MATH 704: PART 1: PRINCIPAL BUNDLES AND CONNECTIONS

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Contents

1. Lie Groups

A Lie group G is a smooth manifold such that the multiplication map $G \times G \to G$, $(g,h) \mapsto gh$, and the inverse map $G \to G$, $g \mapsto g^{-1}$, are smooth maps. A Lie subgroup H of G is a subgroup of G which is at the same time an embedded submanifold. A Lie group homomorphism is a group homomorphism which is a smooth map between the Lie groups. The Lie algebra, denoted by $Lie(G)$, of a Lie group G consists of the set of left-invariant vector fields on G, i.e., $Lie(G) = \{X \in \mathcal{X}(G) | (L_g)_* X = X\}$, where $L_g : G \to G$ is the left translation $L_g(h) = gh$. As a vector space, $Lie(G)$ is naturally identified with the tangent space T_eG via $X \mapsto X(e)$. A Lie group homomorphism naturally induces a Lie algebra homomorphism between the associated Lie algebras. Finally, the universal cover of a connected Lie group is naturally a Lie group, which is in one to one correspondence with the corresponding Lie algebras.

Example 1.1. Here are some important Lie groups in geometry and topology.

- $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, where $GL(n,\mathbb{C})$ can be naturally identified as a Lie subgroup of $GL(2n,\mathbb{R})$.
- $SL(n,\mathbb{R})$, $O(n)$, $SO(n) = O(n) \cap SL(n,\mathbb{R})$, Lie subgroups of $GL(n,\mathbb{R})$.
- $SL(n,\mathbb{C})$, $U(n)$, $SU(n) = U(n) \cap SL(n,\mathbb{C})$, Lie subgroups of $GL(n,\mathbb{C})$.
- $Sp(2n)$, Lie subgroup of $GL(2n,\mathbb{R})$, defined as the subgroup preserving the standard symplectic form on $\mathbb{R}^{2n}$. $Sp(2n) \cap O(2n) = U(n)$, under the natural identification of $GL(n,\mathbb{C})$ as a subgroup of $GL(2n,\mathbb{R})$.
- $\mathbb{S}^1 \subset \mathbb{C}$, $\mathbb{S}^3 \subset \mathbb{H}$, the spin group $Spin(n)$, $n > 2$, which is the universal cover of $SO(n)$. Note that $Spin(n) \to SO(n)$ is a double cover as $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n > 2$; $Spin^c(n) = Spin(n) \times_{\{\pm 1\}} \mathbb{S}^1$.
- Lie groups of low dimensions: $\mathbb{S}^1 = SO(2) = U(1)$, $\mathbb{S}^3/\{\pm 1\} = SO(3)$, $\mathbb{S}^3 = SU(2) = Spin(3)$, $Spin^c(3) = U(2)$, $Spin(4) = \mathbb{S}^3 \times \mathbb{S}^3 = SU(2) \times SU(2)$.

1-parameter subgroups: Given any $X \in Lie(G)$, one can associate a 1-parameter subgroup of G to X as follows. Let $\phi_t^X : G \to G$ denote the flow generated by the vector field X. Then due to X being left-invariant, we have $L_g \circ \phi_t^X = \phi_t^X \circ L_g$ for any $g \in G$. This easily implies the following: (1) the flow ϕ_t^X is complete, i.e., it is defined for all $t \in \mathbb{R}$, (2) for any $s, t \in \mathbb{R}$, $\phi_s^X(e)\phi_t^X(e) = \phi_{s+t}^X(e)$, (3) the flow $\phi_t^X : G \to G$ is given by the right translation $R_{\phi_t^X(e)}$. By (2), $\phi_t^X(e) \in G$ is a 1-parameter subgroup of G, which we associate to $X \in Lie(G)$.

Example 1.2. Consider $G = GL(n,\mathbb{R})$. In this case, $T_eG = M(n,\mathbb{R})$ is the space of all $n \times n$ matrices, and $Lie(G)$ consists of maps $\tilde{A} : GL(n,\mathbb{R}) \to M(n,\mathbb{R})$, where $A \in M(n,\mathbb{R})$, and $\tilde{A} : X \in GL(n,\mathbb{R}) \mapsto XA \in M(n,\mathbb{R})$. Hence the flow ϕ_t^A generated by $A \in Lie(G)$ obeys the ODE $\frac{d}{dt}\phi_t^A = \phi_t^A A$, which implies easily that $\phi_t^A(e) = e^{tA}$. It is clear that one can replace G by any other matrix groups.

Exponential Map: We define $exp : Lie(G) \to G$ by $exp(X) = \phi_1^X(e)$. It is clear that $\phi_t^X(e) = exp(tX)$. One can check easily that $exp : Lie(G) \to G$ is a local diffeomorphism from a neighborhood of $0 \in Lie(G)$ onto a neighborhood of $e \in G$, with $d(exp)$ equaling identity at $0 \in Lie(G)$. By Example 1.2, for $G = GL(n,\mathbb{R})$, the exponential map exp is given by $exp(X) = e^X$ for any $X \in Lie(G)$.

The Adjoint Representation: For any $g \in G$, the map $Ad(g) : G \to G$ defined by $h \mapsto ghg^{-1}$ is an automorphism of G. There is the induced Lie algebra automorphism, which is also denoted by $Ad(g) : Lie(G) \to Lie(G)$ for simplicity. The corresponding homomorphism $Ad : G \to GL(Lie(G))$ sending g to $Ad(g)$ is called the *Adjoint Representation* of G. Note that Ad is a Lie group homomorphism. The corresponding Lie algebra representation is denoted by $ad : Lie(G) \to M(Lie(G))$, where $M(Lie(G))$ stands for the Lie algebra of $GL(Lie(G))$.

To determine Ad, we let $X \in Lie(G)$ and consider the corresponding 1-parameter subgroup $exp(tX)$. Then for any $g \in G$, $Ad(g) : Lie(G) \to Lie(G)$ is given by $X_e \in T_eG \mapsto \frac{d}{dt}(g \cdot exp(tX) \cdot g^{-1})|_{t=0} = \frac{d}{dt}(R_{g^{-1}}(\phi_t^X(g)))|_{t=0} = (R_{g^{-1}})_*(X_g) \in T_eG$. To determine ad, for any $X, Y \in Lie(G)$, $ad(X)(Y_e)$ is given by $\frac{d}{dt}(Ad(exp(tX))(Y_e))|_{t=0} = \frac{d}{dt}(R_{exp(-tX)})_*(Y_{exp(tX)})|_{t=0} = \frac{d}{dt}(\phi_{-t}^X)_*(Y_{\phi_t^X(e)})|_{t=0} = (L_XY)_e = [X,Y]_e$, which implies that $ad(X)(Y) = [X,Y]$ for any $X, Y \in Lie(G)$. Note that ad is a Lie algebra homomorphism, i.e., $ad([X,Y]) = ad(X)ad(Y) - ad(Y)ad(X)$, which is equivalent to the Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

Exercise: For $G = GL(n,\mathbb{R})$, show that $Ad(X)(A) = XAX^{-1}$ for any $X \in GL(n,\mathbb{R})$ and $A \in M(n,\mathbb{R})$, and $ad(A)(B) = AB - BA$ for any $A, B \in M(n,\mathbb{R})$.

The Canonical 1-Form: The canonical 1-form θ on G is the $Lie(G)$-valued, left invariant 1-form determined by $\theta(X) = X$ for any $X \in Lie(G)$. If we pick a basis $\{X_i\}$ of $Lie(G)$ and let $\{\theta^i\}$ be the dual basis of $Lie(G)^*$, then $\theta = \sum_i \theta^i X_i$. Let $[X_j, X_k] = \sum_i c_{jk}^i X_i$ (here c_{jk}^i are called the *structure constants* which completely determine the Lie bracket). Then $d\theta^i = -\frac{1}{2}\sum_{j,k} c_{jk}^i \theta^j \wedge \theta^k$, called the *Maurer-Cartan equation.*

Exercise: Show that for $G = GL(n, \mathbb{R})$, the canonical 1-form θ is given by $A \mapsto A^{-1}dA$, $A \in GL(n, \mathbb{R})$.

Let $\rho : G \to G'$ be any Lie group homomorphism, and let θ, θ' be the canonical 1-forms on G, G' respectively. We denote by $\rho^*\theta'$, $\rho_*\theta$ the $Lie(G')$-valued 1-forms on G defined as follows: for any tangent vector X on G,

$$(\rho^*\theta')(X) = \theta'(\rho_*(X)), \;\; (\rho_*\theta)(X) = \rho_*(\theta(X)).$$

Then by taking X to be a left-invariant vector field on G, it follows easily that

$$\rho^*\theta' = \rho_*\theta.$$

For reference see Chapter 20 of J. Lee [2]

2. Principal Bundles

Fix a smooth manifold M (assume M is connected without loss of generality) and a Lie group G, a *principal G-bundle over M* is a smooth manifold P with a surjective smooth map $\pi : P \to M$ (called *projection*), which satisfies the following conditions:

- The space P is equipped with a smooth, free, right action of G: $P \times G \to P$, denoted by $(p, g) \mapsto pg$.
- For any $(p, g) \in P \times G$, $\pi(pg) = \pi(p)$, and furthermore, π induces a diffeomorphism between the quotient space P/G and M.
- P is *locally trivial*, i.e., for any $x \in M$, there is a neighborhood U and a G-equivariant diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times G$, $\varphi_U(p) = (\pi(p), \psi(p))$, such that $\varphi_U(pg) = (\pi(p), \psi(p)g)$.

The space P is called the *total space*, M is called the *base*, and G is called the *the structure group* of the principal G-bundle. Two principal G-bundles P, P' over M are called *isomorphic* if there is a G-equivariant diffeomorphism from P to P' which induces the identity map on the base M. (Note: P being locally trivial is equivalent to the G-action being proper.)

An alternative definition via transition functions: Suppose $\{U_\alpha\}$ is an open cover of M such that P is trivial over each U_α, with a trivialization $\varphi_{U_\alpha} : \pi^{-1}(U_\alpha) \to U_\alpha \times G$. For any α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, $\varphi_{U_\beta} \circ \varphi_{U_\alpha}^{-1} : U_\alpha \cap U_\beta \times G \to U_\alpha \cap U_\beta \times G$ is given by $(x, g) \mapsto (x, \varphi_{\beta\alpha}(x)g)$ for some smooth map $\varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$. It is easy to check that $\{\varphi_{\beta\alpha}\}$, called the *transition functions associated to* $\{U_\alpha\}$, satisfy the *cocycle conditions* $\varphi_{\gamma\beta}(x)\varphi_{\beta\alpha}(x) = \varphi_{\gamma\alpha}(x)$ for any $x \in U_\alpha \cap U_\beta \cap U_\gamma$. On the other hand, if we are given with an open cover $\{U_\alpha\}$ of M and a set of smooth functions $\{\varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ satisfying the cocycle conditions $\varphi_{\gamma\beta}(x)\varphi_{\beta\alpha}(x) = \varphi_{\gamma\alpha}(x)$ for any $x \in U_\alpha \cap U_\beta \cap U_\gamma$, we can construct a principal G-bundle over M canonically as follows: let $P = \sqcup_\alpha U_\alpha \times G / \sim$, where for any $x \in U_\alpha \cap U_\beta$, $(x, g) \in U_\alpha \times G \sim (x, \varphi_{\beta\alpha}(x)g) \in U_\beta \times G$. Then P is a smooth manifold with a natural projection $\pi : P \to M$, induced by the projections $U_\alpha \times G \to U_\alpha$, and a canonical smooth, free, right G-action on P, induced by the right translations of G on the G-factor of $U_\alpha \times G$, making P a principal G-bundle over M.

Example 2.1. (1) (Trivial bundles) $P = M \times G$.

(2) (Frame bundles) Let $E \to M$ be a smooth rank n real vector bundle. We define the bundle of frames of E as follows: for any $x \in M$, let P_x be the set of bases $(e_1, e_2, \cdots, e_n)$ of the vector space E_x, the fiber of E at x, and set $P := \sqcup_{x \in M} P_x$, with a natural projection $\pi : P \to M$ such that $\pi^{-1}(x) = P_x$, $\forall x \in M$. Let $G = GL(n, \mathbb{R})$. Then P admits a natural free right G-action, sending any basis $(e_1, e_2, \cdots, e_n) \in P_x$ to $(e_1, e_2, \cdots, e_n)A \in P_x$ under the action of $A \in GL(n, \mathbb{R})$. To give a smooth structure to P, for any open set U of M over which E admits a local frame $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$, we define a map $\phi_{U,\sigma} : U \times G \to P$, sending $(x, A) \in U \times G$ to $(\sigma_1(x), \cdots, \sigma_n(x))A \in P_x$, which is clearly one to one, and is onto the image $\pi^{-1}(U) \subset P$. Using $\{\phi_{U,\sigma}\}$, we can define a smooth structure on P with respect to which the G-action is smooth, $\pi : P \to M$ is smooth, inducing a diffeomorphism between the quotient space P/G and M, and furthermore, P is locally trivial, with local trivializations given by $\phi_{U,\sigma}^{-1} : \pi^{-1}(U) \to U \times G$. We remark that in terms of transition functions, P and E correspond to the same set of transition functions. Similar arguments apply to the case where E is a complex vector bundle, in which case the frame bundle is a principal $GL(n, \mathbb{C})$-bundle. If E is a real vector bundle equipped with a metric, then the bundle of orthonormal frames of E is naturally a principal $O(n)$-bundle.

(3) (Hopf fibration) Let $\mathbb{S}^3$ be the unit sphere in $\mathbb{C}^2$, and let $\pi : \mathbb{S}^3 \to \mathbb{CP}^1$ be the map which sends $p \in \mathbb{S}^3$ to the complex line in $\mathbb{C}^2$ which contains p. Then π is a smooth surjective map. There are two free, right $\mathbb{S}^1$-actions on $\mathbb{S}^3$, defined by sending $(p, \lambda) \in \mathbb{S}^3 \times \mathbb{S}^1$ to $p\lambda$ and $p\lambda^{-1}$ in $\mathbb{S}^3$ respectively. These two $\mathbb{S}^1$-actions define $\mathbb{S}^3$ as a principal $\mathbb{S}^1$-bundle over $\mathbb{CP}^1$ in two different ways (note that the local triviality of the principal bundles follows automatically from the fact that $\mathbb{S}^1$ is a compact Lie group).
Question: *Are these two principal $\mathbb{S}^1$-bundles over $\mathbb{CP}^1$ isomorphic?*

Fiber bundles associated to a principal bundle: Let P be a principal G-bundle over M, and let F be a smooth manifold equipped with a smooth, left G-action. Then on $P \times F$ we can define a smooth, free, left G-action as follows: for any $(p, y) \in P \times F$ and $g \in G$, $g \cdot (p, y) = (pg^{-1}, gy) \in P \times F$. One can check easily that this G-action is proper, hence the quotient space is naturally a smooth manifold, denoted by $P \times_G F$. Then $P \times_G F$ is naturally a fiber bundle (locally trivial) over M with fiber diffeomorphic to F. Finally, if F possesses some additional "structure" which is preserved under the G-action on F, then the fibers of $P \times_G F$ will inherit the "structure" from F.

Here are some important examples of this construction.

Example 2.2. (1) Consider the case where $F = V$ is a finite dimensional vector space, with a given linear representation $\rho : G \to GL(V)$. In this case, G naturally acts on V by $(g, v) \mapsto \rho(g)v$. For example, $F = \mathbb{R}^n$ and $\rho : G \to GL(n, \mathbb{R})$. Under this assumption, the associated fiber bundle, denoted by $E = P \times_\rho F$, is a smooth real vector bundle of rank n, and moreover, in terms of transition functions, E is given by $\{\rho \circ \varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$ where $\{\varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ is the set of transition functions of P associated to a cover $\{U_\alpha\}$ of M. Note that in some sense, this is the inverse procedure of the frame bundle construction described in Example 2.1(2).

(2) Suppose $F = G'$ is a Lie group and $\rho : G \to G'$ is a Lie group homomorphism. Then there is an induced smooth left G-action on G' by $(g, g') \mapsto \rho(g)g'$. Note that this G-action on G' commutes with the right G'-action on G' by right translations.

In this case, the associated fiber bundle, denoted by $P' := P \times_\rho F$, is naturally a principal G'-bundle over M, called the *induced bundle of P by ρ*. We remark that if $\{\varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ is the set of transition functions of P associated to a cover $\{U_\alpha\}$, then the set of transition functions of P' associated to $\{U_\alpha\}$ is $\{\rho \circ \varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to G'\}$. Finally, note that there is a canonical smooth map $f : P \to P'$, defined as a composition $P \to P \times G' \to P \times_G G'$, where $P \to P \times G'$ is by $p \mapsto (p, e)$. Then it is easy to see that f is fiber-preserving, equivariant with respect to $\rho : G \to G'$ (i.e., $f(pg) = f(p)\rho(g)$), and induces the identity map on M.

When $\rho : G \to G'$ is some canonical surjective homomorphism, i.e., $\rho : Spin(n) \to SO(n)$, P is called a *lifting* of P', and when $\rho : G \to G'$ defines G as a Lie subgroup of G', P is called a *reduction* of P'. Note that in the latter case, $f : P \to P'$ is an embedding, which is equivariant with respect to the right G-actions on P, P'.

(3) Consider the case $F = G/H$ is a homogeneous space, where P is a principal G-bundle and H is a Lie subgroup of G. Recall that $G/H = \{gH | g \in G\}$ is the set of right H-cosets, so there is a natural smooth left G-action on G/H, given by $(g, g'H) \mapsto gg'H$. Set $E := P \times_G G/H$.

There is a canonically defined smooth, surjective map $\pi : P \to E$, as the composition of maps $P \to P \times G/H \to E$, where the first map is the embedding $p \mapsto (p, eH)$ and the second map is the natural projection $P \times G/H \to P \times_G G/H$ to the quotient space. Since the first map $P \to P \times G/H$ is equivariant with respect to the H-actions (the H-action is from the right on P and from the left on $P \times G/H$), it follows easily that $\pi(ph) = \pi(p)$ for any $p \in P$ and $h \in H$. Note that the right H-action on P is smooth, free and proper, and note that π induces a diffeomorphism between the quotient manifold P/H and the associated fiber bundle E. In other words, with π, P becomes a principal H-bundle over E.

Pull-back bundles: Fix P, a principal G-bundle over M, and let N be a smooth manifold. For any smooth map $f : N \to M$, we can define a principal G-bundle P' over N, called the *pull-back bundle of P by f*, as follows: As a smooth manifold, P' is the submanifold of $N \times P$, consists of points (y, p) such that $f(y) = \pi(p)$ in M, where $\pi : P \to M$ is the bundle projection. Since $\pi : P \to M$ is a submersion, P' is a submanifold of $N \times P$ for any smooth map $f : N \to M$. The dimension of P' equals $\dim N + \dim P - \dim M = \dim N + \dim G$. To see P' is a principal G-bundle over N, we define $\pi' : P' \to N$ by sending $(y, p) \in P'$ to $y \in N$. Then it is clear that $(\pi')^{-1}(y) = P_{f(y)}$, the fiber of P at $f(y) \in M$. We define a right G-action on P' by $((y, p), g) \mapsto (y, pg)$, which is clearly smooth and free, and with $\pi' : P' \to N$ inducing a diffeomorphism between P'/G and N. Finally, note that if P is trivial over $U \subset M$, then P' is trivial over $V := f^{-1}(U) \subset N$. Hence $\pi' : P' \to N$ is a principal G-bundle. We end with the observation that if $\{\varphi_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ is the set of transition functions of P associated to a cover $\{U_\alpha\}$ of M, then $\{\varphi_{\beta\alpha} \circ f : V_\alpha \cap V_\beta \to G\}$ is the set of transition functions of P' associated to the cover $\{V_\alpha\}$ of N, where $V_\alpha = f^{-1}(U_\alpha)$. Finally, we observe that there is a canonical principal G-bundle map $\tilde{f} : P' \to P$, sending $(y, p) \in P'$ to $p \in P$, covering the map $f : N \to M$.

Example 2.3. Consider the fiber bundle $E \to M$ associated to a principal G-bundle P described in Example 2.2(3), where the fiber of E is the homogeneous space G/H.

Suppose there is a smooth section of the bundle $f : M \to E$, and let P' be the principal H-bundle over M which is the pull-back bundle of the principal H-bundle $P \to E$ by f. Then there is a corresponding principal H-bundle map $\tilde{f} : P' \to P$, which defines P' as a reduction of the principal G-bundle P to a principal H-bundle. On the other hand, it is easy to see that any reduction $P' \to P$ of a principal G-bundle to a principal H-bundle gives rise to a smooth section of the associated fiber bundle $E \to M$, where $E = P \times_G G/H$. Thus the problem of existence of a reduction to a principal H-bundle is reduced to the problem of existence of smooth sections of the associated fiber bundle $E \to M$, which is a problem solvable using homotopy theory. For example, for $G = GL(n, \mathbb{R})$, $H = O(n)$, it is known that G/H is a contractible space. This implies that every principal $GL(n, \mathbb{R})$-bundle admits a reduction to a principal $O(n)$-bundle, or equivalently, every real vector bundle has a metric.

3. Connections and curvature

Fix a principal G-bundle P over M. First of all, for any $A \in Lie(G)$, we define a vector field A^* on P, called the *fundamental vector field corresponding to* A, as follows: for any $u \in P$, A^*_u is the tangent vector of the smooth curve ug_t in P where $g_t = exp(tA)$. It is clear that A^* is nowhere vanishing. In what follows, we shall adapt the following notation: for any $g \in G$, let $R_g : P \to P$ be the map $u \mapsto ug$. Then observe that $(R_g)_* A^* = (Ad(g^{-1})(A))^*$.

For any $u \in P$, let G_u be the subspace of T_uP consisting of vectors tangent to the fiber of P through u. Then a *connection in* P, denoted by Γ, is an assignment of a subspace Q_u of T_uP to each $u \in P$ depending smoothly on u, such that

(a) $T_uP = G_u + Q_u$ as a direct sum,
(b) $Q_{ug} = (R_g)_* Q_u$, for any $g \in G$.

A connection Γ in P can be equivalently defined by a $Lie(G)$-valued 1-form ω on P, where ω is uniquely characterized by the properties that (1) $\omega(A^*) = A$ for any $A \in Lie(G)$, (2) $\omega_u(X) = 0$ for any $X \in Q_u$ and any $u \in P$. It is clear that ω obeys the following two conditions:

(a') $\omega(A^*) = A$ for any $A \in Lie(G)$,
(b') $(R_g)^*\omega = Ad(g^{-1})\omega$, where for any tangent vector X on P, $(R_g)^*\omega(X) = \omega((R_g)_*(X))$, and $Ad(g^{-1})\omega(X) = Ad(g^{-1})(\omega(X))$.

Conversely, any $Lie(G)$-valued 1-form ω on P obeying (a'), (b') defines a connection in P by setting $Q_u = \ker \omega_u$, i.e., $Q_u = \{X \in T_uP | \omega_u(X) = 0\}$. Such a form is called a *connection form*.

There is a third way to describe a connection in P, which is through a system of locally defined, $Lie(G)$-valued 1-forms on M satisfying certain compatibility conditions. To see this, let $\{U_\alpha\}$ be an open cover of M such that over each U_α, a localization of P is given by $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$, and we let $\{\varphi_{\beta\alpha}\}$ be the associated transition functions, such that $\psi_\beta \circ \psi_\alpha^{-1}(x, g) = (x, \varphi_{\beta\alpha}(x)g)$. For each α, set $\sigma_\alpha(x) := \psi_\alpha^{-1}(x, e) \in P_x$, $x \in U_\alpha$, which is a local smooth section of P over U_α. Then note that $\sigma_\alpha = \sigma_\beta \varphi_{\beta\alpha}$ over $U_\alpha \cap U_\beta$. With this understood, for any given connection form ω on P, we set $\omega_\alpha := \sigma_\alpha^* \omega$, which is a $Lie(G)$-valued 1-form on U_α. Let θ be the canonical 1-form on G, and for any α, β, set $\theta_{\beta\alpha} = \varphi_{\beta\alpha}^* \theta$, which is a $Lie(G)$-valued 1-form on $U_\alpha \cap U_\beta$.

Then the system of $Lie(G)$-valued 1-forms $\{\omega_\alpha\}$ obeys the following compatibility conditions

$$\omega_\alpha(x) = Ad(\varphi_{\beta\alpha}(x)^{-1})\omega_\beta(x) + \theta_{\beta\alpha}(x), \ \ \forall x \in U_\alpha \cap U_\beta.$$

Conversely, any such a system of $Lie(G)$-valued 1-forms $\{\omega_\alpha\}$ defines a connection form ω on P, such that $\omega_\alpha = \sigma_\alpha^*\omega$, $\forall \alpha$.

Example 3.1. In the case of $G = GL(n, \mathbb{R})$, each ω_α is a $n \times n$ matrix-valued 1-form on U_α, and $\theta_{\beta\alpha} = \varphi_{\beta\alpha}^{-1} d\varphi_{\beta\alpha}$ as $\theta = A^{-1}dA$ for $G = GL(n, \mathbb{R})$. Furthermore, recall that $Ad(X)(A) = XAX^{-1}$ for any $X \in GL(n, \mathbb{R})$ and $A \in M(n, \mathbb{R})$. With this understood, the compatibility conditions for $\{\omega_\alpha\}$ read as

$$\omega_\alpha = \varphi_{\beta\alpha}^{-1}\omega_\beta\varphi_{\beta\alpha} + \varphi_{\beta\alpha}^{-1}d\varphi_{\beta\alpha}.$$

Theorem 3.2. *(Existence of connections) There exists a connection in P. Moreover, the space of connections in P can be identified with the space of smooth sections of the vector bundle $(P \times_{Ad} Lie(G)) \otimes T^*M$, where $P \times_{Ad} Lie(G)$ is the vector bundle associated to P by the adjoint representation $Ad : G \to Lie(G)$.*

Induced connections: Let P' be a principal G'-bundle which is the induced bundle of P by $\rho : G \to G'$, and let $f : P \to P'$ be the corresponding bundle map. Then for any given connection Γ in P, there is an *induced connection* Γ' in P' defined as follows. Suppose $u \in P \mapsto Q_u \subset T_uP$ is the assignment which defines Γ. We define an assignment $u' \in P' \mapsto Q_{u'} \subset T_{u'}P'$ by first setting $Q_{u'} = f_*(Q_u)$ for some $u' = f(u)$, $\forall x \in M$ and $\pi(u) = x$, and then for any other points in the fiber P'_x, which is of the form $u'g'$ for some $g' \in G'$, we define $Q_{u'g'} = (R_{g'})_*Q_{u'}$. This assignment is well-defined due to the fact that $f : P \to P'$ is equivariant with respect to $\rho : G \to G'$, and it automatically satisfies condition (b) in the definition of connections. On the other hand, note that π_* sends the subspace Q_u isomorphically onto T_xM, $x = \pi(u)$, and since f induces the identity map on M, it follows easily that condition (a) is also satisfied by the assignment $u' \mapsto Q_{u'}$. Hence we obtain the induced connection Γ'. We observe that if ω, ω' are the corresponding connection forms of Γ, Γ', then $f^*\omega' = \rho_*\omega$ holds true as $Lie(G')$-valued 1-forms on P. This is because for any $A \in Lie(G)$, f_* sends the fundamental vector field A^* on P to the fundamental vector field corresponding to $\rho_*(A)$ on P'. Finally, we point out that another, equivalent definition of induced connection is as follows: let $\{U_\alpha\}$ be an open cover of M such that over each U_α, a trivialization ψ_α is given for P, with associated transition functions $\{\varphi_{\beta\alpha}\}$. If $\{\omega_\alpha\}$ is a system of locally defined, $Lie(G)$-valued 1-forms on M which gives a connection Γ in P, then the system of $Lie(G')$-valued 1-forms $\{\omega'_\alpha\}$, where $\omega'_\alpha := \rho_*\omega_\alpha$, satisfies the compatibility conditions

$$\omega'_\alpha(x) = Ad(\varphi'_{\beta\alpha}(x)^{-1})\omega'_\beta(x) + \theta'_{\beta\alpha}(x), \ \ \forall x \in U_\alpha \cap U_\beta,$$

where $\{\varphi'_{\beta\alpha} := \rho \circ \varphi_{\beta\alpha}\}$ is the corresponding set of transition functions for P', and $\theta'_{\beta\alpha} := (\varphi'_{\beta\alpha})^*\theta'$. This is because $\rho_* \circ Ad(g) = Ad(\rho(g)) \circ \rho_*$ and $\rho^*\theta' = \rho_*\theta$. Hence $\{\omega'_\alpha\}$ defines a connection in P', which is the induced connection of Γ.

Pull-back connections: Let P' be the pull-back bundle of P by a smooth map $f : N \to M$, and let $\tilde{f} : P' \to P$ denote the corresponding bundle map. Then for any connection Γ in P, the *pull-back connection* of Γ is a connection Γ' in P' defined as follows: let ω be the connection form for Γ, then one can easily check that the pull-back form $\tilde{f}^*\omega$ on P' is also a connection form (i.e., obeying (a') and (b')), which defines the pull-back connection Γ'. Finally, one can easily check that if $\{\omega_\alpha\}$ is a system of locally defined $Lie(G)$-valued 1-forms defining Γ, then the pull-back forms $\{f^*\omega_\alpha\}$ defines Γ'.

Parallel transport: We fix a connection Γ in P, as assignment $u \mapsto Q_u$. For any $x \in M$ and any tangent vector $X \in T_xM$, we can define its *horizontal lift* X^* for any $u \in P_x$, such that X_u^* is the unique vector in Q_u sent to X under π_*. It is clear that $X_{ug}^* = (R_g)_* X_u$, $\forall g \in G$. If X is a vector field on M, then we obtain a vector field X^* on P, called the *horizontal lift* of X.

For any smooth curve $\tau = x_t$, $0 \le t \le 1$, in M, a smooth curve u_t in P is called a *horizontal lift* of τ if $\pi(u_t) = x_t$ for all t and the tangent vectors of u_t are horizontal.

Theorem 3.3. *For any smooth curve $\tau = x_t$, and any $u_0 \in P_{x_0}$, there exists a unique horizontal lift u_t of τ such that the initial point of u_t is u_0.*

Exercise: The problem is essentially local, so prove Theorem 3.3 for the case when $P = M \times G$.

With Theorem 3.3, we can now define *parallel transport*. Given any smooth curve $\tau = x_t$, $0 \le t \le 1$ (in fact it can be more generally a piecewise smooth curve), we define the parallel transport long τ as follows: for any $u \in P_{x_0}$, we let $\tau(u) \in P_{x_1}$ be the end point of the horizontal lift of τ whose initial point is u. Clearly this defines an isomorphism τ from the fiber P_{x_0} to the fiber P_{x_1}. Note that the inverse $\tau^{-1} : P_{x_1} \to P_{x_0}$ is the parallel transport along the curve x_{1-t}, $0 \le t \le 1$.

Application: Here is a nice application of parallel transport: suppose $f_0, f_1 : N \to M$ be two smooth maps which are homotopic. Then the pull-back bundles of P by f_0, f_1 must be isomorphic. To see this, let $f : N \times [0,1] \to M$ be the homotopy between f_0, f_1, and let $P' \to N \times [0,1]$ be the pull-back bundle of P by f. We pick a connection in P'. Then the parallel transport along the curves τ_y in $N \times [0,1]$, $y \in N$, where $\tau_y(t) = (y,t)$, defines an isomorphism between the pull-back bundles of P by f_0, f_1. This fact is fundamental for the classification of principal G-bundles up to isomorphism. In fact, for any Lie group G, there is a *classifying space* of G, denoted by B_G, and a (universal) principal G-bundle E_G over B_G, such that for any principal G-bundle P over M, there is a map f from M to B_G such that P is isomorphic to the pull-back bundle of E_G by f.

Holonomy: For any $x \in M$, let $C(x)$ be the set of piecewise smooth curves whose initial point and end point are x. We define a product structure on $C(x)$ as follows: for any $\mu, \tau \in C(x)$, the product $\mu \cdot \tau$ is the piecewise smooth curve obtained from τ followed by μ. With this understood, the parallel transport defines a map $\phi : C(x) \to Aut(P_x)$, the automorphism group of the fiber P_x. Its image, denoted by $\Phi(x)$, is called the *holonomy group* based at x. One can identify the holonomy group $\Phi(x)$ with a subgroup of G as follows: pick a point $u \in P_x$, then for any $\tau \in C(x)$, $\tau(u) = ug$ for a unique element $g \in G$. The assignment $\phi : \tau \mapsto g$ satisfies $\phi(\mu \cdot \tau) = \phi(\mu)\phi(\tau)$. Its image $\Phi(u)$, which is a subgroup of G, is isomorphic to the holonomy group $\Phi(x)$. Note that $\Phi(u)$ is uniquely determined up to conjugacy. A proof of the following theorem can be found in [1].

Theorem 3.4. *The holonomy group $\Phi(u)$ is a Lie subgroup of G. Moreover, the principal G-bundle P admits a reduction to a principal $\Phi(u)$-bundle.*

If we let $P(u)$ be the subspace of P which consists of points that can be connected to u by a piecewise smooth, horizontal curve in P. Then $P(u)$ is a principal $\Phi(u)$-bundle and the embedding $P(u) \to P$ defines the reduction. Compare Example 2.3.

Curvature: Let ω be a connection form on P. Then $d\omega$ is a $Lie(G)$-valued 2-form on P. On the other hand, for any tangent vectors X, Y of P, $[\omega(X), \omega(Y)]$ also defines a $Lie(G)$-valued 2-form on P, where the bracket is the Lie bracket of $Lie(G)$. We set $\Omega(X,Y) := d\omega(X,Y) + [\omega(X), \omega(Y)]$, called the *curvature* of ω.

Lemma 3.5. *$\Omega(X,Y) = 0$ if one of X or Y is vertical (i.e., tangent to the fibers). Moreover, if X^*, Y^* are horizontal vector fields, then $\Omega(X^*, Y^*) = -\omega([X^*, Y^*])$.*

By the first sentence of Lemma 3.5, Ω can be regarded as a smooth section of $(P \times_{Ad} Lie(G)) \otimes \Lambda^2 M$. If P is given by transition functions $\{\varphi_{\beta\alpha}\}$ and accordingly ω is given by a system of $Lie(G)$-valued 1-forms $\{\omega_\alpha\}$, then Ω is given by a system of $Lie(G)$-valued 2-forms $\{\Omega_\alpha\}$, where $\Omega_\alpha := d\omega_\alpha + \frac{1}{2}[\omega_\alpha, \omega_\alpha]$. One can check directly that $\{\Omega_\alpha\}$ obeys

$$\Omega_\alpha = Ad(\varphi_{\beta\alpha}^{-1})\Omega_\beta.$$

Note that this particularly shows that Ω is a smooth section of $(P \times_{Ad} Lie(G)) \otimes \Lambda^2 M$.

Let P' be a principal G'-bundle which is the induced bundle of P by $\rho : G \to G'$, and let $f : P \to P'$ be the corresponding bundle map. Let ω' be the induced connection of ω. Since $f^*\omega' = \rho_*\omega$, it follows easily that the corresponding curvatures Ω, Ω', as 2-forms on M, obey the equation $\Omega' = \rho_*\Omega$. Likewise, if P' is the pull-back bundle of P by a smooth map $f : N \to M$, and ω' is the pull-back connection of ω, the corresponding curvatures Ω, Ω', as 2-forms on M, are related by $\Omega' = f^*\Omega$.

Exercise: For the case $G = GL(n, \mathbb{R})$, recall that each ω_α is a $n \times n$ matrix-valued 1-form on U_α. Show that $\Omega_\alpha = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha$ as a $n \times n$ matrix-valued 2-form on U_α, and obeys $\Omega_\alpha = \varphi_{\beta\alpha}^{-1}\Omega_\beta\varphi_{\beta\alpha}$.

Flat connections: A connection is called a *flat connection* if its curvature vanishes. By the second statement of Lemma 3.5, if a connection Γ, $u \mapsto Q_u$, is flat, then for any horizontal vector fields X^*, Y^*, its Lie bracket $[X^*, Y^*]$ continues to be horizontal, which means that Γ as a distribution is integrable. By Frobenius Theorem (cf. [2]), for any $u \in P$, there is a maximally defined, connected submanifold $P(u)$ such that for any $u' \in P(u)$, $Q_{u'} = T_{u'}P(u)$. Note that the restriction of $\pi : P \to M$ to the submanifold $P(u)$ is a covering map. It follows that the holonomy groups of a flat connection are discrete subgroups of G, and the corresponding map $\phi : C(x) \to G$ can be factored through to a homomorphism $\pi_1(M, x) \to G$. It follows easily that

$$\{\text{flat connections on principal } G\text{-bundles over } M\}/_{Isom} \longleftrightarrow \{\pi_1(M) \to G\}/\text{conjugacy}.$$

Gauge transformations: Let $\Gamma : u \mapsto Q_u$ be a connection in P and $f : P \to P$ be any automorphism of P. Then the pushforward of f, $f_*\Gamma : u \mapsto f_*Q_{f^{-1}(u)}$, naturally defines a connection in P, called the *gauge transformation* of Γ by f. Note that the gauge transformation of a flat connection is again a flat connection.

Exercise: Show that an automorphism of P can be identified with a smooth section of the associated fiber bundle $P \times_{Ad} G$, where the left action of G on $F = G$ is given by $(g, g') \mapsto Ad(g)(g') = gg'g^{-1}$. Moreover, suppose a connection Γ is given by a system of local $Lie(G)$-valued 1-forms $\{\omega_\alpha\}$ and a bundle automorphism of P is given by a system of local smooth maps to G, $\{g_\alpha\}$, then the gauge transformation of Γ is given by the local $Lie(G)$-valued 1-forms $\{Ad(g_\alpha)(\omega_\alpha - g_\alpha^*\theta)\}$, where θ is the canonical 1-form on G.

Example 3.6. In this example, we revisit the principal $\mathbb{S}^1$-bundles over $\mathbb{CP}^1$ where $P = \mathbb{S}^3$ (see Example 2.1(3)). First consider the case where the $\mathbb{S}^1$-action is given by the first choice in Example 2.1(3). Before we begin, note that $Lie(\mathbb{S}^1) = i\mathbb{R} \subset \mathbb{C}$.

Consider the connection form ω on $\mathbb{S}^3$, where at $x = (x_1, y_1, x_2, y_2) \in \mathbb{S}^3$,

$$\omega(x) = i\sum_{k=1}^{2}(x_k dy_k - y_k dx_k).$$

One can check easily that ω is $\mathbb{S}^1$-equivariant. On the other hand, note that the $\mathbb{S}^1$-action is generated by the vector field $X := \sum_{k=1}^{2}(x_k\partial_{y_k} - y_k\partial_{x_k})$, whose corresponding Lie algebra generater is i (i.e., X is the fundamental vector field corresponding to $i \in Lie(\mathbb{S}^1)$). Note that $\omega(X) = i$. Since the adjoint representation Ad of $\mathbb{S}^1$ is trivial ($\mathbb{S}^1$ is Abelian), it follows that ω is a connection form on $\mathbb{S}^3$.

Next we compute the curvature of ω, $\Omega = d\omega$, as an i-valued 2-form on the base $\mathbb{CP}^1$. To this end, we consider a local coordinate chart (U, ϕ) of $\mathbb{CP}^1$ and a local section to $\mathbb{S}^3$ over U. As a map defined on $\phi(U) = \mathbb{C}$, the local section is given by $z \mapsto (\frac{z}{\sqrt{1+|z|^2}}, \frac{1}{\sqrt{1+|z|^2}})$. Then the pull-back of Ω by this local section is the 2-form we are looking for. An easy calculation shows that it equals $-\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$. Note that the integral of this form over $\mathbb{C}$ equals $2\pi i$. Consequently, the integral of Ω over $\mathbb{CP}^1$ equals $2\pi i$, or $\int_{\mathbb{CP}^1} \frac{i}{2\pi}\Omega = -1$.

Now we make the following three observations: (1) if ω' is any other connection form, then $\omega' = \omega + i\pi^*a$ for some 1-form a on $\mathbb{CP}^1$ (cf. Theorem 3.2). Hence $\Omega' = \Omega + i\pi^* da$, and as a consequence, $\int_{\mathbb{CP}^1} \frac{i}{2\pi}\Omega' = \int_{\mathbb{CP}^1} \frac{i}{2\pi}\Omega = -1$. (2) If we change the $\mathbb{S}^1$-action to the second choice, then note that the vector field generating the $\mathbb{S}^1$-action becomes $-X$, so the connection form changes to $-\omega$. The curvature also changes by sign, so is the integral $\int_{\mathbb{CP}^1} \frac{i}{2\pi}\Omega$. (3) Isomorphic principal $\mathbb{S}^1$-bundles have the same integral $\int_{\mathbb{CP}^1} \frac{i}{2\pi}\Omega$.

Conclusion: The two principal $\mathbb{S}^1$-bundle structures on $\mathbb{S}^3$ are not isomorphic.

This is perhaps the simplest example in the Chern-Weil theory; the integral $\int_{\mathbb{CP}^1} \frac{i}{2\pi}\Omega$ is the so-called *the first Chern number.*

A side note: There is a symplectic geometry aspect in this example. Consider the standard symplectic structure on $\mathbb{R}^4$, $\omega_0 := \sum_{k=1}^2 dx_k \wedge dy_k$, and the vector field $V = \frac{1}{2}\sum_{k=1}^2 x_k \partial_{x_k} + y_k \partial_{y_k}$, which obeys $L_V \omega_0 = \omega_0$ (such a V is called a Liouville vector field). Since V is transverse to $\mathbb{S}^3$, the 1-form $\alpha := i_V \omega_0$ is a contact form on $\mathbb{S}^3$, and $\xi := \ker \alpha$ is called a contact structure on $\mathbb{S}^3$. We observe that in the example, the connection form $\omega = 2i\alpha$, in particular, the connection distribution $\ker \omega = \ker \alpha = \xi$ is the contact structure.

Reference for most of this section and the previous section is [1].

4. Covariant derivatives

Connections in a vector bundle: Let E be a vector bundle of rank n over M, and let $\Gamma(E)$ denote the space of smooth sections of E. A *covariant derivative* on $\Gamma(E)$ is a $\mathbb{R}$-linear map $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Omega^1(M)$ satisfying the following condition: for any $f \in C^\infty(M)$, $\xi \in \Gamma(E)$,

$$\nabla(f\xi) = \xi \otimes df + f\nabla\xi.$$

Note that for any vector field X, there is an associated $\mathbb{R}$-linear map $\nabla_X : \Gamma(E) \to \Gamma(E)$ satisfying $\nabla_X(f\xi) = Xf \cdot \xi + f\nabla_X \xi$.

Exercise: Show that the difference of two covariant derivatives defines a smooth section of $End(E) \otimes T^*M$. Conversely, if ∇ is a covariant derivative on $\Gamma(E)$, A is a smooth section of $End(E) \otimes T^*M$, then $\nabla + A$ is also a covariant derivative on $\Gamma(E)$.

Now we describe a covariant derivative ∇ in terms of a given localization of E. To begin with, let $\{U_\alpha\}$ be an open cover of M, such that E is trivial over each U_α, and we let $\{\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$ be the set of associated transition functions. Since E is trivial over U_α, a smooth section of E over U_α is simply a smooth $\mathbb{R}^n$-valued function on U_α. As such, the usual exterior differential d defines a natural covariant derivative. By the above Exercise, over U_α, $\nabla = d + A_\alpha$ for some A_α, where A_α is a $n \times n$ matrix whose entries are 1-forms on U_α. Thus ∇ corresponds to a system of $n \times n$ matrix valued 1-forms $\{A_\alpha\}$. The crucial observation is that for any α, β, A_α, A_β satisfy the following compatibility condition over $U_\alpha \cap U_\beta$:

$$A_\alpha = \tau_{\beta\alpha}^{-1} A_\beta \tau_{\beta\alpha} + \tau_{\beta\alpha}^{-1} d\tau_{\beta\alpha}.$$

On the other hand, from the discussion in the previous section, $\{A_\alpha\}$ defines a connection in the principal $GL(n, \mathbb{R})$-bundle P, which is the frame bundle of E (cf. Example 2.1(2)). Conversely, given any connection in P. there is a $\{A_\alpha\}$ satisfying the above compatibility condition, which determines a covariant derivative ∇ on $\Gamma(E)$, where over each U_α, $\nabla = d + A_\alpha$. This establishes a one to one correspondence between covariant derivatives on $\Gamma(E)$ and connections in the associated frame bundle P of E.

Exercise: Let Ω be the curvature of the connection in P corresponding to ∇, which is viewed as a smooth section of $End(E) \otimes \Lambda^2 M$ (note that $P \times_{Ad} Lie(G) = End(E)$ here). Show that for any vector fields X, Y on M, and any $\xi \in \Gamma(E)$,

$$\nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]} \xi = \Omega(X, Y)\xi.$$

More generally, let P be any principal G-bundle over M, V be any finite dimensional vector space, and $\rho : G \to GL(V)$ be any given linear representation. Let $E := P \times_\rho V$ be the associated vector bundle. Then for any given connection ω in P, there is an associated covariant derivative ∇ on $\Gamma(E)$. Moreover, for any vector fields X, Y on M, and any $\xi \in \Gamma(E)$,

$$\nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]} \xi = \rho_* \Omega(X, Y)\xi,$$

where $\rho_*\Omega$ is the image of Ω in $End(V)$ under $\rho_* : Lie(G) \to End(V)$.

Parallel transport and holonomy: Let E be a vector bundle of rank n over M, and let P be the corresponding frame bundle. Let ∇ be a covariant derivative on $\Gamma(E)$, and let ω be the corresponding connection form on P, with curvature Ω. The notion of parallel transport and holonomy in P has a corresponding explanation in the setting of vector bundle E. To be more precise, let $\tau = x_t$, $t \in [0, 1]$, be a smooth curve in M, and let $u_t \in P$ be a horizontal lift of x_t. Then the parallel transport τ sends $u_0 \in P_{x_0}$ to $u_1 \in P_{x_1}$. Note that in the current situation, u_t is a basis of the fiber E_{x_t}. Thus τ sending u_0 to u_1 induces naturally an isomorphism, still denoted by $\tau : E_{x_0} \to E_{x_1}$, which is independent of the choice of the basis u_0 of E_{x_0}. This is the parallel transport in the setting of vector bundles. We note that u_t is a horizontal lift of x_t if and only if $\nabla_{\frac{d}{dt}x_t} u_t = 0$, where $\frac{d}{dt}x_t$ denotes the tangent vector of x_t.

Exercise: (Infinitesimal holonomy and curvature) In a local coordinate $\{x^i\}$ of M, suppose $T_{s,t}$ is the holonomy around the rectangular loop in the x^1x^2-plane, which runs from $(x^1, x^2) = (0, 0)$ to $(s, 0)$, followed by $(s, 0)$ to (s, t), then (s, t) to $(0, t)$, then $(0, t)$ back to $(0, 0)$. Prove that

$$T_{s,t} = Id + (-\Omega(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})(0, 0, \cdots, 0) + O(s) + O(t))st.$$

In particular, the infinitesimal holonomy $\frac{\partial^2}{\partial s \partial t} T_{s,t}|_{s=t=0} = -\Omega(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})(0, 0, \cdots, 0)$.

Exterior covariant derivatives: Fix any covariant derivative ∇ on $\Gamma(E)$, we can extend the exterior derivatives $d : \Omega^k(M) \to \Omega^{k+1}(M)$ uniquely to $\mathbb{R}$-linear maps $d_\nabla : \Gamma(E) \otimes \Omega^k(M) \to \Gamma(E) \otimes \Omega^{k+1}(M)$, with $d_\nabla = \nabla$ when $k = 0$, such that the following equation is satisfied: for $\xi \in \Gamma(E) \otimes \Omega^k(M)$, $\eta \in \Omega^l(M)$,

$$d_\nabla(\xi \wedge \eta) = d_\nabla \xi \wedge \eta + (-1)^k \xi \wedge d\eta.$$

The maps d_∇ are called *exterior covariant derivatives.*

Unlike the exterior derivative d, in general $d^2_\nabla \neq 0$. For any $\xi \in \Gamma(E) \otimes \Omega^k(M)$, $f \in C^\infty(M)$, let's compute $d^2_\nabla(f\xi)$:

$$d^2_\nabla(f\xi) = d_\nabla(f d_\nabla \xi + (-1)^k \xi \wedge df) = f d^2_\nabla \xi + (-1)^{k+1} d_\nabla \xi \wedge df + (-1)^k d_\nabla \xi \wedge df + \xi \wedge d^2 f,$$

which implies $d^2_\nabla(f\xi) = f d^2_\nabla \xi$. In other words, d^2_∇ defines a smooth section of $End(E) \otimes \Lambda^2 M$. The following gives another interpretation of curvature.

Exercise: Show that $d^2_\nabla \xi = \Omega_\nabla \wedge \xi$, where Ω_∇ denotes the curvature of ∇, as a smooth section of $End(E) \otimes \Lambda^2 M$.

The Bianchi identity: Let P be any principal G-bundle over M, and let ω be any connection in P, with curvature Ω viewed as a smooth section of $(P\times_{Ad}Lie(G))\otimes\Lambda^2 M$. Let ∇ be the associated covariant derivative on $\Gamma(P\times_{Ad}Lie(G))$. Then

$$d_\nabla \Omega = 0,$$

which is called the *Bianchi identity*.

Exercise: Prove the Bianchi identity.

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