MATH 704: PART 2: THE CHERN-WEIL THEORY

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Contents

1. The fundamental construction

Let G be a Lie group. For any $k > 0$, let $I^k(G)$ be the space of symmetric multilinear maps $f : Lie(G) \times Lie(G) \times \cdots \times Lie(G) \to \mathbb{R}$, which are Ad-invariant, i.e.,

$$f(Ad(a)t_1, Ad(a)t_2, \cdots, Ad(a)t_k) = f(t_1, t_2, \cdots, t_k)$$

for any $a \in G$. Set $I^0(G) = \mathbb{R}$, and $I(G) = \oplus_{k=0}^{\infty} I^k(G)$. Then $I(G)$ can be made into a commutative algebra by defining a product $f \cdot g$ through symmetrization, as follows: for any $f \in I^k(G)$, $g \in I^l(G)$,

$$f \cdot g(t_1, t_2, \cdots, t_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} f(t_{\sigma(1)}, \cdots, t_{\sigma(k)}) g(t_{\sigma(k+1)}, \cdots, t_{\sigma(k+l)}).$$

Fix a principal G-bundle P over M, and consider the associated bundle of Lie algebra $P \times_{Ad} Lie(G)$. Since each $f \in I^k(G)$ is Ad-invariant, it is easy to see that for any smooth sections $s_1, s_2, \cdots, s_k \in \Gamma(P \times_{Ad} Lie(G))$, $f(s_1, s_2, \cdots, s_k)$ defines a smooth function on M. In fact, we can even allow each s_i to be a differential form valued section. This can be done first for decomposable elements and then extend it by linearity. More precisely, let η_i, $i = 1, 2, \cdots, k$, be a differential r_i-form on M, where at most one of r_i's is odd. Then we define

$$f(s_1 \otimes \eta_1, s_2 \otimes \eta_2, \cdots, s_k \otimes \eta_k) = f(s_1, s_2, \cdots, s_k)\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_k.$$

With the preceding understood, we pick a connection form ω on P, and let Ω denote the curvature of ω, here viewed as a smooth section of the bundle $P \times_{Ad} Lie(G) \otimes \Lambda^2 M$. With this understood, for any $f \in I^k(G)$, $f(\Omega, \Omega, \cdots, \Omega)$ is a $2k$-form on M. Here is the main theorem of this section.

Theorem 1.1. *The $2k$-form $f(\Omega, \Omega, \cdots, \Omega)$ is closed, and its de Rham cohomology class is independent of the choice of the connection form ω. Denote the de Rham cohomology class of $f(\Omega, \Omega, \cdots, \Omega)$ by $f(P) \in H^{2k}_{dR}(M)$. Then $f \mapsto f(P)$ defines an algebra homomorphism from $I(G)$ to $H^{even}_{dR}(M)$.*

Proof. We first show that $f(\Omega,\Omega,\cdots,\Omega)$ is closed. To this end, let ∇ be the covariant derivative on $\Gamma(P\times_{Ad} Lie(G))$ associated to the connection form ω, and let d_∇ be the corresponding exterior covariant derivative. Then

$$d\, f(\Omega,\Omega,\cdots,\Omega)=f(d_\nabla\Omega,\cdots,\Omega)+\cdots+f(\Omega,\cdots,d_\nabla\Omega)=0,$$

because of the Bianchi identity $d_\nabla\Omega=0$.

To see that the de Rham cohomology class of $f(\Omega,\Omega,\cdots,\Omega)$ is independent of the choice of the connection form ω, we let ω_0,ω_1 be any two connection forms on P. Then $\omega_1-\omega_0=\alpha$, where α is a smooth section of $P\times_{Ad} Lie(G)\otimes T^*M$. We introduce, for $t\in[0,1]$, $\omega_t=\omega_0+t\alpha$, and let Ω_t be the curvature of ω_t, and denote by d_{∇_t} the exterior covariant derivative associated to ω_t.

With this understood, we claim $\partial_t\Omega_t=d_{\nabla_t}\alpha$, for any $t\in[0,1]$. This can be easily verified by computing locally. Suppose P is trivial over U, and let ω_t be the $Lie(G)$-valued 1-form on U. Then on U, $\Omega_t=d\omega_t+\frac{1}{2}[\omega_t,\omega_t]$. Consequently, on U,

$$\partial_t\Omega_t=d\alpha+\frac{1}{2}[\alpha,\omega_t]+\frac{1}{2}[\omega_t,\alpha]=d\alpha+[\omega_t,\alpha]=d_{\nabla_t}\alpha,$$

which verifies the claim $\partial_t\Omega_t=d_{\nabla_t}\alpha$.

Now we observe

$$\partial_t f(\Omega_t,\cdots,\Omega_t)=f(\partial_t\Omega_t,\cdots,\Omega_t)+\cdots+f(\Omega_t,\cdots,\partial_t\Omega_t)=kf(\Omega_t,\cdots,d_{\nabla_t}\alpha).$$

It follows immediately that

$$f(\Omega_1,\cdots,\Omega_1)-f(\Omega_0,\cdots,\Omega_0)=d\Phi,$$

where $\Phi:=\int_0^1 kf(\Omega_t,\Omega_t,\cdots,\alpha)dt$ is a $(2k-1)$-form on M. This shows that the de Rham cohomology class of $f(\Omega,\Omega,\cdots,\Omega)$ is independent of the choice of the connection form ω. The claim that $f\mapsto f(P)$ defines an algebra homomorphism from $I(G)$ to $H^{even}_{dR}(M)$ is straightforward from the definition.

□

2. Invariant polynomials

In this section, we analyze the algebraic structure of $I(G)$ for the case where G is a compact Lie group. To begin with, we shall first identify elements of $I(G)$ with polynomial functions on the Lie algebra $Lie(G)$ in a canonical way. To this end, we shall fix a basis $t_1,t_2,\cdots,t_n$ of $Lie(G)$, and denote by $\xi_1,\xi_2,\cdots,\xi_n$ the dual basis. Denote by $P^k(G)$ the set of Ad-invariant, homogeneous polynomials in $\xi_1,\xi_2,\cdots,\xi_n$ of degree k. (Elements of $P^k(G)$ are called *invariant polynomials.*) Then for each $f\in I^k(G)$, $P_f(t)=f(t,t,\cdots,t)$, $t\in Lie(G)$, defines an element of $P^k(G)$: if we write $t=a_1t_1+a_2t_2+\cdots+a_nt_n$, where $a_i=\xi_i(t)$, then $P_f(t)$ is a homogeneous polynomial of degree k in $a_1,a_2,\cdots,a_n$. Hence $P_f\in P^k(G)$. Note that we can recover f from P_f by the following formula:

$$f(t_{i_1},t_{i_2},\cdots,t_{i_k})=\frac{1}{k!}\frac{\partial^k}{\partial\xi_{i_1}\partial\xi_{i_2}\cdots\partial\xi_{i_k}}P_f.$$

In particular, the map $f \mapsto P_f$ is injective. This map is also surjective, as for any $P \in P^k(G)$, where

$$P = \sum_{i_1, i_2, \cdots, i_k = 1}^{n} a_{i_1 i_2 \cdots i_k} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k},$$

with $a_{i_1 i_2 \cdots i_k}$ being symmetric in $i_1, i_2, \cdots, i_k$, we can define an element $f \in I^k(G)$, by

$$f(s_1, s_2, \cdots, s_k) := \sum_{i_1, i_2, \cdots, i_k = 1}^{n} a_{i_1 i_2 \cdots i_k} \xi_{i_1}(s_1) \xi_{i_2}(s_2) \cdots \xi_{i_k}(s_k).$$

Then $P = P_f$. Let $P^0(G) = \mathbb{R}$, and $P(G) = \oplus_{k=0}^{\infty} P^k(G)$. Then $I(G)$ and $P(G)$ are isomorphic as commutative algebras under $f \mapsto P_f$.

With the preceding understood, we shall analyze the structure of $I(G)$ by choosing a suitable Lie subgroup G' of G and look at the restriction map $I(G) \to I(G')$. The first question is when the restriction map is injective. It is easy to check that under the following condition, the map $I(G) \to I(G')$ is injective, i.e., if for any $t \in Lie(G)$, there is an element $a \in G$, such that $Ad(a)(t) \in Lie(G')$. This is because, if $f \in I^k(G)$ restricts to $I^k(G')$ to zero, i.e., $f(t', t', \cdots, t') = 0$ for any $t' \in Lie(G')$, then $f(t, t, \cdots, t) = 0$ for any $t \in Lie(G)$, as there is an element $a \in G$ such that $Ad(a)(t) \in Lie(G')$ and f is Ad-invariant. With this understood, we let N be the subgroup of G defined by the following property: for any $n \in N$, $Ad(n)(Lie(G')) = Lie(G')$. If we denote by $I_N(G')$ the subset of $I(G')$ consisting of elements which are invariant under $Ad(n)$, $\forall n \in N$, then it is easy to see that the image of $I(G)$ in $I(G')$ under the restriction map is contained in $I_N(G')$.

For the special case where G is a compact Lie group, we shall take G' to be a maximal torus of G. Recall that a maximal torus T of G is a compact, connected, Abelian Lie subgroup which is maximal in the sense that it is not contained in any larger torus in G. It is known that maximal tori are conjugate to each other in G. We fix a maximal torus T of G, and let N be the normalizer of T in G, i.e.,

$$N = \{g \in G | g \cdot T \cdot g^{-1} = T\}.$$

With this understood, the quotient group $W = N/T$ is called the *Weyl group* of G.

The following two properties of maximal torus are crucial in our consideration here:

(1) For any element $a \in G$, there is a $b \in G$ such that $bab^{-1} \in T$.
(2) For any $t, t' \in T$, if t, t' are conjugate in G, then they are also conjugate by an element of N.

Theorem 2.1. *Suppose G is compact. Let T be a maximal torus of G. Then the restriction map $I(G) \to I_N(T)$ is an isomorphism.*

Note that since T is Abelian, the adjoint representation of T is trivial. Thus if $\xi_1, \xi_2, \cdots, \xi_n$ is a dual basis of $Lie(T)$, then $I(T)$ is identified with the algebra of polynomials in $\xi_1, \xi_2, \cdots, \xi_n$. Finally, $I_N(T)$ is simply the subalgebra consisting of elements of $I(T)$ which are invariant under the Weyl group.

Proof. It follows from property (1) of T, that for any $t \in Lie(G)$, there is an element $a \in G$ such that $Ad(a)(t) \in Lie(T)$. This shows that the restriction map $I(G) \to I(T)$

is injective. Furthermore, by the definition of N, $N = \{n \in G | Ad(n)(Lie(T)) = Lie(T)\}$. Hence the image of $I(G) \to I(T)$ lies in $I_N(T)$.

To see that $I(G) \to I_N(T)$ is surjective, let $f' \in I_N(T)$ be any element, where $f' \in I^k(T)$ for some k. We define $P \in P^k(G)$ as follows: for any $t \in Lie(G)$, we choose an element $a \in G$ such that $Ad(a)(t) \in Lie(T)$, and define $P(t) = f'(Ad(a)(t), Ad(a)(t), \cdots, Ad(a)(t))$. To see that $P(t)$ is independent of the choice of $a \in G$, suppose there is a $b \in G$ such that $Ad(b)(t) \in Lie(T)$. Then $Ad(a)(t), Ad(b)(t)$ are conjugate by an element of G. It follows easily from property (2) of T that there is an element $n \in N$, such that $Ad(n)Ad(a)(t) = Ad(b)(t)$. Since f' is $Ad(n)$-invariant, we have

$$f'(Ad(a)(t), Ad(a)(t), \cdots, Ad(a)(t)) = f'(Ad(b)(t), Ad(b)(t), \cdots, Ad(b)(t)).$$

Hence $P(t)$ is well-defined. Then there is a $f \in I(G)$ such that $P = P_f$, and it follows that f restricts to f' under the map $I(G) \to I_N(T)$. This proves the subjectivity.

□

In the remaining part of this section, we shall examine the cases where $G = U(n)$, $O(n)$, and $SO(n)$.

The case $G = U(n)$: In this case, the maximal torus is the n-fold product $T = U(1) \times U(1) \times \cdots \times U(1)$, i.e., the diagonal matrices $diag(e^{i\xi_1}, e^{i\xi_2}, \cdots, e^{i\xi_n})$. The corresponding Lie algebra consists of diagonal matrices $diag(i\xi_1, i\xi_2, \cdots, i\xi_n)$. With this understood, $I(T)$ can be identified with the algebra of polynomials in $\xi_1, \xi_2, \cdots, \xi_n$. On the other hand, the Weyl group of $U(n)$ is the symmetric group S_n acting on the set $(\xi_1, \xi_2, \cdots, \xi_n)$ as permutations. It follows easily that $I_N(T)$ consists of symmetric polynomials in $\xi_1, \xi_2, \cdots, \xi_n$. With this understood, we introduce polynomial functions $f_1, f_2, \cdots, f_n$ by the following equation

$$\det(\lambda I_n + iX) = \lambda^n - f_1(X)\lambda^{n-1} + f_2(X)\lambda^{n-2} + \cdots + (-1)^n f_n(X), X \in Lie(U(n)).$$

Theorem 2.2. *For $G = U(n)$, the polynomial functions $f_1, f_2, \cdots, f_n$ defined above are algebraically independent elements in $I(G)$ and together they generate $I(G)$.*

Proof. If we restrict $f_1, f_2, \cdots, f_n$ to $Lie(T)$, i.e., assume $X = diag(i\xi_1, i\xi_2, \cdots, i\xi_n)$, then $f_1, f_2, \cdots, f_n$ are precisely the elementary symmetric polynomials in $\xi_1, \xi_2, \cdots, \xi_n$. Since the restriction map $I(G) \to I_N(T)$ is isomorphic and $I_N(T)$ consists of symmetric polynomials in $\xi_1, \xi_2, \cdots, \xi_n$, it follows immediately that $f_1, f_2, \cdots, f_n$ are algebraically independent elements in $I(G)$ and together they generate $I(G)$.

□

The case $G = O(2m+1)$, $O(2m)$, or $SO(2m+1)$: In this case, the maximal torus is the m-fold product $T = SO(2) \times SO(2) \times \cdots \times SO(2)$. Write the Lie algebra elements of the i-th factor as

$$\begin{pmatrix} 0 & -\xi_i \\ \xi_i & 0 \end{pmatrix},$$

where $i = 1, 2 \cdots, m$, then $I(T)$ can be identified with the algebra of polynomials in $\xi_1, \xi_2, \cdots, \xi_m$. The Weyl group is generated by permutations of $\xi_1, \xi_2, \cdots, \xi_m$ plus reflections $(\xi_1, \xi_2, \cdots, \xi_i, \cdots, \xi_m) \mapsto (\xi_1, \xi_2, \cdots, -\xi_i, \cdots, \xi_m)$ for $i = 1, 2, \cdots, m$. It follows easily that $I_N(T)$ consists of symmetric polynomials in $\xi_1^2, \xi_2^2, \cdots, \xi_m^2$. With this understood, we introduce polynomial functions $f_1, f_2, \cdots, f_m$ on $Lie(G)$ by the following formula:

$$\det(\lambda I_n - X) = \lambda^n + f_1(X)\lambda^{n-2} + f_2(X)\lambda^{n-4} + \cdots, \text{ where } X \in Lie(G).$$

Here $G = O(n)$ where $n = 2m+1$ or $2m$, or $G = SO(n)$ for $n = 2m+1$. Note that since $X^T = -X$, the right hand side of the above equation does not contain terms with $\lambda^{n-1}, \lambda^{n-3}, \cdots$. Note that the restrictions of $f_1, f_2, \cdots, f_m$ to $I_N(T)$ are precisely the elementary symmetric polynomials in $\xi_1^2, \xi_2^2, \cdots, \xi_m^2$. The following theorem follows by a similar argument as in the case of $G = U(n)$.

Theorem 2.3. *For $G = O(2m+1)$, $O(2m)$, or $SO(2m+1)$, the polynomial functions $f_1, f_2, \cdots, f_m$ defined above are algebraically independent elements in $I(G)$ and together they generate $I(G)$.*

The case $G = SO(2m)$: In this case, the maximal torus is still the m-fold product $T = SO(2) \times SO(2) \times \cdots \times SO(2)$, however, the Weyl group is different. For $G = SO(2m)$, the Weyl group is generated by permutations of $\xi_1, \xi_2, \cdots, \xi_m$ plus automorphisms $(\xi_1, \xi_2, \cdots, \xi_i, \cdots, \xi_j, \cdots, \xi_m) \mapsto (\xi_1, \xi_2, \cdots, -\xi_i, \cdots, -\xi_j, \cdots, \xi_m)$ for any $i < j$. It follows easily then, that for any $f \in I_N(T)$, there are symmetric polynomials p, q in $\xi_1^2, \xi_2^2, \cdots, \xi_m^2$, such that

$$f = p + \xi_1\xi_2\cdots\xi_m \cdot q.$$

On the other hand, note that $f_m = (\xi_1\xi_2\cdots\xi_m)^2$. It follows easily that $f_1, f_2, \cdots, f_{m-1}$ and $\xi_1\xi_2\cdots\xi_m$ are algebraically independent in $I_N(T)$ and together they generate $I_N(T)$.

We define a polynomial function g on $Lie(SO(2m))$ by the following formula: for $X = (x_{ij})$ where $x_{ij} = -x_{ji}$, we set

$$g(X) = \frac{1}{2^m m!} \sum \epsilon_{i_1 i_2 \cdots i_{2m-1} i_{2m}} x_{i_1 i_2} x_{i_3 i_4} \cdots x_{i_{2m-1} i_{2m}},$$

where the sum is taken over all permutations $(i_1, i_2, \cdots, i_{2m-1} i_{2m})$ of $(1, 2, \cdots, 2m-1, 2m)$, and $\epsilon_{i_1 i_2 \cdots i_{2m-1} i_{2m}} = 1$ or -1 is the sign of the permutation. One can check that g is Ad-invariant, hence defines an element in $I(G)$. Moreover, the restriction of g to $Lie(T)$ equals $(-1)^m \xi_1\xi_2\cdots\xi_m$; in particular, this implies that $f_m = g^2$ in $I(G)$. The following theorem is straightforward.

Theorem 2.4. *For $G = SO(2m)$, the polynomial functions $f_1, f_2, \cdots, f_{m-1}, g$ defined above are algebraically independent elements in $I(G)$ and together they generate $I(G)$.*

3. Chern classes, Pontrjagin classes, and Euler class

Chern classes: Let E be a complex vector bundle of rank n over a smooth manifold M, and let P be the associated frame bundle, which is a principal G-bundle with $G = GL(n, \mathbb{C})$. We define polynomial functions $f_0, f_1, \cdots, f_n$ on $Lie(G)$ by the following formula:

$$\det(\lambda I_n + \frac{i}{2\pi} X) = \sum_{k=0}^{n} f_k(X)\lambda^{n-k}, \text{ where } X \in Lie(G).$$

Here $X \in Lie(G)$ means that it is a $n \times n$ complex-valued matrix. It is clear that $f_0 = 1$, and for $k > 0$, each f_k is Ad-invariant, hence defines an element of $I(G) \otimes \mathbb{C}$. We define the *k-th Chern class* of the complex vector bundle E, denoted by $c_k(E)$, to be the de Rham cohomology class of $f_k(\Omega)$ where Ω is the curvature of any chosen connection form on P. We remark that since P always admits a $U(n)$-reduction, and if we choose a $U(n)$-connection then the curvature Ω obeys $\Omega^T = -\overline{\Omega}$, It follows immediately from Theorem 1.1 that the de Rham cohomology class of $f_k(\Omega)$ is real valued, i.e., $c_k(E) \in H^{2k}_{dR}(M)$. With this understood, we define the *total Chern class* of E to be

$$c(E) := \sum_{k=0}^{n} c_k(E).$$

Note that the total Chern class $c(E)$ is the de Rham cohomology class of

$$\det(I_n + \frac{i}{2\pi}\Omega),$$

where Ω is the curvature of any chosen connection form on P.

The Chern classes satisfy the following axioms (these axioms give an axiomatic definition of Chern classes).

(1) (Naturality) Let E be a complex vector bundle over M, $f : M' \to M$ be a smooth map, and E' be the pull-back bundle of E by f. Then $f^*c(E) = c(E')$.

(2) (Whitney sum formula) Let $E_1, E_2, \cdots, E_n$ be complex line bundles over M and E is the direct sum of $E_1, E_2, \cdots, E_n$. Then

$$c(E) = c(E_1) \wedge c(E_2) \wedge \cdots \wedge c(E_n).$$

For example, suppose E is the direct sum of complex line bundles E_1, E_2. Then the Whitney sum formula implies that $c_1(E) = c_1(E_1) + c_1(E_2)$, $c_2(E) = c_1(E_1) \wedge c_1(E_2)$.

Exercise: Prove the Whitney sum formula.

(3) (Normalization) Let E be the tautological line bundle over $\mathbb{CP}^1$. Then

$$\int_{\mathbb{CP}^1} c_1(E) = -1.$$

(Compare Example 3.6 in Part 1.)

Exercise: Let Σ be a compact Riemann surface, and let E be a complex line bundle over Σ. Suppose $s : \Sigma \to E$ is a smooth section of the bundle E, with only isolated zeros $a_1, a_2, \cdots, a_k$ in Σ. We define the index of each zero a_i, denoted by $Ind(a_i)$, as follows: at each a_i, we pick a small disk neighborhood D_i centered at a_i, over which E is trivial. Note that due to the triviality of E over D_i, the section s may be regarded as a map from D_i to $\mathbb{C}$. With this understood, we define $Ind(a_i)$ to be the degree of the map $\partial D_i \to \mathbb{S}^1$, sending $z \in \partial D_i$ to $s(z)/|s(z)| \in \mathbb{S}^1$. Prove that

$$\sum_{i=1}^{k} Ind(a_i) = \int_\Sigma c_1(E).$$

Remark 3.1. (1) The proof of the above equation is the baby version of the so-called intrinsic proof of the Gauss-Bonnet Theorem due to S.S. Chern.

The same argument applies to a slightly different situation: let E be a holomorphic line bundle over Σ, and let s be a meromorphic section of E. Denote by $a_1, \cdots, a_k$ the set of zeroes and poles of s, and let $Ind(a_i)$ be the multiplicity of a_i if a_i is a zero and the negative of the multiplicity if a_i is a pole. Then the following holds true:

$$\sum_{i=1}^{k} Ind(a_i) = \int_\Sigma c_1(E).$$

(2) If we identify Σ with the zero section of E and let Σ' be the graph of the smooth section s. Then when $Ind(a_i) = 1$ or -1 for all i, Σ and Σ' intersect transversely, and the intersection number $\Sigma \cdot \Sigma' = \sum_i Ind(a_i)$. This shows that the self-intersection number $\Sigma \cdot \Sigma$ of Σ in E is given by

$$\Sigma \cdot \Sigma = \int_\Sigma c_1(E).$$

(3) When $E = T\Sigma$ is the tangent bundle, it is known that the sum of indices $\sum_i Ind(a_i)$ equals the Euler characteristic of Σ, i.e., $\chi(\Sigma) = 2 - 2g_\Sigma = \sum_i Ind(a_i)$, where g_Σ is the genus of Σ. (This is a theorem of Hopf.) Hence

$$2 - 2g_\Sigma = \int_\Sigma c_1(T\Sigma).$$

Example 3.2. Let M be a complex surface and Σ be an embedded holomorphic curve in M. We denote by $TM|_\Sigma$ the pull-back bundle of TM by the embedding $\Sigma \to M$. Then it is easy to see that $TM|_\Sigma$ is the direct sum of $T\Sigma$ and the normal bundle ν_Σ of Σ in M. By the Whitney sum formula,

$$c_1(TM|_\Sigma) = c_1(T\Sigma) + c_1(\nu_\Sigma).$$

Pairing with the fundamental class of Σ, we obtain the so-called *adjunction formula*:

$$g_\Sigma = \frac{1}{2}(\Sigma \cdot \Sigma - c_1(TM) \cdot \Sigma) + 1.$$

There is a symplectic version of adjunction formula.

Exercise: Consider $M = \mathbb{CP}^2$. Use the adjunction formula to show that

$$c_1(T\mathbb{CP}^2) = 3 \cdot PD(\mathbb{CP}^1),$$

where $PD(\mathbb{CP}^1)$ is the Poincare dual of a complex line $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Then prove that if Σ is a smooth algebraic curve of degree d, then its genus g_Σ is given by the formula

$$g_\Sigma = \frac{1}{2}(d-1)(d-2).$$

Now we go back to the Chern-Weil construction in section 1. Suppose P is a principal G-bundle and $\rho : G \to G'$ is a Lie group homomorphism. Let P' be the induced bundle of P by ρ, i.e., $P' = P \times_\rho G'$. Suppose $f \in I^k(G)$ and $f' \in I^k(G')$ such that for any $t \in Lie(G)$,

$$f(t, t, \cdots, t) = f'(\rho_* t, \rho_* t, \cdots, \rho_* t).$$

Then it is clear from the Chern-Weil construction that the de Rham cohomology classes $f(P) = f'(P')$. Use this observation in the following exercise.

Exercise: Let E be a complex vector bundle of rank n, and let $\det : GL(n, \mathbb{C}) \to GL(1, \mathbb{C})$ be the Lie group homomorphism given by the map $A \mapsto \det A$. Then the induced bundle of E by det is called the *determinant line bundle* of E, and is denoted by $\det E$. Show that $c_1(E) = c_1(\det E)$.

Finally, we observe that since every complex vector bundle E admits a Hermitian metric, it can be reduced to a $U(n)$-bundle. It follows easily from Theorem 2.2 that, for every $f \in I(Gl(n, \mathbb{C})) \otimes \mathbb{C}$, the characteristic class $f(P)$ obtained from Chern-Weil construction can be expressed in terms of the Chern classes of E. With this understood, consider the element $ch \in I(Gl(n, \mathbb{C})) \otimes \mathbb{C}$, where for any complex-valued $n \times n$ matrix X,

$$ch(X) = \text{Trace}(e^{\frac{i}{2\pi} X}).$$

The corresponding characteristic class $ch(E)$ is called the *Chern character* of E.

Exercise: Let E be a complex vector bundle of rank n over a 4-dimensional manifold M. Show that

$$ch(E) = n + c_1(E) + \frac{1}{2} c_1(E)^2 - c_2(E).$$

Pontrjagin classes: Let E be a real vector bundle of rank n over M, and let P be the associated frame bundle, which is a principal G-bundle with $G = GL(n, \mathbb{R})$. We define polynomial functions $g_0, g_1, \cdots, g_n$ on $Lie(G)$ by the following formula:

$$\det(\lambda I_n - \frac{1}{2\pi} X) = \sum_{k=0}^{n} g_k(X) \lambda^{n-k}, \text{ where } X \in Lie(G).$$

Here $X \in Lie(G)$ means that it is a $n \times n$ (real-valued) matrix. It is clear that $g_0 = 1$, and for $k > 0$, each g_k is Ad-invariant, hence defines an element of $I(G)$.

For each $k = 1, 2, \cdots, m$, where $n = 2m$ or $n = 2m + 1$, the *k-th Pontrjagin class* of E, denoted by $p_k(E)$, is defined to be the de Rham cohomology class of $g_{2k}(\Omega)$, where Ω is the curvature of any chosen connection form on P. Note that for each k, $p_k(E) \in H^{4k}_{dR}(M)$.

Exercise: Let E be a real vector bundle of rank n over M, and let $E^c := E \otimes \mathbb{C}$ be the complexification of E, which is a complex vector bundle of rank n. Prove that for each k,

$$p_k(E) = (-1)^k c_{2k}(E^c) \in H^{4k}_{dR}(M).$$

The Euler class: Let E be an oriented real vector bundle over M of rank $2m$. We put a metric on E so that E becomes a $SO(2m)$-bundle. Let P be the associated frame bundle, which is a principal G-bundle with $G = SO(2m)$. Note that the isomorphism class of P is independent of the choice of the metric on E that we have chosen.

Pick a connection form ω on P and let Ω be its curvature. Then the *Euler class* of E, denoted by $\chi(E)$, is defined to be the de Rham cohomology class of $g(\frac{1}{2\pi}\Omega)$ in $H^{2m}_{dR}(M)$, where $g \in I(G)$ is the polynomial function in Theorem 2.4. For E an oriented real vector bundle of odd rank, we define $\chi(E) = 0$. One can easily check that the Euler class satisfies the Naturality Axiom and the Whitney sum formula.

Euler class and Chern class: Let E be a complex vector bundle of rank m over M, and let $E_{\mathbb{R}}$ denote the underlying oriented real vector bundle of rank $2m$. Then the Euler class of $E_{\mathbb{R}}$ and the top Chern class of E, i.e., $c_m(E)$ are equal in $H^{2m}_{dR}(M)$. To see this, note that under the canonical identification of $U(m)$ as a subgroup of $SO(2m)$, the maximal torus of $U(m)$ corresponds to the maximal torus of $SO(2m)$, under which the Lie algebras are identified by the maps

$$i\xi_i \mapsto \begin{pmatrix} 0 & -\xi_i \\ \xi_i & 0 \end{pmatrix}.$$

With this understood, the polynomial f_m used to define $c_m(E)$ and the polynomial g used to define $\chi(E_{\mathbb{R}})$ are related by the equation $f_m = \frac{1}{(2\pi)^m} g$. It follows immediately that $c_m(E) = \chi(E_{\mathbb{R}})$.

Example 3.3. Let E be a complex vector bundle of rank 2. Then the equation $(\xi_1 + \xi_2)^2 = 2\xi_1\xi_2 + \xi_1^2 + \xi_2^2$ gives the following relation among the characteristic classes

$$c_1(E)^2 = 2\chi(E_{\mathbb{R}}) + p_1(E_{\mathbb{R}}).$$

Now suppose E is the tangent bundle of a complex surface M. Then the above equation becomes

$$c_1(TM)^2 = 2\chi(TM) + p_1(TM).$$

Pairing with the fundamental class of M, and noting that $\chi(TM)[M] = \chi(M)$ (The Gauss-Bonnet Theorem) and $\frac{1}{3}p_1(TM)[M] = \sigma(M)$ (Hirzebruch's Signature Theorem), we obtain the well-known formula

$$c_1(TM)^2[M] = 2\chi(M) + 3\sigma(M).$$

There is a symplectic version of the above formula.

Exercise: Verify the formula $c_1(TM)^2[M] = 2\chi(M) + 3\sigma(M)$ for $M = \mathbb{CP}^2$.

Exercise: Let M be a compact, connected, oriented smooth 4-manifold with $\sigma(M) \neq 0$. Show that every $f \in \mathrm{Diff}(M)$ is orientation-preserving.

Exercise: Let E be a complex vector bundle. Denote by $\overline{E}$ the complex conjugate of E, which is defined as follows: consider E as the real vector bundle $E_{\mathbb{R}}$ equipped with a complex structure J, then $\overline{E}$ is the complex vector bundle obtained by equipping $E_{\mathbb{R}}$ with the complex structure $-J$. Prove that $c_k(\overline{E}) = (-1)^k c_k(E)$.

Exercise: Show that for any complex vector bundle E, one has $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = E \oplus \overline{E}$. Then use the relations $p_k(E_{\mathbb{R}}) = (-1)^k c_{2k}(E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$ to give a different proof for the formula in Example 3.3.

References

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