MATH 704: PART 3: RIEMANNIAN GEOMETRY

WEIMIN CHEN

Contents

1. The Levi-Civita connection and geodesics

Let M be a smooth n-dimensional manifold (without loss of generality, assume M is connected), and g be a Riemannian metric on M. Let P be the frame bundle of the tangent bundle TM. A fundamental fact in Riemannian geometry is that the metric g determines a unique connection in P, called the *Levi-Civita connection*. We shall describe the corresponding covariant derivative on $\Gamma(TM)$ in terms of the metric g.

To begin with, let ∇ be any covariant derivative on $\Gamma(TM)$. We define the *torsion* of ∇, denoted by T_∇, which is a 2-form valued vector field on M, by the following formula: for any vector fields X, Y on M,

$$T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Theorem 1.1. *Let (M,g) be any Riemannian manifold. There exists a unique covariant derivative ∇ on $\Gamma(TM)$, such that*

(1) *For any X,Y,Z, $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.*
(2) *For any X,Y, $T_\nabla(X,Y) = 0$.*

Proof. Using (1) and (2), one can easily derive the following expression

$$g(\nabla_X Y, Z) = \frac{1}{2}(Xg(Y,Z)+Yg(X,Z)-Zg(X,Y)+g(X,[Z,Y])+g(Y,[Z,X])+g(Z,[X,Y])),$$

which uniquely determines ∇. $\square$

Now we pick a local coordinate system $(U, \{x^i\})$ on M, and denote by $\partial_i := \frac{\partial}{\partial x^i}$ the local coordinate frame on U. We introduce smooth functions Γ_{ij}^k on U by

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^n \Gamma_{ij}^k \partial_k.$$

Furthermore, introduce $n\times n$ matrix functions (g_{ij}) and $(g^{ij}):=(g_{ij})^{-1}$, where $g_{ij}:=g(\partial_i,\partial_j)$. Then we can solve for Γ_{ij}^k:

$$\Gamma_{ij}^k=\frac{1}{2}\sum_{l=1}^n g^{kl}(\partial_i g_{jl}+\partial_j g_{il}-\partial_l g_{ij}).$$

The functions Γ_{ij}^k are called the *Christoffel symbols*.

On the other hand, if $\{E_i\}$ is a local orthonormal frame and we introduce a $n\times n$ matrix of 1-forms (ω_i^j) by the equation $\nabla E_i=\sum_{j=1}^n E_j\otimes\omega_i^j$, then it is easy to check that $(\omega_i^j)=-(\omega_i^j)^T$. The 1-forms ω_i^j are called *connection 1-forms* associated to the local orthonormal frame $\{E_i\}$. Now if $\{\phi^i\}$ denotes the dual coframe of $\{E_i\}$, then ω_i^j can be determined through the equations $d\phi^j=\sum_i\phi^i\wedge\omega_i^j$. (Note that d is the anti-symmetrization of $\nabla\phi^j=-\omega_i^j\otimes\phi^i$.)

The Levi-Civita connection determines a unique covariant derivative, continued to be denoted by ∇, on all the tensor fields, differential forms on M. Note that in this light, (1) in Theorem 1.1 may be stated as the tensor field g being parallel with respect to the Levi-Civita connection, i.e., $\nabla g=0$.

Exercise: Let N be a submanifold of M, with the induced Riemannian metric. Let $pr: T_pM\rightarrow T_pN$, $\forall p\in N$, be the orthogonal projection with respect to the Riemannian metric on M. Denote by ∇^M, ∇^N the covariant derivatives corresponding to the Levi-Civita connections of M and N respectively. Prove that for any vector fields X,Y on N, $\nabla^N_X Y=pr(\nabla^M_X Y)$.

Exercise: Let (M,g) be a Riemannian manifold. Show that the Levi-Civita connection is flat if and only if g is locally isometric to the Euclidean metric, i.e., every point in M is contained in a local chart $(U,\{x^i\})$ such that $g=\sum_i(dx^i)^2$.

Geodesics: Let $x(t)$ be a smooth curve in M. Let $\dot{\mathrm{x}}(\mathrm{t})$ be the tangent vector of $x(t)$. The curve $x(t)$ is called a *geodesic* if $\dot{\mathrm{x}}(\mathrm{t})$ is parallel along $x(t)$, i.e., $\nabla_{\dot{\mathrm{x}}(\mathrm{t})}\dot{\mathrm{x}}(\mathrm{t})=0$.

In a local coordinate system $(U,\{x^i\})$, suppose the smooth curve $x(t)$ is given by

$$x(t)=(x^1(t),x^2(t),\cdots,x^n(t)).$$

Then the geodesic equation $\nabla_{\dot{\mathrm{x}}(\mathrm{t})}\dot{\mathrm{x}}(\mathrm{t})=0$ takes the form of a 2nd order ODE

$$\ddot{\mathrm{x}}^k(t)+\sum_{i,j=1}^n\dot{\mathrm{x}}^i(t)\dot{\mathrm{x}}^j(t)\Gamma_{ij}^k(x(t))=0,\ k=1,2,\cdots,n.$$

Exercise: Let $M=\{(x,y)\in\mathbb{R}^2|y>0\}$ be the upper-half plane, given with the Riemannian metric $g=\frac{1}{y^2}(dx^2+dy^2)$. Show that the vertical lines $x=x_0$ and the semi-circles $(x-x_0)^2+y^2=R^2$, $y>0$, with a suitable parametrization, satisfy the geodesic equation.

Exercise: Let G be a Lie group. For any given inner product g_0 on the tangent space T_eG, we can define a left-invariant Riemannian metric g on G as follows: for any left-invariant vector fields X,Y, we set $g(X,Y):=g_0(X_e,Y_e)$.

- Show that g is right-invariant, i.e., $g((R_a)_*X, (R_a)_*Y) = g(X,Y)$ for all $a \in G$, if and only if the inner product g_0 on T_eG is invariant under the adjoint representation $Ad: G \to GL(Lie(G))$. Such a metric is called *bi-invariant*.
- Let g be a bi-invariant metric on a Lie group G. Show that the Levi-Civita connection obeys $\nabla_X Y = \frac{1}{2}[X,Y]$ for any left-invariant vector fields X, Y. Note that in particular, $\nabla_X X = 0$ for any left-invariant vector field X, which implies that the 1-parameter subgroups $exp(tX)$ are geodesics in (G,g).

The exponential map: By the local existence and uniqueness theorem for 2nd order ODE, for any $p \in M$ and any tangent vector $X \in T_pM$, there is a unique geodesic $x(t)$, defined for $-\epsilon < t < \epsilon$ for some $\epsilon > 0$, such that $x(0) = p$, $\dot{x}(0) = X$. We observe that for any fixed $s > 0$, the curve $y(t) := x(st)$, for $-\epsilon/s < t < \epsilon/s$, is also a geodesic, satisfying $y(0) = p$ and $\dot{y}(0) = sX$. It follows easily that for any $p \in M$, there exists a $\epsilon_p > 0$ such that for any tangent vector $X \in T_pM$ satisfying $g(X,X) < \epsilon_p^2$, the unique geodesic $x(t)$ with $x(0) = p$, $\dot{x}(0) = X$ is defined for $-1 \leq t \leq 1$. With this understood, we set $B(\epsilon_p) = \{X \in T_pM | g(X,X) < \epsilon_p^2\}$, and define $exp_p : B(\epsilon_p) \to M$ by sending X to $x(1)$, where $x(t)$ is the unique geodesic satisfying $x(0) = p$, $\dot{x}(0) = X$. the map exp_p is called the *exponential map* at p. One can show that by choosing $\epsilon_p > 0$ sufficiently small, $exp_p : B(\epsilon_p) \to M$ is a diffeomorphism onto its image.

Example 1.2. Here is an application of exponential map in differential topology. Suppose G is a compact Lie group acting smoothly on M. Then we can always equip M with a G-invariant Riemannian metric. Now for any $p \in M$, let $G_p = \{g \in G | g \cdot p = p\}$ be the isotropy subgroup at p. Then there is an induced linear representation of G_p on the tangent space T_pM. With this understood, note that $exp_p : B(\epsilon_p) \to M$ is G-equivariant because the Riemannian metric is G-invariant. This shows that the G-action on M is locally smoothly conjugate to a linear action.

Normal neighborhoods: For any fixed orthonormal basis $E_1, E_2, \cdots, E_n$ of T_pM, we define $E : T_pM \to \mathbb{R}^n$ by $E : \sum_{i=1}^n v_i E_i \mapsto (v_i)$. On the other hand, let U be the image of $exp_p : B(\epsilon_p) \to M$. Then (U, ϕ) is a local coordinate chart centered at p, where $\phi = E \circ (exp_p)^{-1} : U \to \mathbb{R}^n$. It is called a *normal neighborhood* and has the following properties.

Theorem 1.3. *The normal coordinate chart (U,ϕ) has the following properties:*

- *For any $X = \sum_{i=1}^n X^i E_i \in B(\epsilon_p)$, let $x(t)$ be the geodesic satisfying $x(0) = p$, $\dot{x}(0) = X$. Then $\phi(x(t)) = (tX^1, tX^2, \cdots, tX^n)$.*
- *The metric is Euclidean at p, i.e., let $g_{ij} = g(\partial_i, \partial_j)$, then $g_{ij}(p) = \delta_{ij}$.*
- *The Christoffel symbols vanish at p, i.e., $\Gamma_{ij}^k(p) = 0$, for all i,j,k.*

Exercise: Prove that for any $p \in M$, there exists a local orthonormal frame $\{E_i\}$ near p, such that $\nabla E_i(p) = 0$ for each i; equivalently, the connection 1-forms ω_i^j vanishes at p, i.e., $\omega_i^j(p)$ for all i,j.

Geodesics as length-minimizing curves: Let (M,g) be a Riemannian manifold. Let $x(t)$, $a \leq t \leq b$, be a piecewise smooth curve in M. We can define the length of

$x(t)$ to be

$$L(x(t)) := \int_a^b g(\dot{\mathrm{x}}(t), \dot{\mathrm{x}}(t))^{1/2} dt.$$

Furthermore, we can define a distance function d on M, where for any $p, q \in M$, the distance $d(p,q)$ is defined to be the infimum of the length of all piecewise smooth curves connecting p and q. In this way, M becomes a metric space, whose topology as a metric space can be shown to be the same as the original topology.

Let $x(t)$, $a \leq t \leq b$, be a piecewise smooth curve, where it may have conners at a_i. We set $\Delta_i \dot{\mathrm{x}} := \lim_{t \to a_i, t > a_i} \dot{\mathrm{x}}(t) - \lim_{t \to a_i, t < a_i} \dot{\mathrm{x}}(t)$ to be the difference of the tangent vectors at a_i. Furthermore, we let $x(s,t)$, $-\epsilon < s < \epsilon$, $a \leq t \leq b$, be a variation of $x(t)$, where $x(0,t) = x(t)$, $x(s,a) = x(a)$, $x(s,b) = x(b)$ for all s. Set $V(t) = \frac{\partial}{\partial s} x(s,t)|_{s=0}$.

Lemma 1.4. *Assume $g(\dot{x}(t), \dot{x}(t)) = 1$ for $a \leq t \leq b$. Then*

$$\frac{d}{ds} L(x(s,t))|_{s=0} = -\int_a^b g(V(t), \nabla_{\dot{x}(t)} \dot{x}(t)) dt - \sum_i g(V(a_i), \Delta_i \dot{x}).$$

Corollary 1.5. *If a piecewise smooth curve $x(t)$ is length-minimizing, then it must be smooth and is a geodesic.*

Corollary 1.5 raises a natural question: for any $p, q \in M$, does there exist a geodesic $x(t)$ connecting p and q, such that $d(p,q) = L(x(t))$? It turns out that, locally, the answer to the question is always positive.

Theorem 1.6. *For any $x \in M$, there is a normal neighborhood of x, $exp_x(B(\epsilon))$ for some small $\epsilon > 0$, such that for any points $p, q \in exp_x(B(\epsilon))$, there is an unique geodesic $x(t) \subset exp_x(B(\epsilon))$ connecting p, q, with $d(p,q) = L(x(t))$.*

We remark that such a neighborhood $exp_x(B(\epsilon))$ is called *geodesically convex*. The proof of Theorem 1.6 consists of two steps.

Step 1: Let $(U, \phi = \{x^i\})$ be a normal coordinate chart centered at $p \in M$. Then any $q \in U$ is connected to p by a "radial" geodesic of the form $(tX^1, tX^2, \cdots, tX^n)$. Introducing function $r(q) = (\sum_{i=1}^n x^i(q)^2)^{1/2}$, where $\phi(q) = (x^1(q), x^2(q), \cdots, x^n(q))$, and vector field on $U \setminus \{p\}$, $\partial r := r(q)^{-1} \sum_{i=1}^n x^i(q) \partial_i$. Then (1) ∂r has unit length and is tangent to the "radial" geodesics, (2) ∂r is the gradient vector of the function r with respect to the Riemannian metric g. It follows that $d(p,q) = r(q)$, which is the length of the "radial" geodesic connecting p and q. Moreover, one can show that the "radial" geodesic is the unique geodesic in U which connects p and q.

Step 2: For any $p \in M$, the map $exp : TM \to M \times M$, where $exp(q, X) = (q, exp_q X)$, is defined for $q \in U$, a small neighborhood of p, and $X \in T_q M$ with length less than some small $\delta > 0$. Moreover, choosing U, δ sufficiently small, exp is actually a diffeomorphism onto its image, to be denoted by V. Now choose a sufficiently small $\epsilon > 0$, we have $exp_p(B(\epsilon)) \subset U$ and $exp_p(B(\epsilon)) \times exp_p(B(\epsilon)) \subset V$. It follows easily that for any $q_1, q_2 \in exp_p(B(\epsilon))$, there is a geodesic $x(t)$ connecting q_1, q_2 such that $d(q_1, q_2) = L(x(t))$ (cf. Step 1). Prove that when $\epsilon > 0$ is sufficiently small, $x(t) \subset exp_p(B(\epsilon))$ for all t. (*Hint: $r(x(t))$ is a convex function when $\epsilon > 0$ is small.*)

Completeness: A Riemannian manifold (M, g) is called *geodesically complete*, if every maximally defined geodesic is defined for all $t \in \mathbb{R}$. Recall that a metric space is complete if every Cauchy sequence contains a convergent subsequence.

Theorem 1.7. *(Hopf-Rinow) A Riemannian manifold (M, g) is geodesically complete if and only if it is complete as a metric space. Suppose M is complete and connected. Then for any $p, q \in M$, there exists a geodesic $x(t)$ connecting p and q, such that $d(p, q) = L(x(t))$.*

A proof can be found in Lee [4].

2. The Riemann curvature tensor

Let (M, g) be a Riemannian manifold, and let ∇ denote its Levi-Civita connection. The curvature, which is a 2-form valued endomorphism of TM, is called the *Riemann curvature endomorphism*, and is denoted by R instead of Ω. In other words, for vector fields X, Y, Z on M,

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In local coordinates $(U, \{x^i\})$, if we write $R(\partial_i, \partial_j)\partial_k := \sum_{l=1}^n R_{ijk}^l \partial_l$, then an expression of R_{ijk}^l in terms of the Christoffel symbols Γ_{ij}^k can be easily derived:

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \sum_{t=1}^n (\Gamma_{it}^l \Gamma_{jk}^t - \Gamma_{jt}^l \Gamma_{ik}^t).$$

On the other hand, if $\{E_i\}$ is a local orthonormal frame, with $\{\phi^i\}$ being the dual coframe and ω_i^j being the corresponding connection 1-forms, we introduce R_{kli}^j by the equation $R(E_k, E_l)E_i = \sum_{j=1}^n R_{kli}^j E_j$, and set $\Omega_i^j := \frac{1}{2} \sum_{k,l=1}^n R_{kli}^j \phi^k \wedge \phi^l$. Then Ω_i^j can be expressed in terms of $d\omega_i^j$ and (ω_i^j).

Exercise: Prove that $\Omega_i^j = d\omega_i^j - \sum_{k=1}^n \omega_i^k \wedge \omega_k^j$.

Riemann curvature tensor: We define a covariant 4-tensor field R, called the *Riemann curvature tensor*: for any vector fields X, Y, Z, W,

$$R(X, Y, Z, W) := g(R(X, Y)Z, W).$$

Lemma 2.1. *(Algebraic properties of Riemann curvature tensor)*

(a) $R(X, Y, Z, W) = -R(Y, X, Z, W)$;
(b) $R(X, Y, Z, W) = -R(X, Y, W, Z)$;
(c) $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$;
(d) $R(X, Y, Z, W) = R(Z, W, X, Y)$.

Proof. (a) follows from definition, (b) follows from $\nabla g = 0$, (c) follows from $T_\nabla = 0$ and the Jacobi identity for Lie bracket, (d) follows from (a), (b), (c) combined. ☐

Sectional curvature: Let $\Pi \subset T_pM$ be any 2-dimensional subspace. We define the *sectional curvature*

$$K(\Pi) := \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}, \text{ where } X, Y \text{ is any basis of } \Pi.$$

One can easily check that $K(\Pi)$ is well-defined. Moreover, if we scale the Riemannian metric g by a constant $\lambda > 0$, to a metric λg, then the sectional curvatures change as follows: $K_{\lambda g}(\Pi) = \frac{1}{\lambda} K_g(\Pi)$. When M is 2-dimensional, the sectional curvature is simply the classical Gauss curvature.

Exercise: Let (Σ, g) be a compact closed, connected, oriented 2-dimensional Riemannian manifold. Prove that the Euler class of the tangent bundle $T\Sigma$, $\chi(T\Sigma)$, is represented by $\frac{1}{2\pi} K dVol_g$, where K is the Gauss curvature and $dVol_g$ is the volume form of (Σ, g). Then note that the formula $\chi(\Sigma) = \chi(T\Sigma)[\Sigma]$ gives the so-called Gauss-Bonnet Theorem

$$\chi(\Sigma) = \frac{1}{2\pi} \int_\Sigma K dVol_g.$$

Lemma 2.2. *The sectional curvature K completely determines the Riemann curvature tensor R. (cf. Lemma 8.9, Lee [4]).*

Theorem 2.3. *Let (M, g) be a simply connected, n-dimensional complete Riemannian manifold with constant sectional curvature $K \equiv 1, 0$ or -1. Then (M, g) is isometric to the following models (called* space forms*) respectively.*

- *$K \equiv 1$: the unit sphere $\mathbb{S}^n$ in the standard $(\mathbb{R}^{n+1}, g_0)$ with induced metric.*
- *$K \equiv 0$: the Euclidean space $(\mathbb{R}^n, g_0)$ with standard metric g_0.*
- *$K \equiv -1$: Poincaré half space $\mathbb{R}^n_+ = \{(x^1, x^2, \cdots, x^{n-1}, y) | y > 0\}$, with metric*

$$g_0 = \frac{1}{y^2}((dx^1)^2 + (dx^2)^2 + \cdots + (dx^{n-1})^2 + dy^2)$$

Exercise: In the Poincaré half space model, consider the following orthonormal coframe $\phi^1 := \frac{1}{y} dx^1, \cdots, \phi^{n-1} := \frac{1}{y} dx^{n-1}$, $\phi^n := \frac{1}{y} dy$. Show that $\omega_i^j = 0$ for any $i, j < n$, and $\omega_i^n = \phi^i$ for any $i < n$. Then prove that $\Omega_i^j = \phi^i \wedge \phi^j$, from which one can derive that the sectional curvature $K \equiv -1$.

Exercise: Let G be a Lie group, and let g be a bi-invariant metric on G. Show that the Riemann curvature endomorphism is give by $R(X,Y)Z = \frac{1}{4}[Z, [X,Y]]$, where X, Y, Z are left-invariant vector fields on G. Use this formula in the example of $G = SO(3)$, where g is the bi-invariant metric given by the Ad-invariant inner product g_0 on T_eG, where

$$g_0(A, B) := tr(AB^T), \text{ for any } A, B \in T_eG.$$

Show that $SO(3)$ with this bi-invariant metric has constant sectional curvature.

Exercise: Let $\pi : P \to M$ be a principal G-bundle, and let $\tilde{g}$ be a G-invariant Riemannian metric on P. Then for any $u \in P$, the tangent space T_uP admits an orthogonal decomposition as a sum of G_u and Q_u (recall that G_u denotes the subspace of T_uP consisting of vectors tangent to the orbit at u). For any vector field X on M, there is a unique vector field $\tilde{X} \in Q_u$ on P, the horizontal lift of X, which is G-invariant. We can define a Riemannian metric g on M by $g(X,Y) := \tilde{g}(\tilde{X}, \tilde{Y})$. Let $\tilde{\nabla}$, ∇ be the Levi-Civita connections of $(P, \tilde{g})$ and (M, g) respectively. Prove that for any vector fields X, Y on M,

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}[\tilde{X}, \tilde{Y}]^V,$$

where $[\tilde{X}, \tilde{Y}]^V$ is the vertical component of $[\tilde{X}, \tilde{Y}]$ in G_u. Moreover, for any pair of orthonormal vector fields X, Y on M, prove that the sectional curvatures K, $\tilde{K}$ of the 2-planes spanned by X, Y and $\tilde{X}, \tilde{Y}$ are related by

$$K(X, Y) = \tilde{K}(\tilde{X}, \tilde{Y}) + \frac{3}{4}\tilde{g}([\tilde{X}, \tilde{Y}]^V, [\tilde{X}, \tilde{Y}]^V).$$

Ricci curvature and scalar curvature: We define a covariant 2-tensor Ric, called the *Ricci tensor*, as follows: for any vector $X, Y \in T_pM$,

$$Ric(X, Y) = Trace(Z \mapsto R(Z, X)Y), \text{ where } Z \in T_pM,$$

It is clear that if $\{E_i\}$ is an orthonormal basis of T_pM, then $Ric(X, Y) = \sum_i R(E_i, X, Y, E_i)$. It follows immediately that Ricci tensor is symmetric, as

$$Ric(X, Y) = \sum_i R(E_i, X, Y, E_i) = \sum_i R(Y, E_i, E_i, X) = \sum_i R(E_i, Y, X, E_i) = Ric(Y, X).$$

For any $p \in M$, and any $v \in T_pM$ with unit length, we define the *Ricci curvature at p in the direction of v* to be $Ric_p(v) := Ric(v, v)$. Finally, we define the *scalar curvature* $S(p) := tr_g Ric$, the trace of Ric with respect to the metric g. (Namely, if we set $T : T_pM \to T_pM$ by $g(T(X), Y) = Ric(X, Y)$, then $S(p) = trT$.) It follows easily that if $\{E_i\}$ is an orthonormal basis of T_pM, then $S(p) = \sum_i Ric_p(E_i)$.

Exercise: Verify that for a 2-dimensional Riemannian manifold, $Ric_p(v) = K(p)$ the Gauss curvature for any unit tangent vector $v \in T_pM$, and $S(p) = 2K(p)$.

Exercise: Verify that an n-dimensional Riemannian manifold with constant sectional curvature $K \equiv K_0$ has constant Ricci curvature and scalar curvature as well: $Ric_p(v) = (n-1)K_0$ and $S(p) = n(n-1)K_0$.

Exercise: Let (M, g) be a 3-dimensional Riemannian manifold with constant Ricci curvature. Show that (M, g) must have constant sectional curvature as well.

In dimensions greater than 3, there are examples of Riemannian manifolds which have constant Ricci curvature but non-constant sectional curvature.

A Riemannian manifold (M, g) is called *homogeneous and isotropic* if the following conditions are satisfied:

- There is a transitive Lie group action of G on M by isometries.
- For any $p \in M$, the isotropy subgroup G_p acts on the unit sphere of T_pM transitively.

It is clear that a homogeneous and isotropic Riemannian manifold has constant Ricci curvature.

Example 2.4. Consider $\mathbb{S}^{2n+1}$ given with the induced metric, which has constant sectional curvature $K = 1$. The $\mathbb{S}^1$-action on $\mathbb{S}^{2n+1}$ by complex multiplication is by isometries, thus there is a natural induced metric g on the quotient space $\mathbb{CP}^n$. Then $(\mathbb{CP}^n, g)$ is homogeneous and isotropic, thus has constant Ricci curvature. But clearly it does not have constant sectional curvature when $n > 1$.

Exercise: Let $M = \mathbb{S}^2 \times \mathbb{S}^2$ be given with the product metric, where each $\mathbb{S}^2$-factor is given with the metric with constant Gauss curvature $K = 1$. Prove that M has constant Ricci curvature but non-constant sectional curvature.

Einstein metrics: A Riemannian metric g is called *Einstein* if $Ric = \lambda g$ for some smooth function λ. (This condition is equivalent to the condition that $Ric_p(v)$ depends only on p, not on $v \in T_pM$.) It is clear that any 2-dimensional Riemannian metric is Einstein, and any constant Ricci curvature metric is Einstein.

Theorem 2.5. *For $n \geq 3$, an n-dimensional Einstein metric must have constant Ricci curvature.*

Proof. First of all, if $Ric = \lambda g$ for some function λ, then it is easy to see that $\lambda = \frac{1}{n}S$ where S is the scalar curvature. Hence $Ric = \frac{1}{n}S \cdot g$. Taking the covariant derivative, and noting $\nabla g = 0$, we obtain as covariant 3-tensors

$$\nabla Ric = \frac{1}{n}\nabla S \cdot g.$$

Now let $\{E_i\}$ be an orthonormal basis of T_pM, and $X \in T_pM$ be any vector. Then

$$\sum_i \nabla_{E_i} Ric(X, E_i) = \sum_i \nabla Ric(X, E_i, E_i) = \sum_i \frac{1}{n}\nabla_{E_i} S \cdot g(X, E_i) = \frac{1}{n}\nabla_X S.$$

On the other hand, we claim that $\sum_i \nabla_{E_i} Ric(X, E_i) = \frac{1}{2}\nabla_X S$, which shows that when $n \geq 3$, $\nabla_X S = 0$ for any X. Hence $\lambda = \frac{1}{n}S$ is constant.

In order to derive $\sum_i \nabla_{E_i} Ric(X, E_i) = \frac{1}{2}\nabla_X S$, we need the following so-called Bianchi identity.

Lemma 2.6. *(Bianchi identity)*

The covariant 5-tensor field ∇R obeys the following constraint: for any X, Y, Z, W, V,

$$\nabla_V R(X, Y, Z, W) + \nabla_Z R(X, Y, W, V) + \nabla_W R(X, Y, V, Z) = 0.$$

Proof. We verify the identity *pointwise*. Fix $p \in M$, we choose a normal coordinate chart $(U, \{x^i\})$ centered at p, so that $\Gamma_{ij}^k(p) = 0$. In other words, $\nabla_{\partial_i}\partial_j(p) = 0$ for any i, j. With this in mind, we choose X, Y, Z, W, V to be of the form ∂_i. Then

$$\nabla_V R(X, Y, Z, W) = VR(X, Y, Z, W) - R(\nabla_V X, Y, Z, W) - R(X, \nabla_V Y, Z, W) - \cdots = VR(X, Y, Z, W).$$

On the other hand, $VR(X, Y, Z, W) = VR(Z, W, X, Y) = Vg(R(Z, W)X, Y)$, so that

$$\nabla_V R(X, Y, Z, W) = Vg(R(Z, W)X, Y) = g(\nabla_V \nabla_Z \nabla_W X - \nabla_V \nabla_W \nabla_Z X, Y).$$

Adding these up, we verify the Bianchi identity. □

Now back to the proof of the theorem, in the Bianchi identity we let $X = W = E_i$, $Y = Z = E_j$, and sum up over $i, j = 1, 2, \cdots, n$. After some manipulations, we obtain

$$\nabla_V S - \sum_j \nabla_{E_j} Ric(V, E_j) - \sum_i \nabla_{E_i} Ric(V, E_i) = 0,$$

which gives our claim $\sum_i \nabla_{E_i} Ric(X, E_i) = \frac{1}{2}\nabla_X S$. The proof of Theorem 2.5 is complete. □

Which manifolds (for $n \geq 4$) admit Einstein metrics (i.e., constant Ricci curvature metrics) is still a very interesting open question, with ongoing research. On the other hand, it is known that every compact manifold admits a metric of constant scalar curvature – this is the so-called *Yamabe problem*, eventually solved by Richard Schoen.

Killing vector fields: A vector field X on (M, g) is called a *Killing vector field* if the (local) flow ϕ_t generated by X preserves the metric g, i.e., ϕ_t are (local) isometries.

For any vector field X, we set $A_X := L_X - \nabla_X$, where L_X is the Lie derivative and ∇ is the Levi-Civita connection. Note that $A_X f = 0$ for any $f \in C^\infty(M)$, so that for any tensor field η, $A_X(f\eta) = fA_X\eta$. Furthermore, we note that for any vector field Y, $A_X Y = -\nabla_Y X$, so $A_X \equiv 0$ if and only if X is parallel, i.e. $\nabla X \equiv 0$. Finally, when X is a Killing vector field, we have $L_X g = 0$, which implies that $A_X g = 0$. It follows easily that in this case, A_X as a smooth section of $End(TM)$ is skew-symmetric with respect to g, i.e., for any vector fields Y, Z,

$$g(A_X Y, Z) = -g(Y, A_X Z).$$

Lemma 2.7. *Let X be a Killing vector field and Y be any vector field. Then*

$$div(A_X Y) = -Ric(X, Y) - tr(A_X A_Y).$$

Proof. Let ϕ_t be the flow generated by X. Then for any vector fields V, Z, we have

$$(\phi_t)_* \nabla_V Z = \nabla_{(\phi_t)_* V} (\phi_t)_* Z.$$

Differentiating with respect to t, we obtain $L_X(\nabla_V Z) = \nabla_{L_X V} Z + \nabla_V L_X Z$, which can be written as $L_X \circ \nabla_V - \nabla_V \circ L_X = \nabla_{[X,V]}$. With this understood, we have

$$R(X, V)Y = (\nabla_X \circ \nabla_V - \nabla_V \circ \nabla_X - \nabla_{[X,V]})Y = -(A_X \circ \nabla_V - \nabla_V \circ A_X)Y.$$

Taking trace over V, we obtain

$$-Ric(X, Y) = tr(A_X A_Y) + div(A_X Y).$$

∎

Now in the equation, we take $Y = X$, we obtain

$$div(A_X X) = -Ric(X, X) - tr(A_X A_X),$$

where we note that since A_X is skew-symmetric, $tr(A_X A_X) \leq 0$ with "=" if and only if $A_X = 0$. With this understood, we have the following corollary.

Theorem 2.8. *Suppose (M, g) is a compact Riemannian manifold with negative (resp. vanishing) Ricci curvature. Then every Killing vector field $X \equiv 0$ (resp. is parallel).*

Theorem 2.8 has the following consequences: it is known that the group of isometries of a Riemannian manifold is a Lie group, with its Lie algebra being the space of Killing vector fields. Moreover, when the manifold is compact, the isometry group is also compact (cf. [5]). It follows immediately that the isometry group of a compact Riemannian manifold with negative Ricci curvature must be finite. Furthermore, the identity component of the isometry group of a compact Riemannian manifold with vanishing Ricci curvature must be a compact torus.

3. Geometrization in low dimensions

Broadly speaking, geometrization asks for a given manifold whether there exists a metric of certain specific geometric properties, e.g., of certain constant curvatures. Geometrization that has the most significant implications occurs in low dimensions, i.e., in dimensions 2 and 3.

Geometrization in dimension 2: Here the basic question asks that for a given compact closed, oriented 2-dimensional manifold, does there exist a metric of constant Gauss curvature? Note that the Gauss-Bonnet Theorem already gives some interesting topological constraints regarding the sign of the constant Gauss curvature, i.e., if the sign is positive, the surface has to be a 2-sphere; if the Gauss curvature is zero, the surface has to be a torus, and finally if the sign is negative, the surface has to be of genus greater than 1.

Theorem 3.1. *(Uniformization of 2-dimensional metrics)*

Let (M, g) be a compact closed, oriented 2-dimensional Riemannian manifold. There exists a smooth function u on M such that the new metric $\tilde{g} := e^{2u}g$ has a constant Gauss curvature.

The change of metrics from g to $\tilde{g} := e^{2u}g$ is called a *conformal change*. It has the following implications: recall that M is oriented, so for each $p \in M$, we pick a positively oriented orthonormal basis e_1, e_2 of T_pM, and define an endomorphism $J_p : T_pM \to T_pM$ by $J_p(e_1) = e_2$, $J_p(e_2) = -e_1$. One can check that J_p is independent of the choice of e_1, e_2, and together determines a smooth section J of $End(TM)$, satisfying $J^2 = -Id$. Such a J is called an *almost complex structure* on M, and by its construction, it is said to be *compatible* with the metric g. It is known that for any such J, there always exists a holomorphic coordinate $z = x + iy$ (only locally defined) such that $J(\partial_x) = \partial_y$ and $J(\partial_y) = -\partial_x$, making M a compact Riemann surface. Now note that a conformal change of metrics do not change the almost complex structure J. Hence we arrived at the following statement:

Let Σ be a compact Riemann surface. There exists a metric on Σ, compatible with the holomorphic structure on Σ, which has a constant Gauss curvature.

For a proof of Theorem 3.1, let us analyze how the Gauss curvature changes under a conformal change of metrics. Let $\tilde{g} = e^{2u}g$, and let $\tilde{K}, K$ be the Gauss curvatures of $\tilde{g}, g$ respectively. We pick a local orthonormal frame $\{E_1, E_2\}$ for (M, g), with the dual coframe denoted by $\{\phi^1, \phi^2\}$. Let ω_i^j, Ω_i^j be the corresponding connection 1-forms, and the curvature 2-forms. Then $d\phi^j = \sum_i \phi^i \wedge \omega_i^j$, and $\Omega_i^j = d\omega_i^j = -K\phi^i \wedge \phi^j$.

With this understood, we set $\tilde{\phi}^i := e^u \phi^i$. Then $\{\tilde{\phi}^1, \tilde{\phi}^2\}$ is a local orthonormal coframe with respect to the metric $\tilde{g}$. Let $\tilde{\omega}_i^j$, $\tilde{\Omega}_i^j$ be the corresponding connection 1-forms, and the curvature 2-forms. Then an easy calculation gives

$$\tilde{\omega}_i^j = \nabla_{E_i} u \cdot \phi^j - \nabla_{E_j} u \cdot \phi^i + \omega_i^j,$$

and

$$d\tilde{\omega}_i^j = (\sum_{k=1}^{2} \nabla_{E_k} \nabla_{E_k} u) \cdot \phi^i \wedge \phi^j + (\sum_{k=1}^{2} \nabla_{E_k} u \cdot \phi^k) \wedge \omega_i^j + d\omega_i^j.$$

Now for a pointwise calculation, we may assume $\omega_i^j = 0$, while with this assumption, we note that the expression $\sum_{k=1}^2 \nabla_{E_k}\nabla_{E_k} u$ is nothing but the negative of the Laplacian Δu of the function u (see Part 5: Hodge theory for more details). It follows easily that the function u obeys the following PDE:

$$\Delta u + K - e^{2u}\tilde{K} = 0.$$

It turns out that this equation always has a solution on a compact 2-dimensional manifold M. See [1].

Geometrization in dimension 3: Geometrization has played an essential role in the topological classification of 3-dimensional manifolds. We give a brief account below. (For simplicity, we only consider the orientable case.)

Let M be a compact closed, orientable 3-dimensional smooth manifold. (In dimension 3, the topological, smooth, and piecewise linear categories are all equivalent.) Then by a theorem of Kneser and Milnor, M can be decomposed into a finite connected sum of $M_1, M_2, \cdots, M_k$, where the manifolds M_i are unique up to order. Here the connected sum decomposition is called *prime*, as none of the M_i's can be further decomposed into a nontrivial connected sum. In the orientable case, it is known that each M_i is either $\mathbb{S}^1 \times \mathbb{S}^2$, or M_i is *irreducible*, meaning that any embedded 2-sphere in it bounds a 3-ball. In particular, $\pi_2(M_i) = 0$ if M_i is irreducible. Note that from the fundamental group point of view, $\pi_1(M)$ is a free product of $\pi_1(M_i)$'s.

It is clear then that our understanding of the classification of 3-manifolds hinges on the special case of irreducible 3-manifolds. In this regard, the next crucial piece is the so-called *JSJ decomposition*, named after Jaco, Shalen, and Johannson.

Let M be an irreducible 3-manifold. Then there exists a maximal set (maybe empty) of disjoint, embedded tori $\{S_i\}$ in M such that each S_i is *essential* in the sense that $\pi_1(S_i) \to \pi_1(M)$ is injective, and each S_i is *canonical* in the sense that it can be isotopic off any embedded torus in M. This set of tori $\{S_i\}$ decomposes M into pieces $M_1, M_2, \cdots, M_n$, where each M_k is a 3-manifold with boundary. Maximality of $\{S_i\}$ means that we can not further decompose each M_k nontrivially in the sense that any embedded torus in M_k must bound a product $[0,1] \times T^2$ with a boundary component of M_k. This decomposition is unique up to isotopy of S_i in M, and it is the so-called JSJ decomposition of M. Note that each S_i corresponds to a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ of $\pi_1(M)$ as $\pi_1(S_i) \to \pi_1(M)$ is injective.

So for the classification of 3-manifolds, it remains to understand the pieces in a JSJ decomposition. This is where the *Thurston's Geometrization Conjecture* (now a theorem) comes into play. It claims that each piece in a JSJ decomposition must possess a certain geometric structure, which we will explain below.

A *geometry* (in dimension 3) is a simply connected, complete, 3-dimensional Riemannian manifold X, which is *homogeneous, unimodular*, and *maximal* (we will explain in turn): X is homogeneous means that its isometry group $Isom(X)$ acts transitively, X is unimodular means that it has a quotient (by a discrete subgroup of $Isom(X)$) of finite volume, and finally, the maximality condition means that there is no $Isom(X)$-invariant Riemannian metric on X whose isometry group is strictly larger than $Isom(X)$. With this understood, we say a 3-manifold with boundary M

possess a geometric structure if there is a geometry X such that the interior of M is the quotient of X by a discrete subgroup of $Isom(X)$ acting freely.

With the preceding understood, Thurston's Geometrization reduces the problem of classifying 3-manifolds to that of classifying all the geometries in dimension 3 and understanding the corresponding isometry groups. It turns out that there are exactly 8 geometries: space forms $\mathbb{S}^3, \mathbb{R}^3, \mathbb{H}^3$ with constant sectional curvature $1, 0, -1$, two geometries $\mathbb{R} \times \mathbb{S}^2, \mathbb{R} \times \mathbb{H}^2$ with the product metric where $\mathbb{S}^2, \mathbb{H}^2$ are the space forms with constant Gauss curvature $1, -1$, and three other geometries, denoted by Nil, $\widetilde{SL_2}(\mathbb{R})$, and Sol, see e.g. [2] for more details.

4. Jacobi fields

Let $\gamma(t)$, $a \leq t \leq b$, be a geodesic segment in (M, g), and denote its tangent vector by $V(t)$. We consider a variation of $\gamma(t)$, which is a smooth map $\Gamma(s, t)$ into M, where $-\epsilon < s < \epsilon$, $a \leq t \leq b$, such that $\Gamma(0, t) = \gamma(t)$ for all t. We let $T(s, t)$, $S(s, t)$ to denote the vector fields $\partial_t \Gamma$, $\partial_s \Gamma$ along $\Gamma(s, t)$. (Note that $V(t) = T(0, t)$.) Then because $[T, S] = 0$, it follows easily that

$$R(T, S)T = \nabla_T \nabla_S T - \nabla_S \nabla_T T = \nabla_T \nabla_T S - \nabla_S \nabla_T T.$$

Now if for any fixed s, $\Gamma(s, t)$ is a geodesic (i.e., the variation of $\gamma(t)$ is by geodesics), then we have $\nabla_T T \equiv 0$. Setting $J(t) := S(0, t)$, which is the corresponding *variational vector field* along $\gamma(t)$, it is easy to see that $J(t)$ obeys the following equation:

$$\nabla^2_{V(t)} J(t) + R(J(t), V(t))V(t) = 0, \text{ where } a \leq t \leq b.$$

The above equation, which is a second order linear ODE in $J(t)$, is called the *Jacobi field equation*, and a solution $J(t)$ is called a *Jacobi field* along the geodesic $\gamma(t)$.

Exercise: Suppose X is a Killing vector field, and set $J(t) := X(\gamma(t))$ to be the restriction of X along a geodesic segment $\gamma(t)$. Prove that $J(t)$ is a Jacobi field.

By the existence and uniqueness theorem, for any initial value $J(a) := X \in T_{\gamma(a)}M$, $(\nabla_V J)(a) := Y \in T_{\gamma(a)}M$, there exists a unique Jacobi field $J(t)$ along $\gamma(t)$ satisfying the given initial value. Thus the set of Jacobi fields forms a $2n$-dimensional vector space canonically isomorphic to $T_{\gamma(a)}M \times T_{\gamma(a)}M$. Furthermore, for any real numbers α, β, the variation $\Gamma(s, t) := \gamma(t + s(t - a)\alpha + s\beta)$ is a geodesic variation. The corresponding Jacobi field is easily seen to be $J(t) = ((t - a)\alpha + \beta)V(t)$, in particular, $J(a) = \beta V(a)$, $(\nabla_V J)(a) = \alpha V(a)$.

Exercise: Let $J(t)$ be a Jacobi field. Verify that if $(\nabla_V J)(a)$ is orthogonal to $V(a)$, then $\nabla_V J$ is orthogonal to $V(t)$ for all t, and furthermore, if $J(a)$ is orthogonal to $V(a)$, then $J(t)$ is orthogonal to $V(t)$ for all t.

Conversely, given a Jacobi field $J(t)$ along $\gamma(t)$, we can realize $J(t)$ as the variational vector field of a geodesic variation of $\gamma(t)$ as follows. We pick a smooth curve $\phi(s)$, $-\epsilon < s < \epsilon$, such that $\phi(0) = \gamma(a)$, $\frac{d}{ds}\phi(0) = J(a)$, where $\frac{d}{ds}\phi(s)$ denotes the tangent vector of $\phi(s)$. We let $Y(s)$, $Z(s)$ be the vector fields along $\phi(s)$ which are parallel such that $Y(0) = (\nabla_V J)(a)$, $Z(0) = V(a)$, and consider the map

$$\Gamma(s,t) := exp_{\phi(s)}((t-a)(sY(s) + Z(s))).$$

Note that $\Gamma(0,t) = exp_{\gamma(a)}((t-a)V(a)) = \gamma(t)$, and for each fixed s, $\Gamma(s,t)$ is a geodesic parametrized by the variable t. Hence $\Gamma(s,t)$ is a geodesic variation of $\gamma(t)$. It remains to prove that the variational vector field of $\Gamma(s,t)$, denoted by $\tilde{J}(t)$, is the given Jacobi field $J(t)$. This can be verified by showing that they have the same initial values. To see this, first, we have $\tilde{J}(a) = \partial_s \Gamma(s,a)|_{s=0} = \frac{d}{ds}\phi(0) = J(a)$. Furthermore,

$$\nabla_V \tilde{J}|_{t=a} = \nabla_V(\partial_s \Gamma|_{s=0})|_{t=a} = \nabla_{\frac{d}{ds}\phi}(\partial_t \Gamma|_{t=a})|_{s=0} = \nabla_{\frac{d}{ds}\phi}(sY(s)+Z(s))|_{s=0} = (\nabla_V J)(a).$$

Hence the claim.

Jacobi fields and the exponential map: Suppose the exponential map exp_p, for a given point $p \in M$, is defined at a vector $V \in T_pM$. The pushforward $(exp_p)_* : T_V(T_pM) = T_pM \to T_qM$, where $q = exp_p(V)$, can be described in terms of Jacobi fields along the geodesic $\gamma(t) := exp_p(tV)$. To see this, let $W \in T_pM$ be any tangent vector. Then the curve $s \mapsto V + sW$ in T_pM has tangent vector $W \in T_V(T_pM) = T_pM$ at $s = 0$. Thus the image of W under the pushforward $(exp_p)_* : T_V(T_pM) \to T_qM$ is given by $\frac{d}{ds} exp_p(V + sW)|_{s=0}$. On the other hand, if $J(t)$ is the Jacobi field along $\gamma(t)$ associated to the geodesic variation $exp_p(t(V + sW))$, then

$$J(1) = \partial_s exp_p(t(V + sW))|_{s=0,t=1} = \frac{d}{ds} exp_p(V + sW)|_{s=0}.$$

Thus we obtain the following lemma.

Lemma 4.1. *The pushforward map $(exp_p)_* : T_V(T_pM) = T_pM \to T_qM$, where $q = exp_p(V)$, sends $W \in T_pM$ to $J(1) \in T_qM$, where $J(t)$ is the Jacobi field along $\gamma(t) := exp_p(tV)$ with initial values $J(0) = 0$, $(\nabla_V J)(0) = W$.*

Let $p, q \in M$ be two points connected by a geodesic $\gamma(t)$. We say that q *is conjugate to p along γ* if there is a Jacobi field along γ which vanishes at both p and q but is not identically zero. Lemma 4.1 says that $(exp_p)_* : T_V(T_pM) \to T_qM$ is an isomorphism if and only if q is not conjugate to p along $exp_p(tV)$.

Exercise: Let $\gamma(t)$ be a geodesic segment connecting p and q. Prove that if the sectional curvature is nonpositive along $\gamma(t)$ for any 2-dimensional plane containing the tangent vector of $\gamma(t)$, then p, q are not conjugate points.

With this, Lemma 4.1 implies easily the following theorem.

Theorem 4.2. *Let M be a complete, connected Riemannian manifold with nonpositive sectional curvature. Then for any $p \in M$, the exponential map $exp_p : T_pM \to M$ defines the universal covering map of M.*

Jacobi fields of constant sectional curvature metrics: Let (M, g) be a Riemannian manifold of constant sectional curvature K. It turns out we can explicitly compute the Jacobi fields along a geodesic in this case. Let $\gamma(t)$ be a geodesic segment with initial point $\gamma(0) = p$, defined for $t \geq 0$, such that without loss of generality, assuming its tangent vector $V(t)$ has unit length. Let $J(t)$ be a Jacobi field along

γ, where we assume $J(0) = 0$ and $(\nabla_V J)(0)$ is orthogonal to $V(0)$. Let $\{E_i(t)\}$ be the vector fields along γ which are parallel such that $\{E_i(0)\}$ forms an orthonormal basis of the hyperplane at p which is orthogonal to $V(0)$. Then for each t, $\{E_i(t)\}$ is orthonormal. If we write $J(t) = \sum_i \chi_i(t)E_i(t)$, then the functions $\chi_i(t)$ satisfy the following equations

$$\sum_i \frac{d^2\chi_i}{dt^2}(t) \cdot E_i(t) + \sum_i \chi_i(t) \cdot R(E_i(t), V(t))V(t) = 0.$$

In the case of constant sectional curvature, one can check that for each i,

$$R(E_i(t), V(t))V(t) = KE_i(t),$$

where K is the constant sectional curvature. It follows that for each i, $\chi_i(t)$ satisfies the following ODE

$$\frac{d^2\chi_i}{dt^2}(t) + K\chi_i(t) = 0.$$

With initial conditions $\chi_i(0) = 0$, $\frac{d\chi_i}{dt}(0) = 1$, the solution for $\chi_i(t)$ is given explicitly below:

- $\chi_i(t) = t$, when $K = 0$;
- $\chi_i(t) = \frac{1}{\sqrt{K}} \sin(t\sqrt{K})$, when $K > 0$;
- $\chi_i(t) = \frac{1}{\sqrt{|K|}} \sinh(t\sqrt{|K|})$, when $K < 0$.

The above explicit calculation allows us to determine the metric g explicitly in a normal neighborhood of p.

Theorem 4.3. *Let $(U, \phi = \{x^i\})$ be a normal neighborhood centered at p. Let $\bar{g}$ be the Euclidean metric in (x^i), and let $r = (\sum_{i=1}^n (x^i)^2)^{1/2}$ be the (radial) distance function to p. For any $q \in U \setminus \{p\}$, and any tangent vector $V \in T_qM$, we write $V = V^T + V^\perp$, an orthogonal decomposition where V^T is tangent to the sphere $r = const$ through q. Then the metric g can be written as below, where $r = d(p, q)$:*

- *$g(V, V) = |V^\perp|^2_{\bar{g}} + |V^T|^2_{\bar{g}}$, when $K = 0$;*
- *$g(V, V) = |V^\perp|^2_{\bar{g}} + \frac{1}{Kr^2} \sin^2(r\sqrt{K}) \cdot |V^T|^2_{\bar{g}}$, when $K > 0$;*
- *$g(V, V) = |V^\perp|^2_{\bar{g}} + \frac{1}{|K|r^2} \sinh^2(r\sqrt{|K|}) \cdot |V^T|^2_{\bar{g}}$, when $K < 0$.*

The following is the key to the proof of the above theorem.

Exercise: Let $(U, \phi = \{x^i\})$ be a normal neighborhood centered at p, and let γ be a radial geodesic starting at p. Prove that for any $W = \sum_i W^i\partial_i \in T_pM$, the Jacobi field $J(t)$ along γ with initial values $J(0) = 0$, $(\nabla_V J)(0) = W$ is given by $J(t) = \sum_i tW^i\partial_i$.

References

[1] M. S. Berger, *Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds*, J. Diff. Geom. **5** (1971), 325-332.

[2] M. Boileau, S. Maillot, and J. Porti, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses **15**, Soc. Math. France, Paris (2003).

[3] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I and II, John Wiley & Sons.
[4] J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature*, GTM **176**, Springer.
[5] Myers and Steenrod, *The group of isometries of a Riemannian manifold*, Ann. of Math. (2) **40** (1939), no. 2, 400–416.
[6] C.H. Taubes, *Differential Geometry: Bundles, Connections, Metrics and Curvature*, Oxford Grad. Texts in Math. **23**, Oxford Univ. Press, 2011.

University of Massachusetts, Amherst.
E-mail: wchen@math.umass.edu