# MATH 704: PART 4: GEOMETRY OF COMPLEX MANIFOLDS

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Contents



## 1. Almost complex manifolds

**Algebraic preliminaries:** Let $V$ be a real vector space of dimension $n = 2m$. A *complex structure* on $V$ is an endomorphism $J : V \to V$ such that $J^2 = -Id$. With $J$, $V$ can be made into a complex vector space (of dimension $m$) in a canonical way:

$$(a + ib) \cdot X = aX + bJX, \text{ where } a, b \in \mathbb{R}, X \in V.$$

Conversely, any complex vector space $V$ is naturally a real vector space with a canonical complex structure $J_0$, the one given by multiplication by the complex number $i$, i.e., $J_0X = iX$, for any $X \in V$. Finally, a $\mathbb{R}$-linear map $f : V \to V$ becomes a $\mathbb{C}$-linear map when $V$ is regarded as a complex vector space if and only if $f$ commutes with $J$.

**Example 1.1.** The standard example is $V = \mathbb{R}^{2m}$, with the complex structure $J_0$ given by the matrix $J_0 = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}$. If $(x_1, \cdots, x_m, y_1, \cdots, y_m)$ is the coordinates functions on $V = \mathbb{R}^{2m}$, then with $J_0$, $V$ becomes the complex vector space $\mathbb{C}^m$, with coordinate functions $z_k := x_k + iy_k$, $k = 1, 2, \cdots, m$. Finally, under this identification, the complex general linear group $GL(m, \mathbb{C})$ is identified with a subgroup of the real general linear group $GL(n, \mathbb{R})$ under the correspondence

$$A + iB \mapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix}, \text{ where } A + iB \in GL(m, \mathbb{C}).$$

Let $V^*$ be the dual space of $V$. A complex structure $J$ on $V$ naturally determines a complex structure, still denoted by $J$, on $V^*$ by the following equation:

$$X^*(JX) = JX^*(X), \text{ for any } X \in V, X^* \in V^*.$$

Observe that in terms of bases, if $(e_1, \cdots, e_m, f_1, \cdots, f_m)$ is a basis of $V$ such that $Je_k = f_k$, $Jf_k = -e_k$ for each $k$, and if $(\epsilon_1, \cdots, \epsilon_m, \delta_1, \cdots, \delta_m)$ is the dual basis of $V^*$, then for each $k$,

$$J\epsilon_k = -\delta_k, \quad J\delta_k = \epsilon_k.$$

Let $V^c := V \otimes_{\mathbb{R}} \mathbb{C}$ be the *complexification* of $V$, which is a naturally a complex vector space of dimension $n = 2m$. Then $V$ is contained in $V^c$, via $X \mapsto X \otimes 1$, as a real subspace. There is a $\mathbb{R}$-linear endomorphism on $V^c$, called *complex conjugation*, defined by sending $Z := X + iY$ to $\bar{Z} := X - iY$ for any $X, Y \in V$. Finally, we introduce complex subspaces $V_{1,0}$, $V_{0,1}$, which are the $(+i)$-eigenspace and $(-i)$-eigenspace of $J$ respectively. Then $V^c = V_{1,0} \oplus V_{0,1}$, and the complex conjugation sends $V_{1,0}$ to $V_{0,1}$ as a real isomorphism. Note that

$$V_{1,0} = \{X - iJX |\ X \in V\}, \quad V_{0,1} = \{X + iJX |\ X \in V\}.$$

Similarly, if we denote by $V^{1,0}, V^{0,1}$ the $(+i)$-eigenspace and $(-i)$-eigenspace of $J$ on the dual space $V^*$ respectively, then $(V^*)^c = V^{1,0} \oplus V^{0,1}$. Furthermore, we observe that $V^{1,0}$, $V^{0,1}$ annihilate $V_{0,1}, V_{1,0}$ respectively. Finally, $V$, as a complex vector space, is naturally identified with the complex subspace $V_{1,0}$ of $V^c$ via $X \mapsto \frac{1}{2}(X - iJX)$, the projection of $V$ onto $V_{1,0}$ in $V^c$. Conversely, for a given real vector space $V$, any decomposition of its complexification $V^c = V_{1,0} \oplus V_{0,1}$, where the complex conjugation in $V^c$ sends $V_{1,0}$ to $V_{0,1}$ isomorphically as real vector spaces, defines a complex structure $J$ on $V$ as follows: we note that $V \subset V^c$ is sent to $V_{1,0}$ isomorphically as real vector spaces via the projection $V^c \to V_{1,0}$. With this understood, we simply let $J$ be the pull-back of the complex multiplication of $i$ on $V_{1,0}$.

Recall that a *Hermitian inner product* on a real vector space $V$ with a complex structure $J$ is an inner product $h$ on $V$ such that $h(JX, JY) = h(X, Y)$ for any $X, Y \in V$. It is easy to see that $h(JX, X) = 0$ for any $X \in V$. Note that $h$ and $J$ together determine a skew-symmetric bilinear form $\omega$ on $V$, where $\omega(X, Y) := h(JX, Y)$.

A Hermitian inner product $h$ on $V$ can be extended uniquely to the complexification $V^c$ as a complex symmetric bilinear form, still denoted by $h$, which obeys the following conditions:

- $h(\bar{Z}, \bar{W}) = \overline{h(Z, W)}$, for any $Z, W \in V^c$,
- $h(Z, \bar{Z}) > 0$ for any nonzero $Z \in V^c$,
- $h(Z, \bar{W}) = 0$ for any $Z \in V_{1,0}$, $W \in V_{0,1}$.

Conversely, any complex symmetric bilinear form on $V^c$ obeying the above three conditions determines a Hermitian inner product on $V$. If $(e_1, \cdots, e_m, f_1, \cdots, f_m)$ is a basis of $V$ such that $Je_k = f_k$, $Jf_k = -e_k$ for each $k$, then $z_k := \frac{1}{2}(e_k - if_k)$, $k = 1, 2, \cdots, m$, form a basis of $V_{1,0}$, and $\bar{z}_k := \frac{1}{2}(e_k + if_k)$, $k = 1, 2, \cdots, m$, form a basis of $V_{0,1}$. With this understood, the complex symmetric bilinear form $h$ on $V^c$ is completely determined by the $m \times m$ matrix $(h_{k\bar{l}})$, where

$$h_{k\bar{l}} := h(z_k, \bar{z}_l) = \frac{1}{2}(h(e_k, e_l) - i\omega(e_k, e_l)).$$

**Almost complex manifolds and Nijenhuis tensor:** Let $M$ be a smooth, $2m$-dimensional manifold. An *almost complex structure* on $M$ is a smooth section $J$ of the bundle $End(TM)$ such that $J^2 = -Id$. Equivalently, each tangent space $T_pM$ is equipped with a complex structure $J_p$ which varies smoothly with respect to $p$. Such a manifold $M$ is called an *almost complex manifold*. Each $J_p$ makes $T_pM$ into a complex vector space, and with $J$, $TM$ becomes a complex vector bundle. It is easy to see that the existence of an almost complex structure is equivalent to that the frame bundle of $TM$ admits a $GL(m,\mathbb{C})$-reduction, viewing $GL(m,\mathbb{C})$ as a subgroup of $GL(2m,\mathbb{R})$ in a canonical way. Almost complex manifolds are canonically oriented, as matrices in $GL(m,\mathbb{C})\subset GL(2m,\mathbb{R})$ have positive (real) determinants.

**Example 1.2.** (Complex manifolds)

Let $M$ be a $m$-dimensional complex manifold, that is, $M$ admits an atlas $\{(U_\alpha,\phi_\alpha)\}$, where $\phi_\alpha : U_\alpha \to \mathbb{C}^m$, such that the transition maps $\phi_\beta\circ\phi_\alpha^{-1}$ are bi-holomorphisms. If $\phi_\alpha = (z^k_\alpha)$ and $z^k_\alpha := x^k_\alpha + iy^k_\alpha$, then the endomorphism $J_\alpha$ given by $J_\alpha\frac{\partial}{\partial x^k_\alpha} = \frac{\partial}{\partial y^k_\alpha}$, $J_\alpha\frac{\partial}{\partial y^k_\alpha} = -\frac{\partial}{\partial x^k_\alpha}$, $k = 1,2,\cdots,m$, defines an almost complex structure $J = \{J_\alpha\}$ on $M$. Such an almost complex structure $J$ is called *integrable*.

Let $J$ be an almost complex structure on $M$. The *Nijenhuis tensor* of $J$ is defined as follows: for any vector fields $X,Y$,

$$N(X,Y) := [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY].$$

**Exercise:** Verify the following statements:

(1) $N(X,Y) = -N(Y,X)$, and for any smooth function $f$, $N(fX,Y) = fN(X,Y)$;
(2) $N(X,JX) = 0$ for any $X$; in particular, $N \equiv 0$ when $M$ is 2-dimensional;
(3) $N(X,Y) = 0$ for any $X,Y$ if $J$ is integrable.

**Theorem 1.3.** *(Newlander-Nirenberg)*

*Let $(M,J)$ be any almost complex manifold. If the Nijenhuis tensor of $J$ is zero, then $J$ must be integrable.*

For any almost complex manifold $(M,J)$, the complexification of the tangent bundle and cotangent bundle admit canonical decompositions: $TM\otimes_{\mathbb{R}}\mathbb{C} = T_{1,0}M\oplus T_{0,1}M$ and $T^*M\otimes_{\mathbb{R}}\mathbb{C} = T^{1,0}M\oplus T^{0,1}M$. Furthermore, for each $r$, the exterior bundle $\Lambda^r M\otimes_{\mathbb{R}}\mathbb{C} = \oplus_{p+q=r}\Lambda^{p,q}M$, where $\Lambda^{p,q}M = (\Lambda^p T^{1,0}M)\Lambda(\Lambda^q T^{0,1}M)$. Consequently, the space of complex-valued $r$-forms on $M$ is decomposed into a direct sum of $r$-forms of type $(p,q)$, which is a smooth section of the bundle $\Lambda^{p,q}M$:

$$\Omega^r(M)\otimes_{\mathbb{R}}\mathbb{C} = \oplus_{p+q=r}\Omega^{p,q}(M), \text{ where } \Omega^{p,q}(M) = \Gamma(\Lambda^{p,q}M).$$

**Exercise:** Note that for any complex-valued 1-form $\omega$, the exterior differential $d\omega$ has a canonical decomposition into 2-forms of type $(2,0)$, $(1,1)$, and $(0,2)$. Prove that the vanishing of the Nijenhuis tensor of $J$ is equivalent to the following statement: for any $\omega\in\Omega^{1,0}(M)$, the $(0,2)$-component of $d\omega$ is zero.

*Remarks:* Note that an almost complex structure $J$ on a manifold $M$ can be implicitly defined by specifying a decomposition of the complexification of the cotangent bundle $T^*M\otimes_{\mathbb{R}}\mathbb{C} = T^{1,0}M\oplus T^{0,1}M$ where $T^{0,1}M$ is the complex conjugate of $T^{1,0}M$. With this definition of $J$, we can verify $J$ is integrable by showing that near any point $p\in M$, there is a local frame $\{\phi^i\}$ of $T^{1,0}M$ such that for each $i$, $d\phi^i$ contains no $(0,2)$-component.

It follows easily that if the Nijenhuis tensor of $J$ is zero, then for any $(p,q)$-form $\omega$, the exterior differential $d\omega$ is a sum of a $(p+1,q)$-form and a $(p,q+1)$-form. We denote the $(p+1,q)$-form and the $(p,q+1)$-form by $\partial\omega$, $\bar\partial\omega$ respectively. Hence in this case, $d=\partial+\bar\partial$, where $\partial:\Omega^{p,q}(M)\rightarrow\Omega^{p+1,q}(M)$ and $\bar\partial:\Omega^{p,q}(M)\rightarrow\Omega^{p,q+1}(M)$. The equation $d^2=0$ becomes

$$\partial^2=\bar\partial^2=0, \text{ and } \partial\circ\bar\partial+\bar\partial\circ\partial=0.$$

Let $M$ be a complex manifold. Then the holomorphic tangent and cotangent bundles of $M$ are naturally identified with $T_{1,0}M$ and $T^{1,0}M$ respectively. More precisely, let $\{z^k=x^k+iy^k\}$ be a system of local holomorphic coordinates. Then for each $k$, one has $\frac{\partial}{\partial z^k}=\frac{1}{2}(\frac{\partial}{\partial x^k}-i\frac{\partial}{\partial y^k})$, $dz^k=dx^k+idy^k$, which form a local frame or coframe of the corresponding bundles. Note that $\frac{\partial}{\partial \bar z^k}=\frac{1}{2}(\frac{\partial}{\partial x^k}+i\frac{\partial}{\partial y^k})$, $d\bar z^k=dx^k-idy^k$, which form a local frame and coframe of $T_{0,1}M$ and $T^{0,1}M$ respectively.

**Exercise:** Let $M$ be a complex manifold. Prove that a complex valued smooth function $f$ is holomorphic if and only if the $(0,1)$-component of $df$ is zero, i.e., $\bar\partial f=0$, and if and only if $f$ is annihilated by any vector field of type $(0,1)$, i.e., a smooth section of $T_{0,1}M$. Furthermore, a smooth map $f:M\rightarrow M'$ between complex manifolds is holomorphic if and only if $f_*:TM\otimes_{\mathbb{R}}\mathbb{C}\rightarrow TM'\otimes_{\mathbb{R}}\mathbb{C}$ preserves the decomposition into $(1,0)$, $(0,1)$-components.

Note that locally, for a $(p,q)$-form $\omega=fdz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar z^{j_1}\wedge\cdots\wedge d\bar z^{j_q}$, the decomposition of $d\omega$ into $\partial\omega$, $\bar\partial\omega$ are given by

$$\partial\omega=\sum_k\frac{\partial f}{\partial z^k}dz^k\wedge dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar z^{j_1}\wedge\cdots\wedge d\bar z^{j_q},$$

and

$$\bar\partial\omega=(-1)^p\sum_k\frac{\partial f}{\partial \bar z^k}dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar z^k\wedge d\bar z^{j_1}\wedge\cdots\wedge d\bar z^{j_q}.$$

Next we consider covariant derivatives on an almost complex manifold $(M,J)$. Note that if $\nabla$ is a covariant derivative which is associated to a connection in the frame bundle of $TM$ as a complex vector bundle, it must commute with $J$, or equivalently, $J$ is parallel with respect to $\nabla$, i.e., $\nabla J=0$.

**Exercise:** Let $\nabla$ be a covariant derivative such that $\nabla J=0$. Suppose $\nabla$ is torsion-free, i.e., $T_\nabla(X,Y)=\nabla_XY-\nabla_YX-[X,Y]=0$ for any vector fields $X,Y$. Prove that the Nijenhuis tensor of $J$ must be zero.

Every almost complex manifold $(M,J)$ admits a Hermitian metric, i.e., a Riemannian metric $h$ such that $h(JX,JY)=h(X,Y)$ for any $X,Y$. The above fact implies that if $J$ is parallel with respect to the Levi-Civita connection of the Riemannian manifold $(M,h)$ (which is torsion-free), $M$ must be a complex manifold. In other words, as far as Riemannian geometry is concerned, an almost complex structure does not add anything interesting unless it is integrable. In the next section, we will see that there is another condition we need to impose on the Hermitian metric: the *Kähler condition*.

**Holomorphic vector fields:** Let $M$ be a complex manifold. In this case, $T_{1,0}M$ is simply the holomorphic tangent bundle. A *holomorphic vector field* on $M$ is by definition a holomorphic section of $T_{1,0}M$, which locally can be written as $\sum_k f_k \frac{\partial}{\partial z^k}$ for a local holomorphic coordinates $\{z^k\}$, where $f_k$ are local holomorphic functions. Holomorphic vector fields are closely related to infinitesimal automorphisms of the complex structure of $M$.

More generally, consider an almost complex manifold $(M, J)$. Let $X$ be a vector field such that the flow generated by $X$ preserves $J$ (such a $X$ is called an infinitesimal automorphism of $J$). This happens if and only if the Lie derivative $L_X J = 0$, which is equivalent to $[X, JY] = J[X, Y]$. Plug this into the expression of $N(X, Y)$, we obtain $N(X, Y) = [JX, JY] - J[JX, Y]$. It follows immediately that if $J$ is integrable, then $X$ is an infinitesimal automorphism of $J$ if and only if $JX$ is. It is known that the space of infinitesimal automorphisms forms a Lie algebra under the Lie bracket, and it is the Lie algebra of the automorphism group of $(M, J)$. So when $M$ is a complex manifold, the Lie algebra of infinitesimal automorphisms is a complex Lie algebra under $J$.

**Exercise:** Let $M$ be a complex manifold. Prove that $X$ is an infinitesimal automorphism of the complex structure if and only if the projection of $X$ onto $T_{1,0}M$, i.e., $X \mapsto \frac{1}{2}(X - iJX)$, is a holomorphic vector field.

**Symplectic manifolds, a side note:** A *symplectic structure* on a (necessarily even-dimensional) manifold $M$ is a closed, non-degenerate 2-form, usually denoted by $\omega$. Every symplectic manifold is almost complex; in fact, one can always find an almost complex structure $J$ such that $g(X, Y) := \omega(X, JY)$ defines a Riemannian matric (in fact, $g$ will be a Hermitian metric on $(M, J)$). Such a $J$ is called *$\omega$-compatible*, which forms an infinite-dimensional contractible space. It is clear that most of these $J$'s will not be integrable at all, so the corresponding Riemannian geometry is not interesting. However, introducing such an almost complex structure is fundamental in Gromov's pseudo-holomorphic curve theory for symplectic manifolds.

## 2. Kähler manifolds

**The Kähler condition:** We begin with a lemma for almost complex manifolds.

**Lemma 2.1.** *Let $(M, J)$ be an almost complex manifold, and let $h$ be any Hermitian metric on $(M, J)$. Denote by $\nabla$ the Levi-Civita connection, and let $\omega$ be the 2-form associated to $h$, i.e., for any vector fields $X, Y$, $\omega(X, Y) = h(JX, Y)$. Then the following equation holds true: for any vector fields $X, Y, Z$,*

$$2h((\nabla_X J)Y, Z) = d\omega(X, Y, Z) - d\omega(X, JY, JZ) + h(JX, N(Y, Z)).$$

*Proof.* First of all, $(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y$, which implies that

$$h((\nabla_X J)Y, Z) = h(\nabla_X JY, Z) + h(\nabla_X Y, JZ).$$

Then for the two terms on the right-hand of the above, use the following formula:

$$h(\nabla_X Y, Z) = \frac{1}{2}(Xh(Y, Z) + Yh(X, Z) - Zh(X, Y) + h(X, [Z, Y]) + h(Y, [Z, X]) + h(Z, [X, Y])).$$

On the other hand, recall that for a 2-form $\alpha$,

$$d\alpha(U, V, W) = U\alpha(V, W) - V\alpha(U, W) + W\alpha(U, V) - \alpha([U, V], W) + \alpha([U, W], V) - \alpha([V, W], U).$$

The lemma follows easily by putting these equations together.

□

If $\nabla J = 0$, then $\nabla\omega = 0$ because $\nabla h = 0$. This implies that $d\omega = 0$ because $d\omega$ equals the anti-symmetrization of $\nabla\omega$. The other direction of the following corollary follows from Lemma 2.1.

**Corollary 2.2.** *The almost complex structure $J$ is parallel with respect to the Levi-Civita connection if and only if the Nijenhuis tensor is zero and the associated 2-form $\omega$ is closed.*

**Definition 2.3.** A Hermitian metric on a complex manifold is called a *Kähler metric* if the associated 2-form is closed. Such a complex manifold is called a *Kähler manifold.* The associated 2-form of a Kähler metric is called the *Kähler form.*

**Kähler metrics in local coordinates:** Let $M$ be a complex manifold, $h$ be a Hermitian metric on $M$, which is extended to the complexification $TM \otimes_{\mathbb{R}} \mathbb{C}$ canonically. Suppose $z_1, z_2, \cdots, z_n$ is a local holomorphic coordinate system. We set $h_{k\bar{l}} := h(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l})$. Then $(h_{k\bar{l}})$ is a $n \times n$ Hertimian matrix with entries smooth local functions, i.e., $\overline{h_{l\bar{k}}} = h_{k\bar{l}}$. The Hermitian metric $h$ is completely determined by the matrix $(h_{k\bar{l}})$, and is customarily written as $\sum_{k,l} h_{k\bar{l}} dz^k \otimes d\bar{z}^l$. Note that for any (real) vector fields $X, Y$, $h(X,Y)$ equals the real part of $2\sum_{k,l} h_{k\bar{l}} dz^k(X) d\bar{z}^l(Y)$:

$$h(X,Y) = Re\ (2\sum_{k,l} h_{k\bar{l}} dz^k(X) d\bar{z}^l(Y)).$$

On the other hand, the associated 2-form $\omega$ is given by

$$\omega = i \cdot (\sum_{k,l} h_{k\bar{l}} dz^k \wedge d\bar{z}^l).$$

It follows easily that the Kähler condition $d\omega = 0$ is equivalent to

$$\frac{\partial}{\partial z^j} h_{k\bar{l}} - \frac{\partial}{\partial z^k} h_{j\bar{l}} = 0, \text{ for any } j,k,l.$$

**Lemma 2.4.** *A Hermitian metric $h$ on $M$ is Kähler if and only if for any point $p \in M$, there is a local holomorphic coordinate system centered at $p$ such that the corresponding Hermitian matrix $(h_{k\bar{l}})$ obeys the conditions $dh_{k\bar{l}}(p) = 0$ for any $k, l$.*

*Proof.* If $dh_{k\bar{l}}(p) = 0$ for any $k, l$, then $\frac{\partial}{\partial z^j} h_{k\bar{l}} - \frac{\partial}{\partial z^k} h_{j\bar{l}} = 0$ holds true at $p$ for any $k, l$, so $d\omega(p) = 0$. Since $p$ is arbitrary, $\omega$ is closed.

On the other hand, suppose the metric is Kähler. We choose a local holomorphic coordinate system centered at $p$, and consider the corresponding Hermitian matrix $(h_{k\bar{l}})$. After a linear change of coordinates, we may write near $p$

$$h_{k\bar{l}} = \delta_{k\bar{l}} + \sum_s a_{k\bar{l}s} z_s + a_{k\bar{l}\bar{s}} \bar{z}_s + \text{ terms of order at least 2}.$$

Then note that the condition $\frac{\partial}{\partial z^j} h_{k\bar{l}} - \frac{\partial}{\partial z^k} h_{j\bar{l}} = 0$ implies that $a_{s\bar{l}k} = a_{k\bar{l}s}$, and $\overline{h_{l\bar{k}}} = h_{k\bar{l}}$ implies that $\bar{a}_{k\bar{l}s} = a_{l\bar{k}\bar{s}}$. With this understood, let $z_k = w_k - \frac{1}{2}\sum_{i,j} a_{i\bar{k}j} w_i w_j$. Then one can easily check that $w_1, \cdots, w_n$ define a local holomorphic coordinate system near $p$, and furthermore, $\omega = i \cdot (\sum_{k,l} \hat{h}_{k\bar{l}} dw^k \wedge d\bar{w}^l)$, where

$$\hat{h}_{k\bar{l}} = \delta_{k\bar{l}} + \text{ terms of order at least 2.}$$

Note that $d\hat{h}_{k\bar{l}}(p) = 0$. See [4] for more details. $\square$

**Exercise:** Show that a Hermitian metric is Kähler if and only if for any point $p \in M$, there is a local unitary coframe $\phi^1, \cdots, \phi^n$ (local sections of $T^{1,0}M$) centered at $p$ such that $d\phi^i(p) = 0$ for any $i$.

**Examples of Kähler manifolds:** A fundamental question is what complex manifolds are Kähler. Before we list some examples, observe that any complex submanifold of a Kähler manifold is Kähler. On the other hand, it is easy to see that a Kähler form must be non-degenerate, hence it is particularly a symplectic structure on the manifold. Consequently, if $M$ is a compact Kähler manifold, then $H^2(M, \mathbb{R}) \neq 0$. As an example, $M = \mathbb{S}^1 \times \mathbb{S}^3$ is a complex surface which can not be Kähler.

**Example 2.5.** Let $M = \mathbb{C}^n$. Then the Euclidean metric on $M = \mathbb{R}^{2n}$ is Hermitian with respect to the complex structure on $M = \mathbb{C}^n$. Furthermore, it is Kähler, as the associated 2-form $\omega = \frac{i}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k$, which is closed. (Note that if $z_k = x_k + iy_k$, then $\omega = \sum_{k=1}^n dx_k \wedge dy_k$, which is the standard symplectic structure on $\mathbb{R}^{2n}$. )

**Example 2.6.** (Stein manifolds) A *Stein manifold* is a properly embedded complex submanifold of $\mathbb{C}^N$. So Stein manifolds are Kähler. Stein manifolds can also be intrinsically described, which we will explain below.

Let $M$ be a complex manifold. A real-valued (smooth) function $\phi$ on $M$ is called *plurisubharmonic* (resp. *strictly plurisubharmonic*) if for any local holomorphic coordinate system $\{z_k\}$, the Hermitian matrix $(\frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l})$ is positive semi-definite (resp. positive definite). In a more intrinsic formulation, let $J$ be the complex structure on $M$, and set $d^{\mathbb{C}}\phi := d\phi \circ J$. Then $\phi$ is strictly plurisubharmonic if and only if the 2-form $\omega_\phi := -dd^{\mathbb{C}}\phi$ is a Kähler form. A function $\phi$ is called *exhausting* if it is proper and bounded from below. With this understood, the following is true: *a complex manifold which admits an exhausting strictly plurisubharmonic function is Stein* (cf. [2]). Note that the function $\phi := |z|^2$ on $\mathbb{C}^N$ is an exhausting strictly plurisubharmonic function, and its restriction to any properly embedded complex submanifold $M \subset \mathbb{C}^N$ is an exhausting strictly plurisubharmonic function on $M$. Finally, we remark that a Stein manifold must be non-compact because a strictly plurisubharmonic function can not attain a local maximum at an interior point.

**Exercise:** Show that $d^{\mathbb{C}} = i(\partial - \bar{\partial})$, and $\omega_\phi = 2i\partial\bar{\partial}\phi$. So in a local holomorphic coordinates $\{z_k\}$, we have $\omega_\phi = 2i(\sum_{k,l} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} dz_k \wedge d\bar{z}_l)$.

**Example 2.7.** (The Fubini-Study metric on $\mathbb{CP}^n$) Let $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ be the Hopf fibration, where $\mathbb{S}^{2n+1}$ is considered as the principal $\mathbb{S}^1$-bundle with the action of $\lambda \in \mathbb{S}^1$ given by the multiplication of $\lambda$. As in Example 3.6 in Part 1, there is a canonical connection 1-form $\omega$ on $\mathbb{S}^{2n+1}$:

$$\omega = i\sum_{k=1}^{n+1}(x_k dy_k - y_k dx_k).$$

Since $\mathbb{S}^1$ is Abelian, the curvature $\Omega = d\omega$ can be written as $\Omega = i\pi^*\eta$ for some (real-valued) 2-form $\eta$ on $\mathbb{CP}^n$. We set $\omega_{FS} := \frac{1}{2\pi}\eta$.

**Claim:** $\omega_{FS}$ is a Kähler form on $\mathbb{CP}^n$. (The corresponding Kähler metric is called the *Fubini-Study metric.*)

It is clear that $\omega_{FS}$ is closed. To show $\omega_{FS}$ defines a Hermitian metric, we calculate it in local coordinates. As in Example 3.6 of Part 1, we consider the pull-back of $\Omega$ via the local section of $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$, $(z_1, z_2, \cdots, z_n) \mapsto \frac{1}{\sqrt{1+|z|^2}}(z_1, z_2, \cdots, z_n, 1)$, where $|z|^2 = \sum_k |z_k|^2$. A easy calculation shows that

$$\omega_{FS} = \frac{i}{2\pi}\cdot\frac{(1+|z|^2)\sum_k dz_k\wedge d\bar{z}_k - (\sum_k \bar{z}_k dz_k)\wedge(\sum_k z_k d\bar{z}_k)}{(1+|z|^2)^2}.$$

**Observation:** Let $E_0$ be the dual of the tautological line bundle over $\mathbb{CP}^n$. Then $[\omega_{FS}] = c_1(E_0)$.

**Exercise:** There is another formulation for $\omega_{FS}$. Let $\pi : \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{CP}^n$ be the natural projection. Then note that $(z_1, z_2, \cdots, z_n) \mapsto (z_1, z_2, \cdots, z_n, 1)$ is a local holomorphic section of $\pi$, and

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\ln(1+|z_1|^2+|z_2|^2+\cdots+|z_n|^2).$$

Prove that for any local holomorphic section $Z : U \to \mathbb{C}^{n+1}\setminus\{0\}$ of $\pi$, we have

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\ln|Z|^2, \text{ where } |Z| \text{ is the norm of } Z \in \mathbb{C}^{n+1}.$$

Let $M \subset \mathbb{CP}^N$ be an embedded complex submanifold. Then a theorem of Chow says that $M$ is algebraic, i.e., $M$ is the zero set of polynomial equations. A natural question is: what compact complex manifold is a smooth algebraic variety?

**Theorem 2.8.** *(Kodaira embedding theorem, cf. e.g.* [4]*) Let $M$ be a compact complex manifold. Then $M$ can be embedded in $\mathbb{CP}^N$ as a complex submanifold for some $N > 0$, if and only if there is a complex line bundle $E$ on $M$ and a Kähler metric on $M$ such that the corresponding Kähler form $\omega$ obeys $[\omega] = c_1(E)$.*

**Example 2.9.** (The 2-dimensional case) Let $M$ be an orientable 2-dimensional manifold, and let $g$ be any Riemannian metric on $M$. Then fixing any orientation on $M$, the metric $g$ determines a unique complex structure $J$ on $M$, with respect to which $g$ is Hermitian. Note that $g$ is automatically Kähler because $M$ is 2-dimensional.

## 3. Curvature of a Kähler metric

**Algebraic properties:** Let $\nabla$ be the Levi-Civita connection of a Kähler metric $h$, $J$ be the complex structure. Then $\nabla\circ J = J\circ\nabla$. This has the following easy consequences on the curvature endomorphism $R$ and the Ricci tensor *Ric*.

**Lemma 3.1.** (1) *$R(X,Y)(JZ) = JR(X,Y)Z$, $R(JX,JY)Z = R(X,Y)Z$, and*
$$R(JX,JY,Z,W) = R(X,Y,JZ,JW) = R(X,Y,Z,W).$$
(2) *$Ric(JX,JY) = Ric(X,Y)$;*
(3) *$Ric(X,Y) = \frac{1}{2} Trace(J \circ R(X,JY))$.*

*Proof.* First, $R(X,Y)(JZ) = JR(X,Y)Z$ follows trivially from $\nabla \circ J = J \circ \nabla$. Secondly,
$$R(Z,W,JX,JY) = h(R(Z,W)JX,JY) = h(JR(Z,W)X,JY) = h(R(Z,W)X,Y) = R(Z,W,X,Y),$$
which implies easily
$$R(JX,JY,Z,W) = R(X,Y,JZ,JW) = R(X,Y,Z,W).$$
Finally, $R(JX,JY)Z = R(X,Y)Z$ follows from
$$h(R(JX,JY)Z,W) = R(JX,JY,Z,W) = R(X,Y,Z,W) = h(R(X,Y)Z,W).$$
To see $Ric(JX,JY) = Ric(X,Y)$, note that
$$Ric(JX,JY) = Trace(V \mapsto R(V,JX)JY) = Trace(JV \mapsto R(JV,JX)JY).$$
But $R(JV,JX)JY = JR(V,X)Y$, so $Ric(JX,JY) = Trace(JV \mapsto JR(V,X)Y) = Trace(V \mapsto R(V,X)Y) = Ric(X,Y)$.

Finally, to see $Ric(X,Y) = \frac{1}{2} Trace(J \circ R(X,JY))$, we note that
$$Ric(X,Y) = Trace(V \mapsto R(V,X)Y) = Trace(V \mapsto -JR(V,X)JY),$$
and on the other hand,
$$R(V,X)JY + R(JY,V)X + R(X,JY)V = 0.$$
One can easily check that $Ric(X,Y) = Trace(V \mapsto -JR(JY,V)X)$, hence the claim $Ric(X,Y) = \frac{1}{2} Trace(J \circ R(X,JY))$. □

**Remark 3.2.** By (1), we note that the curvature $R$, as a 2-form valued endomorphism of $TM \otimes_{\mathbb{R}} \mathbb{C}$, preserves the decomposition $TM \otimes_{\mathbb{R}} \mathbb{C} = T_{1,0}M \oplus T_{0,1}M$, and moreover, the 2-form value is of type $(1,1)$. The latter means that for any $X,Y \in \Gamma(T_{1,0}M)$ or $X,Y \in \Gamma(T_{0,1}M)$, we have $R(X,Y) = 0$.

By (2), we can introduce *Ricci form* $\rho$, where $\rho(X,Y) := Ric(JX,Y)$. Then it follows easily $\rho(X,Y) = -\rho(Y,X)$. Moreover, by (3), $\rho(X,Y) = \frac{1}{2} Trace(J \circ R(X,Y))$, which is a $(1,1)$-form.

**Holomorphic sectional curvature:** Let $\Pi$ be a 2-plane in $T_pM$ which is invariant under $J$ (i.e., it is a complex line), then the sectional curvature $K(\Pi)$ is called the *holomorphic sectional curvature*. Strengthening the fact that sectional curvature completely determines the Riemann curvature tensor (cf. Lemma 2.2 in Part 3), in the Kähler case it is known that the holomorphic sectional curvature completely determines the Riemann curvature tensor. More concretely, we have the following lemma, see [5], Chapter IX, section 7 for a proof.

**Lemma 3.3.** *Let $V$ be a real vector space with complex structure $J$, let $T_1, T_2 : V \times V \times V \times V \to \mathbb{R}$ be multilinear maps satisfying the following 4 conditions:*

- *$T(X,Y,Z,W) = -T(Y,X,Z,W) = -T(X,Y,W,Z)$;*

- *$T(X, Y, Z, W) = T(Z, W, X, Y)$;*
- *$T(X, Y, Z, W) + T(Y, Z, X, W) + T(Z, X, Y, W) = 0$;*
- *$T(JX, JY, Z, W) = T(X, Y, JZ, JW) = T(X, Y, Z, W)$.*

*Then if $T_1(X, JX, JX, X) = T_2(X, JX, JX, X)$ for any $X \in V$, then $T_1 = T_2$.*

This algebraic fact has the following easy consequence.

**Proposition 3.4.** *Let $(M, h)$ be a Kähler manifold. Suppose for any $p \in M$, the holomorphic sectional curvature $K(\Pi)$ for any $\Pi \subset T_pM$ depends only on $p$. Then the Kähler metric $h$ must be an Einstein metric. In particular, when $\dim_{\mathbb{C}} M \geq 2$, $(M, h)$ has constant holomorphic sectional curvature.*

*Proof.* For each $p \in M$, we define a quadrilinear map $R_0$ on $T_pM$ by

$$\begin{aligned} R_0(X, Y, Z, W) \quad := \quad & \frac{1}{4}(h(X, W)h(Y, Z) + h(X, JW)h(Y, JZ) - h(X, Z)h(Y, W) \\ & -h(X, JZ)h(Y, JW) - 2h(X, JY)h(Z, JW)). \end{aligned}$$

It is easy to check that $R_0$ obeys the 4 conditions in Lemma 3.3, and moreover, $R_0(X, JX, JX, X) = h(X, X)^2$. It follows from Lemma 3.3, that if $c(p)$ denotes the holomorphic sectional curvature at $p$, then the Riemann curvature tensor $R$ of $(M, h)$ obeys $R(p) = c(p)R_0$. With this understood, it follows easily that

$$Ric(X, Y)(p) = \frac{1}{2}(n + 1)c(p)h(X, Y)(p), \text{ where } n = \dim_{\mathbb{C}} M.$$

It follows immediately that when $\dim_{\mathbb{C}} M \geq 2$, $c(p)$ is constant in $p$. □

**Example 3.5.** The $U(n + 1)$-action on $\mathbb{C}^{n+1}$ induces a holomorphic action on $\mathbb{CP}^n$, which is obviously transitive. Furthermore, the holomorphic action on $\mathbb{CP}^n$ preserves the Fubini-Study metric (Explain why). With this understood, note that for any $p \in \mathbb{CP}^n$, the isotropy subgroup at $p$ is isomorphic to $U(n)$, which acts transitively on the space of complex lines in $T_pM$. This implies immediately that $\mathbb{CP}^n$ with the Fubini-Study metric has constant holomorphic sectional curvature.

Kähler manifolds with constant holomorphic sectional curvature are completely determined, see [5] for a proof of the following theorem.

**Theorem 3.6.** *Two simply connected, complete Kähler manifolds with the same constant holomorphic sectional curvature are isometric by a bi-holomorphism. The following is a complete list of the model spaces (where $c > 0$ is a constant):*

- *(positive holomorphic sectional curvature) $M = \mathbb{CP}^n$, with a Kähler form*

$$\omega = \frac{ic}{2\pi} \cdot \frac{(1 + |z|^2) \sum_k dz_k \wedge d\bar{z}_k - (\sum_k \bar{z}_k dz_k) \wedge (\sum_k z_k d\bar{z}_k)}{(1 + |z|^2)^2} = \frac{ic}{2\pi} \partial\bar{\partial} \ln(1 + |z|^2).$$

- *(zero holomorphic sectional curvature) $M = \mathbb{C}^n$, with the standard flat metric.*
- *(negative holomorphic sectional curvature) $M = \{z \in \mathbb{C}^n \mid |z|^2 < 1\}$, with a Kähler form*

$$\omega = \frac{ic}{2\pi} \cdot \frac{(1 - |z|^2) \sum_k dz_k \wedge d\bar{z}_k + (\sum_k \bar{z}_k dz_k) \wedge (\sum_k z_k d\bar{z}_k)}{(1 - |z|^2)^2} = -\frac{ic}{2\pi} \partial\bar{\partial} \ln(1 - |z|^2).$$

*(The corresponding Kähler metric is called the Bergmann metric.)*

**Curvature in local coordinates:** Let $\{z^k\}$ be a local holomorphic coordinate system. Note that $\nabla_{\frac{\partial}{\partial \bar{z}^i}} \frac{\partial}{\partial z^j}$ is of type $(1,0)$ and $\nabla_{\frac{\partial}{\partial z^j}} \frac{\partial}{\partial \bar{z}^i}$ is of type $(0,1)$, and $\nabla_{\frac{\partial}{\partial \bar{z}^i}} \frac{\partial}{\partial z^j} = \nabla_{\frac{\partial}{\partial z^j}} \frac{\partial}{\partial \bar{z}^i}$ because $\nabla$ is torsion-free. It follows that they must be both zero. On the other hand, $\overline{\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j}} = \nabla_{\frac{\partial}{\partial \bar{z}^i}} \frac{\partial}{\partial \bar{z}^j}$, so $\nabla$ is completely determined by $\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j}$.

We shall write $\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial z^k}$. Then

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l h^{k\bar{l}} \left( \frac{\partial h_{j\bar{l}}}{\partial z^i} + \frac{\partial h_{i\bar{l}}}{\partial z^j} - \frac{\partial h_{ij}}{\partial \bar{z}^l} \right) = \sum_l h^{k\bar{l}} \cdot \frac{\partial h_{j\bar{l}}}{\partial z^i}.$$

Since the curvature is a 2-form of type $(1,1)$, it follows easily that

$$R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^l}\right) = -\sum_s \frac{\partial \Gamma^s_{ik}}{\partial \bar{z}^j} h_{s\bar{l}} = -\frac{\partial^2 h_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + \sum_{s,t} h^{s\bar{t}} \frac{\partial h_{k\bar{t}}}{\partial z^i} \frac{\partial h_{s\bar{l}}}{\partial \bar{z}^j}.$$

**Exercise:** Note that the holomorphic sectional curvature of the complex line determined by $\frac{\partial}{\partial z^i}$ (i.e., the 2-plane spanned by $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}$) equals $R(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}) / h(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i})^2$. Use the above formula of curvature to show that the holomorphic sectional curvature of the Fubini-Study metric in Example 2.7 equals $4\pi$.

**Kähler-Einstein metrics:** A Kähler metric which is Einstein as a Riemannian metric is called a *Kähler-Einstein metric*. Every Kähler metric in complex 1-dimension is Einstein, however, things are much more complex and interesting in dimensions $\geq 2$.

**Lemma 3.7.** *The Ricci form $\rho(X,Y) = i \cdot Trace(R(X,Y)|_{T_{1,0}})$; in particular, the de Rham cohomology class of the Ricci form of a Kähler metric on $M$ equals $2\pi \cdot c_1(TM)$.*

*Proof.* Recall that the Ricci form

$$\rho(X,Y) = \frac{1}{2} Trace(J \circ R(X,Y)) = \frac{1}{2}(Trace(J \circ R(X,Y)|_{T_{1,0}}) + Trace(J \circ R(X,Y)|_{T_{0,1}}).$$

On the other hand, on $T_{1,0}$, $J = i$, and on $T_{0,1}$, $J = -i$, and furthermore, $\overline{R(X,Y)|_{T_{0,1}}} = R(X,Y)|_{T_{1,0}}$. With this understood,

$$Trace(J \circ R(X,Y)|_{T_{0,1}}) = -i \cdot Trace(R(X,Y)^T|_{T_{0,1}}) = i \cdot Trace\overline{R(X,Y)|_{T_{0,1}}} = i \cdot Trace(R(X,Y)|_{T_{1,0}}).$$

It follows immediately that $\rho(X,Y) = i \cdot Trace(R(X,Y)|_{T_{1,0}})$, and the de Rham cohomology class $[\rho] = 2\pi \cdot c_1(T_{1,0}M) = 2\pi \cdot c_1(TM)$. ☐

In a local holomorphic coordinates $\{z^k\}$, if we let $G = \det(h_{k\bar{l}})$, then

$$\frac{\partial G}{\partial z^i} = G \cdot \sum_{k,l} h^{k\bar{l}} \cdot \frac{\partial h_{k\bar{l}}}{\partial z^i} = G \cdot \sum_k \Gamma^k_{ik}.$$

This gives a local formula for the Ricci form:

$$\rho = i \cdot \sum_k -\frac{\partial \Gamma^k_{ik}}{\partial \bar{z}^j} dz^i \wedge d\bar{z}^j = -i \partial \bar{\partial} \ln G.$$

**Example 3.8.** Back to the example of Fubini-Study metric on $\mathbb{CP}^n$. We showed that the holomorphic sectional curvature equals $4\pi$. Then in the formula

$$Ric(X,Y)(p) = \frac{1}{2}(n+1)c(p)h(X,Y)(p), \text{ where } n = \dim_{\mathbb{C}} M,$$

we have $c(p) = 4\pi$, which implies that $c_1(T\mathbb{CP}^n) = (n+1)[\omega_{FS}]$. Note that $\int_L \omega_{FS} = 1$ for any complex line $L \subset \mathbb{CP}^n$, so under the Poincaré duality, $[\omega_{FS}]$ equals the homology class of a hyperplane $H \subset \mathbb{CP}^n$. Hence

$$c_1(T\mathbb{CP}^n) = (n+1)\cdot H.$$

Note that for $n = 2$, we obtain $c_1(T\mathbb{CP}^2) = 3 \cdot PD(\mathbb{CP}^1)$ (cf. Example 3.2 in Part 2).

**Calabi's Conjecture:** Let $M$ be a compact complex manifold, $h$ be a Kähler metric with Kähler form $\omega$. For any closed $(1,1)$-form $\eta$ which represents $c_1(TM)$, there exists a unique Kähler metric $\tilde{h}$ with Kähler form $\tilde{\omega}$, such that

- the Ricci form of $\tilde{h}$ equals $2\pi\eta$,
- $[\tilde{\omega}] = [\omega] \in H^2_{dR}(M)$.

Calabi's Conjecture was solved by S.T. Yau in 1977 (the uniqueness part due to Calabi). Note that in the case of $c_1(TM) = 0$, we can take $\eta = 0$, and Calabi's Conjecture implies that $M$ admits a Kähler-Einstein metric with vanishing Ricci curvature.

Let $M$ be a complex manifold. The *canonical line bundle* of $M$, denoted by $K_M$, is the determinant line bundle of the holomorphic cotangent bundle $T^*M = T^{1,0}M$. Its dual $K_M^*$ is called the *anti-canonical line bundle*. Note that $c_1(K_M^*) = c_1(TM)$.

**Definition 3.9.** A complex line bundle $E$ over a complex manifold $M$ is called positive (resp. negative) if $c_1(E)$ is represented by a positive (resp. negative) multiple of a Kähler form on $M$.

An immediate corollary of Lemma 3.7 is

**Corollary 3.10.** *If a complex manifold $M$ (of dimension $\geq 2$) admits a Kähler-Einstein metric, then its anti-canonical line bundle $K_M^*$ must be either positive, or zero (i.e., torsion), or negative. Correspondingly, the Ricci curvature is positive, zero, or negative.*

The question of the existence of Kähler-Einstein metrics is completely solved. In the case of $K_M^*$ being negative or torsion, the necessary topological condition in Corollary 3.10 turns out to be also sufficient. The case where $K_M^*$ is negative is due to Aubin and Yau (independently), and the case where $K_M^*$ is torsion is a consequence of Yau's solution of Calabi's conjecture. Additional conditions are required for the case where $K_M^*$ is positive, which was resolved by Chen-Donaldson-Sun and Tian.

**Example 3.11.** Let $M$ be a compact complex surface. If $K_M^*$ is positive, zero, or negative, then $c_1(TM)^2[M] \geq 0$, with "=" if and only if $c_1(TM) = 0$ (i.e., torsion). On the other hand, if we blow up $M$ at one point, then one can easily check that the number $c_1(TM)^2[M]$ decreases by 1. It follows then that after blowing up sufficiently many points, $M$ can not admit any Kähler-Einstein metric. For example, $\mathbb{CP}^2$ blown up at more than 8 points does not admit any Kähler-Einstein metric.

Suppose $M$ is not minimal, i.e., there exists a holomorphic 2-sphere $C$ with $C^2 = -1$. If $M$ admits a Kähler-Einstein metric, then by the adjunction formula, $c_1(K_M) \cdot C = -C^2 - 2 = -1$, which implies that the Ricci curvature of $M$ must be positive.

**Example 3.12.** Here we discuss the classification of compact complex surfaces $M$ with $b_1(M) = 0$, $b_2(M) = 1$, where the existence of Kähler-Einstein metrics plays a crucial role.

First, $c_1(TM)^2[M] = 2\chi(M)+3\sigma(M) = 2(2+b_2)+3(b_2-2b_2^-) = 9-6b_2^- > 0$. By the complex surface theory, $c_1(TM)^2[M] > 0$ implies that $M$ is algebraic, in particular, $M$ has a Kähler metric with Kähler form $\omega$. Furthermore, $[\omega]^2 > 0$ implies that $b_2^+ = b_2 = 1$, so that $c_1(TM)^2[M] = 9$.

Secondly, since $b_2 = 1$, we have $c_1(K_M^*) = \lambda[\omega]$ for some $\lambda \neq 0$. When $\lambda > 0$, the complex surface theory implies that $M$ is biholomorphic to $\mathbb{CP}^2$. For the case of $\lambda < 0$, $M$ admits a Kähler-Einstein metric with negative Ricci curvature. Now observing that $c_1(TM)^2[M] = 3c_2(TM)[M]$, as $c_2(TM) = \chi(TM)$, so that $c_2(TM)[M] = \chi(M) = 3$. On the other hand, by a theorem of S.S. Chern, if a complex surface $M$ admits a Kähler-Einstein metric, then $c_1(TM)^2[M] \leq 3c_2(TM)[M]$, with " = " if and only if the Kähler-Einstein metric has a constant holomorphic sectional curvature. It follows immediately that in the case of $c_1(K_M) = \lambda[\omega]$ where $\lambda < 0$, $M$ has a metric of negative constant holomorphic sectional curvature, hence its universal cover is the open unit ball in $\mathbb{C}^2$. In particular, $\pi_1(M)$ is infinite. Such a complex surface $M$ is called a *fake* $\mathbb{CP}^2$; it's known that there are exactly 100 such complex surfaces (50 as smooth 4-manifolds). See [1].

**A side note:** Let $(M, \omega)$ be a symplectic 4-manifold with $b_1(M) = 0$, $b_2(M) = 1$. Analogously, one has $c_1(K_M^*) = \lambda[\omega]$ for some $\lambda \neq 0$. When $\lambda > 0$, a deep theorem of Taubes says that $M$ must be diffeomorphic to $\mathbb{CP}^2$. The case where $\lambda < 0$ remains widely open; we know nothing except when $M$ is a complex surface. In particular, it is not known if there is a simply connected $M$ with $c_1(K_M^*) = \lambda[\omega]$ where $\lambda < 0$. Such a $M$ is homeomorphic to $\mathbb{CP}^2$ but carries an exotic smooth structure. We know it can not be a complex surface, whose only proof is through the existence of Kähler-Einstein metric, which can not be extended to the symplectic setting in any obvious way.

## 4. Connections in Hermitian vector bundles

Let $E$ be a holomorphic vector bundle over a complex manifold $M$. We can always endow $E$ with a Hermitian metric, that is, for any $p \in M$, there is a Hermitian inner product $h_p$ on the fiber $E_p$ such that $h_p$ depends on $p$ smoothly. It turns out that there is a unique connection in $E$ (i.e., a covariant derivative on $\Gamma(E)$) which is compatible to both structures on $E$. Such a connection is called the *Hermitian connection* (or *Chern connection*) of the Hermitian vector bundle $(E, h)$.

**Theorem 4.1.** *Let $E$ be a holomorphic vector bundle over a complex manifold $M$, and let $h$ be any Hermitian metric on $E$. There is a unique covariant derivative $\nabla$ on $\Gamma(E)$ which obeys the following conditions:*

(1) *If we decompose $\nabla = \nabla^{1,0} + \nabla^{0,1}$ where $\nabla^{0,1}$ is the $(0,1)$-component of $\nabla$, then $\nabla^{0,1} = \bar{\partial}$ in any holomorphic trivialization of $E$.*

(2) *The covariant derivative is unitary, i.e., for any vector field $X$, we have*

$$Xh(\eta,\xi) = h(\nabla_X\eta,\xi) + h(\eta,\nabla_X\xi).$$

*Moreover, the curvature $\Omega$ of $\nabla$, as a 2-form valued endomorphism of $E$, is of type $(1,1)$, i.e., $\Omega$ has no $(2,0)$ and $(0,2)$ components.*

*Proof.* We first show that there is a covariant derivative $\nabla$ obeying (1). To this end, we fix an open cover $\{U_\alpha\}$ of $M$ such that $E$ admits a holomorphic trivialization $\Phi_\alpha$ over $U_\alpha$. Let $\{f_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$. Let $\nabla^\alpha$ be the covariant derivative over $U_\alpha$, such that with respect to the holomorphic trivialization $\Phi_\alpha$, $\nabla^\alpha = d$. Then we define $\nabla\xi = \sum_\alpha \nabla^\alpha(f_\alpha\xi)$.

Next we show that there is a unique $\nabla$ which obeys both (1) and (2). First, we observe that for any covariant derivative which obeys (1), its $(0,1)$-component is uniquely determined. Let $\bar\partial + \alpha$ be the uniquely determined $(0,1)$-component in a local unitary trivialization of $E$. Then we define $\nabla$ so that in the same local unitary trivialization, $\nabla = d + A$ where $A = \alpha - \bar\alpha^T$. Note that $\nabla^{0,1} = \bar\partial + \alpha$ so that it obeys (1). On the other hand, $\bar A^T = -A$ so $\nabla$ also obeys (2). □

As an example for illustration, suppose $E$ is a holomorphic line bundle. Let $s$ be a local nonzero holomorphic section, let $|s|$ be the norm of $s$ with respect to the Hermitian metric $h$. We pick any covariant derivative $\hat\nabla$ which obeys (1) in Theorem 4.1. Then $\hat\nabla s = a \cdot s$ for some complex valued $(1,0)$-form $a$. Now consider the unitary frame $u := |s|^{-1}\cdot s$. Then

$$\hat\nabla u = (-\frac{d|s|}{|s|} + a)\cdot u.$$

It follows easily that the $(0,1)$-component of $\hat\nabla$ is $-\frac{\bar\partial|s|}{|s|}$. Consequently, for the Hermitian connection $\nabla$, we have $\nabla u = \frac{1}{|s|}(\partial|s| - \bar\partial|s|)u = ((\partial - \bar\partial)\ln|s|)\cdot u$. The curvature $\Omega$ of $\nabla$ is given by

$$\Omega = d(\partial - \bar\partial)\ln|s| = -2\partial\bar\partial\ln|s| = -\partial\bar\partial\ln|s|^2.$$

It is easy to check if we choose a different holomorphic section $s'$, $\Omega$ remains the same.

**Observation:** *The Fubini-Study form $\omega_{FS}$ is $-\frac{i}{2\pi}$ times the curvature of the Hermitian connection in the tautological line bundle over $\mathbb{CP}^n$ for a natural Hermitian metric.*

Finally, we remark that when $M$ is a Kähler manifold and $E$ is its holomorphic tangent bundle (given with the Kähler metric), the Hermitian connection of $E$ is simply the Levi-Civita connection. With this understood, note that the Levi-Civita connection induces a connection on the determinant line bundle of $E$, namely $K_M^*$. This induced connection is easily seen the Hermitian connection. With this understood, suppose $\{z^k\}$ is a local holomorphic coordinate system. Then $\frac{\partial}{\partial z^1}\wedge\cdots\wedge\frac{\partial}{\partial z^n}$ is a local nonzero holomorphic section of $K_M^*$, with its norm given by $G^{1/2} = \sqrt{\det(h_{k\bar l})}$. Thus the curvature of the Hermitian connection on $K_M^*$ is $\Omega = -\partial\bar\partial\ln G$ by the formula in the previous paragraph. On the other hand, as the curvature of the induced connection of Levi-Civita, $\Omega$ is equal to $Trace(R|_{T_{1,0}M})$, which is $-i\rho$ where $\rho$ is the Ricci form. With $\rho = -i\partial\bar\partial\ln G$, we see $\Omega = -\partial\bar\partial\ln G$ as well.

We end with the following fact.

**Theorem 4.2.** *A complex vector bundle $E$ over a complex manifold $M$ admits a holomorphic vector bundle structure if and only if $E$ admits a connection whose curvature has no $(0,2)$-component.*

A proof can be found in [6] which is based on the Newlander-Nirenberg theorem, or in [3] which is "elementary", based on the idea of "gauge fixing".

## References

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