 1)$. The moment map is

$$H(z_0, z_1, \cdots, z_n) = -\frac{1}{2}(|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2).$$

For any $\lambda < 0$ the level surface $H^{-1}(\lambda)$ is the sphere of radius -2λ, and the $\mathbb{S}^1$-action on $H^{-1}(\lambda)$ is given by the Hopf fibration, with the quotient being $\mathbb{CP}^n$. We have claimed in Example 4.4 that the symplectic form ω_λ at $\lambda = -\frac{1}{2}$ is π times the Fubini-Study form on $\mathbb{CP}^n$, which is normalized so that the integral of its n-th power over $\mathbb{CP}^n$ equals 1. Note that this implies that $\int_{\mathbb{CP}^n} \omega_{-1/2}^n = \pi^n$. We shall next give an independent verification of this fact using Proposition 4.9.

First note that as $\lambda \to 0$ the form ω_λ converges to 0. This gives, by Proposition 4.9,

$$[\omega_{-\frac{1}{2}}] = 0 - 2\pi(-\frac{1}{2} - 0) \cdot c = \pi \cdot c,$$

where c is the first Chern class of the Hopf fibration. It is known that $c \in H^2(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}$ is the positive generator, so that $c^n[\mathbb{CP}^n] = 1$. This implies that

$$\int_{\mathbb{CP}^n} \omega^n_{-\frac{1}{2}} = \pi^n$$

as we claimed.

(3) Consider the $\mathbb{S}^1$-action on $\mathbb{CP}^2$ in Example 4.5 with weights $\mathbf{m} = (0, -1, -2)$. There are three fixed points $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$, where the moment map has values 0, $\frac{1}{2}$ and 1 respectively. Using the standard model for the moment map near a critical point as in the proof of Proposition 4.7, it is easy to check that for any regular value λ, the reduced space B_λ is the weighted projective space $\mathbb{CP}^1(1, 2)$, which is the quotient space $(\mathbb{C}^2 \setminus \{(0, 0)\})/\sim$, where $(z_1, z_2) \sim (zz_1, z^2 z_2)$. (Note that $\mathbb{CP}^1(1, 2)$ is a 2-dimensional orbiford, with one singular point of order 2.) However, for $\lambda \in (0, \frac{1}{2})$, the first Chern class of the (orbifold) $\mathbb{S}^1$-pricipal bundle $H^{-1}(\lambda) \to B_\lambda$ equals $-\frac{1}{2} \in H^2(\mathbb{CP}^1(1, 2); \mathbb{Q})$ and for $\lambda \in (\frac{1}{2}, 1)$, it equals $\frac{1}{2} \in H^2(\mathbb{CP}^1(1, 2); \mathbb{Q})$. Note that the first Chern class changes by 1 when passing the critical value $\lambda = \frac{1}{2}$.

For a Hamiltonian $\mathbb{S}^1$-action with at most isolated fixed points, the moment map is a Morse function. Propositions 4.7 and 4.9 show that the weights of the induced action on the tangent space of each fixed point contain vital information about the equivariant symplectic geometry of the manifold. There are certain constraints amongst the weights of the fixed points, as shown in the following beautiful theorem of Duistermaat and Heckman.

Theorem 4.11. *(Duistermaat-Heckman). Assume a Hamiltonian $\mathbb{S}^1$-action on a compact $2n$-dimensional symplectic manifold (M,ω) has only isolated fixed points. Let $H : M \to \mathbb{R}$ be a moment map, and let $e(p)$ denote the product of the weights at a fixed point p. Then*

$$\int_M e^{-2\pi\hbar H}\frac{\omega^n}{n!} = \sum_p \frac{e^{-2\pi\hbar H(p)}}{\hbar^n e(p)}$$

for every $\hbar \in \mathbb{C}$, where the sum on the right-hand side runs over all fixed points of the $\mathbb{S}^1$-action.

Example 4.12. If one expands both sides of the Duistermaat-Heckman formula as power series in $\hbar$ and then compares the coefficients, the following set of constraints are obtained:

$$\sum_p \frac{H(p)^k}{e(p)} = 0, \text{ for } k = 0, 1, \cdots, n-1,$$

and

$$\int_M \omega^n = (-2\pi)^n \sum_p \frac{H(p)^n}{e(p)}.$$

We check this out on an example of $\mathbb{S}^1$-action on $\mathbb{CP}^2$ as discussed in Example 4.5, with weights $\mathbf{m} = (0,-2,-5)$. It is easy to check that there are three isolated fixed points $p_1 = [1,0,0]$, $p_2 = [0,1,0]$, $p_3 = [0,0,1]$, which have weights $(-2,-5)$, $(2,-3)$, and $(5,3)$ respectively. Let H be the standard moment map as given in Example 4.5. Then $H(p_1) = 0$, $H(p_2) = 1$, and $H(p_3) = \frac{5}{2}$. We also know (see Example 4.10 (2)) that $\int_{\mathbb{CP}^2} \omega^2 = \pi^2$. With the preceding understood, the set of constraints obtained from the Duistermaat-Heckman formula are the following for this example:

$$\frac{1}{(-2)\cdot(-5)} + \frac{1}{2\cdot(-3)} + \frac{1}{5\cdot 3} = 0, \quad \frac{0}{(-2)\cdot(-5)} + \frac{1}{2\cdot(-3)} + \frac{\frac{5}{2}}{5\cdot 3} = 0$$

and

$$(-2\pi)^2\left(\frac{0^2}{(-2)\cdot(-5)} + \frac{1^2}{2\cdot(-3)} + \frac{(\frac{5}{2})^2}{5\cdot 3}\right) = \pi^2.$$

4.2. **Hamiltonian torus actions.** We denote by $\mathbb{T}^n = (\mathbb{S}^1)^n$ the n-torus. The corresponding Lie algebra and its dual are denoted by $\mathbf{t}^n$ and $(\mathbf{t}^n)^*$ respectively. Since $\mathbb{T}^n$ is abelian, the Lie bracket is trivial, and $\mathbf{t}^n$ and $(\mathbf{t}^n)^*$ can be canonically identified with $\mathbb{R}^n$, with the pairing between $\mathbf{t}^n$ and $(\mathbf{t}^n)^*$ given by the inner product on $\mathbb{R}^n$.

Let (M,ω) be a connected symplectic manifold, and let $\mathbb{T}^n$ act on M via symplectomorphisms. Then for any $\xi \in \mathbf{t}^n$, one has a 1-parameter group of symplectomorphisms $\exp(t\xi)$. We denote by X_ξ the vector field on M which generates the flow $\exp(t\xi)$. Note that for any $\xi,\eta \in \mathbf{t}^n$, $[X_\xi, X_\eta] = X_{[\xi,\eta]} = 0$ since $\mathbb{T}^n$ is abelian. On the other hand, each X_ξ is a symplectic vector field, i.e., $i_{X_\xi}\omega$ is closed. We say that a symplectic

$\mathbb{T}^n$-action on (M,ω) is **weakly Hamiltonian** if for any $\xi \in \mathbf{t}^n$, $i_{X_\xi}\omega = dH_\xi$ for some smooth function H_ξ on M.

In order to define Hamiltonian actions, we recall the concept of **Poisson bracket**. Let F,H be smooth functions on M. We denote by X_F, X_H the corresponding Hamiltonian vector fields, i.e., $i_{X_F}\omega = dF$, $i_{X_H}\omega = dH$. Then the Poisson bracket of F,H is defined and denoted by

$$\{F,H\} \equiv \omega(X_F,X_H) = dF(X_H) = -dH(X_F).$$

In particular, $\{F,H\}=0$ means that the Hamiltonian function F is constant under the flow generated by X_H (and vice versa). The set of smooth functions on (M,ω) becomes a Lie algebra under the Poisson bracket.

With the above understood, a weakly Hamiltonian $\mathbb{T}^n$-action is called **Hamiltonian** if for any $\xi,\eta \in \mathbf{t}^n$, the Poisson bracket $\{H_\xi,H_\eta\}=0$. (In general, a weakly Hamiltonian Lie group action is called Hamiltonian if $\xi \mapsto H_\xi$ can be chosen to be a Lie algebra homomorphism.)

Exercise: Show that for any weakly Hamiltonian $\mathbb{T}^n$-action on (M,ω), the Poisson bracket $\{H_\xi,H_\eta\}$ is a constant function on M for any $\xi,\eta\in\mathbf{t}^n$. In particular, a weakly Hamiltonian $\mathbb{T}^n$-action is Hamiltonian if $\exp(t\xi)$ has a fixed point for any $\xi\in\mathbf{t}^n$ (e.g., when M is compact, closed).

The **moment map** of a Hamiltonian $\mathbb{T}^n$-action on (M,ω) is a smooth map

$$\mu : M \to (\mathbf{t}^n)^* = \mathbb{R}^n,$$

such that for any $\xi \in \mathbf{t}^n = \mathbb{R}^n$,

$$H_\xi(p) = \langle \mu(p), \xi\rangle, \forall p \in M,$$

is a Hamiltonian function for $\exp(t\xi)$, i.e., $i_{X_\xi}\omega = dH_\xi$. Note that the assignment $\xi \mapsto H_\xi$ is linear.

Remark 4.13. (1) The moment map always exists. For example, let $\xi_1,\cdots,\xi_n \in \mathbf{t}^n$ be a basis, and let $\xi_1^*,\cdots,\xi_n^* \in (\mathbf{t}^n)^*$ be the corresponding dual basis. Then

$$\mu(p) = H_{\xi_1}(p)\xi_1^* + \cdots + H_{\xi_n}(p)\xi_n^*, \;\; \forall p \in M,$$

is a moment map.

(2) The moment map is uniquely defined up to a constant vector in $(\mathbf{t}^n)^*$.

(3) Because $\{H_\xi,H_\eta\}=0$ for any $\xi,\eta\in\mathbf{t}^n$ and the fact that $\mathbb{T}^n$ is connected, the moment map $\mu : M \to \mathbb{R}^n$ is $\mathbb{T}^n$-invariant, i.e., $\mu(g\cdot p) = \mu(p)$ for any $g \in \mathbb{T}^n$.

Let p be a point in M. We next give a description of the image of $d\mu_p : T_pM \to (\mathbf{t}^n)^* = \mathbb{R}^n$. Let us consider the subspace of $\mathbf{t}^n = \mathbb{R}^n$ which annihilates the image, i.e., the set of $\xi \in \mathbf{t}^n = \mathbb{R}^n$ such that $\langle d\mu_p(Y),\xi\rangle = 0$ for all $Y \in T_pM$. Observe the identity $\langle d\mu_p(Y),\xi\rangle = (dH_\xi)_p(Y) = \omega_p(X_\xi, Y)$. Since ω is nondegenerate, we see immediately that the set of ξ which annihilates the image of $d\mu_p : T_pM \to (\mathbf{t}^n)^* = \mathbb{R}^n$ is the subspace $\{\xi \in \mathbf{t}^n | X_\xi = 0 \text{ at } p\}$, or equivalently, the subspace $\{\xi \in \mathbf{t}^n | p \text{ is a fixed point of the subgroup } \exp(t\xi)\}$. In particular, since the principal orbit, i.e., the set of points in M which has trivial isotropy, is open and dense for an effective action, we see that the set of regular values of the moment map $\mu : M \to (\mathbf{t}^n)^* = \mathbb{R}^n$ is open and dense in the image $\mu(M)$.

Let $\lambda \in (\mathbf{t}^n)^* = \mathbb{R}^n$ be a regular value of μ. Since μ is $\mathbb{T}^n$-invariant, we see that the level surface $\mu^{-1}(\lambda)$ is $\mathbb{T}^n$-invariant. The quotient space $B_\lambda \equiv \mu^{-1}(\lambda)/\mathbb{T}^n$, which is an orbifold in general of dimension $\dim M - 2n$, has a natual symplectic structure ω_λ. The space $(B_\lambda, \omega_\lambda)$ is called the reduced space at λ (its proof is similar to the case of $\mathbb{S}^1$-action, cf. Theorem 4.1). Note that $\dim M - 2n \geq 0$, namely, the dimension of the torus is at most half of the dimension of the symplectic manifold which the torus acts on. When the dimension of the torus equals half of the dimension of the symplectic manifold, the reduced spaces consist of single points, and the preimages $\mu^{-1}(\lambda)$ are orbits of the torus action, which are easily seen to be embedded Lagrangian tori (they are Lagrangian because of the conditon $\{H_\xi, H_\eta\} = \omega(X_\xi, X_\eta) = 0$ for any $\xi, \eta \in \mathbf{t}^n$).

The fundamental result concerning Hamiltonian torus actions is the following convexity theorem, due to Atiyah and Guillemin-Sternberg independently.

Theorem 4.14. *(Atiyah-Guillemin-Sternberg). Let (M, ω) be a compact, connected symplectic manifold which is equipped with a Hamiltonian $\mathbb{T}^n$-action of moment map $\mu : M \to \mathbb{R}^n$. Then the fixed points of the $\mathbb{T}^n$-action form a finite union of connected symplectic submanifolds $Q_1, \cdots, Q_N$, such that on each Q_j, the moment map μ has a constant value $\lambda_j \in \mathbb{R}^n$, and the image of μ is the convex hull of λ_j, i.e.,*

$$\mu(M) = \{\sum_{j=1}^N x_j \lambda_j | \sum_{j=1}^N x_j = 1, x_j \geq 0\} \subset \mathbb{R}^n.$$

Example 4.15. (1) Consider the following Hamiltonian $\mathbb{T}^n$-action on $\mathbb{CP}^n$

$$(t_1, t_2, \cdots, t_n) \cdot [z_0, z_1, z_2, \cdots, z_n] = [z_0, e^{-it_1} z_1, e^{-it_2} z_2, \cdots, e^{-it_n} z_n],$$

which has moment map

$$\mu([z_0, z_1, \cdots, z_n]) = \frac{1}{2}(\frac{|z_1|^2}{|z_0|^2 + \cdots + |z_n|^2}, \cdots, \frac{|z_n|^2}{|z_0|^2 + \cdots + |z_n|^2}).$$

Clearly $\mu(\mathbb{CP}^n) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | \sum_{i=1}^n x_i \leq \frac{1}{2}, x_i \geq 0\}$. The fixed points are $p_0 = [1, 0, \cdots, 0]$, $p_1 = [0, 1, \cdots, 0], \cdots$, $p_n = [0, 0, \cdots, 1]$, which are mapped under μ to $\lambda_0 = (0, 0, \cdots, 0)$, $\lambda_1 = (\frac{1}{2}, 0, \cdots, 0), \cdots$, $\lambda_n = (0, 0, \cdots, \frac{1}{2})$ respectively. $\mu(\mathbb{CP}^n)$ is the n-simplex with vertices $\lambda_0, \lambda_1, \cdots, \lambda_n$.

(2) A non-effective $\mathbb{T}^2$-action on $\mathbb{CP}^2$. Consider the following non-effective action

$$(t_1, t_2) \cdot [z_0, z_1, z_2] = [z_0, e^{-it_1} z_1, e^{-2it_2} z_2],$$

which has moment map

$$\mu([z_0, z_1, z_2]) = \frac{1}{2}(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{2|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}).$$

The image $\mu(\mathbb{CP}^2)$ is the triangle with vertices $(0, 0), (\frac{1}{2}, 0)$ and $(0, 1)$.

(3) $\mathbb{T}^2$-actions on $\mathbb{CP}^1 \times \mathbb{CP}^1$. (i) Consider the following $\mathbb{T}^2$-action on $\mathbb{CP}^1 \times \mathbb{CP}^1$:

$$(t_1, t_2) \cdot ([z_0, z_1], [w_0, w_1]) = ([z_0, e^{-it_1} z_1], [w_0, e^{-it_2} w_1]).$$

The moment map is

$$\mu([z_0,z_1],[w_0,w_1]) = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2}, \frac{|w_1|^2}{|w_0|^2+|w_1|^2}\right),$$

and the fixed points are $([1,0],[1,0])$, $([1,0],[0,1])$, $([0,1],[1,0])$ and $([0,1],[0,1])$, with values under μ being $(0,0)$, $(0,\frac{1}{2})$, $(\frac{1}{2},0)$ and $(\frac{1}{2},\frac{1}{2})$ respectively. The image of the moment map is

$$\mu(\mathbb{CP}^1\times\mathbb{CP}^1) = \{(x_1,x_2)\in\mathbb{R}^2 | 0\leq x_1,x_2\leq \frac{1}{2}\}.$$

(ii) Consider the following $\mathbb{T}^2$-action on $\mathbb{CP}^1\times\mathbb{CP}^1$:

$$(t_1,t_2)\cdot([z_0,z_1],[w_0,w_1]) = ([z_0,e^{-it_1}z_1],[w_0,e^{-it_1-it_2}w_1]).$$

The moment map is

$$\mu([z_0,z_1],[w_0,w_1]) = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2}+\frac{|w_1|^2}{|w_0|^2+|w_1|^2}, \frac{|w_1|^2}{|w_0|^2+|w_1|^2}\right),$$

and the fixed points are $([1,0],[1,0])$, $([1,0],[0,1])$, $([0,1],[1,0])$ and $([0,1],[0,1])$, with values under μ being $(0,0)$, $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},0)$ and $(1,\frac{1}{2})$ respectively. The image of μ is the parallelagram with vertices $(0,0)$, $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},0)$ and $(1,\frac{1}{2})$.

(iii) Consider the following $\mathbb{T}^2$-action on $\mathbb{CP}^1\times\mathbb{CP}^1$:

$$(t_1,t_2)\cdot([z_0,z_1],[w_0,w_1]) = ([z_0,e^{-it_1}z_1],[w_0,e^{-2it_1-it_2}w_1]).$$

The moment map is

$$\mu([z_0,z_1],[w_0,w_1]) = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2}+\frac{2|w_1|^2}{|w_0|^2+|w_1|^2}, \frac{|w_1|^2}{|w_0|^2+|w_1|^2}\right),$$

and the fixed points are $([1,0],[1,0])$, $([1,0],[0,1])$, $([0,1],[1,0])$ and $([0,1],[0,1])$, with values under μ being $(0,0)$, $(1,\frac{1}{2})$, $(\frac{1}{2},0)$ and $(\frac{3}{2},\frac{1}{2})$ respectively. The image of μ is the parallelagram with vertices $(0,0)$, $(1,\frac{1}{2})$, $(\frac{1}{2},0)$ and $(\frac{3}{2},\frac{1}{2})$.

(4) A $\mathbb{T}^2$-action on a Hirzebruch surface $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. Here the Hirzebruch surface is given as the complex surface

$$M\equiv\{([a,b],[x,y,z])\in\mathbb{CP}^1\times\mathbb{CP}^2 | ay=bx\}.$$

The $\mathbb{T}^2$-action on M is the restriction of the following $\mathbb{T}^2$-action on $\mathbb{CP}^1\times\mathbb{CP}^2$

$$(t_1,t_2)\cdot([a,b],[x,y,z]) = ([e^{-it_1}a,b],[e^{-it_1}x,y,e^{-it_2}z]),$$

which leaves M invariant.

The moment map is

$$\mu([a,b],[x,y,z]) = \frac{1}{2}\left(\frac{|a|^2}{|a|^2+|b|^2}+\frac{|x|^2}{|x|^2+|y|^2+|z|^2}, \frac{|z|^2}{|x|^2+|y|^2+|z|^2}\right),$$

and there are four fixed points on M, which are

$$([1,0],[1,0,0]),([1,0],[0,0,1]),([0,1],[0,1,0]),([0,1],[0,0,1]).$$

The corresponding values under the moment map are $(1,0)$, $(\frac{1}{2},\frac{1}{2})$, $(0,0)$ and $(0,\frac{1}{2})$. So the image of μ is $\{(x_1,x_2)\in\mathbb{R}^2 | x_1+x_2\leq 1, x_1\geq 0, 0\leq x_2\leq\frac{1}{2}\}$.

Let (M, ω) be a symplectic manifold with a Hamiltonian $\mathbb{T}^n$-action and let $\mu : M \to (\mathbf{t}^n)^*$ be the corresponding moment map. For any $k < n$ let $\mathbb{T}^k \subset \mathbb{T}^n$ be a sub-torus. Then there is naturally an induced Hamiltonian $\mathbb{T}^k$-action on M. The moment map of the induced action is $\mu : M \to (\mathbf{t}^n)^*$ composed with the projection $(\mathbf{t}^n)^* \to (\mathbf{t}^k)^*$.

Example 4.16. Consider the $\mathbb{S}^1$-action on $\mathbb{CP}^2$ which is induced from the standard $\mathbb{T}^2$-action on $\mathbb{CP}^2$ considered in Example 4.15 (1) by the embedding $\mathbb{S}^1 \subset \mathbb{T}^2$ given by $t \mapsto (t, 2t)$. This is the same $\mathbb{S}^1$-action we considered in Example 4.10 (3).

Note that the image of the moment map of the $\mathbb{T}^2$-action on $\mathbb{CP}^2$ is the triangle with vertices $(0,0)$, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Its projection onto the line $\mathbb{R}\langle 1, 2\rangle \subset \mathbb{R}^2$ is the line segment between the points $(0,0)$ and $(\frac{1}{5}, \frac{2}{5})$, which are the images of the vertices $(0,0)$ and $(0, \frac{1}{2})$ under the projection. Note that the image of the vertex $(\frac{1}{2}, 0)$ is the middle point $(\frac{1}{10}, \frac{1}{5})$ of the line segment. Compare with the moment map in Example 4.10 (3) and notice that the length of the vector $\langle 1, 2\rangle$ is $\sqrt{5}$.

In general the set of regular values of the moment map is divided into several chambers. We illustrate this with the following example of a $\mathbb{T}^2$-action on $\mathbb{CP}^3$.

Example 4.17. Consider the Hamiltonian $\mathbb{T}^2$-action on $\mathbb{CP}^3$

$$(t_1, t_2) \cdot [z_0, z_1, z_2, z_3] = [z_0, e^{-it_1} z_1, e^{-2it_1} z_2, e^{-it_2} z_3],$$

which has moment map

$$\mu([z_0, z_1, z_2, z_3]) = \frac{1}{2}\Big(\frac{|z_1|^2 + 2|z_2|^2}{\sum_{j=0}^3 |z_j|^2}, \frac{|z_3|^2}{\sum_{j=0}^3 |z_j|^2}\Big).$$

The image of μ is the triangle with vertices $(0,0)$, $(1,0)$ and $(0, \frac{1}{2})$. The set of regular values of μ is the interior of the triangle with the line segment between the points $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ removed. So it is divided into two chambers by the line segment. Notice that the "wall" that divides the two chambers is the image of $\{[0, z, 0, w] \in \mathbb{CP}^3\}$ under μ, which is fixed by the diagonal sub-torus $\{(t,t)\} \subset \mathbb{T}^2$.

A compact, connected symplectic manifold of dimension $2n$ is called **toric** if it admits an effective Hamiltonian $\mathbb{T}^n$-action. Delzant showed that such a space together with the $\mathbb{T}^n$-action is uniquely determined by the image of the moment map, and moreover, there exists a $\mathbb{T}^n$-invariant complex structure with respect to which the symplectic form is Kähler.

Compact, connected symplectic 4-manifolds with a Hamiltonian $\mathbb{S}^1$-action have been classified, from which it is known that such spaces are all Kähler and the $\mathbb{S}^1$-actions are holomorphic. However, S. Tolman constructed an example of a Hamiltonian $\mathbb{T}^2$-action on a compact, connected 6-dimensional symplectic manifold which does not admit any $\mathbb{S}^1$-invariant holomorphic Kähler structure.

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