MATH 703: PART 4: INTEGRAL CURVES, FLOWS AND TANGENT DISTRIBUTIONS

WEIMIN CHEN

Contents

1. Integral curves and flows

Let $X \in \mathcal{X}(M)$ be a smooth vector field on a smooth manifold M. An **integral curve** of X is a smooth curve $\gamma : (a,b) \to M$ such that

$$\gamma'(t) = X(\gamma(t)), \;\; \forall t \in (a,b).$$

More generally, let $X_t \in \mathcal{X}(M)$ be a time-dependent smooth vector field, where $t \in J$ lies in an interval J (of any type), where we assume X_t depends on t smoothly. An **integral curve** of X_t is a smooth curve $\gamma : (a,b) \cap J \to M$, such that

$$\gamma'(t) = X_t(\gamma(t)), \;\; \forall t \in (a,b) \cap J.$$

Let's examine the integral curve equation in local coordinates. Suppose $\gamma(t) \subset U$ for $t \in (a,b)$, and let $x^1, x^2, \cdots, x^n$ be a system of local coordinate functions on U. Then there are smooth functions $\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t)$ on (a,b) such that the smooth curve $\gamma(t)$ is given by $x^i = \gamma_i(t)$, $i = 1, 2, \cdots, n$. With this understood, note that $\gamma'(t) = \sum_{i=1}^n \gamma_i'(t) \frac{\partial}{\partial x^i}|_{\gamma(t)}$. On the other hand, $X_t(p) = \sum_{i=1}^n X^i(t,p) \frac{\partial}{\partial x^i}|_p$ for some functions $X^i(t,p)$ which are smooth in both t and p, where $t \in J$, $p \in U$. With this understood, the time-dependent integral curve equation $\gamma'(t) = X_t(\gamma(t))$, $\forall t \in (a,b) \cap J$, with initial value condition $\gamma(s) = p$, where $s \in J$, $p \in U$, becomes

$$\gamma_i'(t) = X^i(t, \gamma_1(t), \cdots, \gamma_n(t)), \;\; \gamma_i(s) = x^i(p), \;\; i = 1, 2, \cdots, n,$$

which is a system of (time-dependent) ODEs in $\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t)$. The corresponding theory of ODEs implies the following results which are relevant in our discussion:

Local Existence: For any $p_0 \in M$, $s_0 \in J$, there are open neighborhoods U_0 of p_0, $J_0 \subset J$ of s_0, such that for any $p \in U_0$, $s \in J_0$, there is an interval (a,b), with $J_0 \subset (a,b) \cap J$, and a smooth curve $\gamma_{s,p} : (a,b) \cap J \to M$, which satisfies $\gamma_{s,p}'(t) = X_t(\gamma_{s,p}(t))$, with initial value condition $\gamma_{s,p}(s) = p$.

Uniqueness: For fixed $p \in U_0$, $s \in J_0$, if $\gamma : (a,b) \cap J \to M$, $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \cap J \to M$ are two solutions of the integral curve equation with the same initial value condition, i.e., $\gamma(s) = \tilde{\gamma}(s) = p$, then $\gamma = \tilde{\gamma}$ on the overlap $(a,b) \cap (\tilde{a}, \tilde{b}) \cap J$.

Smoothness: Let $\theta : ((a,b) \cap J) \times J_0 \times U_0 \to M$ be the map sending (t,s,p) to $\gamma_{s,p}(t)$. Then θ is a smooth map.

Exercise: Let $M = \mathbb{R}$, and let $\gamma : \mathbb{R} \to M$ defined by $\gamma(t) = t^3$. Explain why the tangent vectors $\gamma'(t)$ define only a continuous, not smooth, vector field on M.

Fundamental Question: Concerning global properties of integral curves on smooth manifolds, the fundamental question is under what conditions on the vector field X_t, the integral curve $\gamma_{s,p}(t)$ is maximally defined, i.e., is defined for all $t \in J$, for any $s \in J$ and any $p \in M$ (resp. a subset U of M). We will call such a X_t **complete** (resp. over subset U). A special case worth of attention is the following: let $K \subset M$ be the subset such that $X_t(p) = 0$ for all $p \in K, t \in J$. Then for any $p \in K$, the constant curve $\gamma_{s,p}(t) \equiv p$, where $s \in J$, $t \in J$, is the unique solution to the integral curve equation with the initial value condition determined by s and p.

Suppose X_t is a complete vector field. Then for any fixed $s_0 \in J$, we define a smooth family (in t) of smooth maps $\theta_t^{s_0} : M \to M$, where $t \in J$, by setting

$$\theta_t^{s_0}(p) := \theta(t, s_0, p) = \gamma_{s_0,p}(t).$$

Note that $\theta_{s_0}^{s_0} = Id_M$, and for each $t \in J$, $\theta_t^{s_0}$ is a diffeomorphism, as $\theta_{s_0}^t \circ \theta_t^{s_0} = Id_M$. (Intuitively speaking, the map $\theta_t^{s_0}$ is defined by moving along the integral curves of X_t from time s_0 to time t.) Furthermore, if $X_t \equiv 0$ on a subset $K \subset M$ for all $t \in J$, $\theta_t^{s_0} \equiv Id$ on K for any $t \in J$. This construction allows us, by choosing X_t properly, to construct diffeomorphisms on M with various geometrical or topological significance.

Example 1.1. Consider $M = \mathbb{R}^2 \setminus \{(0,0)\}$, and let x, y be the standard coordinate functions on $\mathbb{R}^2$. Let $X_t = \frac{\partial}{\partial x}$, where $t \in J = \mathbb{R}$, which is time-independent. It is easy to see that for any initial value condition $s \in J$, $p \in M$, if p lies on the x-axis, the corresponding integral curve can not be defined for all $t \in J = \mathbb{R}$, because of the existence of a "hole" $(0,0)$ in M. Thus the vector field X_t in this example is not complete, and its non-completeness is related to certain non-compactness, which will be made more precise in the following lemma (i.e., **Escape Lemma**).

First, note that for any fixed initial value condition $s \in J, p \in M$, there is a maximal sub-interval $J_{s,p} \subset J$, over which the integral curve $\gamma_{s,p}$ is defined (i.e., $\gamma_{s,p}$ can not be extended over a sub-interval of J strictly containing $J_{s,p}$), due to the **Uniqueness** property mentioned earlier.

Lemma 1.2. *(Escape Lemma) For any $s \in J$, $p \in M$, if $J_{s,p} \neq J$, then $\gamma_{s,p}(J_{s,p})$ can not lie in a compact subset of M.*

Proof. Since $J_{s,p} \neq J$, there is a sequence $t_i \in J_{s,p}$ which converges to $t_0 \in J$, such that $t_0 \in J \setminus J_{s,p}$. Suppose to the contrary that $\gamma_{s,p}(J_{s,p})$ lies in a compact subset of M. Then the sequence $p_i := \gamma_{s,p}(t_i) \in M$ has a convergent subsequence, which is still denoted by $\{p_i\}$ without loss of generality. We denote the limit of $\{p_i\}$ by $p_0 \in M$. By the **Local Existence**, there are open neighborhoods U_0 of p_0, J_0 of t_0, such that for any initial value condition $s \in J_0$, $q \in U_0$, there is an integral curve $\gamma_{s,q} : J_0 \to M$ such that $\gamma_{s,q}(s) = q$. With this understood, we pick a $t_i \in J_0$ for i large enough, such that $p_i \in U_0$. Then both γ_{t_i,p_i} and $\gamma_{s,p}$ satisfies $\gamma_{t_i,p_i}(t_i) = p_i$, $\gamma_{s,p}(t_i) = p_i$. By the **Uniqueness** property, $\gamma_{t_i,p_i} = \gamma_{s,p}$ on the overlap of their domains. Consequently, we may extend $\gamma_{s,p}$ over to the sub-interval $J_{s,p} \cup J_0$ of J, which is strictly larger than $J_{s,p}$. This contradiction shows that $\gamma_{s,p}(J_{s,p})$ can not lie in a compact subset of M. □

For a time-dependent vector field X_t, $t \in J$, we define and denote its support by

$$\text{supp } X_t := \overline{\{p \in M \mid \text{there is a } t \in J \text{ such that } X_t(p) \neq 0\}}.$$

Corollary 1.3. *If supp X_t is compact, in particular, if M is compact, then X_t is complete.*

Proof. For any $s \in J$, $p \in M$, if $\gamma_{s,p}(J_{s,p}) \subset \text{supp } X_t$, then $J = J_{s,p}$ by Lemma 1.2. If $\gamma_{s,p}(J_{s,p})$ is not contained in supp X_t, then there is a $t_0 \in J_{s,p}$ such that $p_0 := \gamma_{s,p}(t_0) \in M \setminus \text{supp } X_t$. In this case, $X_t(p_0) = 0$ for all $t \in J$. Consequently, the integral curve $\gamma_{t_0,p_0}(t) = p_0$ for all $t \in J$. Since $\gamma_{s,p}(t_0) = p_0$, we see immediately $\gamma_{s,p}(t) = p_0$ for all $t \in J$ by the **Uniqueness** property. In particular, $J = J_{s,p}$ as well. Hence X_t is complete. □

In general, for non-complete vector fields X_t, we can still define the smooth maps $\theta_t^{s_0}$ locally, for t sufficiently close to s_0. The following lemma makes this precise.

Lemma 1.4. *Fix any $s_0 \in J$. Then for any $p_0 \in M$, there is an open neighborhood V of p_0 and an $\epsilon > 0$, such that for $t \in (s_0 - \epsilon, s_0 + \epsilon) \cap J$, the maps $\theta_t^{s_0}$ defined by $\theta_t^{s_0}(p) := \gamma_{s_0,p}(t)$ form a smooth family of smooth open embeddings from V into M.*

We remark that one may regard the maps $\theta_t^{s_0}$ as germs of open embeddings centered at points of M, defined for $|t - s_0|$ sufficiently small. With this in mind, note that

$$\theta_{s_2}^{s_1} \circ \theta_{s_1}^{s_0} = \theta_{s_2}^{s_0}$$

holds true for s_0, s_1, s_2 sufficiently close.

Proof. By the **Local Existence**, there is an open neighborhood U_0 of p_0, an open neighborhood J_0 of s_0 in J, such that the integral curve $\gamma_{s,p} : J_0 \to M$ exists for any $s \in J_0$ and $p \in U_0$. Moreover, $\gamma_{s,p}$ is unique, and the map $\theta : J_0 \times J_0 \times U_0 \to M$, sending (t, s, p) to $\gamma_{s,p}(t)$ is smooth. We consider the restriction of θ on $J_0 \times \{s_0\} \times U_0$, which is also smooth, and choose an $\epsilon > 0$ sufficiently small, such that $\theta(t, s_0, p_0) \in U_0$ when $|t - s_0| \leq \epsilon$ and $t \in J_0$. In other words, $([-\epsilon + s_0, s_0 + \epsilon] \cap J_0) \times \{p_0\} \subset \theta^{-1}(U_0)$. Since $([-\epsilon+s_0, s_0+\epsilon] \cap J_0) \times \{p_0\}$ is compact, there is an open neighborhood V of p_0, such that $([-\epsilon + s_0, s_0 + \epsilon] \cap J_0) \times V \subset \theta^{-1}(U_0)$. In particular, for any $t \in (s_0 - \epsilon, s_0 + \epsilon) \cap J \subset J_0$, $\gamma_{s_0,p}(t) \subset U_0$ for any $p \in V$. Since there is a unique integral curve $\gamma_{s,p} : J_0 \to M$ for any $s \in J_0$ and $p \in U_0$, the map $\theta_t^{s_0}$ defined by $\theta_t^{s_0}(p) := \gamma_{s_0,p}(t)$, for $p \in V$, has a smooth inverse, which is defined by moving along the integral curves backwards. This proves that each $\theta_t^{s_0}$ is a smooth open embedding. □

Besides local versions of $\theta_t^{s_0}$ for $|t-s_0|$ small, $\theta_t^{s_0}$ can be defined for all $t\in J$ under certain additional conditions. The following result is frequently used in the proof of various "neighborhood theorems" in symplectic/contact geometry.

Theorem 1.5. *Let $J=[0,1]$, and let $K\subset M$ be a compact subset. Suppose X_t is a time-dependent vector field on M, such that $X_t\equiv 0$ on K for any $t\in J$. Then there is an open neighborhood V of K such that for any $p\in V$, $\theta_t(p):=\gamma_{0,p}(t)$ is defined for all $t\in J$ (here $\gamma_{0,p}(t)$ denotes an integral curve of X_t). Moreover, $\theta_t: V\rightarrow M$, for $t\in J$, form a smooth family of open smooth embeddings, with $\theta_0=Id$, and $\theta_t=Id$ on K for all $t\in J$.*

In applications, K is either a point in M or a compact closed embedded submanifold of M.

Proof. Since K is compact, there is an open neighborhood U of K such that the closure $\overline{U}$ is compact. We first show that there is an open neighborhood $U_0\subset U$ of K, such that for any $s\in J$, $p\in U_0$, the integral curve $\gamma_{s,p}(t)\subset U$ of X_t is defined for all $t\in J$. Suppose to the contrary this is not true. Then there is a sequence $\{p_i\}$ converging to a point $p_0\in K$, a sequence $s_i\in J$, such that $\gamma_{s_i,p_i}(t)\subset U$ is not defined for all $t\in J$. Let (a_i,b_i) be the maximal interval over which $\gamma_{s_i,p_i}(t)\subset U$. Then since $\overline{U}$ is compact, by the same argument as in Lemma 1,2, either $\gamma_{s_i,p_i}(a_i)$ or $\gamma_{s_i,p_i}(b_i)$ must lie on $\overline{U}\setminus U$. Suppose s_i converges to $s_0\in J$. Then it is easy to see that the integral curves $\gamma_{s_i,p_i}(t)$ converges to the integral curve $\gamma_{s_0,p_0}(t)$. But since $X_t(p_0)=0$ for all $t\in J$, $\gamma_{s_0,p_0}(t)\equiv p_0$, which contradicts the fact that either $\gamma_{s_i,p_i}(a_i)$ or $\gamma_{s_i,p_i}(b_i)$ must lie on $\overline{U}\setminus U$. Hence the claim that there is an open neighborhood $U_0\subset U$ of K, such that for any $s\in J$, $p\in U_0$, the integral curve $\gamma_{s,p}(t)\subset U$ of X_t is defined for all $t\in J$.

Consider the smooth map $\theta: J\times U_0\rightarrow U$, where $\theta(t,p)=\gamma_{0,p}(t)$. Since $\gamma_{0,p}(t)=p$ for all $t\in J$ if $p\in K$, we see that $J\times K\subset\theta^{-1}(U_0)$. Since $J\times K$ is compact, there is an open neighborhood V of K such that $J\times V\subset\theta^{-1}(U_0)$, which means that for any $p\in V$, $\gamma_{0,p}(t)\subset U_0$ for all $t\in J$. This implies that for each $t\in J$, the map $\theta_t: V\rightarrow U_0$, defined by $\theta_t(p):=\gamma_{0,p}(t)$, has a smooth inverse, which is defined by moving along the integral curves backwards. This shows that θ_t, $t\in J$, form a smooth family of smooth open embeddings from V into M. Other claims about θ_t are straightforward. □

Exercise: Prove an extension of Theorem 1.5, where we assume K is a compact closed embedded submanifold of M, and $\forall t\in J,\forall p\in K$, $X_t(p)\in T_pK$.

Flows on manifolds – the time-independent case: Consider the case where $X_t=X$, which is a time-independent vector field on M. In this case, $J=\mathbb{R}$, and X is complete if the integral curves of X are defined for all $t\in\mathbb{R}$, for any initial value condition $s\in\mathbb{R}$ and $p\in M$.

Lemma 1.6. *(Translation Invariance) Let $\gamma:(a,b)\rightarrow M$ be an integral curve of a time-independent vector field X. Then for any $c\in\mathbb{R}$, the curve $\gamma_c(t):=\gamma(t-c)$, defined on $(a+c,b+c)$, is also an integral curve of X.*

Proof. $\gamma_c'(t)=\gamma'(t-c)=X(\gamma(t-c))=X(\gamma_c(t))$. □

Due to Lemma 1.6, we can always set the initial value of the time to $t = 0$ without loss of generality.

A smooth map $\theta : \mathbb{R} \times M \to M$ is called a **global flow** on M, if the maps $\theta_t : M \to M$ defined by $\theta_t(p) := \theta(t, p)$, $p \in M$, obeys

(i) $\theta_0 = Id$ on M,
(ii) $\theta_s \circ \theta_t = \theta_{s+t}$, for any $s, t \in \mathbb{R}$.

Note that by (i) and (ii), θ_{-t} is the inverse of θ_t, so that each θ_t is a diffeomorphism.

Theorem 1.7. *There is a one to one correspondence between global flows on M and time-independent, complete, vector fields on M, in which a global flow θ corresponds to a vector field X such that $\theta(t, p)$ is the integral curve $\gamma_{0,p}(t)$ of X.*

Proof. Given a global flow θ, we define an $X \in \mathcal{X}(M)$ as follows: for any $p \in M$, let $X(p)$ be the tangent vector $\frac{\partial}{\partial t}\theta(t, p)|_{t=0}$. We claim that for each $p \in M$, $\gamma_p(t) := \theta(t, p)$ is an integral curve of X with the initial value condition $\gamma_p(0) = p$. To see this, we compute $\gamma_p'(t)$. By the property $\theta_s \circ \theta_t = \theta_{s+t}$, for any $s, t \in \mathbb{R}$, we have

$$\theta(s, \theta(t, p)) = \theta(s + t, p).$$

It follows that $\gamma_p'(t) = \frac{\partial}{\partial s}\theta(s + t, p)|_{s=0} = X(\theta(t, p)) = X(\gamma_p(t))$. This in particular implies that X is complete. The claim $\gamma_p(0) = p$ follows from $\theta_0 = Id$.

Let X be a complete vector field. Then $\theta : \mathbb{R} \times M \to M$ defined by $\theta(t, p) = \gamma_{0,p}(t)$, where $\gamma_{0,p}(t)$ is the integral curve of X with initial value condition determined by $s = 0$ and p, is a smooth map. Clearly $\theta_0 = Id$. To see $\theta_s \circ \theta_t = \theta_{s+t}$, for any $s, t \in \mathbb{R}$, we let $q = \theta(t, p) = \gamma_{0,p}(t)$. Then

$$\theta(s, \theta(t, p)) = \gamma_{0,q}(s) = \gamma_{t,q}(s + t) = \gamma_{0,p}(s + t) = \theta(s + t, p).$$

Note that $\gamma_{0,q}(s) = \gamma_{t,q}(s + t)$ by Lemma 1.6, and $\gamma_{t,q}(s + t) = \gamma_{0,p}(s + t)$ by the **Uniqueness** property. □

In general for any $X \in \mathcal{X}(M)$, a certain local version of flows θ is well-defined. More precisely, in Lemma 1.4, we choose $s_0 = 0$, and set $\theta_t^X := \theta_t^{s_0}$. Then θ_t^X is a germ of local smooth open embeddings defined for $|t|$ sufficiently small, which obeys

$$\theta_s^X \circ \theta_t^X = \theta_{s+t}^X$$

for $|s|, |t|$ sufficiently small.

In the next few examples, we outline some applications of flows on manifolds in differential topology. We leave the details out as exercises.

Example 1.8. Let $p, q \in M$ be two distinct points, which can be connected by a continuous path. Then it is easy to show that there is a smooth curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$, and furthermore, γ can be arranged to be an embedding. With this understood, we extend the tangent vectors $\gamma'(t)$ to a smooth vector field over a compact neighborhood of the curve $\gamma(t)$, $t \in [0, 1]$, and then use a partition of unity argument to extend it over to the whole manifold M by zero. Call this vector field X, which is of compact support, hence complete by Lemma 1.2. Then observe that the global flow θ associated to X as given in Theorem 1.7 has the property that $\theta_1(p) = q$. As a consequence, the diffeomorphism group of a connected manifold acts on the manifold transitively. As a topological application, the connected sum operation of smooth manifolds is well-defined.

Example 1.9. Let $f \in C^\infty(M)$, such that for any $c \in \mathbb{R}$, the level set $M_c := f^{-1}(c)$ is compact. Pick a Riemannian metric g on M, we define the gradient vector field of f, denoted by grad f, by the following formula

$$g(\text{grad } f, Y) = df(Y), \quad \forall Y \in \mathcal{X}(M).$$

Suppose $[a,b]$ is an interval which contains no critical values of f. Then in an open neighborhood U of $f^{-1}([a,b])$, we define $X := \text{grad } f / g(\text{grad } f, \text{grad } f) \in \mathcal{X}(U)$. We observe that $df(X) = 1$ on U. Using the flow associated to X, one can show that there is a diffeomorphism $\psi : f^{-1}([a,b]) \to M_a \times [a,b]$, such that for any $c \in [a,b]$, the map ψ restricts to a diffeomorphism from $M_c := f^{-1}(c)$ to $M_a \times \{c\}$.

More generally, let $F : M \to N$ be a smooth map such that for any $q \in N$, the level set $F^{-1}(q)$ is compact. Suppose the set of regular values of F in N is path-connected. Then for any regular values $q_0, q_1 \in N$ of F, we can choose a smooth embedding $\gamma : [0,1] \to N$, such that $\gamma(0) = q_0$, $\gamma(1) = q_1$, and $\gamma(t) \in N$ is a regular value of F for any $t \in [0,1]$. Then there is a diffeomorphism $\psi : F^{-1}(\gamma([0,1])) \to F^{-1}(q_0) \times [0,1]$, such that ψ maps $F^{-1}(\gamma(t))$ diffeomorphically onto $F^{-1}(q_0) \times \{t\}$ for any $t \in [0,1]$.

Example 1.10. Let M be a compact closed, connected, 1-dimensional smooth manifold. We shall explain how to prove that M is diffeomorphic to $\mathbb{S}^1$. First, by passing to a double cover if necessary, we may assume M is orientable, which means TM is trivial as $\dim M = 1$. Pick a $X \in \mathcal{X}(M)$ which is nowhere vanishing. Since M is compact, X is complete. Then the global flow θ associated to X gives rise to a diffeomorphism from $\mathbb{S}^1$ to M. Finally, if a double cover of M is diffeomorphic to $\mathbb{S}^1$, so is M.

2. The Lie derivative, Cartan's formula, and Lie groups revisited

Let $X \in \mathcal{X}(M)$, and for simplicity, let θ_t denote the local flows generated by X (i.e., θ_t^X in Section 1), which are germs of local open smooth embeddings defined for $|t|$ sufficiently small, and which satisfy $\theta_s \circ \theta_t = \theta_{s+t}$.

Definition 2.1. Let ω denote a covariant tensor field on M, and $Y \in \mathcal{X}(M)$. We define the Lie derivative L_X of ω and Y as follows: for any $p \in M$,

$$(L_X\omega)(p) = \lim_{t\to 0} \frac{\theta_t^*(\omega(\theta_t(p))) - \omega(p)}{t}, \quad (L_X Y)(p) = \lim_{t\to 0} \frac{(\theta_{-t})_*(Y(\theta_t(p))) - Y(p)}{t}.$$

The following are straightforward from the definitions.

Lemma 2.2. *(1)* $\frac{d}{dt}(\theta_t^*\omega) = \theta_t^*(L_X\omega)$, *(2)* $\frac{d}{dt}((\theta_{-t})_*Y) = (\theta_{-t})_*(L_X Y)$.

Proof. (1) Note that $L_X\omega = \frac{d}{dt}(\theta_t^*\omega)|_{t=0}$. It follows that

$$\frac{d}{dt}(\theta_t^*\omega) = \frac{d}{ds}(\theta_{s+t}^*\omega)|_{s=0} = \frac{d}{ds}(\theta_t^* \circ \theta_s^*\omega)|_{s=0} = \theta_t^*(\frac{d}{ds}(\theta_s^*\omega)|_{s=0}) = \theta_t^*(L_X\omega).$$

The proof for (2) is similar. ∎

Proposition 2.3. *(Properties of L_X) Let $X \in \mathcal{X}(M)$.*

(1) $L_X f = Xf$, *for any* $f \in C^\infty(M)$.

(2) $L_X(\omega_1 \otimes \omega_2) = L_X\omega_1 \otimes \omega_2 + \omega_1 \otimes L_X\omega_2$.
(3) $L_X(\omega_1 \wedge \omega_2) = L_X\omega_1 \wedge \omega_2 + \omega_1 \wedge L_X\omega_2$.
(4) $L_X(\omega(Y_1, \cdots, Y_k)) = (L_X\omega)(Y_1, \cdots, Y_k) + \omega(L_XY_1, \cdots, Y_k) + \cdots + \omega(Y_1, \cdots, L_XY_k)$.
(5) $L_X(d\omega) = d(L_X\omega)$.
(6) $L_X(i_Y\omega) = i_{L_XY}\omega + i_YL_X\omega$.
(7) $L_XY = [X, Y]$.

Proof. (1) $L_Xf(p) = \lim_{t\to 0} \frac{f(\theta_t(p))-f(p)}{t} = \lim_{t\to 0} \frac{f(\gamma_p(t))-f(\gamma_p(0))}{t} = X_p(f)$, $\forall p \in M$, because $X_p = \gamma_p'(0)$. Hence $L_Xf = Xf$ for any $f \in C^\infty(M)$.

(2)-(4) are consequences of the Leibniz Rule. (5) follows from the commutativity of d and θ_t^*. (6) uses the identity $i_Y\theta_t^*\omega = \theta_t^*(i_{(\theta_t)_*Y}\omega)$.

For (7), we apply (4) with $\omega = df$, $\forall f \in C^\infty(M)$, which gives

$$L_X(df(Y)) = L_X(df)(Y) + df(L_XY).$$

Applying (1) and (5), we obtain $X(Yf) = Y(Xf) + L_XY(f)$, which implies that $L_XY(f) = [X, Y](f)$ for any $f \in C^\infty(M)$. Hence $L_XY = [X, Y]$. □

The fact that $L_XY = [X, Y]$, $\forall X, Y \in \mathcal{X}(M)$, has some important consequences. The following are straightforward from the properties of the Lie bracket.

Corollary 2.4. *Let $X, Y, Z \in \mathcal{X}(M)$, $f \in C^\infty(M)$.*
(1) $L_XY = -L_YX$.
(2) $L_X[Y, Z] = [L_X, Y] + [Y, L_XZ]$.
(3) $L_{[X,Y]}Z = L_X(L_YZ) - L_Y(L_XZ)$.
(4) $L_X(fY) = (L_Xf)Y + fL_XY$.
(5) *If X, Y are F-related to $\tilde{X}, \tilde{Y}$ respectively, then so is L_XY to $L_{\tilde{X}}\tilde{Y}$.*

On the other hand, we observe that Lemma 2.2(1) holds true more generally.

Lemma 2.5. *Let X_t be a time-dependent vector field, and let $\theta_t^{s_0}$ be the local open embeddings associated to X_t (cf. Lemma 1.4). Then for any covariant tensor field ω,*
(1) $L_{X_t}\omega = \frac{d}{ds}(\theta_s^t)^*\omega|_{s=t}$.
(2) $\frac{d}{dt}(\theta_t^{s_0})^*\omega = (\theta_t^{s_0})^*L_{X_t}\omega$.

Proof. For (1), we observe that Proposition 2.3 (1)-(3), (5) determines the Lie derivative L_X on covariant tensor fields ω uniquely by the local expression of ω. With this understood, $L_{X_t}\omega = \frac{d}{ds}(\theta_s^t)^*\omega|_{s=t}$ follows easily by checking that $\frac{d}{ds}(\theta_s^t)^*|_{s=t}$ obeys Proposition 2.3 (1)-(3), (5) on the covariant tensor fields.

For (2), we note $\theta_s^t \circ \theta_t^{s_0} = \theta_s^{s_0}$, and apply the argument in the proof of Lemma 2.2. □

Theorem 2.6. *(Cartan's Formula). For any $\omega \in \Omega^k(M)$, $X \in \mathcal{X}(M)$,*

$$L_X\omega = i_X(d\omega) + d(i_X\omega).$$

In other words, $L_X = i_X \circ d + d \circ i_X$.

Proof. Set $Q_X := i_X \circ d + d \circ i_X$. We need to check that Q_X obeys Proposition 2.3 (1),(3),(5). Then Cartan's formula follows from the local expression of ω.

For (1), for any $f \in C^\infty(M)$, $Q_X f = i_X(df) + d(i_X f) = Xf$, as $i_X f = 0$ trivially. For (2), let $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$, we compute

$$i_X(d(\omega\wedge\eta)) = i_X(d\omega\wedge\eta+(-1)^k\omega\wedge d\eta) = i_X d\omega\wedge\eta+(-1)^{k+1}d\omega\wedge i_X\eta+(-1)^k i_X\omega\wedge d\eta+\omega\wedge i_X(d\eta),$$

and

$$d(i_X(\omega\wedge\eta)) = d(i_X\omega\wedge\eta+(-1)^k\omega\wedge i_X\eta) = d(i_X\omega)\wedge\eta+(-1)^{k-1}i_X\omega\wedge d\eta+(-1)^k d\omega\wedge i_X\eta+\omega\wedge d(i_X\eta).$$

It follows immediately that $Q_X(\omega \wedge \eta) = Q_X\omega \wedge \eta + \omega \wedge Q_X\eta$, proving (2). Finally,

$$Q_X(d\omega) = d(i_X(d\omega)) = d(i_X(d\omega) + d(i_X\omega)) = d(Q_X\omega),$$

which verifies (5) of Proposition 2.3. Hence $L_X = Q_X = i_X \circ d + d \circ i_X$. ☐

We give some applications in symplectic geometry.

Example 2.7. Let ω be a symplectic structure on M, and let $H \in C^\infty(M)$. We define a smooth vector field X by the formula $i_X\omega = dH$, where we used the fact that ω is non-degenerate. Note that supp X = supp dH, which we assume is compact. Then X is complete.

Let θ_t be the global flow associated to X. Observe that $L_X\omega = i_X(d\omega)+d(i_X\omega) = 0$. This implies that $\frac{d}{dt}\theta_t^*\omega = \theta_t^*(L_X\omega) = 0$, so that $\theta_t^*\omega = \theta_0^*\omega = \omega$. In other words, the flow θ_t preserves the symplectic structure ω, which is called a **Hamiltonian flow**.

More generally, let H_t, $t \in [0,1]$, be a time-dependent smooth family of smooth functions on M. We let X_t, $t \in [0,1]$, be the time-dependent vector field defined by $i_{X_t}\omega = dH_t$. Then we continue to have $L_{X_t}\omega = i_{X_t}(d\omega) + d(i_{X_t}\omega) = 0$. Denote by ψ_t the smooth family of maps $\theta_t^{s_0}$ associated to X_t, with $s_0 = 0$, we have by Lemma 2.5 that $\frac{d}{dt}\psi_t^*\omega = \psi_t^*(L_{X_t}\omega) = 0$. Consequently, $\psi_t^*\omega = \psi_0^*\omega = \omega$, as $\psi_0 = Id$. The maps ψ_t are called **Hamiltonian symplectomorphisms**.

A (time-dependent) smooth vector field X_t is called a **symplectic vector field** if $L_{X_t}\omega = 0$, or equivalently, the maps ψ_t associated to X_t preserves the symplectic structure ω. Cartan's formula implies that X_t is a symplectic vector field if and only if $i_{X_t}\omega$ is closed for each t, i.e., $d(i_{X_t}\omega) = 0$.

Theorem 2.8. *(Darboux's Theorem) Let ω be a symplectic structure on M. For any $p \in M$, there is an open neighborhood U of p, and a system of local coordinate functions on U, denoted by $x^1, y^1, x^2, y^2, \cdots, x^m, y^m$, where $\dim M = 2m$, such that*

$$\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2 + \cdots + dx^m \wedge dy^m.$$

Proof. First, we can choose a local coordinate system $x^1, y^1, x^2, y^2, \cdots, x^m, y^m$ near p, such that $\omega(p) = \omega_0(p)$, where $\omega_0 = dx^1 \wedge dy^1 + dx^2 \wedge dy^2 + \cdots + dx^m \wedge dy^m$. We consider a smooth family of differential 2-forms $\omega_t := (1-t)\omega_0 + t\omega$. Note that $\omega_t = \omega_0$ at $t = 0$, $\omega_t = \omega$ at $t = 1$, and $d\omega_t = 0$ for all $t \in [0,1]$. Furthermore, note that $\omega_t(p) = \omega_0(p)$ for all t, so that there is an open neighborhood U of p such that ω_t is non-degenerate for all $t \in [0,1]$. Finally, observe that $\frac{d}{dt}\omega_t = \omega - \omega_0 = d\alpha$, because $d\omega = d\omega_0 = 0$ and locally the de Rham cohomology is trivial. We can even choose α such that $\alpha(p) = 0$.

We let X_t be the time-dependent vector field defined by $i_{X_t}\omega_t = -\alpha$. Observe that $X_t(p) = 0$ for all $t \in [0,1]$, as $\alpha(p) = 0$. Let θ_t, $t \in [0,1]$, be the smooth family of open embeddings associated to X_t in Theorem 1.5 (with $K = \{p\}$), where $\theta_t : V \to U$ for a neighborhood V of p. Then $\frac{d}{dt}\theta_t^*\omega_t = \theta_t^*(L_{X_t}\omega_t + \frac{d}{dt}\omega_t) = \theta_t^*(d(i_{X_t}\omega_t) + d\alpha) = 0$. Consequently, $\theta_t^*\omega_t = \theta_0^*\omega_0 = \omega_0$, as $\theta_0 = Id$. Now if we set $\psi := \theta_1^{-1}$, then

$$\omega = \psi^*\omega_0 = d\tilde{x}^1 \wedge d\tilde{y}^1 + d\tilde{x}^2 \wedge d\tilde{y}^2 + \cdots + d\tilde{x}^m \wedge d\tilde{y}^m,$$

where $\tilde{x}^i := \psi^*(x^i)$, $\tilde{y}^i := \psi^*(y^i)$, $i = 1, 2, \cdots, m$, is a new system of local coordinate functions near p. □

Lie groups revisited: Let G be a Lie group. We first consider the flows generated by the left-invariant vector fields on G. But first, observe

Lemma 2.9. *Let $F : M \to \tilde{M}$ be any smooth map, and X_t, $\tilde{X}_t$ be time-dependent smooth vector fields on M, $\tilde{M}$ respectively, such that for each t, X_t and $\tilde{X}_t$ are F-related. Let $\theta_t^{s_0}$, $\tilde{\theta}_t^{s_0}$ denote the open embeddings associated to X_t, $\tilde{X}_t$ respectively (cf. Lemma 1.4). Then $F \circ \theta_t^{s_0} = \tilde{\theta}_t^{s_0} \circ F$ holds for any t.*

Proof. Let $\tilde{p} := F(p)$, $\forall p \in M$, let $\gamma_{s_0,p}(t)$, $\tilde{\gamma}_{s_0,\tilde{p}}(t)$ be the integral curves of X_t, $\tilde{X}_t$, with initial value conditions $\gamma_{s_0,p}(s_0) = p$, $\tilde{\gamma}_{s_0,\tilde{p}}(s_0) = \tilde{p}$. Then note that

$$\frac{d}{dt}F(\gamma_{s_0,p}(t)) = F_*(\gamma'_{s_0,p}(t)) = F_*(X_t(\gamma_{s_0,p}(t))) = \tilde{X}_t(F(\gamma_{s_0,p}(t))).$$

With $F(\gamma_{s_0,p}(s_0)) = F(p) = \tilde{p}$, it follows by uniqueness that $\tilde{\gamma}_{s_0,\tilde{p}}(t) = F(\gamma_{s_0,p}(t))$. By the definition of $\theta_t^{s_0}$ and $\tilde{\theta}_t^{s_0}$, this is the same as $F \circ \theta_t^{s_0} = \tilde{\theta}_t^{s_0} \circ F$. □

Lemma 2.10. *Let $X \in Lie(G)$ be a left-invariant vector field on G. Then X is complete. Moreover, if θ_t^X denotes the global flow generated by X, then for any $g \in G$, $L_g \circ \theta_t^X = \theta_t^X \circ L_g$.*

Proof. Let $\gamma_e(t)$, $-\epsilon_0 < t < \epsilon_0$, be the integral curve of X with initial value condition $\gamma_e(0) = e$. Suppose to the contrary that X is not complete. Then there is an $h \in G$ such that the integral curve $\gamma_h(t)$ of X with initial value condition $\gamma_h(0) = h$ is only defined maximally on an interval $(a, b) \neq \mathbb{R}$. Without loss of generality, assume $b < \infty$. We let $c := b - \epsilon_0/2$, and let $g := \gamma_h(c)$. Then we observe that, as $(L_g)_*X = X$, $\gamma_g(t) = g\gamma_e(t)$, $-\epsilon_0 < t < \epsilon_0$, is an integral curve of X with initial value condition $\gamma_g(0) = g$ by Lemma 2.9. By translation invariance (cf. Lemma 1.6) and the **Uniqueness** property, we can extend $\gamma_h(t)$ over to the larger interval $(a, b + \epsilon_0/2)$, which is a contradiction to the maximality of (a, b). Hence X is complete. $L_g \circ \theta_t^X = \theta_t^X \circ L_g$ for any $g \in G$ follows from $(L_g)_*X = X$ for any $g \in G$ by Lemma 2.9. □

Corollary 2.11. *Let $X \in Lie(G)$.*

(1) $\theta_t^X(e)$, $t \in \mathbb{R}$, is a 1-parameter subgroup of G, i.e., for any $s, t \in \mathbb{R}$,

$$\theta_s^X(e)\theta_t^X(e) = \theta_{s+t}^X(e).$$

(2) The flow $\theta_t^X : G \to G$ is given by the right translation $R_{\theta_t^X(e)}$.

Example 2.12. Consider $G = GL(n, \mathbb{R})$. For any $A \in M(n, \mathbb{R})$, the left-invariant vector field $\tilde{A}$ determined by A, i.e., $\tilde{A}(e) = A$, is given by $X \in GL(n, \mathbb{R}) \mapsto XA \in M(n, \mathbb{R})$. With this understood, it is easy to check that e^{tA} is the integral curve which passes through e at $t=0$, i.e., $\theta_t^{\tilde{A}}(e) = e^{tA}$. For any $B \in GL(n,\mathbb{R})$, the integral curve which passes through B at $t=0$ is Be^{tA}, i.e., $\theta_t^{\tilde{A}}(B) = Be^{tA} = R_{e^{tA}}(B)$.

The above example motivates the definition of the exponential map of G, denoted by $exp: Lie(G) \to G$, defined by $exp(X) = \theta_1^X(e)$, $\forall X \in Lie(G)$. It is easy to check that $\theta_t^X(e) = exp(tX)$.

For any $g \in G$, the automorphism $h \mapsto ghg^{-1}$ of G induces an automorphism of the Lie algebra $Lie(G)$, to be denoted by $Ad(g): Lie(G) \to Lie(G)$. The corresponding Lie group representation $Ad: G \to GL(Lie(G))$ is called the **Adjoint Representation** of G. The corresponding Lie algebra representation is denoted by $ad: Lie(G) \to End(Lie(G))$.

Lemma 2.13. *(1) For any $g \in G$, $Ad(g)(X) = (R_{g^{-1}})_*(X)$, $\forall X \in Lie(G)$.*
(2) For any $X \in Lie(G)$, $ad(X)(Y) = [X,Y]$, $\forall Y \in Lie(G)$.

Proof. For (1), $Ad(g)(X)(e) = \frac{d}{dt}(g\theta_t^X(e)g^{-1})|_{t=0} = (R_{g^{-1}})_*(X(g)) = (R_{g^{-1}})_*(X)(e)$. Hence $Ad(g)(X) = (R_{g^{-1}})_*(X)$.

For (2), $ad(X)(Y)(e) = \frac{d}{dt}(Ad(\theta_t^X(e))(Y)(e))|_{t=0} = \frac{d}{dt}(R_{\theta_{-t}^X(e)})_*(Y(\theta_t^X(e)))|_{t=0}$. By Corollary 2.11(2), $\frac{d}{dt}(R_{\theta_{-t}^X(e)})_*(Y(\theta_t^X(e)))|_{t=0} = (L_XY)(e) = [X,Y]_e$. This proves that $ad(X)(Y) = [X,Y]$. □

Example 2.14. For the case of $G = GL(n,\mathbb{R})$, $Ad: GL(n,\mathbb{R}) \to GL(M(n,\mathbb{R}))$ is given by $Ad(X)(A) = XAX^{-1}$, $\forall X \in GL(n,\mathbb{R})$, $A \in M(n,\mathbb{R})$. Furthermore, $ad: M(n,\mathbb{R}) \to End(M(n,\mathbb{R}))$ is given by $ad(A)(B) = AB - BA$, $\forall A, B \in M(n,\mathbb{R})$.

Exercise: Note that $ad: Lie(G) \to End(Lie(G))$ is a Lie algebra homomorphism, which means $ad([X,Y]) = ad(X)ad(Y) - ad(Y)ad(X)$. Check that this is equivalent to the Jacobi identity.

3. Tangent distributions, Frobenius theorem, and contact structures

For any $X \in \mathcal{X}(M)$, let θ_t^X denote the local flow generated by X. Then the identity $L_XY = [X,Y]$, $\forall X, Y \in \mathcal{X}(M)$, has the following important consequence:

$$\theta_s^X \circ \theta_t^Y = \theta_t^Y \circ \theta_s^X, \ \ \forall s,t, \quad \text{if } [X,Y] = 0.$$

To see this, we note that $\frac{d}{ds}((\theta_{-s}^X)_*Y) = (\theta_{-s}^X)_*(L_XY) = 0$, which implies that $(\theta_{-s}^X)_*Y = Y$, $\forall s$. Our claim follows easily from Lemma 2.9.

Lemma 3.1. *Let M be a smooth manifold of dimension n. Let $p \in M$ be a point, and $X_1, X_2, \cdots, X_k$, $k \leq n$, be smooth vector fields defined in a neighborhood of p such that $X_1(p), X_2(p), \cdots, X_k(p)$ are linearly independent in T_pM. If furthermore $[X_i, X_j] = 0$ for any $i, j = 1, 2, \cdots, k$, then there is a neighborhood W of p with a system of local coordinate functions $x^1, x^2, \cdots, x^n$ such that $X_i = \frac{\partial}{\partial x^i}$ for $i = 1, 2, \cdots, k$.*

Proof. First, we choose a local coordinate chart (U, ϕ) centered at p, with local coordinate functions $y^1, y^2, \cdots, y^n$ such that $\{X_1(p), X_2(p), \cdots, X_k(p), \frac{\partial}{\partial y^{k+1}}|_p, \cdots, \frac{\partial}{\partial y^n}|_p\}$ form a basis of T_pM. For each $i = 1, 2, \cdots, k$, we inductively define maps θ_i as follows: there is a neighborhood $U_1 \subset U$ and $\epsilon_1 > 0$ such that $\theta_1: (-\epsilon_1, \epsilon_1) \times U_1 \to U$, where

$\theta_1(t,q) = \theta_t^{X_1}(q)$ is defined, and there is a neighborhood $U_2 \subset U_1$ and $\epsilon_2 > 0$ such that $\theta_2 : (-\epsilon_2, \epsilon_2) \times U_2 \to U_1$, where $\theta_2(t,q) = \theta_t^{X_2}(q)$ is defined. Continuing in this process, at the last step there is a neighborhood $U_k \subset U_{k-1}$ and $\epsilon_k > 0$ such that $\theta_k : (-\epsilon_k, \epsilon_k) \times U_k \to U_{k-1}$, where $\theta_k(t,q) = \theta_t^{X_k}(q)$ is defined. We set $V = U_k$ and $\epsilon = \min(\epsilon_i)$. Moreover, let $S \subset V$ be the slice defined by $y^1 = y^2 = \cdots = y^k = 0$.

With the proceeding understood, for any $i = 1, 2, \cdots, k$, and any $t \in (-\epsilon, \epsilon)$, let $\theta_i^t : U_i \to U_{i-1}$ be the restriction of θ_i to $\{t\} \times U_i$. Define $\psi : \prod_{i=1}^k (-\epsilon, \epsilon) \times \phi(S) \to U$ by the following formula:

$$\psi(t_1, t_2, \cdots, t_k, t_{k+1}, \cdots, t_n) = \theta_1^{t_1} \circ \theta_2^{t_2} \circ \cdots \circ \theta_k^{t_k} \circ \phi^{-1}(0, 0, \cdots, 0, t_{k+1}, \cdots, t_n).$$

So far we have not used the assumptions in the lemma. Now consider the fact that $\{X_1(p), X_2(p), \cdots, X_k(p), \frac{\partial}{\partial y^{k+1}}|_p, \cdots, \frac{\partial}{\partial y^n}|_p\}$ is a basis of T_pM. It is easy to see that this implies that ψ_* is an isomorphism from the tangent space at $(0, 0, \cdots, 0) \in \prod_{i=1}^k (-\epsilon, \epsilon) \times \phi(S)$ onto T_pM. By the inverse function theorem, ψ is a local diffeomorphism from a neighborhood of $(0, 0, \cdots, 0)$ onto a neighborhood W of p. Secondly, the assumption that $[X_i, X_j] = 0$ for any $i, j = 1, 2, \cdots, k$ implies the commutativity $\theta_i^{t_i} \circ \theta_j^{t_j} = \theta_j^{t_j} \circ \theta_i^{t_i}$ for any $i, j = 1, 2, \cdots, k$, which in turn implies easily that $\psi_*(\frac{\partial}{\partial t_i}) = X_i$ for any $i = 1, 2, \cdots, k$. It follows easily that (W, ψ^{-1}) is a local coordinate chart centered at p with the desired property. ∎

Definition 3.2. Let M be a smooth manifold of dimension n.

(1) A k-dimensional tangent distribution D on M, $k \leq n$, is an assignment $p \mapsto D_p$, for $p \in M$, where $D_p \subset T_pM$ is a k-dimensional subspace. D is said to be smooth if for any $p \in M$, there are smooth vector fields $X_1, X_2, \cdots, X_k$ defined in a neighborhood U of p such that for any $q \in U$, $X_1(q), X_2(q), \cdots, X_k(q)$ is a basis of D_q. Furthermore, such a D is called **integrable** (or **involutive**) if $[X_i, X_j](q) \in D_q$ for any $q \in D$, $i, j = 1, 2, \cdots, k$.

(2) An immersed submanifold (i.e., an injective immersion) $N \subset M$ is called an **integral manifold** of D if for any $p \in N$, $T_pN = D_p$.

Note that a smooth tangent distribution $D = \sqcup_{p \in M} D_p$ on M forms a (smooth) sub-bundle of TM.

It is easy to see that if for any $p \in M$, there is an integral manifold N of D such that $p \in N$, then D must be integrable. The converse is given by the following

Theorem 3.3. *(The Frobenius Theorem) Let D be a k-dimensional, integrable, smooth tangent distribution on M of dimension n. Then for any $p \in M$, there exists a local coordinate chart U centered at p, with local coordinate functions (x^i), such that each slice in U defined by $x^{k+1} \equiv c_1, \cdots, x^n \equiv c_{n-k}$ is an integral manifold of D.*

Proof. We will show that for any $p \in M$, there is a neighborhood W of p and a set of smooth vector fields $X_1, X_2, \cdots, X_k$ on W, such that $X_1(q), X_2(q), \cdots, X_k(q)$ is a basis of D_q for any $q \in W$, and $[X_i, X_j] = 0$ on W for any $i, j = 1, 2, \cdots, k$. Then the theorem follows from Lemma 3.1.

To this end, we fix a local coordinate chart (U, ϕ) centered at p, with local coordinate functions (x^i) on U. Without loss of generality, we assume $T_pM = D_p \oplus$

Span $(\frac{\partial}{\partial x^{k+1}}|_p, \cdots, \frac{\partial}{\partial x^n}|_p)$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first k coordinates. We consider the smooth map $F : U \to \mathbb{R}^k$, where $F := \pi \circ \phi$. We observe that at p, $F_* : D_p \to T_0\mathbb{R}^k$ is an isomorphism by the assumption that $T_pM = D_p \oplus \text{Span}\,(\frac{\partial}{\partial x^{k+1}}|_p, \cdots, \frac{\partial}{\partial x^n}|_p)$. Consequently, there is a smaller neighborhood W of p such that for any $q \in W$, $F_* : D_q \to T_{F(q)}\mathbb{R}^k$ is an isomorphism.

For each $i = 1, 2, \cdots, k$, we let $X_i \in D$ be the inverse image of $\frac{\partial}{\partial x^i}$ under F_*. Then clearly, for any $q \in W$, $X_1(q), X_2(q), \cdots, X_k(q)$ is a basis of D_q. Furthermore, for any $i, j = 1, 2, \cdots, k$, note that $F_*([X_i, X_j]) = [F_*(X_i), F_*(X_j)] = [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$. On the other hand, by the assumption that D is integrable, $[X_i, X_j] \in D$, we must have $[X_i, X_j] = 0$ because F_* is injective when restricted to D. Hence the theorem.

□

Example 3.4. We consider a 2-dimensional tangent distribution D on $\mathbb{R}^3$, $D = \text{Span}\,(X, Y)$, where $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$. D is integrable as $[X, Y] = -Y$ can be easily verified. Let's find the integral manifolds of D.

At $0 = (0,0,0) \in \mathbb{R}^3$, $X = \frac{\partial}{\partial y}$, $Y = \frac{\partial}{\partial x}$, so $D_0 \oplus \text{Span}\,(\frac{\partial}{\partial z}) = T_0\mathbb{R}^3$. Consider $F : \mathbb{R}^3 \to \mathbb{R}^2$, given by $F(x, y, z) = (x, y)$, then $F_* : D_q \to T_{F(q)}\mathbb{R}^2$ is an isomorphism for q near 0. Let $X_1, X_2 \in D$ such that $F_*(X_1) = \frac{\partial}{\partial x}$, $F_*(X_2) = \frac{\partial}{\partial y}$. We shall determine X_1, X_2 explicitly.

Let a, b, c, d, u, v be smooth functions such that

$$X_1 = \frac{\partial}{\partial x} + u\frac{\partial}{\partial z} = aX + bY, \quad X_2 = \frac{\partial}{\partial y} + v\frac{\partial}{\partial z} = cX + dY.$$

Then $a = 0$, $b = 1$, $c = 1$, $d = -x$, and $X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$, $X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$.

Next we determine the flows of X_1, X_2. Let θ_1^t be the flow generated by X_1, and write $\theta_1^t(x, y, z) = (x_1(t), y_1(t), z_1(t))$. Then $x_1'(t) = 1$, $y_1'(t) = 0$, and $z_1'(t) = y_1(t)$. It follows easily that

$$x_1(t) = x + t, \quad y_1(t) = y, \quad z_1(t) = z + yt.$$

A similar calculation shows that the flow generated by X_2 is given by $\theta_2^t(x, y, z) = (x, y + t, z + xt)$. Now computing the map ψ, we have

$$\psi(t_1, t_2, t_3) = \theta_1^{t_1} \circ \theta_2^{t_2}(0, 0, t_3) = \theta_1^{t_1}(0, t_2, t_3) = (t_1, t_2, t_3 + t_2 t_1) = (x, y, z).$$

Consequently, $\psi^{-1}(x, y, z) = (x, y, z - xy)$, and the integral manifolds of D are given by $z - xy = \textit{constant}$. (One easily check that $X(z - xy) = 0$, $Y(z - xy) = 0$.)

We remark that there is an alternative way to determine the integral manifolds of D. We consider 1-forms α such that $\alpha(X) = \alpha(Y) = 0$. If we write $\alpha = f dx + g dy + h dz$ for some smooth functions f, g, h. Then $\alpha(X) = \alpha(Y) = 0$ gives

$$fx + g + hx(y+1) = 0, \quad f + hy = 0.$$

Combining the two equations, we get $g + hx = 0$. Choosing $h = 1$, we have $f = -y$, $g = -x$, and $\alpha = -y dx - x dy + dz = d(z - xy)$. It follows that the integral manifolds of D are given by $z - xy = \textit{constant}$.

Set $\Omega^*(M) := \oplus_{k=0}^n \Omega^k(M)$. Then $\Omega^*(M)$ is an algebra under the wedge product $\wedge$.

Definition 3.5. (1) A linear subspace $\mathcal{J} \subset \Omega^*(M)$ is called an **ideal** if for any $\omega \in \mathcal{J}$, $\eta \wedge \omega \in \mathcal{J}$ for any $\eta \in \Omega^*(M)$. $\mathcal{J}$ is called a **differential ideal** if $d(\mathcal{J}) \subset \mathcal{J}$, i.e, for any $\omega \in \mathcal{J}$, $d\omega \in \mathcal{J}$.

(2) Let D be a smooth tangent distribution on M. A k-form $\omega \in \Omega^k(M)$ is said to **annihilate** D if $\omega(X_1, X_2, \cdots, X_k) = 0$ for any $X_1, X_2, \cdots, X_k \in D$. Denote by $\mathcal{J}^k(D)$ the space of annihilating k-forms, and set $\mathcal{J}(D) = \oplus_{k=0}^n \mathcal{J}^k(D)$.

Lemma 3.6. *Let D be a smooth tangent distribution of co-dimension k on M (i.e., $k = \dim M - \dim D$).*

1. *For any $p \in M$, there is a neighborhood U of p, and a set of 1-forms $\alpha_1, \alpha_2, \cdots, \alpha_k$ on U, such that*
$$D_q = \ker \alpha_1(q) \cap \ker \alpha_2(q) \cap \cdots \cap \ker \alpha_k(q), \quad \forall q \in U.$$
2. *A r-form $\omega \in \mathcal{J}^r(D)|_U$ if and only if there are $(r-1)$-forms $\beta_1, \beta_2, \cdots, \beta_k$ on U, such that $\omega = \sum_{i=1}^k \beta_i \wedge \alpha_i$ on U.*
3. *D is integrable if and only if $d\alpha_i \in \mathcal{J}^2(D)|_U$ for any $i = 1, 2, \cdots, k$.*

The 1-forms $\alpha_1, \alpha_2, \cdots, \alpha_k$ are called **local defining 1-forms of** D.

Proof. (1) Let $X_1, X_2, \cdots, X_{n-k}$ be smooth vector fields defined in a neighborhood U of p such that for any $q \in U$, $X_1(q), X_2(q), \cdots, X_{n-k}(q)$ is a basis of D_q. We add smooth vector fields $X_{n-k+1}, \cdots, X_n$ to them, and shrink U if necessary, so that $X_1, X_2, \cdots, X_n$ is a local frame of TM over U. Let $\omega_1, \omega_2, \cdots, \omega_n$ be the corresponding local frame of the cotangent bundle T^*M over U which is dual to $X_1, X_2, \cdots, X_n$. We set $\alpha_1 := \omega_{n-k+1}, \cdots, \alpha_k := \omega_n$. Then $\alpha_i(X_j) = 0$ for any $i = 1, 2, \cdots, k$, $j = 1, 2, \cdots, n-k$. This implies that $D_q \subset \ker \alpha_1(q) \cap \ker \alpha_2(q) \cap \cdots \cap \ker \alpha_k(q)$, $\forall q \in U$. On the other hand, for any $X \in \mathcal{X}(U)$ such that
$$X(q) \in \ker \alpha_1(q) \cap \ker \alpha_2(q) \cap \cdots \cap \ker \alpha_k(q), \quad \forall q \in U,$$
i.e., $\alpha_i(X) = 0$ for any $i = 1, 2, \cdots, k$, if we write $X = \sum_{j=1}^n a_j X_j$, where $a_j \in C^\infty(U)$, we must have $a_j \equiv 0$ for $j = n-k+1, \cdots, n$, which implies $X(q) \in D_q$ for any $q \in U$. This proves $D_q = \ker \alpha_1(q) \cap \ker \alpha_2(q) \cap \cdots \cap \ker \alpha_k(q)$, $\forall q \in U$.

(2) Since $\omega_1, \omega_2, \cdots, \omega_n$ is a local frame of T^*M over U, for any $\omega \in \Omega^r(U)$, we can write
$$\omega = \sum_{i_1 < i_2 < \cdots < i_r} \omega_{i_1 i_2 \cdots i_r} \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_r}, \quad \omega_{i_1 i_2 \cdots i_r} \in C^\infty(U).$$
Note that $\omega(X_{i_1}, X_{i_2}, \cdots, X_{i_r}) = \omega_{i_1 i_2 \cdots i_r}$. If there is a multi-index $(i_1, i_2, \cdots, i_r)$ which does not contain any $j \in \{n-k+1, \cdots, n\}$, such that $\omega_{i_1 i_2 \cdots i_r}(q) \neq 0$ for some $q \in U$, then $\omega(X_{i_1}, X_{i_2}, \cdots, X_{i_r})(q) \neq 0$ for some $X_{i_1}, X_{i_2}, \cdots, X_{i_r} \in D_q$, implying ω does not belong to $\mathcal{J}^r(D)|_U$. This shows that if $\omega \in \mathcal{J}^r(D)|_U$, then $\omega = \sum_{i=1}^k \beta_i \wedge \alpha_i$ on U for some $\beta_i \in \Omega^{r-1}(U)$. Conversely, if $\omega = \sum_{i=1}^k \beta_i \wedge \alpha_i$ on U for some $\beta_i \in \Omega^{r-1}(U)$, then $\omega(X_1, X_2, \cdots, X_r) = 0$ for any $X_1, X_2, \cdots, X_r \in D$, hence $\omega \in \mathcal{J}^r(D)|_U$.

(3) For any $X, Y \in \mathcal{X}(U)$, where $X, Y \in D$,
$$d\alpha_i(X, Y) = X(\alpha_i(Y)) - Y(\alpha_i(X)) - \alpha_i([X, Y]) = -\alpha_i([X, Y]).$$
It follows immediately that D is integrable, which means $[X, Y] \in D$, if and only if $d\alpha_i \in \mathcal{J}^2(D)|_U$ for any $i = 1, 2, \cdots, k$.

□

As an easy corollary, we obtain the following characterization of integrability of tangent distributions.

Theorem 3.7. *Let D be a smooth tangent distribution on M. Then $\mathcal{J}(D)$ is an ideal of $\Omega^*(M)$, and moreover, D is integrable if and only if $\mathcal{J}(D)$ is a differential ideal.*

Exercise: Prove Theorem 3.7.

Consider a smooth tangent distribution D on M of co-dimension 1, and let α be a local defining 1-form of D. Then D is integrable if and only if $d\alpha = 0$ on D. At the other end of the spectrum, i.e., the 2-form $d\alpha$ is non-degenerate on D, we arrive at the notion of contact structures. Note that if $d\alpha$ is non-degenerate on D, then D must be even-dimensional, so that M must be odd-dimensional.

Definition 3.8. Let M be a smooth manifold of dimension $2m+1$, and let D be a co-dimension 1 smooth tangent distribution on M. D is called a **contact structure** if for any local defining 1-form α of D, $d\alpha$ is non-degenerate on D. Equivalently, $(d\alpha)^m \wedge \alpha$ is nowhere vanishing. Such a 1-form α is called a **contact form**.

Exercise: Show that $d\alpha$ is non-degenerate on D if and only if $(d\alpha)^m \wedge \alpha$ is nowhere vanishing.

Exercise: Suppose m is odd in Definition 3.8. Show that if M admits a contact structure, then M must be orientable.

Example 3.9. Consider $M = \mathbb{R}^{2n+1}$. Let $D = \ker \alpha$, where $\alpha = dz - \sum_{i=1}^n y_i dx_i$, where $x_1, y_1, \cdots, x_n, y_n, z$ are a system of standard coordinates on $\mathbb{R}^{2n+1}$. Computing $d\alpha$, we get $d\alpha = \sum_{i=1}^n dx_i \wedge dy_i$. It follows easily that

$$(d\alpha)^n \wedge \alpha = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \wedge dz.$$

D is called the **standard contact structure on** $\mathbb{R}^{2n+1}$.

Example 3.10. Consider $M = \mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1}$, the unit sphere. Let D be the co-dimension 1 smooth tangent distribution on $\mathbb{S}^{2m+1}$ which is invariant under the complex structure J on $\mathbb{C}^{m+1}$. Then D is a contact structure on $\mathbb{S}^{2m+1}$, called the **standard contact structure of** $\mathbb{S}^{2m+1}$.

To see this, let $\phi = |z_0|^2 + |z_1|^2 + \cdots + |z_m|^2$. Then $D = \ker \alpha$, where $\alpha := d\phi \circ J$. Furthermore, $d\alpha = -2i \sum_{j=0}^m dz_j \wedge d\bar{z}_j$, which is non-degenerate on the tangent spaces of $\mathbb{C}^{m+1}$. Since D is invariant under J, D_q, $\forall q \in \mathbb{S}^{2m+1}$, is a complex subspace of the tangent space $T_q\mathbb{C}^{m+1}$. It follows that $d\alpha$ must be non-degenerate on D as well. Hence D is a contact structure.

Exercise: Let $\omega = d\lambda$ be a symplectic structure on M, and let $N = \mathbb{R} \times M$. Let z be the coordinate on $\mathbb{R}$. Show that $\alpha := dz + \lambda$ is a contact form on N.

Let α be a contact form on M. It is easy to check that $\omega := d(e^t\alpha)$ is a symplectic structure on $\mathbb{R} \times M$, which is called the **symplectization** of (M, α). Note that $L_{\frac{\partial}{\partial t}} \omega = \omega$. The flow θ_t generated by $\frac{\partial}{\partial t}$ is simply the translations in t. It is easy to see that $\theta_t^* \omega = e^t \omega$.

Exercise: More generally, let ω be a symplectic structure on M. A **Liouville vector field on** M is an $X \in \mathcal{X}(M)$ such that $L_X\omega = \omega$. Show that

(1) The set of Liouville vector fields X on M is in one to one correspondence with the set of 1-forms λ on M such that $\omega = d\lambda$, in the sense that $i_X\omega = \lambda$.

(2) Let $S \subset M$ be a co-dimension 1 embedded submanifold. Then the restriction of λ on S is a contact form if and only if X is a normal vector field along S.

Exercise: Let $S \subset M$ be a compact closed, embedded submanifold of co-dimension 1. Suppose X is a Liouville vector field (i.e., $L_X\omega = \omega$) defined in a neighborhood of S, which is a normal vector field along S. Let $\alpha := i_X\omega$ be the contact form on S. Prove that there is an $\epsilon > 0$, and a smooth map $\psi : (-\epsilon, \epsilon) \times S \to M$, which is a diffeomorphism onto a neighborhood of S in M, such that (i) $\psi(0, p) = p$, $\forall p \in S$, (ii) $\psi^*\omega = d(e^t\alpha)$.

Exercise: Let $S \subset M$ be a compact closed, embedded submanifold of co-dimension 1. S is said to be of **contact type** if there is a contact form α on S such that $\omega|_S = d\alpha$. Show that there is a Liouville vector field X defined in a neighborhood of S such that $i_X\omega = \alpha$. (Note that X is necessarily normal along S.)

Example 3.11. Consider $M = \mathbb{R}^{2n}$, with the standard symplectic structure $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. Let $X := \frac{1}{2}\sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i})$. Then $L_X\omega_0 = \omega_0$. Note that X is normal along $\mathbb{S}^{2n-1}$. Let $\alpha_0 := i_X\omega_0$ be the contact form on $\mathbb{S}^{2n-1}$. Then $D := \ker \alpha_0$ is the standard contact structure on $\mathbb{S}^{2n-1}$ discussed in Example 3.10.

Let α be a contact form on M. The **Reeb vector field** associated to α, denoted by R_α, is the vector field on M uniquely determined by

$$\alpha(R_\alpha) = 1, \;\; i_{R_\alpha} d\alpha = 0.$$

Exercise: Show that R_α exists for any contact form α. Moreover, the flow θ_t generated by R_α (called the **Reeb flow**) preserves α, i.e., $\theta_t^*\alpha = \alpha$.

Exercise: Let M be a compact closed manifold, and let α_t, $t \in [0, 1]$, be a smooth family of contact forms on M. Let $D_t := \ker \alpha_t$ be the corresponding contact structures on M. Show that there is a time-dependent vector field X_t, where $X_t \in D_t$ for each $t \in [0, 1]$, such that the family of diffeomorphisms $\theta_t : M \to M$ generated by X_t (with $\theta_0 = Id$) have the property that

$$\theta_t^*\alpha_t = e^{f_t}\alpha_0$$

for some smooth family f_t of smooth functions on M. Note that $(\theta_t)_*(D_0) = D_t$ for each $t \in [0, 1]$. (This is usually referred to as Gray's Theorem.)

References

[1] John M. Lee, *Introduction to Smooth Manifolds*, GTM **218**, Springer.
[2] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, GTM, **94**.

University of Massachusetts, Amherst.
E-mail: wchen@math.umass.edu