MATH 703: PART 3: DIFFERENTIAL FORMS AND INTEGRATION

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Contents

1. Differential forms and the exterior derivative

Alternating tensors: Let S_k be the symmetric group of k letters. Then any element $\sigma \in S_k$ defines a permutation $\sigma : (1, 2, \cdots, k) \mapsto (\sigma(1), \sigma(2), \cdots, \sigma(k))$. We let sign $\sigma = 1$ if σ is even (i.e., a product of even number transpositions) and sign $\sigma = -1$ if σ is odd.

Definition 1.1. Let V be a real vector space of dimension n. For any $T \in T^k(V)$, $\sigma \in S_k$, we define

$$T^\sigma(X_1, X_2, \cdots, X_k) := T(X_{\sigma(1)}, X_{\sigma(2)}, \cdots, X_{\sigma(k)}), \ \ \forall X_1, X_2, \cdots, X_k \in V.$$

We say T is **alternating** if for any $\sigma \in S_k$, $T^\sigma = \text{sign } \sigma \cdot T$, and the subspace of $T^k(V)$ consisting of alternating tensors is denoted by $\Lambda^k(V)$.

We fix a basis $e_1, e_2, \cdots, e_n$ of V, and let $\epsilon^1, \epsilon^2, \cdots, \epsilon^n$ be the dual basis of V^*.

Example 1.2. (1) $\Lambda^0(V) = T^0(V) = \mathbb{R}$, $\Lambda^1(V) = T^1(V) = V^*$.

(2) Consider the case of $k = 2$. For any $T \in \Lambda^2(V)$, we write $T = T_{ij}\epsilon^i \otimes \epsilon^j$. Then T is alternating means that $T_{ij} = -T_{ji}$ for any i, j. Consequently,

$$T = \sum_{i<j} T_{ij}(\epsilon^i \otimes \epsilon^j - \epsilon^j \otimes \epsilon^i).$$

Set $\epsilon^{(i,j)} := \epsilon^i \otimes \epsilon^j - \epsilon^j \otimes \epsilon^i$. It follows that $\{\epsilon^{(i,j)} | i < j\}$ is a basis of $\Lambda^2(V)$; in particular, $\dim \Lambda^2(V) = \frac{1}{2}n(n-1)$.

(3) Consider the case of $k = n$. For any $T \in \Lambda^n(V)$,

$$T = T_{i_1 i_2 \cdots i_n}\epsilon^{i_1} \otimes \epsilon^{i_2} \otimes \cdots \otimes \epsilon^{i_n} = T_{12\cdots n} \cdot T_0,$$

where $T_0 := \sum_{\sigma \in S_n} \text{sign } \sigma \cdot (\epsilon^1 \otimes \epsilon^2 \otimes \cdots \otimes \epsilon^n)^\sigma$. Consequently, $\Lambda^n(V)$ is 1-dimensional, generated by the element T_0. Furthermore, observe that for any $X_1, X_2, \cdots, X_n \in V$,

$$T_0(X_1, X_2, \cdots, X_n) = \det(\epsilon^i(X_j)).$$

As a corollary, observe that if $F : V \to V$ is a linear map, then the induced map (i.e., the pull-back) $F^* : \Lambda^n(V) \to \Lambda^n(V)$ is given by the multiplication by $\det F$.

Definition 1.3. (1) For any k, we define a linear map $\text{Alt} : T^k(V) \to T^k(V)$ by

$$\text{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign } \sigma \cdot T^\sigma, \ \ \forall T \in T^k(V).$$

(2) For any multi-index $(i_1, i_2, \cdots, i_k)$, we define

$$\epsilon^{(i_1,i_2,\cdots,i_k)} := k!\text{Alt}(\epsilon^{i_1} \otimes \epsilon^{i_2} \otimes \cdots \otimes \epsilon^{i_k}).$$

It is clear from the definition that for any $T \in \Lambda^k(V)$, $\text{Alt}(T) = T$.

Lemma 1.4. *(1) For any $T \in T^k(V)$, $\tau \in S_k$,*

$$(i) \ \ Alt(T^\tau) = sign\ \tau \cdot Alt(T), \ \ and\ (ii) \ \ (Alt(T))^\tau = sign\ \tau \cdot Alt(T).$$

In particular, the latter implies that $Alt(T)$ is an alternating tensor for any T.

(2) $\epsilon^{(i_1,i_2,\cdots,i_k)} = 0$ if there is a repeated index in $i_1, i_2, \cdots, i_k$. Furthermore, the set $\{\epsilon^{(i_1,i_2,\cdots,i_k)} | i_1 < i_2 < \cdots < i_k\}$ is a basis of $\Lambda^k(V)$. In particular, $\Lambda^k(V) = \{0\}$ if $k > n$.

Proof. (1). For any $T \in T^k(V)$, $\tau \in S_k$,

$$\text{Alt}(T^\tau) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign } \sigma \cdot (T^\tau)^\sigma = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign } \sigma \cdot T^{\tau\sigma} = \text{sign } \tau \cdot \text{Alt}(T),$$

and

$$(\text{Alt}(T))^\tau = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign } \sigma \cdot (T^\sigma)^\tau = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign } \sigma \cdot T^{\sigma\tau} = \text{sign } \tau \cdot \text{Alt}(T).$$

(2). It follows easily from $\text{Alt}(T^\tau) = \text{sign } \tau \cdot \text{Alt}(T)$ that if there is a repeated index in $i_1, i_2, \cdots, i_k$, $\epsilon^{(i_1,i_2,\cdots,i_k)} = 0$. On the other hand, for each multi-index $(i_1, i_2, \cdots, i_k)$ where there is no repeated index, $\epsilon^{(i_1,i_2,\cdots,i_k)} \in \Lambda^k(V)$ and is nonzero, as $\epsilon^{(i_1,i_2,\cdots,i_k)}(e_{i_1}, e_{i_2}, \cdots, e_{i_k}) = 1$. To see that $\{\epsilon^{(i_1,i_2,\cdots,i_k)} | i_1 < i_2 < \cdots < i_k\}$ is a basis of $\Lambda^k(V)$, we note that for any $T \in \Lambda^k(V)$,

$$T = T_{i_1 i_2 \cdots i_k} \epsilon^{i_1} \otimes \epsilon^{i_2} \otimes \cdots \otimes \epsilon^{i_k} = \sum_{i_1 < i_2 < \cdots < i_k} T_{i_1 i_2 \cdots i_k} \cdot \epsilon^{(i_1,i_2,\cdots,i_k)}.$$

□

The wedge product: For any $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, we define the wedge product of ω and η, denoted by $\omega \wedge \eta \in \Lambda^{k+l}(V)$, by the formula

$$\omega \wedge \eta := \frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \eta).$$

Lemma 1.5. *For any multi-indices $(i_1, i_2, \cdots, i_k)$ and $(j_1, j_2, \cdots, j_l)$,*

$$\epsilon^{(i_1,i_2,\cdots,i_k)} \wedge \epsilon^{(j_1,j_2,\cdots,j_l)} = \epsilon^{(i_1,i_2,\cdots,i_k,j_1,j_2,\cdots,j_l)}.$$

Proof. For any $\sigma \in S_k$, $\tau \in S_l$, we note that

$$\text{sign } \sigma \cdot (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k})^\sigma \cdot \text{sign } \tau \cdot (\epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_l})^\tau = \text{sign } (\sigma, \tau) \cdot (\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k} \otimes \epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_l})^{(\sigma,\tau)},$$

where $(\sigma, \tau) \in S_{k+l}$. It follows immediately from Lemma 1.4(1)(i) that

$$\text{Alt}(\epsilon^{(i_1,i_2,\cdots,i_k)} \otimes \epsilon^{(j_1,j_2,\cdots,j_l)}) = k!l!\text{Alt}(\epsilon^{i_1} \otimes \epsilon^{i_2} \otimes \cdots \otimes \epsilon^{i_k} \otimes \epsilon^{j_1} \otimes \epsilon^{j_2} \otimes \cdots \otimes \epsilon^{j_l}),$$

from which Lemma 1.5 follows.

□

The following follows easily from Lemmas 1.4 and 1.5.

Proposition 1.6. *(1)* $(a\omega + a'\omega') \wedge \eta = a\omega \wedge \eta + a'\omega' \wedge \eta$, $\forall a, a' \in \mathbb{R}$.

(2) $(\omega \wedge \eta) \wedge \xi = \omega \wedge (\eta \wedge \xi)$.

(3) For any $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$,

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

(4) For any $w_1, w_2, \cdots, w_k \in V^*$, $X_1, X_2, \cdots, X_k \in V$,

$$w_1 \wedge w_2 \wedge \cdots \wedge w_k(X_1, X_2, \cdots, X_k) = \det(w_i(X_j)).$$

In particular, for any multi-index $(i_1, i_2, \cdots, i_k)$, $\epsilon^{(i_1,i_2,\cdots,i_k)} = \epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \cdots \wedge \epsilon^{i_k}$.

Exercise: Prove Proposition 1.6.

With Proposition 1.6 at hand, we let $\Lambda^*(V) := \oplus_{k=0}^n \Lambda^k(V)$. Then under the wedge product, $(\Lambda^*(V), \wedge)$ is a graded, anticommutative algebra, called the **exterior algebra** of V. Note $\dim \Lambda^*(V) = 2^n$.

Interior multiplication: For any $X \in V$, we define the interior multiplication $i_X : \Lambda^k(V) \to \Lambda^{k-1}(V)$, for any $k > 0$, by the following formula

$$i_X\omega(X_1, X_2, \cdots, X_{k-1}) := \omega(X, X_1, X_2, \cdots, X_{k-1}), \forall \omega \in \Lambda^k(V), X_1, X_2, \cdots, X_{k-1} \in V.$$

Proposition 1.7. *(1) For any* $X \in V$, $i_X \circ i_X = 0$.

(2) $i_{aX+bY}\omega = ai_X\omega + bi_Y\omega$, $\forall \omega \in \Lambda^k(V), X, Y \in V$ *and* $a, b \in \mathbb{R}$.

(3) For any $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, $X \in V$,

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^k \omega \wedge (i_X\eta).$$

Proof. (1) and (2) are straightforward from the definition. For (3), it follows easily from the following formula:

$$i_{e_l}(\epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \cdots \wedge \epsilon^{i_k}) = (-1)^{s+1} \epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \cdots \widehat{\epsilon^{i_s}} \wedge \cdots \wedge \epsilon^{i_k} \text{ if } l = i_s,$$

and $i_{e_l}(\epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \cdots \wedge \epsilon^{i_k}) = 0$ if $l \neq i_s$ for any s.

□

Differential forms: Let M be a smooth manifold of dimension n. For any $0 \leq k \leq n$, let $\Lambda^k M := \sqcup_{p \in M} \Lambda^k(T_pM)$. By standard construction, $\Lambda^k M$ is a smooth vector bundle over M. More concretely, let (x^i) be a system of local coordinate functions over U. Then $\{dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} | i_1 < i_2 < \cdots < i_k\}$ is a local frame of $\Lambda^k M$ over U. A smooth section of $\Lambda^k M$ is called a **differential** k**-form** on M, and the space of differential k-forms is denoted by $\Omega^k(M)$, which is a $C^\infty(M)$-module. Note that $\Omega^k(M)$ is a sub-module of $\mathcal{T}^k M$, the space of covariant k-tensor fields. Locally, a differential k-form can be written as

$$\omega = \sum_{i_1 < i_2 < \cdots < i_k} \omega_{i_1 i_2 \cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \text{ where } \omega_{i_1 i_2 \cdots i_k} \in C^\infty(U).$$

Note that for any smooth map $F : M \to N$, the pull-back map $F^* : \mathcal{T}^k N \to \mathcal{T}^k M$ induces a map $F^* : \Omega^k(N) \to \Omega^k(M)$.

For any $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$, we define the wedge product of ω and η, denoted by $\omega \wedge \eta \in \Omega^{k+l}(M)$, by

$$\omega \wedge \eta(p) := \omega(p) \wedge \eta(p), \ \ \forall p \in M.$$

Then the analog of Proposition 1.6 holds true (with $\mathbb{R}$ replaced by $C^\infty(M)$). Let $\Omega^*(M) := \oplus_{k=0}^n \Omega^k(M)$. Then under the wedge product $\wedge$, $\Omega^*(M)$ is a graded, anti-commutative algebra, called the **exterior algebra** of M. Finally, it is easy to check that for any smooth map $F : M \to N$, $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$.

We observe that in Theorem 3.1 of Part 2, if the multilinear map ψ is alternating, then the resulting tensor field σ is a differential k-form (assuming the case of $l = 0$). This observation allows us to define the interior multiplication for differential forms. More concretely, for any smooth vector field $X \in \mathcal{X}(M)$, we define $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ by the following formula: for any $\omega \in \Omega^k(M)$,

$$(i_X\omega)(X_1, X_2, \cdots, X_{k-1}) := \omega(X, X_1, X_2, \cdots, X_{k-1}), \ \ \forall X_1, X_2, \cdots, X_{k-1} \in \mathcal{X}(M).$$

We note that the analog of Proposition 1.7 holds true (with $\mathbb{R}$ replaced by $C^\infty(M)$). Moreover, for any smooth map $F : M \to N$, $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$, if X, Y are F-related, i.e., $F_*(X_p) = Y_{F(p)}$, $\forall p \in M$, then

$$i_X(F^*\omega) = F^*(i_Y\omega), \ \ \forall \omega \in \Omega^k(N).$$

The exterior derivative: Recall for smooth vector fields, there is an operation called Lie bracket. For differential forms, the corresponding operation is the so-called **exterior derivative**.

Theorem 1.8. *Let M be a smooth manifold. There exists unique $\mathbb{R}$-linear maps, called the exterior derivative, $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for $k \geq 0$, such that*

1. *for any $f \in C^\infty(M) = \Omega^1(M)$, $df \in \Omega^1(M) = \mathcal{T}^1 M$ is the differential of f, i.e., for any $p \in M$, $X \in T_pM$, $df(p)(X) = X(f)$,*
2. *for any $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$,*
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta,$$
3. *$d \circ d = 0 : \Omega^k(M) \to \Omega^{k+2}(M)$ for any $k \geq 0$, and*
4. *for any smooth map $F : M \to N$, $d(F^*\omega) = F^*(d\omega)$, $\forall \omega \in \Omega^k(N)$.*

Proof. We first address the existence part. To this end, we choose a smooth atlas $\{(U_\alpha, \phi_\alpha)\}$, and fix a smooth partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$, and then write any $\omega \in \Omega^k(M)$ as $\omega = \sum_\alpha \omega_\alpha$, where $\omega_\alpha := f_\alpha \omega$, with supp $\omega_\alpha \subset U_\alpha$. We shall define $d\omega := \sum_\alpha d\omega_\alpha$.

With this understood, we shall deal with the special case of $M = U$, where U is a local coordinate chart on M, with local coordinate functions (x^i). In this case, for any differential k-form $\omega \in \Omega^k(U)$, we can write

$$\omega = \sum_{i_1<i_2<\cdots<i_k} \omega_{i_1 i_2 \cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \text{ where } \omega_{i_1 i_2 \cdots i_k} \in C^\infty(U).$$

We define $d\omega$ by the following formula:

$$d\omega := \sum_{i_1<i_2<\cdots<i_k} d\omega_{i_1 i_2 \cdots i_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \in \Omega^{k+1}(U),$$

where $d\omega_{i_1 i_2 \cdots i_k} \in \Omega^1(U)$ is the differential of the smooth function $\omega_{i_1 i_2 \cdots i_k}$. It is straightforward to check that (1)-(3) are satisfied in this case.

Lemma 1.9. *Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$, for $k \geq 0$, be $\mathbb{R}$-linear maps satisfying (1)-(3) in Theorem 1.8. Then the following are true.*

(i) *For any $p \in M$, the value $d\omega(p)$ depends only on the values of ω on any open neighborhood of p; in particular, for any open subset W of M, the restriction of $d\omega$ on W only depends on the restriction of ω on W.*

(ii) *Let U be any open subset of M and let (x^i) be any local coordinate functions on U. If $\omega = \sum_{i_1<i_2<\cdots<i_k} \omega_{i_1 i_2 \cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$, then $d\omega = \sum_{i_1<i_2<\cdots<i_k} d\omega_{i_1 i_2 \cdots i_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$.*

As a consequence of (i) and (ii), the maps d are unique.

Proof. For (i), it suffices to show that if $\omega = 0$ on an open neighborhood W of p, then $d\omega(p) = 0$. To see this, we pick a smooth partition of unity $\{\phi, \psi\}$ subordinate to $\{W, M \setminus \{p\}\}$. Then

$$d\omega = d(\phi\omega + \psi\omega) = d(\phi\omega) + d\psi \wedge \omega + \psi d\omega.$$

Note that $\phi\omega = 0$ on M, so that $d(\phi\omega) = 0$. Furthermore, $\psi = 0$ in a neighborhood of p so that $d\psi(p) = 0$ and $\psi(p) = 0$, which implies that $d\omega(p) = 0$.

For (ii), we observe that by property (2),

$$d\omega = \sum_{i_1<i_2<\cdots<i_k} d\omega_{i_1 i_2 \cdots i_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} + \omega_{i_1 i_2 \cdots i_k} d(dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}).$$

Further application of (2), together with (1) and (3), easily implies that

$$d(dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}) = 0.$$

Hence $d\omega = \sum_{i_1<i_2<\cdots<i_k} d\omega_{i_1 i_2 \cdots i_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$.

□

Next we show that the definition $d\omega := \sum_\alpha d\omega_\alpha$ satisfies (1)-(3), hence establish the existence part. For (1), consider the case $\omega = f \in C^\infty(M)$. Then

$$d\omega = \sum_\alpha d(f_\alpha f) = \sum_\alpha (f \cdot df_\alpha + f_\alpha \cdot df) = f \cdot (\sum_\alpha df_\alpha) + (\sum_\alpha f_\alpha) \cdot df = df,$$

verifying (1). Observe that by the same argument, for any $\omega \in \Omega^k(M)$ such that $\text{supp } \omega \subset U$, where U is a local coordinate chart, and for any local coordinate functions (x^i) on U such that $\omega = \sum_{i_1<i_2<\cdots<i_k} \omega_{i_1 i_2\cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$,

$$d\omega := \sum_\alpha d\omega_\alpha = \sum_{i_1<i_2<\cdots<i_k} d\omega_{i_1 i_2\cdots i_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}.$$

With this understood, we verify (2) and (3). For (2), let $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$. Then writing $\omega = \sum_\alpha \omega_\alpha$, $\eta = \sum_\beta \eta_\beta$, we have $\omega \wedge \eta = \sum_{\alpha,\beta} \omega_\alpha \wedge \eta_\beta$. By the above observation, we have

$$\begin{aligned} d(\omega \wedge \eta) &= \sum_{\alpha,\beta} d(\omega_\alpha \wedge \eta_\beta) \\ &= \sum_{\alpha,\beta} d\omega_\alpha \wedge \eta_\beta + (-1)^k \omega_\alpha \wedge d\eta_\beta \\ &= (\sum_\alpha d\omega_\alpha) \wedge (\sum_\beta \eta_\beta) + (-1)^k (\sum_\alpha \omega_\alpha) \wedge (\sum_\beta d\eta_\beta) \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

For (3), note that $d\omega = \sum_\alpha d\omega_\alpha$, and $d \circ d\omega = d(\sum_\alpha d\omega_\alpha) = \sum_\alpha d \circ d\omega_\alpha$ (using the observation). Since each $\text{supp } \omega_\alpha \subset U_\alpha$ which is a local coordinate chart, $d \circ d = 0$ holds true there. This shows that $d \circ d\omega = 0$, verifying (3).

Finally, (4) follows easily from the following facts: (i) $F^*(df) = d(F^* f)$ for any smooth function f, (ii) $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$, (iii) the local expression for $d\omega$ (cf. Lemma 1.9), and (iv) $d \circ d = 0$.

□

Exterior derivative and Lie bracket:

Theorem 1.10. *For any $\omega \in \Omega^1(M)$, $X, Y \in \mathcal{X}(M)$,*

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

More generally, for any $\omega \in \Omega^k(M)$, $X_1, X_2, \cdots, X_{k+1} \in \mathcal{X}(M)$,

$$\begin{aligned} d\omega(X_1, X_2, \cdots, X_{k+1}) &= \sum_{1\le i\le k+1} (-1)^{i+1} X_i(\omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{k+1})) \\ &+ \sum_{1\le i<j\le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{k+1}). \end{aligned}$$

Proof. We shall only prove for the case of $k = 1$; the general case is completely similar.

First, set $\Omega(X,Y) := X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$. Then $\Omega(X,Y) = -\Omega(Y,X)$. Moreover, for any $f \in C^\infty(M)$,

$$\Omega(fX,Y) = fX(\omega(Y)) - Y(f)\omega(X) - fY(\omega(X)) - \omega(f[X,Y] - Y(f)X) = f\Omega(X,Y).$$

Consequently, $\Omega(X,Y)$ defines a differential 2-form. It follows easily that it suffices to check the identity $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$ locally for the special case of $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ from a local coordinate frame $(\frac{\partial}{\partial x^i})$.

To this end, write $\omega = \omega_k dx^k$. Then $d\omega = d\omega_k \wedge dx^k$, which gives

$$d\omega(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = \Omega(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}).$$

□

Symplectic structures: A differential 2-form $\omega \in \Omega^2(M)$ is said to be **non-degenerate** if for any point $p \in M$, the map $X \mapsto i_X\omega(p)$ defines an isomorphism between T_pM and T_p^*M. Locally, we can write $\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$. Let $A := (\omega_{ij})$ where $\omega_{ij} = -\omega_{ji}$, be the skew-symmetric matrix. Then with respect to the bases $(\frac{\partial}{\partial x^i})$ and (dx^i), the map $X \mapsto i_X\omega$ is given by the matrix A. Consequently, if ω is non-degenerate, then M must be even-dimensional.

Definition 1.11. (1) A **symplectic structure** on a smooth manifold M is a differential 2-form $\omega \in \Omega^2(M)$ such that (i) ω is non-degenerate, (ii) $d\omega = 0$.

(2) A half-dimensional embedded submanifold $L \subset M$ is called **Lagrangian** if for any $p \in L$, $\omega(p)(X, Y) = 0$ for any $X, Y \in T_pL$.

Example 1.12. Consider $M = \mathbb{R}^{2n} = \mathbb{C}^n$, and let $z_k = x_k + iy_k$, $k = 1, 2, \cdots, n$, be the complex coordinates on $\mathbb{C}^n$. Note that x_k, y_k, $k = 1, 2, \cdots, n$, are the real coordinates on $\mathbb{R}^{2n}$. It is easy to check the following is a symplectic structure, called the **standard symplectic structure**:

$$\omega_0 := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$

Let L be an affine subspace defined by either $y_k = c_k$, $\forall k$, or $x_k = c_k$, $\forall k$. Then L is a Lagrangian submanifold. More generally, for any smooth function $f(x_1, x_2, \cdots, x_n)$, we consider the graph of df, i.e., $L = \{y_k = \frac{\partial f}{\partial x_k} | k = 1, 2, \cdots, n\}$. Then L is Lagrangian.

Example 1.13. For any smooth manifold M, the cotangent bundle T^*M has a canonical symplectic structure ω_0.

Let $\pi : T^*M \to M$ be the projection sending (p, v) to p, where $p \in M$ and $v \in T_p^*M$. We define a 1-form τ on T^*M as follows: for any $(p, v) \in T^*M$, $\tau(p, v) := \pi^*_{(p,v)}(v)$, where $\pi^*_{(p,v)} : T_p^*M \to T^*_{(p,v)}(T^*M)$ is the dual of $\pi_{*,(p,v)} : T_{(p,v)}(T^*M) \to T_pM$.

We compute τ locally. Let (x^i) be a system of local coordinate functions on M. Then each cotangent vector v can be uniquely written as $v = \sum_i y_i dx^i$. Consequently, (x^i, y_i) is a system of local coordinate functions on T^*M. Moreover, the projection $\pi : T^*M \to M$ is given by $(x^i, y_i) \mapsto x^i$. It follows immediately that at (p, v) where $v = \sum_i y_i dx^i$, $\tau = \sum_i y_i dx^i$. The canonical symplectic structure on T^*M is defined to be $\omega_0 := -d\tau$. In local coordinates (x^i, y_i),

$$\omega_0 = -d\tau = -d(\sum_i y_i dx^i) = \sum_i dx^i \wedge dy_i.$$

As for Lagrangian submanifolds, let $\alpha \in \Omega^1(M)$ be a differential 1-form on M. Then as α is a smooth section of T^*M, its graph $L \subset T^*M$ is a half-dimensional submanifold. We observe that L is a Lagrangian if and only if the pull-back of ω_0 via $\alpha : M \to T^*M$ is zero. With this understood, note that $\alpha^*\omega_0 = -d\alpha$, which implies that L is Lagrangian if and only if α is closed, i.e., $d\alpha = 0$. In particular, for any $f \in C^\infty(M)$, the graph of df is a Lagrangian submanifold of T^*M.

2. Orientation, integration and Stokes Theorem

Orientation: Let M be a smooth manifold of dimension n. Observe that the bundle of n-forms $\Lambda^n M$ is of rank 1, i.e., a line bundle.

Definition 2.1. The smooth manifold M is called **orientable** if $\Lambda^n M$ is trivial, which is equivalent to M admitting a nowhere vanishing n-form. Moreover, if M is orientable, then an **orientation** on M is the equivalence class of nowhere vanishing n-forms on M in the following sense: let Ω_1, Ω_2 be two nowhere vanishing n-forms on M, then there exists a $\lambda \in C^\infty(M)$ such that $\Omega_2 = \lambda\Omega_1$, where $\lambda(p) \neq 0$ for any $p \in M$. We say Ω_1, Ω_2 are **equivalent**, and write $[\Omega_1] = [\Omega_2]$, if $\lambda(p) > 0$ for any $p \in M$. An **oriented manifold** is a manifold equipped with a specific orientation.

Exercise: Suppose M is orientable. Show that if M is connected, then there are precisely two orientations on M.

Lemma 2.2. *A smooth manifold M is orientable if and only if TM admits a set of local trivializations over an open cover $\{U_\alpha\}$, such that the associated transition functions $\{\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$ satisfy the following condition:* $\det \tau_{\beta\alpha}(p) > 0$ *for any α, β and $p \in U_\alpha \cap U_\beta$.*

Proof. First, suppose TM admits a set of local trivializations with the said property. For each α, let $e_1^\alpha, e_2^\alpha, \cdots, e_n^\alpha$ be the local frame defining the trivialization of TM over U_α, and let $\epsilon_\alpha^1, \epsilon_\alpha^2, \cdots, \epsilon_\alpha^n$ be the dual frame. Then

$$\Omega_\alpha := \epsilon_\alpha^1 \wedge \epsilon_\alpha^2 \wedge \cdots \wedge \epsilon_\alpha^n$$

is a nowhere vanishing n-form on U_α. On the overlap $U_\alpha \cap U_\beta$, $\Omega_\beta = \det \tau_{\beta\alpha}\Omega_\alpha$ holds. Now let $\{f_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$. Then $\Omega := \sum_\alpha f_\alpha \Omega_\alpha$ is a differential n-form on M, which is nowhere vanishing due to the fact that $\det \tau_{\beta\alpha}(p) > 0$ for any α, β and $p \in U_\alpha \cap U_\beta$.

On the other hand, let Ω be a nowhere vanishing n-form on M. Given any set of local trivializations of TM over an open cover $\{U_\alpha\}$, let $e_1^\alpha, e_2^\alpha, \cdots, e_n^\alpha$ be the local frame defining the trivialization of TM over U_α. Then by re-arranging the order of $e_1^\alpha, e_2^\alpha, \cdots, e_n^\alpha$, we may assume without loss of generality that $\Omega(e_1^\alpha, e_2^\alpha, \cdots, e_n^\alpha) > 0$ for each α. Note that for any α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, $\Omega(e_1^\alpha, e_2^\alpha, \cdots, e_n^\alpha) = \det \tau_{\beta\alpha}\Omega(e_1^\beta, e_2^\beta, \cdots, e_n^\beta)$, implying $\det \tau_{\beta\alpha}(p) > 0$ for any α, β and $p \in U_\alpha \cap U_\beta$.

□

Recall that TM is always an $O(n)$-bundle. Lemma 2.2 implies that M is orientable if and only if TM is a $SO(n)$-bundle.

Example 2.3. Every complex manifold M is orientable. This is because TM is a $GL(n, \mathbb{C})$-bundle, and the determinant function is positive on the subgroup $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$.

More generally, recall that a mixed tensor field of type $(1, 1)$ defines an endomorphism of TM (cf. Theorem 3.1 of part 2). A tensor field $J \in \mathcal{T}_1^1 M$ is called an **almost**

complex structure if $J^2 = -Id$. Every complex manifold admits a canonical almost complex structure J_0, i.e., if $z^k = x^k + iy^k$ is a system of local holomorphic coordinate functions, then $J_0(\frac{\partial}{\partial x^k}) = \frac{\partial}{\partial y^k}$, $J_0(\frac{\partial}{\partial y^k}) = -\frac{\partial}{\partial x^k}$.

Now let J be an almost complex structure on M. Then for any $p \in M$, $J_p : T_pM \to T_pM$ obeys $J_p^2 = -Id_{T_pM}$. With this understood, TM can be made into a smooth complex vector bundle as follows: for each $p \in M$, we define a complex multiplication on T_pM by $z \cdot v := av + bJ_p(v)$, where $z = a + ib$, $v \in T_pM$. Consequently, TM is a $GL(n, \mathbb{C})$-bundle. Thus if M admits an almost complex structure, M is orientable.

Exercise: Let V be a real vector space of dimension $n = 2m$. Let $\omega \in \Lambda^2(V)$.

(1) Show that if ω is non-degenerate, i.e., $X \mapsto i_X\omega$ defines an isomorphism between V and V^*, then there exists a basis $\epsilon^1, \delta^1, \epsilon^2, \delta^2, \cdots, \epsilon^m, \delta^m$ of V^*, such that

$$\omega = \epsilon^1 \wedge \delta^1 + \epsilon^2 \wedge \delta^2 + \cdots + \epsilon^m \wedge \delta^m.$$

(2) Show that ω is non-degenerate if and only if $\omega \wedge \omega \wedge \cdots \wedge \omega$ (m-fold wedge product) is nonzero.

Example 2.4. Every symplectic manifold is orientable. More concretely, let ω be a symplectic structure on M of dimension $2m$. Then the $2m$-form $\Omega := \omega \wedge \omega \wedge \cdots \wedge \omega$ is nowhere vanishing.

Example 2.5. A smooth manifold M is called **parallelizable** if TM is trivial. An important class of parallelizable manifolds is given by Lie groups; a basis of left-invariant vector fields on a Lie group gives rise to a global frame of its tangent bundle. Clearly, every parallelizable manifold is orientable.

Proposition 2.6. *Let $S \subset M$ be a co-dimension 1 submanifold. Suppose M is orientable. Then S is orientable if and only if the normal bundle of S in M is trivial.*

Proof. First, we show that if the normal bundle is trivial, S must be orientable. Since S is of co-dimension 1, the normal bundle is of rank 1. Thus triviality of the bundle implies that there is a global frame, i.e., a smooth, non-zero section of the normal bundle. This smooth nonzero section is given by a smooth vector field X along S, such that for any $p \in S$, X_p is not in T_pS. With this understood, let Ω_M be a nowhere vanishing n-form on M, where $n = \dim M$. Then $\Omega_S := i_X\Omega_M$ is a differential $(n-1)$-form on S, which is nowhere vanishing. Hence S is orientable.

Conversely, suppose S is orientable, and let Ω_S be a nowhere vanishing $(n-1)$-form on S. We cover S by a smooth atlas $\{U_\alpha\}$ of slice charts, where if $x_\alpha^1, x_\alpha^2, \cdots, x_\alpha^n$ are the local coordinate functions on U_α, $S \cap U_\alpha$ is given by $x_\alpha^n = constant$. With this understood, for each α, we let $X_\alpha = \frac{\partial}{\partial x_\alpha^n}$ or $-\frac{\partial}{\partial x_\alpha^n}$, where there is a unique choice such that $i_{X_\alpha}\Omega_M = \lambda_\alpha \Omega_S$ for a positive smooth function λ_α on $S \cap U_\alpha$. Let $\{f_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$. Then $X := \sum_\alpha f_\alpha X_\alpha$ defines a smooth nonzero section of the normal bundle of S. □

Example 2.7. The n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has a trivial normal bundle. Hence each $\mathbb{S}^n$ is orientable.

Let M, N be oriented manifolds, with Ω_M, Ω_N defining the orientation respectively. For simplicity, we assume both M, N are connected. Let $F : M \to N$ be a local diffeomorphism. Then $F^*\Omega_N$ is nowhere vanishing on M, and there are two possibilities: (i) $[F^*\Omega_N] = [\Omega_M]$, or (ii) $[F^*\Omega_N] = [-\Omega_M]$. In case (i), F is called **orientation-preserving** and in case (ii), **orientation-reversing**. We remark that when $M = N$ and $\Omega_M = \Omega_N$, F is orientation-preserving or not is independent of the choice of the orientation Ω_M itself. Furthermore, note that if F is an odd order periodic diffeomorphism of M, F is always orientation-preserving.

Example 2.8. We consider $\tau = -Id : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, which leaves $\mathbb{S}^n$ invariant. We claim that the involution $\tau : \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving if and only if n is odd.

To see this, we let $x_0, x_1, \cdots, x_n$ be the standard coordinates on $\mathbb{R}^{n+1}$, and let $\hat{\Omega} := dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n$, which defines an orientation on $\mathbb{R}^{n+1}$. On the other hand, consider the normal vector field X on $\mathbb{S}^n$, where at $p \in \mathbb{S}^n$, X_p is the vector from the origin of $\mathbb{R}^{n+1}$ to $p \in \mathbb{S}^n$. It is easy to see that $\tau_*(X) = X$. With this understood, we let $\Omega := i_X\hat{\Omega}$. Then Ω is nowhere vanishing on $\mathbb{S}^n$, thus defining an orientation on $\mathbb{S}^n$. Finally, we compute the action of τ on Ω:

$$\tau^*\Omega = \tau^*(i_X\hat{\Omega}) = \tau^*(i_{\tau_* X}\hat{\Omega}) = i_X(\tau^*\hat{\Omega}) = i_X((-1)^{n+1}\hat{\Omega}) = (-1)^{n+1}\Omega.$$

It follows that $\tau : \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving if and only if n is odd. As a consequence, $\mathbb{RP}^n$ is orientable if and only if n is odd (cf. Prop. 2.10 below).

Lemma 2.9. *Let G be a finite group acting on M smoothly and freely, and let $N = M/G$ be the quotient manifold. For any $\omega \in \Omega^k(M)$, there is an $\eta \in \Omega^k(N)$ such that $\omega = \pi^*\eta$, where $\pi : M \to N$ is the natural projection, if and only if for any $g \in G$, $g^*\omega = \omega$ (here $g : M \to M$ is the map $p \mapsto g \cdot p$, $\forall p \in M$).*

Exercise: Prove Lemma 2.9.

Proposition 2.10. *Let G be a finite group acting on M smoothly and freely, where M is connected and orientable, and let $N = M/G$ be the quotient manifold. Then N is orientable if and only if for any $g \in G$, the map $g : M \to M$ by $p \mapsto g \cdot p$, $\forall p \in M$, is orientation-preserving.*

Proof. First, suppose N is orientable, and pick an orientation form Ω_N of N. Then the form $\Omega_M := \pi^*\Omega_N$, where $\pi : M \to N$ is the natural projection, is nowhere vanishing on M, thus defining an orientation on M. With this understood, observe that for any $g \in G$, $\pi \circ g = \pi$, which implies that $g^*\Omega_M = g^*\pi^*\Omega_N = (\pi \circ g)^*\Omega_N = \pi^*\Omega_N = \Omega_M$.

On the other hand, suppose that for any $g \in G$, the map $g : M \to M$ is orientation-preserving. We pick an orientation form Ω_M on M. Then $[g^*\Omega_M] = [\Omega_M]$, which implies that $g^*\Omega_M = \lambda_g\Omega_M$ for some smooth function $\lambda_g > 0$. Let $\hat{\Omega}_M := \sum_{g\in G} g^*\Omega_M = (\sum_{g\in G}\lambda_g)\Omega_M$. Then $g^*\hat{\Omega}_M = \hat{\Omega}_M$ for any $g \in G$, hence by Lemma 2.9, there is an Ω_N such that $\pi^*\Omega_N = \hat{\Omega}_M$. On the other hand, observe that $\hat{\Omega}_M$ is nowhere vanishing, which implies that Ω_N is nowhere vanishing. It follows that N is orientable. □

Exercise: Prove that the lens spaces $L(p, q)$ are orientable.

Proposition 2.11. *Let M is connected and non-orientable. Then there is a unique $2 : 1$ covering $\tilde{M} \to M$ such that $\tilde{M}$ is orientable. In particular, a smooth manifold M is orientable if there is no epimorphism $\pi_1(M) \to \mathbb{Z}_2$ (e.g. $\pi_1(M) = 0$).*

Exercise: Prove Proposition 2.11.

Manifolds with boundary: Let $\mathbb{H}^n := \{(x_1, x_2, \cdots, x_n) | x_n \geq 0\}$ be the upper half-space of $\mathbb{R}^n$, where we denote by $\partial\mathbb{H}^n$ its boundary $x_n = 0$. A smooth manifold of boundary M is a Hausdorff and second countable topological space M with a smooth atlas $\{(U_\alpha, \phi_\alpha)\}$, where the map $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ or $\mathbb{H}^n$. An important issue to clarify here is the smoothness of the map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$. If the domain $\phi_\alpha(U_\alpha \cap U_\beta)$ contains a point $x \in \partial\mathbb{H}^n$, then $\phi_\beta \circ \phi_\alpha^{-1}$ is smooth means that it admits an extension to an open neighborhood of x which is smooth. Note that while the smooth extensions are not unique, all the partial derivatives of the extensions at the point x are uniquely determined –these values are what really matter. With this understood, it is easy to see that all the things we have developed so far concerning smooth manifolds can be extended to smooth manifolds with boundary.

Let M be a smooth manifold with boundary. We let

$$\partial M := \{p \in M | \exists \text{ a chart } (U, \phi) \text{ such that } \phi(p) \in \partial\mathbb{H}^n\}$$

and

$$\text{Int } M := \{p \in M | \exists \text{ a chart } (U, \phi) \text{ such that } \phi(p) \in \mathbb{R}^n \text{ or } \mathbb{H}^n \setminus \partial\mathbb{H}^n\}.$$

Theorem 2.12. *Let M be a smooth manifold with boundary of dimension n. Then*

1. *$M = \text{Int } M \sqcup \partial M$ (disjoint union).*
2. *Int M is an open submanifold of M without boundary.*
3. *∂M is an embedded submanifold of M without boundary, of dimension $n-1$.*

Exercise: Prove Theorem 2.12.

A smooth vector field X along ∂M is said to be an **inward-pointing normal vector field** if for any $p \in \partial M$, X_p is not in $T_p(\partial M)$ and there is a smooth curve $\gamma : [0, \epsilon) \to M$ with $\gamma(0) = p$ such that $\gamma'(0) = X_p$.

Lemma 2.13. *There exists an inward-pointing normal vector field along ∂M. Moreover, a smooth vector field X along ∂M is an inward-pointing normal vector field if and only if for any $p \in \partial M$ and any local chart (U, ϕ) containing p, with local coordinate functions (x^i), $X_p = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}|_p$ for some $a_i \in \mathbb{R}$ where $a_n > 0$.*

Proof. Let X be an inward-pointing normal vector field along ∂M, and let $p \in \partial M$ be any point. Let $\gamma : [0, \epsilon) \to M$ be a smooth curve with $\gamma(0) = p$ such that $\gamma'(0) = X_p$. Then for any local chart (U, ϕ) containing p, with local coordinate functions (x^i), we write $X_p = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}|_p$ where $a_i \in \mathbb{R}$. Then $a_n = X_p(x_n) = \frac{d}{dt}(x_n(\gamma(t)))|_{t=0} \geq 0$ because the function $x_n(\gamma(t)) \geq 0$ and $x_n(\gamma(0)) = x_n(p) = 0$. Furthermore, if $a_n = 0$, then X_p is in $T_p(\partial M)$ which is a contradiction. Hence $a_n > 0$. Conversely, if $X_p = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}|_p$ where $a_n > 0$. Then we let $\gamma(t)$ be the smooth curve $\phi^{-1}(\phi(p) + (a_1 t, a_2 t, \cdots, a_n t)) \subset U$ where $t \in [0, \epsilon)$, $\epsilon > 0$ is small. It is clear that $\gamma(0) = p$ and $X_p = \gamma'(0)$, which shows that X is an inward-pointing normal vector field.

For the existence part, we simply cover ∂M by local charts $\{(U_\alpha, \phi_\alpha)\}$ where each $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Let (x_α^i) be the local coordinate functions on U_α. We pick a smooth partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$. Then $X := \sum_\alpha f_\alpha \frac{\partial}{\partial x_\alpha^n}$ is an inward-pointing normal vector field along ∂M. ∎

Corollary 2.14. *(1) The normal bundle of ∂M in M is trivial.*

(2) For any two inward-pointing normal vector fields X_0, X_1, the vector field $X_t = (1-t)X_0 + tX_1$ is an inward-pointing normal vector field for every $t \in [0,1]$.

Definition 2.15. Let M be an oriented manifold with boundary. There is a canonical orientation on ∂M, called **boundary orientation**, which is defined as follows: pick an orientation form Ω_M on M, and an inward-pointing normal vector field X along ∂M, we let $\Omega_{\partial M} := -i_X \Omega_M$. The canonical orientation on ∂M is defined to be $[\Omega_{\partial M}]$.

We remark that by Corollary 2.14 (2), $[\Omega_{\partial M}]$ is independent of the choice of X. On the other hand, note that $[\Omega_{\partial M}]$ depends only on $[\Omega_M]$, not on the choice of Ω_M.

Integration of differential forms and Stokes Theorem: Let M be an oriented smooth manifold of dimension n, with or without boundary. Let $\omega \in \Omega^n(M)$ such that $\operatorname{supp} \omega$ is compact (e.g. M is compact). We will define the integration of ω over M, to be denoted by $\int_M \omega$.

We first consider the special case of $M = U$, an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. Let $x_1, x_2, \cdots, x_n$ be the standard coordinates such that $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ defines the orientation on U. Let $\omega = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ where $\operatorname{supp} f \subset U$ is compact. Then we define

$$\int_U \omega := \int_U f dx_1 dx_2 \cdots dx_n.$$

Suppose V is another open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and $F: V \to U$ is a diffeomorphism. Assuming $y_1, y_2, \cdots, y_n$ are the standard coordinates on V, we have

$$F^*\omega = f \circ F \det(DF) dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n.$$

Without loss of generality, we may assume $dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n$ defines the orientation on V. Then

$$\int_V F^*\omega = \int_V f \circ F \det(DF) dy_1 dy_2 \cdots dy_n.$$

Observation: $\int_U \omega = \int_V F^*\omega$ if and only if F is orientation-preserving; otherwise, $\int_U \omega = -\int_V F^*\omega$.

Now we define $\int_M \omega$ for any $\omega \in \Omega^n(M)$ such that $\operatorname{supp} \omega$ is compact. We fix an orientation form Ω_M on M. Since $\operatorname{supp} \omega$ is compact, we can choose a finite cover of $\operatorname{supp} \omega$ by local coordinates charts $\{(U_\alpha, \phi_\alpha)\}$. For each α, let $\phi_\alpha(p) = (x_\alpha^1(p), \cdots, x_\alpha^n(p))$, $p \in U_\alpha$, such that $\Omega_M(\frac{\partial}{\partial x_\alpha^1}, \cdots, \frac{\partial}{\partial x_\alpha^n}) > 0$. Moreover, pick a smooth partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$. Then we define

$$\int_M \omega := \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(f_\alpha \omega).$$

Proposition 2.16. *The integral $\int_M \omega$ is well-defined. Moreover,*

- $\int_M (a\omega + a'\omega') = a \int_M \omega + a' \int_M \omega'$, $\forall a, a' \in \mathbb{R}$.
- *Let Ω_M be an orientation form on M, $\omega = f\Omega_M$ for some compactly supported smooth function f, where $f \geq 0$ and $f(p) > 0$ at some $p \in M$. Then $\int_M \omega > 0$.*

- *Let $F : M \to N$ be a diffeomorphism. Then $\int_M F^*\omega = \int_N \omega$ if F is orientation-preserving, and $\int_M F^*\omega = -\int_N \omega$ if F is orientation-reversing. In particular, $\int_{-M} \omega = -\int_M \omega$, where $-M$ is M with the opposite orientation.*

Lemma 2.17. *Let $\omega \in \Omega^{n-1}(\mathbb{H}^n)$ where supp ω is compact. Then $\int_{\mathbb{H}^n} d\omega = \int_{\partial\mathbb{H}^n} \omega$, where $\partial\mathbb{H}^n$ is given the boundary orientation.*

Proof. Let $x_1, x_2, \cdots, x_n$ be the standard coordinates on $\mathbb{H}^n$ such that $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ defines the orientation on $\mathbb{H}^n$. Then it is easy to see that the boundary orientation on $\partial\mathbb{H}^n$ is given by $(-1)^n dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$.

We write $w = \sum_{i=1}^n w_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$. Then

$$d\omega = \sum_{i=1}^n (-1)^{i-1} \frac{\partial w_i}{\partial x_i} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

Consequently,

$$\int_{\mathbb{H}^n} d\omega = \sum_{i=1}^n (-1)^{i-1} \int_{\mathbb{H}^n} \frac{\partial w_i}{\partial x_i} dx_1 dx_2 \cdots dx_n = (-1)^{n-1} \int_{\partial\mathbb{H}^n} (-w_n) dx_1 dx_2 \cdots dx_{n-1} = \int_{\partial\mathbb{H}^n} \omega.$$

□

With Lemma 2.17, a straightforward argument involving partition of unity gives the following

Stokes Theorem: *$\int_M d\omega = \int_{\partial M} \omega$ for any compactly supported ω.*

Integration of functions: Let M be an oriented smooth manifold (with or without boundary). Fix any orientation form Ω_M on M, we can define the integral of a compactly supported smooth function f over M, denoted by $\int_M f$, by

$$\int_M f := \int_M f\Omega_M.$$

Integration on Riemannian manifolds: Let g be a Riemannian metric on M. Let $e_1, e_2, \cdots, e_n$ be any local orthonormal frame of TM, which is positively oriented in the sense that $\Omega_M(e_1, e_2, \cdots, e_n) > 0$. Let $\epsilon^1, \epsilon^2, \cdots \epsilon^n$ be the dual frame. Then it is easy to see that

$$dV_g := \epsilon^1 \wedge \epsilon^2 \wedge \cdots \wedge \epsilon^n$$

is independent of the choice of the orthonormal frame $e_1, e_2, \cdots, e_n$. Furthermore, note that $[dV_g] = [\Omega_M]$. dV_g is called the **volume form** associated to g. With this understood, for any compactly supported smooth function f on M, we define

$$\int_M f := \int_M f dV_g.$$

Suppose ∂M is nonempty, and let $\tilde{g}$ be the induced Riemannian metric on ∂M. Observe that there is a unique unit vector field N along ∂M, such that (i) $-N$ is inward-pointing, (ii) N is orthogonal to ∂M. It is easy to see that the volume form on ∂M, $dV_{\tilde{g}}$, is given by $dV_{\tilde{g}} = i_N dV_g$.

Let $X \in \mathcal{X}(M)$. Then the divergence of X, denoted by $\operatorname{div}_g(X)$, is defined to be the smooth function determined by the equation

$$\operatorname{div}_g(X) \cdot dV_g = d(i_X dV_g).$$

Then the following theorem is an easy consequence of the Stokes Theorem.

Divergence Theorem: $\int_M \operatorname{div}_g(X) = \int_{\partial M} g(X, N)$ for any compactly supported $X \in \mathcal{X}(M)$.

Integration on Lie groups: Let G be a Lie group. Fix an orientation on G, we let $\epsilon^1, \epsilon^2, \cdots \epsilon^n$ be the dual basis of a positively oriented basis of $Lie(G)$. Then

$$\Omega := \epsilon^1 \wedge \epsilon^2 \wedge \cdots \wedge \epsilon^n$$

is an orientation form on G. Observe that for any $g \in G$, $L_g^* \Omega = \Omega$.

For any compactly supported smooth function f on G, we define

$$\int_G f := \int_G f\Omega.$$

We note that the integral $\int_G f$ is left-invariant, i.e., for any $g \in G$, $\int_G L_g^* f = \int_G f$.

Theorem 2.18. *When G is compact, the integral $\int_G f$ is bi-invariant, i.e., for any $g \in G$, $\int_G L_g^* f = \int_G R_g^* f = \int_G f$.*

Proof. It suffices to show that $\int_G R_g^* f = \int_G f$ for any $g \in G$. To this end, we note that for any $g \in G$, $R_g^* \Omega$ is left-invariant, so that there exists a $\lambda(g) \neq 0$ such that $R_g^* \Omega = \lambda(g)\Omega$. In particular, $R_g : G \to G$ is orientation-preserving if and only if $\lambda(g) > 0$. As G is compact, it follows that

$$\int_G \lambda(g)\Omega = \int_G R_g^* \Omega = \text{sign } \lambda(g) \int_G \Omega,$$

implying $|\lambda(g)| = 1$ for any $g \in G$. Now for any $f \in C^\infty(G)$, $g \in G$,

$$\int_G R_g^* f = \int_G (R_g^* f)\Omega = \text{sign } \lambda(g) \int_G R_g^*(f\Omega) = \int_G f\Omega = \int_G f.$$

□

Lemma 2.19. *Let G be a compact Lie group acting smoothly on M. For any $f \in C^\infty(M)$, let $\bar{f}$ be the function on M defined by*

$$\bar{f}(p) := \int_G f(g \cdot p), \text{ where } f(g \cdot p) \text{ is regarded as a function on } G, \ \forall p \in M.$$

Then $\bar{f} \in C^\infty(M)$, and $\bar{f}$ is G-invariant, i.e., $\forall h \in G$, $h^* \bar{f} = \bar{f}$.

Proof. First, $\bar{f} \in C^\infty(M)$ because $f(g \cdot p)$ is smooth in both g and p. To see that $\bar{f}$ is G-invariant, we let $h \in G$ be any element. Then

$$(h^* \bar{f})(p) = \bar{f}(h \cdot p) = \int_G f(gh \cdot p) = \int_G (R_h^* f)(g \cdot p) = \int_G f(g \cdot p) = \bar{f}(p), \ \forall p \in M.$$

□

Exercise: Let M be a smooth manifold equipped with a smooth action of a compact Lie group G. Show that M admits a Riemannian metric which is G-invariant.

3. De Rham cohomology

Definition and homotopy invariance: Let M be a smooth manifold. The p-th de Rham cohomology group of M is defined to be the vector space over $\mathbb{R}$:

$$H^p_{dR}(M) := \frac{\{\ker d : \Omega^p(M) \to \Omega^{p+1}(M)\}}{\{\text{Image } d : \Omega^{p-1}(M) \to \Omega^p(M)\}}.$$

Note that in the above definition we used the fact that $d \circ d = 0$. Clearly, $H^p_{dR}(M) = 0$ if $p < 0$ or $p > \dim M$. Since d commutes with pullback maps, for any smooth map $F : M \to N$, there is an induced homomorphism $F^* : H^p_{dR}(N) \to H^p_{dR}(M)$.

Recall that smooth maps $F_0, F_1 : M \to N$ are homotopic if there is a smooth map $H : M \times [0,1] \to N$ such that for any $x \in M$, $F_i(x) = H(x,i)$ for $i = 0,1$.

Theorem 3.1. *If $F_0, F_1 : M \to N$ are homotopic, then $F_0^* = F_1^* : H^p_{dR}(N) \to H^p_{dR}(M)$. As a consequence, homotopy equivalent manifolds have isomorphic de Rham cohomology groups.*

Proof. The key is the existence of $\mathbb{R}$-linear maps $h : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$, which are defined as follows: for any $\omega \in \Omega^k(M \times [0,1])$, $h(\omega) := \int_0^1 i_{\frac{\partial}{\partial t}} \omega dt$. For $i = 0,1$, let $I_i : M \to M \times [0,1]$ be the embedding sending x to (x,i). Then the maps h obey the following equation: $d \circ h + h \circ d = I_1^* - I_0^*$.

To verify this, it suffices to compute locally. We write

$$\omega = f_{i_1 i_2 \cdots i_k}(t) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} + g_{j_1 j_2 \cdots j_{k-1}}(t) dt \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_{k-1}}.$$

Then it is easy to check that

$$\begin{aligned} h \circ d(\omega) &= (f_{i_1 i_2 \cdots i_k}(1) - f_{i_1 i_2 \cdots i_k}(0)) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\ &- (\int_0^1 dg_{j_1 j_2 \cdots j_{k-1}}(t) dt) \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_{k-1}} \end{aligned}$$

and

$$d \circ h(\omega) = (\int_0^1 dg_{j_1 j_2 \cdots j_{k-1}}(t) dt) \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_{k-1}},$$

which implies that $d \circ h + h \circ d = I_1^* - I_0^*$. Consequently, $d \circ h \circ H^* + h \circ d \circ H^* = F_1^* - F_0^*$. Since $d \circ H^* = H^* \circ d$, we have $d \circ (h \circ H^*) + (h \circ H^*) \circ d = F_1^* - F_0^*$. For any $\omega \in \Omega^p(N)$ such that $d\omega = 0$, we let $\eta := h \circ H^*(\omega) \in \Omega^{p-1}(M)$, then $F_1^*(\omega) - F_0^*(\omega) = d\eta$. It follows that $F_0^* = F_1^* : H^p_{dR}(N) \to H^p_{dR}(M)$. ☐

The following are straightforward from the definition or homotopy invariance, where part (4) also uses Stokes Theorem.

Proposition 3.2. *(1) If $M = M_1 \sqcup M_2$, then $H^p_{dR}(M) = H^p_{dR}(M_1) \times H^p_{dR}(M_2)$.*

(2) Let M be connected. Then $H^0_{dR}(M) = \mathbb{R}$, identified with constant functions.

(3) If M is contractable, e.g., $M = \mathbb{R}^n$, $\mathbb{B}^n$, $H^p_{dR}(M) = 0$ for $p \neq 0$.

(4) If $\pi_1(M)$ is finite, then $H^1_{dR}(M) = 0$.

Exercise: Prove Proposition 3.2.

De Rham cohomology under regular finite coverings:

Proposition 3.3. *Let G be a finite group acting smoothly and freely on M, and let $\pi : M \to N = M/G$ be the quotient map. Then $\pi^* : H^p_{dR}(N) \to H^p_{dR}(M)$ is injective, with its image being the G-invariant part of $H^p_{dR}(M)$, i.e.,*

$$H^p_{dR}(M)^G := \{\alpha \in H^p_{dR}(M) | g^*\alpha = \alpha, \forall g \in G\}.$$

Proof. First of all, note that for any $g \in G$, $\pi \circ g = \pi$, so that for any $\omega \in \Omega^p(N)$, $g^*(\pi^*\omega) = \pi^*\omega$. It follows easily that $\pi^* : H^p_{dR}(N) \to H^p_{dR}(M)^G$.

Now we show that $\pi^* : H^p_{dR}(N) \to H^p_{dR}(M)$ is injective. Let $\alpha \in H^p_{dR}(N)$ such that $\pi^*\alpha = 0$. Representing α by $\omega \in \Omega^p(N)$, we have $\pi^*\omega = d\eta$ for some $\eta \in \Omega^{p-1}(M)$. Setting $\tilde{\eta} := \frac{1}{|G|}\sum_{g\in G} g^*\eta$, it is easy to see, as $\pi \circ g = \pi$, that $\pi^*\omega = d\tilde{\eta}$. Note that $g^*\tilde{\eta} = \tilde{\eta}$ for any $g \in G$, so that by Lemma 2.9, there is an $\eta' \in \Omega^{p-1}(N)$ such that $\pi^*\eta' = \tilde{\eta}$. It follows that $\pi^*(\omega - d\eta') = 0$. Since $\pi : M \to N$ is a local diffeomorphism, $\pi^* : \Omega^p(N) \to \Omega^p(M)$ is injective. This implies that $\omega = d\eta'$ and $\alpha = [\omega] = 0$ in $H^p_{dR}(N)$. Hence $\pi^* : H^p_{dR}(N) \to H^p_{dR}(M)$ is injective.

To see that $\pi^* : H^p_{dR}(N) \to H^p_{dR}(M)^G$ is surjective, we let $\alpha \in H^p_{dR}(M)^G$ be any element. Representing α by $\omega \in \Omega^p(M)$, we note that for any $g \in G$, there is an $\eta_g \in \Omega^{p-1}(M)$ such that $g^*\omega = \omega + d\eta_g$ (as $g^*\alpha = \alpha$). Let $\tilde{\omega} := \frac{1}{|G|}\sum_{g\in G} g^*\omega$, $\eta := \frac{1}{|G|}\sum_{g\in G}\eta_g$. Then $\tilde{\omega} = \omega + d\eta$. On the other hand, by Lemma 2.9, there is an $\omega' \in \Omega^p(N)$ such that $\pi^*\omega' = \tilde{\omega}$. Note that $\pi^* d\omega' = d\tilde{\omega} = 0$, which implies that $d\omega' = 0$. Let $\alpha' \in H^p_{dR}(N)$ be the class of ω'. Then $\pi^*\alpha' = \alpha$, which shows that $\pi^* : H^p_{dR}(N) \to H^p_{dR}(M)^G$ is surjective.

□

Cup product and Poincaré duality: There is a natural $\mathbb{R}$-bilinear map, called the **cup product**, $H^p_{dR}(M) \times H^q_{dR}(M) \to H^{p+q}_{dR}(M)$, $(\alpha, \beta) \mapsto \alpha \cup \beta$, which is defined as follows: represent α, β by $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(M)$ respectively. Then as $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta = 0$, we define $\alpha \cup \beta \in H^{p+q}_{dR}(M)$ to be the de Rham cohomology class of $\omega \wedge \eta$. It is easy to check that the cup product is well-defined.

Example 3.4. Let M be a compact closed manifold, ω be a symplectic structure on M. Then M must be even-dimensional, say $\dim M = 2m$. Since ω is non-degenerate, $\omega \wedge \omega \wedge \cdots \wedge \omega$ (m-fold) is nowhere vanishing. As M is compact, $\int_M \omega \wedge \omega \wedge \cdots \wedge \omega \neq 0$. On the other hand, $d\omega = 0$, so ω defines a de Rham cohomology class $[\omega] \in H^2_{dR}(M)$. We note that by Stokes Theorem, $\omega \wedge \omega \wedge \cdots \wedge \omega \neq d\eta$ for any η, as $\partial M = \emptyset$. Consequently, $[\omega]^m := [\omega] \cup [\omega] \cup \cdots \cup [\omega] \in H^{2m}_{dR}(M)$ is non-zero. This implies that for any $0 < k \leq m$, $[\omega]^k \in H^{2k}_{dR}(M)$ is non-zero as well. In conclusion, a necessary condition for a compact closed manifold M of dimension $2m$ to admit a symplectic structure is that for any p, where $0 \leq p \leq 2m$, p is even, $H^p_{dR}(M) \neq 0$.

Theorem 3.5. *Let M be compact closed (i.e., $\partial M = \emptyset$) and oriented, and let $n = \dim M$. Then each $H^p_{dR}(M)$ is finite dimensional. Moreover, the $\mathbb{R}$-bilinear map $H^p_{dR}(M) \times H^{n-p}_{dR}(M) \to \mathbb{R}$, $(\alpha, \beta) \mapsto \int_M \alpha \cup \beta$, is non-degenerate, implying the following duality (called* **Poincaré duality***): $H^p_{dR}(M) \cong (H^{n-p}_{dR}(M))^*$.*

Suppose M is compact closed and connected, which is non-orientable. Then there is a double cover $\tilde{M} \to M$ such that $\tilde{M}$ is orientable. It follows from Proposition 3.3 and Theorem 3.5 that each $H^p_{dR}(M)$ is finite dimensional as well. Similarly,

Exercise: Let M be compact closed and connected, of dimension n. Show that M is non-orientable if and only if $H^n_{dR}(M) = 0$.

Theorem 3.6. *(The Künneth formula)* $H^n_{dR}(M \times N) = \oplus_{p+q=n} H^p_{dR}(M) \otimes H^q_{dR}(N)$, *where for any* $\alpha \in H^p_{dR}(M)$, $\beta \in H^q_{dR}(N)$, $\alpha \otimes \beta = \pi_1^* \alpha \cup \pi_2^* \beta$. *(Here* $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$.*)*

The Mayer-Vietoris Theorem: The most useful tool for computing the de Rham cohomology groups is the Mayer-Vietoris Theorem. We shall illustrate it with some fundamental examples.

Let M be a smooth manifold, U, V be open subsets such that $M = U \cup V$. Let $k : U \to M$, $l : V \to M$, $i : U \cap V \to U$, $j : U \cap V \to V$ denote the inclusion maps. Then there are $\mathbb{R}$-linear maps $\delta : H^p_{dR}(U \cap V) \to H^{p+1}_{dR}(M)$ (called **connecting homomorphisms**), such that the following exact sequence holds:

$$\cdots \xrightarrow{\delta} H^p_{dR}(M) \xrightarrow{k^* \oplus l^*} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{i^* - j^*} H^p_{dR}(U \cap V) \xrightarrow{\delta} H^{p+1}_{dR}(M) \xrightarrow{k^* \oplus l^*} \cdots$$

The above sequence is called the **Mayer-Vietoris sequence**.

Example 3.7. Here we use the Mayer-Vietoris Theorem to compute the de Rham cohomology groups of $\mathbb{S}^n$, for $n > 0$. To this end, note that $\mathbb{S}^n = U \cup V$, where U, V are the complement of the north pole and south pole respectively. In particular, U, V are diffeomorphic to $\mathbb{R}^n$, and $U \cap V$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$, which is homotopy equivalent to $\mathbb{S}^{n-1}$. Consequently, $H^p_{dR}(U) = H^p_{dR}(V) = 0$ for $p > 0$, and $H^p_{dR}(U \cap V) = H^p_{dR}(\mathbb{S}^{n-1})$ for any p. Looking at the $p > 0$ part of the Mayer-Vietoris sequence, we obtain immediately that

$$H^p_{dR}(\mathbb{S}^{n-1}) = H^{p+1}_{dR}(\mathbb{S}^n), \quad \forall p > 0.$$

Looking at the $p = 0$ part of the Mayer-Vietoris sequence, we note that $H^0_{dR}(\mathbb{S}^n) = H^0_{dR}(U) = H^0_{dR}(V) = \mathbb{R}$. Moreover, when $n = 1$, $H^0_{dR}(\mathbb{S}^{n-1}) = \mathbb{R} \oplus \mathbb{R}$, which implies that $H^1_{dR}(\mathbb{S}^n) = \mathbb{R}$ for $n = 1$. If $n > 1$, $H^0_{dR}(\mathbb{S}^{n-1}) = \mathbb{R}$, which implies that $H^1_{dR}(\mathbb{S}^n) = 0$ for $n > 1$. Now using $H^p_{dR}(\mathbb{S}^{n-1}) = H^{p+1}_{dR}(\mathbb{S}^n)$, $\forall p > 0$ inductively, we obtain that $H^p_{dR}(\mathbb{S}^n) = 0$ for $0 < p < n$, and $H^n_{dR}(\mathbb{S}^n) = \mathbb{R}$.

Example 3.8. In this example we compute the de Rham cohomology groups of $\mathbb{CP}^n$ for $n > 1$. Let $l_0 \in \mathbb{CP}^n$ be the complex line through the point $(0, 0, \cdots, 1) \in \mathbb{C}^{n+1}$. Let U be an open ball centered at l_0 and $V = \mathbb{CP}^n \setminus \{l_0\}$. Clearly, $\mathbb{CP}^n = U \cup V$.

Now the key observation is that V is a complex line bundle over $\mathbb{CP}^{n-1}$ (in fact, it is the dual of the tautological line bundle over $\mathbb{CP}^{n-1}$). In particular, $H^p_{dR}(V) = H^p_{dR}(\mathbb{CP}^{n-1})$ for any p. On the other hand, $U \cap V$ is homotopy equivalent to $\mathbb{S}^{2n-1}$, so that $H^p_{dR}(U \cap V) = H^p_{dR}(\mathbb{S}^{2n-1})$ for any p. Finally, $H^p_{dR}(U) = 0$ for $p > 0$.

With $H^p_{dR}(U \cap V) = H^p_{dR}(\mathbb{S}^{2n-1}) = 0$ for $1 \leq p \leq 2n-2$, the Mayer-Vietoris sequence implies that $H^p_{dR}(\mathbb{CP}^n) = H^p_{dR}(\mathbb{CP}^{n-1})$ for $2 \leq p \leq 2n-2$, and $H^{2n-1}_{dR}(\mathbb{CP}^n) = 0$.

(Here we also use the fact that $\dim_{\mathbb{R}} \mathbb{CP}^{n-1} = 2n-2$.) Inductively, we obtain that $H^p_{dR}(\mathbb{CP}^n) = 0$ if p is odd and $2 \leq p \leq 2n$. Looking at the $p=0$ part of the Mayer-Vietoris sequence, it follows that $H^1_{dR}(\mathbb{CP}^n) = H^1_{dR}(\mathbb{CP}^{n-1})$ as well. As $\mathbb{CP}^1 = \mathbb{S}^2$, we conclude that $H^1_{dR}(\mathbb{CP}^n) = 0$. Finally, Looking at the $p = 2n-1$ part of the Mayer-Vietoris sequence, we obtain $H^{2n}_{dR}(\mathbb{CP}^n) = H^{2n-1}_{dR}(\mathbb{S}^{2n-1}) = \mathbb{R}$. In summary, we have for $0 \leq p \leq 2n$,

$$H^p_{dR}(\mathbb{CP}^n) = \mathbb{R} \text{ if } p \text{ is even, } H^p_{dR}(\mathbb{CP}^n) = 0 \text{ if } p \text{ is odd.}$$

Exercise: Note that by the calculation of $H^p_{dR}(\mathbb{S}^n)$ in Example 3.7 and applying Proposition 3.3, we conclude that

$$H^p_{dR}(\mathbb{RP}^n) = 0 \text{ for } 0 < p < n,\ H^n_{dR}(\mathbb{RP}^n) = 0 \text{ if } n \text{ is even, } H^n_{dR}(\mathbb{RP}^n) = \mathbb{R} \text{ if } n \text{ is odd.}$$

Using a similar argument as in Example 3.8, give an independent proof of the above.

Exercise: Let M be compact closed, connected, and of dimension n. Using the idea in Example 3.8, show that

$$H^p_{dR}(M \setminus \{pt\}) = H^p_{dR}(M) \text{ for } 0 \leq p \leq n-2, \text{ and } p = n-1 \text{ and } M \text{ is orientable,}$$

and

$$H^p_{dR}(M \setminus \{pt\}) = H^p_{dR}(M) \oplus \mathbb{R} \text{ if } p = n-1 \text{ and } M \text{ is non-orientable,}$$

and $H^n_{dR}(M \setminus \{pt\}) = 0$.

Exercise: Let M be the quotient space of a free smooth involution τ on $\mathbb{S}^1 \times \mathbb{S}^2$, where τ acts on the $\mathbb{S}^1$-factor by $z \mapsto \bar{z}$ and on the $\mathbb{S}^2$-factor as the antipodal map.

(1) Show that $M = \mathbb{RP}^3 \# \mathbb{RP}^3$. Then use the Mayer-Vietoris Theorem to compute the de Rham cohomology groups of M.

(2) Use Proposition 3.3 and the Künneth formula to compute the de Rham cohomology groups of M alternatively.

Example 3.9. (**Mapping Torus**) Here we compute the de Rham cohomology groups of a mapping torus. Let M be compact closed, connected, and of dimension n, and let $\tau : M \to M$ be a diffeomorphism. The mapping torus of τ is the smooth manifold

$$N := (M \times [0,1])/(x,1) \sim (\tau(x), 0).$$

It is easy to see that $N = U \cup V$, where U, V are a product of M with an interval, such that $U \cap V$ is a disjoint union of two products of M with an interval. Moreover, we can identify both $H^p_{dR}(U) \oplus H^p_{dR}(V)$ and $H^p_{dR}(U \cap V)$ with $H^p_{dR}(M) \oplus H^p_{dR}(M)$, such that the map $\phi_p := i^* - j^* : H^p_{dR}(U) \oplus H^p_{dR}(V) \to H^p_{dR}(U \cap V)$ is given by $(x, y) \mapsto (x - y, x - \tau^*(y))$, where $x, y \in H^p_{dR}(M)$. With this understood, it follows from the Mayer-Vietoris Theorem that $H^p_{dR}(N) = \text{Coker } \phi_{p-1} \oplus \ker \phi_p$. Identifying $\ker \phi_p$ and Coker ϕ_p, we obtain for any p,

$$H^p_{dR}(N) = \text{Coker } (Id - \tau^* : H^{p-1}_{dR}(M) \to H^{p-1}_{dR}(M)) \oplus \ker(Id - \tau^* : H^p_{dR}(M) \to H^p_{dR}(M)).$$

Note that if $\tau = Id$, we recover the Künneth formula for $N = M \times \mathbb{S}^1$.

Exercise: The Klein bottle K is the quotient space of an involution κ on $\mathbb{S}^1 \times \mathbb{S}^1$, where $\kappa(z_1, z_2) = (\bar{z}_1, -z_2)$. Compute the de Rham cohomology groups of K from the following three viewpoints:

(1) Use Proposition 3.3 and the Künneth formula, seeing $K = \mathbb{S}^1 \times \mathbb{S}^1/\kappa$.

(2) Show that $K = \mathbb{RP}^2 \# \mathbb{RP}^2$, and use the Mayer-Vietoris Theorem.

(3) Show that K is the mapping torus of $\tau : \mathbb{S}^1 \to \mathbb{S}^1$ where $\tau(z) = \bar{z}$, and use the formula in Example 3.9.

Exercise: Let $M = \mathbb{S}^1 \times \mathbb{S}^2/\kappa$, where $\kappa(x, y) = (-x, -y)$.

(1) Compute the de Rham cohomology groups of M using Proposition 3.3 and the Künneth formula.

(2) Compute the de Rham cohomology groups of M, seeing M as a mapping torus.

(3) Show that M is the non-trivial $\mathbb{S}^1$-bundle over $\mathbb{RP}^2$ (even though M has the same de Rham cohomology groups of the trivial $\mathbb{S}^1$-bundle over $\mathbb{RP}^2$).

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