MATH 704: PART 5: HODGE THEORY

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Contents

1. The Hodge star and a variational viewpoint

Let (M, g) be an oriented Riemannian manifold of dimension n (for the most part we will assume M is compact without boundary). For simplicity, we shall denote the metric g_p by $\langle , \rangle_p$ for each $p \in M$.

The Hodge star operator: First of all, for any $m > 0$, there is an induced metric on $\Lambda^m M$. To see this, for each $p \in M$, consider multilinear map $(T_p^* M \times \cdots \times T_p^* M) \times (T_p^* M \times \cdots \times T_p^* M) \to \mathbb{R}$, given by $((v_1, \cdots, v_m), (w_1, \cdots, w_m)) \mapsto \det(\langle v_i, w_j \rangle_p)$. This induces a symmetric bilinear form $\Lambda_p^m M \times \Lambda_p^m M \to \mathbb{R}$, which is positive definite. We shall denote it by $\langle , \rangle_p$ as well. It is easy to check that if $\epsilon^1, \epsilon^2, \cdots, \epsilon^n$ form a local orthonormal coframe of M, then

$$\{\epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \cdots \wedge \epsilon^{i_m} | i_1 < i_2 < \cdots < i_m\}$$

form a local orthonormal frame for the bundle $\Lambda^m M$ with respect to the induced metric. With this understood, we shall define the Hodge star $* : \Lambda^m M \to \Lambda^{n-m} M$.

Fix a positively oriented orthonormal coframe $\epsilon^1, \epsilon^2, \cdots, \epsilon^n$, for any multi-index $(i_1, i_2, \cdots, i_m)$, we let $(j_1, j_2, \cdots, j_{n-m})$ be the multi-index where $\{j_1, \cdots, j_{n-m}\} = \{1, 2, \cdots, n\} \setminus \{i_1, \cdots, i_m\}$. With this understood, we define

$$*(\epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \cdots \wedge \epsilon^{i_m}) = \pm \epsilon^{j_1} \wedge \epsilon^{j_2} \wedge \cdots \wedge \epsilon^{j_{n-m}},$$

where the $\pm$-sign is chosen such that the following holds:

$$(\epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \cdots \wedge \epsilon^{i_m}) \wedge *(\epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \cdots \wedge \epsilon^{i_m}) = \epsilon^1 \wedge \epsilon^2 \wedge \cdots \wedge \epsilon^n.$$

By linearity, this extends to a map $* : \Lambda^m M \to \Lambda^{n-m} M$, characterized by the property

$$\langle v, w \rangle \cdot dVol_g = v \wedge *w,$$

where $dVol_g = \epsilon^1 \wedge \epsilon^2 \wedge \cdots \wedge \epsilon^n$ is the positive volume form. One can easily check that the square of the Hodge star, $** : \Lambda^m M \to \Lambda^m M$ is given by $** = (-1)^{m(n-m)}$.

From now on, we assume M is compact without boundary. We define a L^2-product on $\Omega^m M$ as follows: for any $\alpha, \beta \in \Omega^m M$, we define

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle dVol_g = \int_M \alpha \wedge *\beta.$$

We denote the L^2-norm of α by $||\alpha||$. With the L^2-product as above, we can define the adjoint of $d : \Omega^{m-1}M \to \Omega^m M$, which is denoted by $d^* : \Omega^m M \to \Omega^{m-1}M$, by the equation $(d\alpha, \beta) = (\alpha, d^*\beta)$ for any $\alpha \in \Omega^{m-1}M$, $\beta \in \Omega^m M$.

Lemma 1.1. *$d^* = - * d *$ when n is even, and $d^* = (-1)^m * d *$ when n is odd.*

Proof. For $\alpha \in \Omega^{m-1}M$, $\beta \in \Omega^m M$, we have

$$d(\alpha\wedge*\beta) = d\alpha\wedge*\beta+(-1)^{m-1}\alpha\wedge d(*\beta) = d\alpha\wedge*\beta+(-1)^{m-1}\cdot(-1)^{(n-m+1)(m-1)}\alpha\wedge**d(*\beta).$$

Now observe that $(-1)^{(n-m+1)(m-1)+m} = -1$ when n is even, and $(-1)^{(n-m+1)(m-1)+m} = (-1)^m$ when n is odd. The lemma follows from Stokes's theorem. $\square$

Harmonic forms: Let $\omega_0 \in \Omega^p M$ be a closed p-form, i.e., $d\omega_0 = 0$. We consider the class of ω_0 in the de Rham cohomology group $H^p_{dR}M$, $[\omega_0] = \{\omega_0 + d\eta | \eta \in \Omega^{p-1}M\}$.

Lemma 1.2. *Suppose ω_0 has the minimal L^2-norm among the closed p-forms in the class $[\omega_0]$, i.e.,*

$$(\omega_0, \omega_0) = \inf_{\omega\in[\omega_0]} (\omega, \omega).$$

Then ω_0 must be co-closed, i.e., $d^*\omega_0 = 0$. Moreover, such a p-form is unique (if exists).

Proof. First, under the assumption of minimality of (ω_0, ω_0), for any $\eta \in \Omega^{p-1}M$, we have

$$2(\omega_0, d\eta) = \frac{d}{dt}(\omega_0 + td\eta, \omega_0 + td\eta)|_{t=0} = 0.$$

This gives $(d^*\omega_0, \eta) = 0$ for any $\eta \in \Omega^{p-1}M$. In particular, we take $\eta = d^*\omega_0$, we obtain $(d^*\omega_0, d^*\omega_0) = 0$, which implies that $d^*\omega_0 = 0$.

Next we prove uniqueness. Suppose $\alpha, \beta \in [\omega_0]$ such that both $d^*\alpha = d^*\beta = 0$. We shall prove that $\alpha = \beta$ must be true. Note that $\alpha - \beta = d\eta$ for some η. Then $d^*(d\eta) = d^*(\alpha - \beta) = 0$, which implies that $(d\eta, d\eta) = (\eta, d^*d\eta) = 0$. Hence $d\eta = 0$, and $\alpha = \beta$ is proved.

$\square$

Definition 1.3. A p-form ω is called a *harmonic form* if $d\omega = d^*\omega = 0$.

The above discussions show that each de Rham cohomology class in $H^p_{dR}M$ can have at most one harmonic representative. The question is

Is there always a harmonic representative for each de Rham cohomology class?

2. The Hodge decomposition theorem

The existence question of harmonic reprsentative of a de Rham cohomology class was solved not via the variational approach in the last section, but rather using the standard techniques of elliptic PDEs, which we shall explain in this section.

The Laplace-Beltrami operator: For any $0 \leq p \leq n$, we let $\Delta := dd^* + d^*d : \Omega^p M \to \Omega^p M$, called the *Laplace-Beltrami operator*. It is known a nonlinear elliptic partial differential operator, and it is obviously a self-adjoint operator with respect to the L^2-product on $\Omega^p M$, i.e., $(\Delta v, w) = (v, \Delta w)$, for any $v, w \in \Omega^p M$. We let $\mathcal{H}^p(M) = \{\omega \in \Omega^p M | \Delta\omega = 0\}$ be the kernel of Δ. The relevance of $\mathcal{H}^p(M)$ is clear from the following lemma.

Lemma 2.1. *$\omega \in \mathcal{H}^p(M)$ if and only if ω is a harmonic p-form, i.e., $d\omega = d^*\omega = 0$.*

Exercise: Let $M = \mathbb{R}^3$, given with the Euclidean metric. Compute $\Delta : \Omega^p M \to \Omega^p M$ for $p = 0, 1$.

Exercise: Let $\{E_i\}$ be a local orthonormal frame such that the corresponding connection 1-forms ω_i^j vanish at a given point $p \in M$. Show that for any $f \in C^\infty(M)$, $\Delta f = -\sum_i \nabla_{E_i}\nabla_{E_i} f$ holds at the point p.

The Hodge decomposition theorem: We are interested in solving the equation $\Delta\omega = \eta$, where $\omega, \eta \in \Omega^p M$. In order to make use of techniques in functional analysis, we shall formulate it differently. Note that $\Delta\omega = \eta$ holds true if and only if for any $\phi \in \Omega^p M$,

$$(\Delta\omega, \phi) = (\omega, \Delta\phi) = (\eta, \phi).$$

If we view ω as a bounded linear functional l on $\Omega^p M$ via $l(\beta) = (\omega, \beta), \forall \beta \in \Omega^p M$, then the equation $\Delta\omega = \eta$ can be regarded as $l(\Delta\phi) = (\eta, \phi)$. A bounded linear functional l satisfying the above equation is called a *weak solution* of $\Delta\omega = \eta$. Then the idea is to find classical solutions via weak solutions.

The following two facts (stated as theorems) are crucial in this approach; their proofs are based on the theory of elliptic PDEs which is beyond the scope here.

Theorem 2.2. *(Regularity)*
Let $\eta \in \Omega^p M$, and let l be a weak solution of $\Delta\omega = \eta$. Then there exists a $\omega \in \Omega^p M$ such that $l(\beta) = (\omega, \beta), \forall \beta \in \Omega^p M$. Consequently, $\Delta\omega = \eta$.

Theorem 2.3. *(Pre-compactness)*
Let $\{\alpha_n\}$ be a sequence in $\Omega^p M$ such that $||\alpha_n|| < c$ and $||\Delta\alpha_n|| < c$ for all n and for some constant $c > 0$. Then there is a subsequence of α_n which is a Cauchy sequence in the L^2-norm.

Assuming these two theorems, we shall prove the following

Theorem 2.4. *(The Hodge decomposition theorem)*
For each $0 \leq p \leq n$, $\mathcal{H}^p(M)$ is finite dimensional, and we have the following orthogonal direct sum decomposition of $\Omega^p M$:

$$\Omega^p M = \mathcal{H}^p(M) \oplus \Delta(\Omega^p M).$$

Proof. The statement that $\mathcal{H}^p(M)$ is finite dimensional follows immediately from Theorem 2.3 (pre-compactness).

It remains to show that $\Omega^p M = \mathcal{H}^p(M) \oplus \Delta(\Omega^p M)$. Let $\mathcal{H}^p(M)^\perp$ be the orthogonal complement of $\mathcal{H}^p(M)$ in $\Omega^p M$. Then $\Omega^p M = \mathcal{H}^p(M) \oplus \mathcal{H}^p(M)^\perp$ as $\mathcal{H}^p(M)$ is finite dimensional, and we shall prove $\mathcal{H}^p(M)^\perp = \Delta(\Omega^p M)$ below, which finishes off the proof of Theorem 2.4.

First of all, note that $\Delta(\Omega^p M) \subset \mathcal{H}^p(M)^\perp$. So it remains to show $\mathcal{H}^p(M)^\perp \subset \Delta(\Omega^p M)$. The following inequality plays the key role:

There exists a constant $c > 0$ such that for any $\beta \in \mathcal{H}^p(M)^\perp$, $||\beta|| \leq c||\Delta\beta||$.

Suppose to the contrary, there is no such constant $c > 0$. Then there is a sequence $\{\beta_i\}$ in $\mathcal{H}^p(M)^\perp$ such that $||\beta_i|| = 1$ and $||\Delta\beta_i|| \to 0$. By the pre-compactness (Theorem 2.3), a subsequence of $\{\beta_i\}$ (still denoted by $\{\beta_i\}$ for simplicity) converges in L^2-norm. We can use it to define a bounded linear functional l by

$$l(\psi) = \lim_{i\to\infty} (\beta_i, \psi), \forall \psi \in \Omega^p M.$$

Note that $l(\Delta\phi) = \lim_{i\to\infty}(\beta_i, \Delta\phi) = \lim_{i\to\infty}(\Delta\beta_i, \phi) = 0$, so that l is a weak solution of $\Delta\omega = 0$. Then by Theorem 2.2 (Regularity), there must be a $\omega \in \Omega^p M$, such that $l(\psi) = (\omega, \psi)$ for any $\psi \in \Omega^p M$, and $\Delta\omega = 0$. By the definition of l, it is easy to see that $\omega \in \mathcal{H}^p(M)^\perp$ and $||\omega|| = 1$ as $\{\beta_i\}$ is in $\mathcal{H}^p(M)^\perp$ and $||\beta_i|| = 1$. But $\Delta\omega = 0$ implies that $\omega \in \mathcal{H}^p(M)$, which is a contradiction.

Now we complete the proof for $\mathcal{H}^p(M)^\perp \subset \Delta(\Omega^p M)$. For any $\eta \in \mathcal{H}^p(M)^\perp$, we define a linear functional l on $\Delta(\Omega^p M)$ as follows:

$$l(\Delta\phi) = (\eta, \phi), \ \ \forall \phi \in \Omega^p M.$$

To see that l is bounded, we let ψ be the component of ϕ in $\mathcal{H}^p(M)^\perp$. Then

$$|l(\Delta\phi)| = |l(\Delta\psi)| \leq ||\eta|| \cdot ||\psi|| \leq c||\eta|| \cdot ||\Delta\psi|| = c||\eta|| \cdot ||\Delta\phi||.$$

Now by Hahn-Banach theorem, l can be extended to a bounded linear functional on $\Omega^p M$, which continues to be denoted by l. Clearly, l is a weak solution of $\Delta\omega = \eta$. $\mathcal{H}^p(M)^\perp \subset \Delta(\Omega^p M)$ follows immediately from the Regularity theorem (Theorem 2.2). $\square$

As an immediate corollary, we have

Corollary 2.5. *The de Rham group $H^p_{dR}M$ is isomorphic to $\mathcal{H}^p(M)$; in particular, $H^p_{dR}M$ is finite dimensional.*

Some applications: We list some immediate topological consequences of the Hodge decomposition theorem.

Theorem 2.6. *(Poincaré duality). Let M be a compact closed, orientable manifold of dimension n. Then $H^p_{dR}M$ is isomorphic to the dual space of $H^{n-p}_{dR}M$. Equivalently, the pairing $H^p_{dR}M \times H^{n-p}_{dR}M \to \mathbb{R}$ defined by $([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta$ is non-degenerate.*

Theorem 2.7. *Let M be a compact closed manifold of dimension n. Assume M is connected. Then $H^n_{dR}M = \mathbb{R}$ if M is orientable, and $H^n_{dR}M = 0$ if otherwise.*

Theorem 2.8. *Let M be a compact closed oriented manifold, equipped with a smooth, finite, free action of G. Let $N = M/G$ be the quotient manifold. Then $H^p_{dR}N$ is isomorphic to $(H^p_{dR}M)^G$, i.e., the subspace of $H^p_{dR}M$ which is fixed under the induced action of G.*

We end with a useful technique due to Bochner (called *Bochner's technique*).

Lemma 2.9. *Let $\{E_i\}$ be a local orthonormal frame, $\{\phi^i\}$ be the dual coframe, such that the corresponding connection 1-forms ω_i^j vanishes at a given point p. Let α be any 1-form, for simplicity, assume $\alpha = f\phi^1$. Then at the point p, the following holds:*

$$\langle \Delta\alpha, \alpha\rangle_p = -(\sum_j \nabla_{E_j}\nabla_{E_j} f)\cdot f + Ric(E_1, E_1)f^2.$$

Proof. Recall that the curvature $\Omega_i^j = \sum_{k<l} R^j_{kli}\phi^k\wedge\phi^l$, where $R^j_{kli} = R(E_k, E_l, E_i, E_j)$. On the other hand, at point p, $\Omega_i^j = d\omega_i^j$. It follows easily that

$$Ric(E_i, E_j) = \sum_k R^k_{kij} = \sum_k d\omega_j^k(E_k, E_i) = \sum_k (\nabla_{E_k}\omega_j^k(E_i) - \nabla_{E_i}\omega_j^k(E_k)).$$

In particular, $Ric(E_1, E_1) = \sum_k(\nabla_{E_k}\omega_1^k(E_1) - \nabla_{E_1}\omega_1^k(E_k))$.

On the other hand, for 1-forms $\Delta\alpha = -((-1)^n * d * d + d * d\, *)\alpha$. We compute separately:

$$\begin{aligned}
d*d*(f\phi^1) &= d*d(f\phi^2\wedge\phi^3\wedge\cdots\wedge\phi^n)\\
&= d*(\nabla_{E_1}f*1 + f\sum_{i>1}(-1)^i\phi^2\wedge\cdots\wedge d\phi^i\wedge\cdots\wedge\phi^n)\\
&= d*(\nabla_{E_1}f*1 + f\sum_{i>1}(-1)^i\omega_1^i(E_i)\phi^2\wedge\cdots\wedge\phi^1\wedge\phi^i\wedge\cdots\wedge\phi^n)\\
&= \sum_k(\nabla_{E_k}\nabla_{E_1}f + f\sum_i\nabla_{E_k}\omega_1^i(E_i))\phi^k.
\end{aligned}$$

$$\begin{aligned}
(-1)^n*d*d(f\phi^1) &= (-1)^n*d*(\sum_j(\nabla_{E_j}f)\phi^j\wedge\phi^1 + fd\phi^1)\\
&= (-1)^n*d(\sum_{j>1}\nabla_{E_j}f*(\phi^j\wedge\phi^1) + \sum_{k,l}f\omega_k^1(E_l)*(\phi^k\wedge\phi^l)),
\end{aligned}$$

where

$$d(\sum_{j>1}\nabla_{E_j}f*(\phi^j\wedge\phi^1) = \sum_{j>1}(\nabla_{E_j}\nabla_{E_j}f(-*\phi^1) + \nabla_{E_1}\nabla_{E_j}f*\phi^j),$$

and

$$d(f\omega_k^1(E_l)*(\phi^k\wedge\phi^l)) = f(\nabla_{E_l}\omega_k^1(E_l)*\phi^k - \nabla_{E_k}\omega_k^1(E_l)*\phi^l).$$

Combing these equations. we obtain

$$\begin{aligned}\Delta\alpha &= -((\sum_j \nabla_{E_j}\nabla_{E_j} f)\phi^1 + \sum_{j>1}(\nabla_{E_j}\nabla_{E_1} f - \nabla_{E_1}\nabla_{E_j} f)\phi^j \\ &\quad + f(\sum_{k,i}\nabla_{E_k}\omega_1^i(E_i)\phi^k - \sum_{k,l}(\nabla_{E_l}\omega_k^1(E_l)\phi^k - \nabla_{E_k}\omega_k^1(E_l)\phi^l))).\end{aligned}$$

(Note further that the term $\sum_{j>1}(\nabla_{E_j}\nabla_{E_1} f - \nabla_{E_1}\nabla_{E_j} f)\phi^j$ actually disappears because $\nabla_{E_j}\nabla_{E_1} f - \nabla_{E_1}\nabla_{E_j} f = [E_j, E_1]f = 0$ at the point p.) Picking up the ϕ^1-component, we obtain easily that

$$\langle\Delta\alpha,\alpha\rangle_p = -(\sum_j \nabla_{E_j}\nabla_{E_j} f)\cdot f + \sum_k(\nabla_{E_k}\omega_1^k(E_1) - \nabla_{E_1}\omega_1^k(E_k))f^2,$$

which is what we claimed: $\langle\Delta\alpha,\alpha\rangle_p = -(\sum_j \nabla_{E_j}\nabla_{E_j} f)\cdot f + Ric(E_1,E_1)f^2$.

□

Theorem 2.10. *(Bochner) Let (M,g) be a compact connected orientable Riemannian manifold. If the Ricci tensor is semi-positive definite, then $b_1(M) \le \dim M$. If furthermore, the Ricci tensor is positive definite at some point, then $b_1(M) = 0$.*

Proof. In the equation in Lemma 2.9, if we let $\tilde{\alpha} = fE_1$ be the vector field dual to α, then $Ric(E_1,E_1)f^2 = Ric(\tilde{\alpha},\tilde{\alpha})$. To understand the term $-(\sum_j \nabla_{E_j}\nabla_{E_j} f)\cdot f$, we note that at the point p,

$$-(\sum_j \nabla_{E_j}\nabla_{E_j} f)\cdot f\phi^1\wedge\cdots\wedge\phi^n = \langle\nabla\alpha,\nabla\alpha\rangle\phi^1\wedge\cdots\wedge\phi^n - \frac{1}{2}d(*d\langle\alpha,\alpha\rangle).$$

It follows easily that for any 1-form α, we have

$$(\Delta\alpha,\alpha) = \int_M \langle\nabla\alpha,\nabla\alpha\rangle dVol_g + \int_M Ric(\tilde{\alpha},\tilde{\alpha})dVol_g.$$

In particular, if $Ric(\tilde{\alpha},\tilde{\alpha}) \ge 0$, then $\Delta\alpha = 0$ implies that $Ric(\tilde{\alpha},\tilde{\alpha}) = 0$ and $\nabla\alpha = 0$. The latter implies that α is determined by its value at one point, so that the space of harmonic 1-forms $\mathcal{H}^1(M)$ may be regarded as a subspace of T_p^*M for any given point p. It follows that $b_1(M) \le \dim M$. If in addition the Ricci tensor is positive definite at some point, then $Ric(\tilde{\alpha},\tilde{\alpha}) = 0$ (with $\nabla\alpha = 0$) implies that $\alpha = 0$. Hence $b_1(M) = 0$.

□

As an example, let M be a compact Kähler manifold whose canonical line bundle K_M is torsion. Then by Yau's solution of Calabi's conjecture, M admits a Kähler-Einstein metric, whose Ricci curvature is necessarily constant zero. Theorem 2.10 implies that $b_1(M) \le \dim_{\mathbb{R}} M$.(e.g. M is a complex torus.)

3. Hodge theory on complex manifolds

Hodge theory of Dolbeault cohomology: Let M be a complex manifold of complex dimension n. For any $p \ge 0$, the complex $\{\bar{\partial} : \Omega^{p,q}M \to \Omega^{p,q+1}M | q \ge$

$0\}$ is exact, i.e., $\bar\partial^2 = 0$. The cohomology groups are called *the (p,q)-th Dolbeault cohomology*:

$$H^{p,q}_{\bar\partial}M = \frac{\{\omega \in \Omega^{p,q}M \mid \bar\partial\omega = 0\}}{\{\omega \mid \omega = \bar\partial\eta, \eta \in \Omega^{p,q-1}M\}}.$$

The relevance of Dolbeault cohomology groups in complex geometry is due to

Theorem 3.1. *(Dolbeault Theorem)*
$H^{p,q}_{\bar\partial}M = H^q(M,\Omega^p)$, where $H^q(M,\Omega^p)$ is the q-th Čech cohomology group of the sheaf Ω^p of holomorphic p-forms (i.e., $(p,0)$-forms) on M.

Hodge theory can be similarly developed for Dolbeault cohomology. More concretely, let M be compact and fix a Hermitian metric h on M. Then h induces a Hermitian inner product $\langle,\rangle_x$ on $\Lambda^{p,q}_x M$ for each $x \in M$. By integrating over M, we define a L^2-Hermitian product on $\Omega^{p,q}M$:

$$(\psi,\eta) := \int_M \langle\psi,\eta\rangle_x dVol_h.$$

We define the adjoint operator $\bar\partial^* : \Omega^{p,q}M \to \Omega^{p,q-1}M$ of $\bar\partial$ by

$$(\bar\partial^*\psi,\eta) = (\psi,\bar\partial\eta),\ \forall\psi \in \Omega^{p,q}M,\ \eta \in \Omega^{p,q-1}M.$$

By the same argument, one can show that if a $\bar\partial$-closed form ψ has minimal norm among the Dolbeault cohomology class of ψ, then ψ must be $\bar\partial$-coclosed, i.e., $\bar\partial^*\psi = 0$, and moreover, such a ψ, if exists, is unique in the Dolbeault cohomology class. In order to prove existence, we form the $\bar\partial$-Laplacian $\Delta_{\bar\partial} := \bar\partial^*\bar\partial + \bar\partial\bar\partial^* : \Omega^{p,q}M \to \Omega^{p,q}M$. If we let $\mathcal{H}^{p,q}_{\bar\partial}M = \{\eta \in \Omega^{p,q}M \mid \Delta_{\bar\partial}\eta = 0\}$, then $\mathcal{H}^{p,q}_{\bar\partial}M = \{\eta \in \Omega^{p,q}M \mid \bar\partial\eta = \bar\partial^*\eta = 0\}$.

Next we introduce the Hodge star $* : \Lambda^{p,q}_x M \to \Lambda^{n-p,n-q}_x M$, $\forall x \in M$, such that

$$\langle\psi,\eta\rangle_x dVol_h(x) = \psi \wedge *\eta,\ \ \forall\psi,\eta \in \Lambda^{p,q}_x M.$$

Let $\{\phi^i\}$ be a local unitary coframe. Then it is easy to check that

$$dVol_h = (-1)^{n(n-1)/2} i^n \phi^1 \wedge \phi^2 \wedge \cdots \wedge \phi^n \wedge \bar\phi^1 \wedge \bar\phi^2 \wedge \cdots \wedge \bar\phi^n.$$

With this understood, if locally we write $\eta = \sum_{I,J} \eta_{I\bar J}\phi^I \wedge \bar\phi^J$, where $I = (i_1, i_2, \cdots, i_p)$, $J = (j_1, j_2, \cdots, j_q)$ are ascending multi-indices, $\phi^I = \phi^{i_1} \wedge \cdots \wedge \phi^{i_p}$, $\bar\phi^J = \bar\phi^{j_1} \wedge \cdots \wedge \bar\phi^{j_q}$, then

$$*\eta = (-1)^{n(n-1)/2} i^n \cdot \sum_{I,J} \epsilon_{IJ}\overline{\eta_{I\bar J}}\phi^{I_0} \wedge \bar\phi^{J_0},$$

where I_0, J_0 are ascending multi-indices complementary to I, J, and $\epsilon_{IJ} = 1$ or -1, which is the sign of the permutation $(1,2,\cdots,n,\bar1,\bar2,\cdots,\bar n) \mapsto (I,\bar J,I_0,\bar J_0)$. This local formula easily implies that for any $\eta \in \Omega^{p,q}M$,

$$**\eta = (-1)^{p+q}\eta.$$

Finally, with this expression the Stokes's theorem easily implies that $\bar\partial^* = -*\bar\partial*$.

Example 3.2. Let $M = \mathbb{C}^2$ with the standard Hermitian metric (i.e., Euclidean metric). Let z_1, z_2 be the coordinates, and let $\phi^i = \frac{1}{\sqrt2}dz_i$. Then $\{\phi^1,\phi^2\}$ form a unitary coframe. Furthermore, note that when $n=2$, $dVol_h=\phi^1\wedge\phi^2\wedge\bar{\phi}^1\wedge\bar{\phi}^2$. With this understood, we compute $\Delta_{\bar\partial}f$ for a smooth complex valued function f:

$$\begin{aligned}
\Delta_{\bar\partial}f &= \bar\partial^*\bar\partial f\\
&= -*\bar\partial*(\sqrt{2}\frac{\partial f}{\partial\bar z_1}\bar\phi^1+\sqrt{2}\frac{\partial f}{\partial\bar z_2}\bar\phi^2)\\
&= -\sqrt{2}*\bar\partial(\overline{\frac{\partial f}{\partial\bar z_1}}\phi^1\wedge\phi^2\wedge\bar\phi^2-\overline{\frac{\partial f}{\partial\bar z_2}}\phi^1\wedge\phi^2\wedge\bar\phi^1)\\
&= -2*(\frac{\partial}{\partial\bar z_1}\overline{\frac{\partial f}{\partial\bar z_1}}\bar\phi^1\wedge\phi^1\wedge\phi^2\wedge\bar\phi^2-\frac{\partial}{\partial\bar z_2}\overline{\frac{\partial f}{\partial\bar z_2}}\bar\phi^2\wedge\phi^1\wedge\phi^2\wedge\bar\phi^1)\\
&= -2(\frac{\partial^2 f}{\partial z_1\partial\bar z_1}+\frac{\partial^2 f}{\partial z_2\partial\bar z_2})\\
&= -\frac{1}{2}(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial y_2^2})f\\
&= \frac{1}{2}\Delta f.
\end{aligned}$$

The key point is that the $\bar\partial$-Laplacian is an elliptic differential operator, which implies the following Hodge decomposition theorem.

Theorem 3.3. *For any $p,q\geq 0$, $\mathcal{H}^{p,q}_{\bar\partial}M$ is finite dimensional, and moreover, we have $\Omega^{p,q}M=\mathcal{H}^{p,q}_{\bar\partial}M\oplus\Delta_{\bar\partial}(\Omega^{p,q}M)$ as an orthogonal decomposition with respect to the L^2-Hermitian product.*

Corollary 3.4. *For any $p,q\geq 0$, the Dolbeault cohomology group $H^{p,q}_{\bar\partial}M$ is isomorphic to $\mathcal{H}^{p,q}_{\bar\partial}M$; in particular, it is finite dimensional.*

Theorem 3.5. *(Kodaira-Serre duality)*
The pairing $H^q(M,\Omega^p)\otimes H^{n-q}(M,\Omega^{n-p})\rightarrow H^n(M,\Omega^n)\cong\mathbb{C}$ is non-degenerate.

The dimensions $h^{p,q}(M):=\dim_{\mathbb{C}}H^{p,q}_{\bar\partial}M$ are called the **Hodge numbers** of M; a priori they are not topological invariants of M as they depend on the complex structure. Note that the Kodaira-Serre duality implies that $h^{p,q}(M)=h^{n-p,n-q}(M)$.

Extension to holomorphic vector bundles: Let E be a holomorphic vector bundle over M. We can extend the definition of Dolbeault cohomology groups to (p,q)-forms with values in E. To this end, recall from Section 4 in Part 4 that there is a special class of covariant derivatives ∇ on $\Gamma(E)$ which obeys condition (1) in Theorem 4.1. In particular, the $(0,1)$-component of ∇ is uniquely determined and the $(0,2)$-component of its curvature is zero. With this understood, the $(0,1)$-component of the exterior covariant derivative d_∇ defines an operator $\bar\partial:\Omega^{p,q}(E)\rightarrow\Omega^{p,q+1}(E)$, where $\Omega^{p,q}(E)=\Gamma(E)\otimes\Omega^{p,q}M$, which is independent of the choice of ∇. Furthermore, since the curvature has no $(0,2)$-part, $\bar\partial^2=0$. The corresponding cohomology groups are denoted by $H^{p,q}_{\bar\partial}(E)$. If $H^q(M,\Omega^p(E))$ denotes the q-th Čech cohomology group of the sheaf of E-valued holomorphic p-forms, then the Dolbeault theorem claims that $H^q(M,\Omega^p(E))$ is isomorphic to $H^{p,q}_{\bar\partial}(E)$.

The Hodge theory can be extended to the case of $H^{p,q}_{\bar\partial}(E)$. To this end, we fix a Hermitian metric on E, which gives rise to a L^2-product on $\Omega^{p,q}(E)$. This allows to define the adjoint operator $\bar\partial^*$ and form the $\bar\partial$-Laplacian $\Delta_{\bar\partial} := \bar\partial^*\bar\partial + \bar\partial\bar\partial^* : \Omega^{p,q}(E) \to \Omega^{p,q}(E)$. If we let $\mathcal{H}^{p,q}(E) = \{\eta \in \Omega^{p,q}(E) | \Delta_{\bar\partial}\eta = 0\}$, then one has similarly $\mathcal{H}^{p,q}(E) = \{\eta \in \Omega^{p,q}(E) | \bar\partial\eta = \bar\partial^*\eta = 0\}$. We remark that in this case, the Hodge star takes the following form: $* : \Lambda^{p,q}_x(E) \to \Lambda^{n-p,n-q}_x(E^*)$, $x \in M$, where E^* is the dual of E.

Theorem 3.6. *(1) $H^{p,q}_{\bar\partial}(E)$ is isomorphic to $\mathcal{H}^{p,q}(E)$; in particular, it is finite dimensional.*

(2) (Kodaira-Serre duality) The pairing $H^q(M, \Omega^p(E)) \otimes H^{n-q}(M, \Omega^{n-p}(E^*)) \to H^n(M, \Omega^n) \cong \mathbb{C}$ is non-degenerate.

See [2] for more details.

Hodge theory on Kähler manifolds: On a complex manifold M with a Hermitian metric h, there are two Laplacian operators: the Laplace-Beltrami operator Δ and the $\bar\partial$-Laplacian $\Delta_{\bar\partial}$. In general they are unrelated, however, when the metric is Kähler, we have the following important fact:

Lemma 3.7. *(cf. [2]) If the Hermitian metric is Kähler, then $\Delta = 2\Delta_{\bar\partial}$.*

As an immediate consequence, note that the orthogonal decomposition $\Omega^r M = \oplus_{r=p+q}\Omega^{p,q}M$ is preserved by Δ. If we let $\mathcal{H}^{p,q}M = \{\eta \in \Omega^{p,q}M | \Delta\eta = 0\}$, then

$$\mathcal{H}^r M = \oplus_{r=p+q}\mathcal{H}^{p,q}M, \quad \mathcal{H}^{p,q}M = \overline{\mathcal{H}^{q,p}M}.$$

More intrinsically, we consider the following subgroups of $H^r_{dR}(M) \otimes \mathbb{C}$:

$$H^{p,q}M := \frac{\{\omega \in \Omega^{p,q}M | d\omega = 0\}}{\{\omega \in \Omega^{p,q}M | \omega = d\eta\}}.$$

Then it is easy to see that $H^{p,q}M$ is isomorphic to $\mathcal{H}^{p,q}M$, and $H^{p,q}M = \overline{H^{q,p}M}$. Moreover, since $\mathcal{H}^{p,q}_{\bar\partial}M = \mathcal{H}^{p,q}M$, $H^{p,q}M$ is isomorphic to $H^{p,q}_{\bar\partial}M$.

Theorem 3.8. *Let M be a compact Kähler manifold. Then*

$$H^r_{dR}(M) \otimes \mathbb{C} \cong \oplus_{r=p+q}H^{p,q}M, \text{ and } H^{p,q}M \cong H^{p,q}_{\bar\partial}M.$$

Corollary 3.9. *The odd Betti numbers of a compact Kähler manifold must be even.*

Example 3.10. Let M be a compact complex surface. In this case, the Hodge numbers of M are completely determined by $h^{1,0}(M)$, $h^{0,1}(M)$, $h^{1,1}(M)$, $h^{0,2}(M)$. The number $q(M) := h^{0,1}(M)$ is called the *irregularity* of M, and the number $p_g(M) := h^{0,2}(M)$ is called the *geometric genus*. Finally, it is known that M is Kähler if and only if $b_1(M)$ is even.

Now suppose M is Kähler. Then $q(M) = \frac{1}{2}b_1(M)$, and $b_2(M) = 2p_g(M) + h^{1,1}(M)$. We will show that, furthermore, $b_2^+(M) = 2p_g(M) + 1$, as a consequence of the Index Theorem (called the Riemann-Roch). More concretely, we have from the index theorem that

$$1 - h^{0,1}(M) + h^{0,2}(M) = Todd(TM)[M],$$

where for any complex vector bundle E of rank r, its Todd class $Todd(E)$ is defined as follows: let the total Chern class $c(E) = \prod_{i=1}^r(1 + \delta_i)$, then $Todd(E) = \prod_{i=1}^r \frac{\delta_i}{1-e^{-\delta_i}}$.

Consider the case where E is of rank 2. Then

$$Todd(E)=\prod_{i=1}^{2}\frac{\delta_i}{1-e^{-\delta_i}}=\prod_{i=1}^{2}(1+(\frac{1}{2}\delta_i-\frac{1}{6}\delta_i^2)+\frac{1}{4}\delta_i^2+\cdots)=1+\frac{1}{2}(\delta_1+\delta_2)+\frac{1}{12}(3\delta_1\delta_2+\delta_1^2+\delta_2^2)+\cdots$$

Note that $\chi(TM)=\delta_1\delta_2$, $p_1(TM)=\delta_1^2+\delta_2^2$, it follows that

$$1-q(M)+p_g(M)=Todd(TM)[M]=\frac{1}{4}(\chi(M)+\sigma(M))=\frac{1}{2}(1-b_1(M)+b_2^+(M)),$$

which gives $b_2^+(M)=2p_g(M)+1$.

If M is non-Kähler, it is known that $b_2^+(M)=2p_g(M)$, $q(M)=h^{0,1}(M)=h^{1,0}(M)+1$, and $b_1(M)=h^{1,0}(M)+h^{0,1}(M)$. So in any case (i.e., Kähler or non-Kähler), the Hodge numbers of M are topological invariants. See [1].

In the Kähler case, it is useful to also consider the operator $\partial:\Omega^{p,q}M\rightarrow\Omega^{p+1,q}M$. Let $\partial^*:\Omega^{p+1,q}M\rightarrow\Omega^{p,q}M$ be the adjoint operator. Then we can form the ∂-Laplacian $\Delta_\partial:=\partial^*\partial+\partial\partial^*:\Omega^{p,q}M\rightarrow\Omega^{p,q}M$. As part of the set of relations among the various operators as in Lemma 3.7, we also have

Lemma 3.11. *(cf.* [2]*) In the Kähler case, $\Delta=2\Delta_\partial$, and $\partial\bar\partial^*+\bar\partial^*\partial=0$.*

Note that $\Delta_\partial=\Delta_{\bar\partial}=\frac{1}{2}\Delta$. We set $\mathcal{H}^{p,q}_\partial M=\{\eta\in\Omega^{p,q}M|\Delta_\partial\eta=0\}$. Then $\mathcal{H}^{p,q}M=\mathcal{H}^{p,q}_\partial M=\mathcal{H}^{p,q}_{\bar\partial}M$. We have the following useful fact.

Lemma 3.12. *(The $\partial\bar\partial$-Lemma) Let η be a (p,q)-form such that η is d-exact. Then there exists a $(p-1,q-1)$-form γ such that $\eta=\partial\bar\partial\gamma$.*

Proof. First of all, since η is d-exact, $0=d\eta=\partial\eta+\bar\partial\eta$, which implies that $\bar\partial\eta=\partial\eta=0$ since η is a (p,q)-form. Secondly, note that η is orthogonal to $\mathcal{H}^{p,q}M=\mathcal{H}^{p,q}_\partial M=\mathcal{H}^{p,q}_{\bar\partial}M$. In particular, $\eta\in\Delta_{\bar\partial}(\Omega^{p,q}M)$. Let $\omega=\Delta_{\bar\partial}^{-1}\eta$. Then

$$\eta=\Delta_{\bar\partial}\omega=\bar\partial\bar\partial^*\omega+\bar\partial^*\bar\partial\omega=\bar\partial\bar\partial^*\omega.$$

Note that $\bar\partial^*\bar\partial\omega=0$ because $\bar\partial$ commutes with $\Delta_{\bar\partial}^{-1}$ and $\bar\partial\eta=0$.

Now consider $\bar\partial^*\omega$. Since it is $\bar\partial$-co-exact, it is also orthogonal to $\mathcal{H}^{p,q}M=\mathcal{H}^{p,q}_\partial M=\mathcal{H}^{p,q}_{\bar\partial}M$. In particular, it lies in the image of Δ_∂. On the other hand, note that $\partial\bar\partial^*\omega=-\bar\partial^*\partial\omega=0$ because ∂ commutes with $\Delta_\partial=\Delta_{\bar\partial}$, so that $\partial\omega=\Delta_{\bar\partial}^{-1}(\partial\eta)=0$, as $\partial\eta=0$. With this understood, let $\xi=\Delta_\partial^{-1}(\bar\partial^*\omega)$. Then

$$\bar\partial^*\omega=\partial\partial^*\xi+\partial^*\partial\xi=\partial\partial^*\xi,$$

as $\partial\xi=\Delta_\partial^{-1}(\partial\bar\partial^*\omega)=0$. The lemma follows by taking $\gamma=-\partial^*\xi$.

□

References

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