MATH 703: PART 2: VECTOR BUNDLES

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Contents

1. Smooth vector bundles

Definition 1.1. Let M be a smooth manifold. A **smooth real vector bundle of rank n over M** consists of a smooth manifold E together with a surjective smooth map $\pi : E \to M$ with the following properties:

(i) For each $p \in M$, $E_p := \pi^{-1}(p)$, called the **fiber** at p, is a n-dimensional vector space over $\mathbb{R}$.

(ii) There exists an open cover $\{U_\alpha\}$ of M, such that for each α, there is a diffeomorphism $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$, sending each E_p, $p \in U_\alpha$, isomorphically to $\{p\} \times \mathbb{R}^n$ (as vector spaces). Ψ_α is called a **trivialization** over U_α.

An **isomorphism** between two vector bundles over M is a diffeomorphism which sends fibers isomorphically to fibers and induces the identity map on M.

Remarks: (1) For any smooth manifold M, $E := M \times \mathbb{R}^n$ with the projection onto the factor M is a smooth real vector bundle of rank n over M, called a **trivial bundle** or **product bundle**.

(2) For any α, β where $U_\alpha \cap U_\beta \neq \emptyset$, the map $\Psi_\beta \circ \Psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ sends (p, v) to $(p, \tau_{\beta\alpha}(q)(v))$ for some smooth map $\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$. The maps $\{\tau_{\beta\alpha}\}$ are called the associated **transition functions**, which obeys

$$\tau_{\gamma\alpha}(q) = \tau_{\gamma\beta}(q)\tau_{\beta\alpha}(q),\ \forall q \in U_\alpha \cap U_\beta \cap U_\gamma.$$

(3) In Definition 1.1, if we replace $\mathbb{R}$ by $\mathbb{C}$ and $\mathbb{R}^n$ by $\mathbb{C}^m$, we get the notion of **smooth complex vector bundle of rank m over M**. If in addition, M and E are complex manifolds and each Ψ_α is a biholomorphism, then we get the notion of **holomorphic vector bundle over M**.

In order to get examples other than trivial bundles, we need the following

Theorem 1.2. *Let M be a smooth manifold, E be a set, with a surjective map $\pi : E \to M$, such that for any $p \in M$, the fiber $E_p := \pi^{-1}(p)$ at p is a n-dimensional vector space over $\mathbb{R}$. Suppose there is a smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ of M such that*

(i) *for each* α, *there is a bijection* $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$, *sending each* E_p, $p \in U_\alpha$, *isomorphically to* $\{p\} \times \mathbb{R}^n$ *(as vector spaces);*

(ii) *for any* α, β *where* $U_\alpha \cap U_\beta \neq \emptyset$, *the map* $\Psi_\beta \circ \Psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ *sends* (p, v) *to* $(p, \tau_{\beta\alpha}(q)(v))$ *for some smooth map* $\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$.

Then E *is naturally a smooth manifold, making it a smooth real vector bundle of rank* n *over* M, *with transition functions* $\{\tau_{\beta\alpha}\}$.

Proof. It follows directly from Theorem 1.3 in Part I.

□

Remarks: (1) Theorem 1.2 has analogs for complex/holomorphic vector bundles.

(2) It follows easily from Theorem 1.2 that given any smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ of M with a set of smooth maps $\{\tau_{\beta\alpha}\}$, where $\tau_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$, which obeys

$$\tau_{\gamma\alpha}(q) = \tau_{\gamma\beta}(q)\tau_{\beta\alpha}(q), \ \forall q \in U_\alpha \cap U_\beta \cap U_\gamma,$$

one can construct a smooth real vector bundle of rank n over M having $\{\tau_{\beta\alpha}\}$ as the associated transition functions.

Example 1.3. (1) Let M be a smooth manifold of dimension n. We let

$$TM := \sqcup_{p \in M} T_pM$$

be the disjoint union of tangent spaces of M, which comes with a natural surjective map $\pi : TM \to M$ sending each tangent vector in T_pM to $p \in M$. Then TM is a smooth real vector bundle of rank n over M, called the **tangent bundle** of M.

We verify (i)-(ii) of Theorem 1.2 for TM. Fix a smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ of M. Then for each α, let x_α^i, $i = 1, 2, \cdots, n$, be the local coordinate functions on U_α. Then for each $p \in U_\alpha$, $(\frac{\partial}{\partial x_\alpha^i}|_p)$ is a basis of T_pM. Thus for any tangent vector $X \in T_pM$, we can write $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_\alpha^i}|_p$, where each $X_i \in \mathbb{R}$. This gives rise to a bijection $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$, sending $X \in T_pM$, $p \in U_\alpha$, to $(p, X_1, X_2, \cdots, X_n) \in U_\alpha \times \mathbb{R}^n$. Clearly, it is an isomorphism from T_pM to $\{p\} \times \mathbb{R}^n$ for each $p \in U_\alpha$. By Proposition 2.5 in Part I, for any α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, the map $\Psi_\beta \circ \Psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ is given by $(p, v) \mapsto (p, D(\phi_\beta \circ \phi_\alpha^{-1})(\phi_\alpha(p))(v))$, where D is the Jacobian. Clearly, the map $D(\phi_\beta \circ \phi_\alpha^{-1}) \circ \phi_\alpha : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ is a smooth map. This verifies (i)-(ii) in Theorem 1.2.

(2) Consider the set $E := \{(x, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} | v \in x\}$, with $\pi : E \to \mathbb{RP}^n$ sending (x, v) to x. For each $x \in \mathbb{RP}^n$, the fiber $E_x := \pi^{-1}(x)$ is simply the line in $\mathbb{R}^{n+1}$ corresponding to the point $x \in \mathbb{RP}^n$, which is a 1-dimensional vector space over $\mathbb{R}$. We will show that E is a smooth real vector bundle of rank 1 over $\mathbb{RP}^n$, which is called the **tautological line bundle** (rank 1 bundles are called line bundles).

For simplicity, we assume $n = 2$. Recall from Example 1.4(1) in Part 1, $\mathbb{RP}^2$ has a canonical smooth atlas $\{(U_\alpha, \phi_\alpha) | \alpha = 1, 2, 3\}$, where

$$U_\alpha = \{l(x_1, x_2, x_3) \in \mathbb{RP}^2 | x_\alpha \neq 0\}.$$

We define $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}$ by sending (x, v) to (x, v_α), where the vector $v \in x$ has coordinates (v_1, v_2, v_3). It follows easily from the fact that $x_\alpha \neq 0$ for $x \in U_\alpha$ that

Ψ_α is a bijection. It remains to determine $\Psi_\beta \circ \Psi_\alpha^{-1}$. For simplicity, we examine the case of $\alpha = 1$, $\beta = 2$. Note that Ψ_1^{-1} sends $(l(x_1, x_2, x_3), t)$ to $(l(x_1, x_2, x_3), v)$, where $v = (t, tx_2/x_1, tx_3/x_1)$, and Ψ_2 sends $(l(x_1, x_2, x_3), v)$ to $(l(x_1, x_2, x_3), tx_2/x_1)$. Thus $\Psi_2 \circ \Psi_1^{-1} = Id \times \tau_{21}$, where $\tau_{21} : U_1 \cap U_2 \to GL(1, \mathbb{R})$ is the map sending $l(x_1, x_2, x_3)$ to x_2/x_1. In general, $\tau_{\beta\alpha}(l(x_1, x_2, x_3)) = x_\beta/x_\alpha$.

Exercise: Define the tautological bundles over $\mathbb{CP}^m$ and Grassmannians $G_{k,n}$.

Regarding Remark(2) following Theorem 1.2, we illustrate it with the following

Example 1.4. ("Infinite" Möbius Band) Consider $\mathbb{S}^1 \subset \mathbb{R}^2$, which has a canonical smooth atlas $\{(U_N, \phi_N), (U_S, \phi_S)\}$, where $U_N = \mathbb{S}^1 \setminus \{(0, 1)\}$, $U_S = \mathbb{S}^1 \setminus \{(0, -1)\}$. Note that $U_N \cap U_S$ is a disjoint union of $U_+ := \{x_1 > 0\}$ and $U_- := \{x_1 < 0\}$. We define a transition function $\tau_{SN} : U_N \cap U_S \to GL(1, \mathbb{R})$, by setting it equal 1 on U_+ and -1 on U_-. Note that τ_{SN} is a smooth map. We shall construct a smooth line bundle over $\mathbb{S}^1$ with transition function τ_{SN} as follows.

Let E be the quotient space $U_N \times \mathbb{R} \sqcup U_S \times \mathbb{R}/\sim$, where for any $x \in U_N \cap U_S$, $(x, t) \in U_N \times \mathbb{R}$ is $\sim$ to $(x, \tau_{SN} t) \in U_S \times \mathbb{R}$. We define $\pi : E \to \mathbb{S}^1$ by sending (x, t) to x. For each $x \in \mathbb{S}^1$, the fiber E_x is a copy of $\mathbb{R}$, hence a 1-dimensional vector space over $\mathbb{R}$. Note that the inclusions $U_N \times \mathbb{R} \to U_N \times \mathbb{R} \sqcup U_S \times \mathbb{R}$, $U_S \times \mathbb{R} \to U_N \times \mathbb{R} \sqcup U_S \times \mathbb{R}$ induce injective maps from $U_N \times \mathbb{R}$, $U_S \times \mathbb{R}$ to E, with image $\pi^{-1}(U_N)$, $\pi^{-1}(U_S)$ respectively. We define $\Psi_N : \pi^{-1}(U_N) \to U_N \times \mathbb{R}$, $\Psi_S : \pi^{-1}(U_S) \to U_S \times \mathbb{R}$ to the inverses of them. Then it is easy to see that $\Psi_S \circ \Psi_N^{-1} = Id \times \tau_{SN}$ on $(U_N \cap U_S) \times \mathbb{R}$. By Theorem 1.2, E is a smooth line bundle over $\mathbb{S}^1$ with transition function τ_{SN}.

Pull-back bundles: Let $\pi : E \to M$ be a smooth vector bundle of rank n, and let $F : N \to M$ be any smooth map. We define the **pull-back bundle of E via F**, denoted by F^*E, as follows. As a set, we consider $E' := \sqcup_{p \in N} E_{F(p)}$, with the surjective map $\pi : E' \to N$ defined by sending any $v \in E_{F(p)}$ to p. It is easy to see that for each $p \in N$, the fiber of E' at p, denoted by E'_p, is given by $E_{F(p)}$, i.e., the fiber of E at the image $F(p)$ of p under the map F. Clearly, $E'_p = E_{F(p)}$ is a n-dimensional vector space over $\mathbb{R}$.

We shall apply Theorem 1.2 to show that E' is a smooth vector bundle of rank n over N. To this end, we choose a smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ of M such that over each U_α, E has a trivialization $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$. We set $V_\alpha := F^{-1}(U_\alpha)$. Then $\{V_\alpha\}$ is an open cover of N. For simplicity, we assume $\{V_\alpha\}$ comes from a smooth atlas of N. Then we define a map $\Psi'_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^n$ as follows: note that $\pi^{-1}(V_\alpha) = \sqcup_{p \in V_\alpha} E'_p = \sqcup_{p \in V_\alpha} E_{F(p)}$. With this understood, Ψ'_α sends each $E'_p = E_{F(p)}$ to $\{p\} \times \mathbb{R}^n$ isomorphically by Ψ_α. It is easy to see that Ψ'_α is a bijection. Moreover, for any α, β such that $V_\alpha \cap V_\beta \neq \emptyset$, $\Psi'_\beta \circ \Psi'_\alpha = Id \times (\tau_{\beta\alpha} \circ F)$, where $\{\tau_{\beta\alpha}\}$ is the associated transition functions for E. The transition functions of E' are $\tau'_{\beta\alpha} := \tau_{\beta\alpha} \circ F : V_\alpha \cap V_\beta \to GL(n, \mathbb{R})$, which are clearly smooth because both $\tau_{\beta\alpha}$ and F are smooth maps. Hence our claim.

Remarks: (1) In terms of transition functions, the pull-back bundles are the bundles determined by the pull-back of the corresponding transition functions.

(2) Note that if the image $F(N)$ lies entirely in a U_α over which E is trivial, then the pull-back bundle F^*E is easily seen trivial. In particular, any pull-back bundle of a trivial bundle is trivial.

(3) If $S \subset M$ is an embedded submanifold of lower dimension or open subset, $i : S \to M$ the inclusion map, the pull-back bundle i^*E is called the **restriction** of E to S, also denoted by $E|_S$.

Example 1.5. Let $\pi : E \to \mathbb{RP}^n$ be the tautological line bundle. For any $k < n$, consider the smooth embedding $F : \mathbb{RP}^k \to \mathbb{RP}^n$ induced by $\mathbb{R}^{k+1} \to \mathbb{R}^{k+1} \times \{0\} \subset \mathbb{R}^{n+1}$. Then the pull-back bundle F^*E is the tautological line bundle over $\mathbb{RP}^k$.

Sections, frames, and trivializations:

Definition 1.6. Let $\pi : E \to M$ be a smooth vector bundle of rank n, and let $U \subset M$ an open subset.

(1) A **smooth section** of E over U is a smooth map $s : U \to E$ such that $\pi \circ s = Id$ on U (equivalently, for any $p \in U$, $s(p) \in E_p$). When $U = M$, s is called a **global section** of E; otherwise, s is called a **local section**.

(2) A set $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ of n smooth sections is called a **local frame** over U, if for any $p \in U$, $(\sigma_1(p), \sigma_2(p), \cdots, \sigma_n(p))$ is a basis of E_p. When $U = M$, σ is called a **global frame** of E.

Example 1.7. Consider the tangent bundle $\pi : TM \to M$. Over any local coordinate chart (U, ϕ), with local coordinate functions (x^i), the map $\frac{\partial}{\partial x^i}$, for each i, sending $p \in U$ to $\frac{\partial}{\partial x^i}|_p \in T_pM$, is a smooth section over U. Moreover, the set $(\frac{\partial}{\partial x^i})$ is a local frame of TM over U, called a **local coordinate frame**.

Recall from Example 1.3(1) that in proving TM is a smooth vector bundle, we used local coordinate frames to define local trivializations of TM. In fact, this is true in general, as we see below.

Proposition 1.8. *Local frames and local trivializations correspond to each other in a canonical way. More precisely, if $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ is a local frame of E over U, then the map $\Psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ defined by sending $\sum_{i=1}^n v_i \sigma_i(p) \in E_p$ to $(p, (v_1, v_2, \cdots, v_n)) \in U \times \mathbb{R}^n$ is a local trivialization of E over U. On the other hand, given any trivialization $\Psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$, the set $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$, where for each i, $\sigma_i(p) := \Psi^{-1}(p, e_i)$ where $e_1, e_2, \cdots, e_n$ is the standard basis of $\mathbb{R}^n$, is a local frame of E over U. In particular, E is a trivial bundle iff it admits a global frame.*

Exercise: (1) Prove that a section s of E over U (i.e., a map $s : U \to E$ such that $\pi \circ s = Id$ on U) is smooth if and only if for any local frame $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ of E over U, $s(p) = \sum_{i=1}^n a_i(p)\sigma_i(p)$ for some smooth functions $a_1, a_2, \cdots, a_n$ on U.

(2) Prove Proposition 1.8.

Example 1.9. We will show that the tautological line bundle over $\mathbb{RP}^n$ is not trivial. For illustration, we shall give two proofs here.

Proof 1: The tautological line bundle E over $\mathbb{RP}^1$ is the pull-back bundle of that over $\mathbb{RP}^n$. Hence it suffices to show that $\pi : E \to \mathbb{RP}^1$ is not trivial. To this end, recall that $\mathbb{RP}^1$ has a canonical smooth atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$, where $U_i = \{l(x_1, x_2) \in$

$\mathbb{RP}^1 | x_i \neq 0\}$, $i = 1, 2$. Moreover, there are canonical trivializations $\Psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}$ such that the transition function $\tau_{21}(l(x_1, x_2)) = x_2/x_1$, for $l(x_1, x_2) \in U_1 \cap U_2$. (See Example 1.3(2).)

With this understood, we note that $U_1 \cap U_2 = U_+ \sqcup U_-$, a disjoint union of two connected components, where $U_+ = \{x_1x_2 > 0\}$ and $U_- = \{x_1x_2 < 0\}$. Now suppose to the contrary that E is trivial, and let σ be a global frame of E. Then for $i = 1, 2$, $f_i := \Psi_i \circ \sigma$ is a smooth, nonzero function over U_i, which obeys $\tau_{21} f_1 = f_2$ over $U_+ \sqcup U_-$. Since f_1 is nonzero on U_1, it must have the same sign over U_+ and U_-. However, $\tau_{21}(l(x_1, x_2)) = x_2/x_1$, which has a different sign over U_+ and U_-, so that f_2 must have a different sign over U_+ and U_-. But this contradicts the fact that f_2 is nonzero on U_2, hence E is nontrivial.

Proof 2: In this proof, we consider the complement of the zero-section of the tautological line bundle E over $\mathbb{RP}^n$, $E \setminus \{0\} := \sqcup_{x \in \mathbb{RP}^n} E_x \setminus \{0\}$. Recall that $E = \{(x, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} | v \in x\}$. Under the projection $(x, v) \mapsto v$, it is easy to see that $E \setminus \{0\}$ is mapped homeomorphically onto $\mathbb{R}^{n+1} \setminus \{0\}$. In particular, $E \setminus \{0\}$ is a connected space. On the other hand, if E were trivial, then $E \setminus \{0\}$ is diffeomorphic to $\mathbb{RP}^n \times (\mathbb{R} \setminus \{0\})$, which is disconnected. Hence the proof.

The associated sphere bundles: The idea in Proof 2 in the previous example can be generalized. Let $\pi : E \to M$ be a smooth real vector bundle of rank n. Consider the smooth Lie group action of $\mathbb{R}$ on $E \setminus \{0\}$, sending $(t, (p, v))$ to $(p, e^t v)$ for any $v \in E_p \setminus \{0\}$, $p \in M$. Since $v \neq 0$, the $\mathbb{R}$-action is free. One can also check that the $\mathbb{R}$-action is proper as well. By the Quotient Manifold Theorem, the quotient space $E \setminus \{0\}/\mathbb{R}$ is a smooth manifold, which we will denote by $S(E)$. In fact, it is easy to see that $\pi : E \to M$ factors through $S(E)$, which induces a smooth, surjective map $\tilde{\pi} : S(E) \to M$, where for each $p \in M$, the fiber $\tilde{\pi}^{-1}(p)$ is a smooth manifold diffeomorphic to $\mathbb{S}^{n-1}$. This is an example of **fiber bundle**; it is locally trivial, as for any local trivialization of E, $\Psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$, there is an induced diffeomorphism from $\tilde{\pi}^{-1}(U)$ to $U \times \mathbb{S}^{n-1}$. The bundle $\tilde{\pi} : S(E) \to M$ is called the associated **sphere bundle** of E. We remark that $S(E)$ is homotopy equivalent to $E \setminus \{0\}$, so it captures all the topological information of $E \setminus \{0\}$, but is easier to handle.

Example 1.10. For simplicity, let M be connected. We claim that the set of smooth line bundles (up to isomorphism) over M is in one to one correspondence with $H^1(M, \mathbb{Z}_2)$. More concretely, we will show that each nontrivial smooth line bundle over M determines a nonzero element in $H^1(M, \mathbb{Z}_2)$, and vice versa.

To see this, let $\pi : E \to M$ be a smooth line bundle. Consider the associated sphere bundle $\tilde{\pi} : S(E) \to M$, which in this case has fibers $\mathbb{S}^0 = \{\pm\}$. It is easy to see that E is nontrivial if and only if $S(E)$ is connected and $\tilde{\pi} : S(E) \to M$ is a double cover. So for each nontrivial E, $\tilde{\pi} : S(E) \to M$ corresponds to an epimorphism $\pi_1(M) \to \mathbb{Z}_2$, which factors through $H_1(M)$, giving rise to an epimorphism $H_1(M) \to \mathbb{Z}_2$, which corresponds to a nonzero element in $H^1(M, \mathbb{Z}_2)$. Conversely, given any nonzero element in $H^1(M, \mathbb{Z}_2)$, we get an epimorphism $\pi_1(M) \to \mathbb{Z}_2$, which corresponds to a double cover $\tilde{M}$ of M. We consider $\tilde{M} \times \mathbb{R}$, with a free smooth $\mathbb{Z}_2$-action on it, which is the deck transformation on the $\tilde{M}$-factor and is given by multiplication by -1 on the $\mathbb{R}$-factor. The quotient $E := (\tilde{M} \times \mathbb{R})/\mathbb{Z}_2$ is the corresponding nontrivial smooth line bundle over M, with $S(E) = \tilde{M}$.

Example 1.11. In this example, let $M = \mathbb{S}^2$. Let $\pi : E \to M$ be a smooth real vector bundle of rank 2. The associated sphere bundle $\tilde{\pi} : S(E) \to M$ is a $\mathbb{S}^1$-bundle. Note that in this case, E is trivial if and only if $S(E) = \mathbb{S}^2 \times \mathbb{S}^1$. Observe $\pi_1(S(E)) = \mathbb{Z}$ when E is trivial.

Now let E be the complex tautological line bundle over $\mathbb{CP}^1$. Since $\mathbb{CP}^1 = \mathbb{S}^2$, E can be regarded as a smooth real vector bundle of rank 2 over $M = \mathbb{S}^2$. As we argued in Proof 2 of Example 1.9, it is easy to see that $S(E) = \mathbb{S}^3$. In particular, $\pi_1(S(E)) = 0$ if E is the tautological bundle over $\mathbb{CP}^1$.

Exercise: Let $E = T\mathbb{S}^2$. Prove, using the Seifert-Van Kampen theorem, that $\pi_1(S(E)) = \mathbb{Z}_2$.

Hence $T\mathbb{S}^2$ is nontrivial, and also, $T\mathbb{S}^2$ is not isomorphic to the tautological line bundle over $\mathbb{CP}^1$ (as a rank 2 real bundle).

Induced bundles: We shall only discuss the case of real vector bundles in details; the case of complex/holomorphic vector bundles is analogous.

The dual bundle: Let $\pi : E \to M$ be a smooth real vector bundle of rank n. Let $E^* := \sqcup_{p\in M} E_p^*$, where E_p^* is the dual space of E_p, with $\pi^* : E^* \to M$ the surjective map so that each fiber $(\pi^*)^{-1}(p) = E_p^*$. Then $\pi^* : E^* \to M$ can be made into a smooth real vector bundle of rank n, called the **dual bundle** of E.

More concretely, let $\{U_\alpha\}$ be an open cover of M such that over each U_α, there is a trivialization of E, $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$. Let $\sigma_\alpha = (\sigma_{i,\alpha})$ be the corresponding local frame over U_α, and let $\{\tau_{\beta\alpha}\}$ be the associated transition functions. Then

$$\sigma_{j,\alpha}(p) = \sum_{i=1}^{n} [\tau_{\beta\alpha}(p)]_{ij}\sigma_{i,\beta}(p), \;\; \forall p \in U_\alpha \cap U_\beta,$$

where $[\tau_{\beta\alpha}(p)]_{ij}$ is the (i,j)-entry of the matrix $\tau_{\beta\alpha}(p)$. For each α, let $(\sigma^*_{i,\alpha}(p))$ be the dual basis of the basis $(\sigma_{i,\alpha}(p))$. Note that, if

$$\sigma^*_{j,\alpha}(p) = \sum_{i=1}^{n} [\tau^*_{\beta\alpha}(p)]_{ij}\sigma^*_{i,\beta}(p), \;\; \forall p \in U_\alpha \cap U_\beta,$$

then the matrix $\tau^*_{\beta\alpha}(p)$ with (i,j)-entry $[\tau^*_{\beta\alpha}(p)]_{ij}$ equals $(\tau_{\beta\alpha}(p)^T)^{-1}$. Consequently, if we use $\sigma^*_\alpha := (\sigma^*_{i,\alpha})$ as a local frame to define a local trivialization of E^* over U_α, then the associated transition functions are $\{\tau^*_{\beta\alpha}\}$, which are smooth. By Theorem 1.2, $\pi^* : E^* \to M$ is a smooth real vector bundle of rank n. This motivates

Definition 1.12. Let $G \subset GL(n,\mathbb{R})$ be a Lie subgroup, and $\rho : G \to GL(m,\mathbb{R})$ be a Lie group homomorphism. Suppose $\pi : E \to M$ is a smooth real vector bundle of rank n with the property that there exist local trivializations of E over $\{U_\alpha\}$ such that the associated transition functions $\{\tau_{\beta\alpha}\}$ have $\tau_{\beta\alpha}(p) \in G$ for any $p \in U_\alpha \cap U_\beta$. We define $\tilde{\tau}_{\beta\alpha} = \rho \circ \tau_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(m,\mathbb{R})$. Then

$$\tilde{\tau}_{\gamma\alpha}(p) = \tilde{\tau}_{\gamma\beta}(p)\tilde{\tau}_{\beta\alpha}(p), \; \forall p \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Let $\tilde{\pi} : \tilde{E} \to M$ be the smooth vector bundle of rank m determined by $\{\tilde{\tau}_{\beta\alpha}\}$. We call it the **induced bundle** of E via $\rho : G \to GL(m, \mathbb{R})$.

So the dual bundle of E is the induced bundle of E via $\rho : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, where $\rho(A) = (A^T)^{-1}$, $A \in GL(n, \mathbb{R})$.

Example 1.13. (1) The dual bundle of TM is called the **cotangent bundle** of M, denoted by T^*M. Note that the dual of a local coordinate frame $(\frac{\partial}{\partial x^i})$ is (dx^i), where dx^i is the local smooth section of T^*M sending p to $dx^i(p) \in T_p^*M$. (dx^i) is called a **local coordinate coframe**.

(2) Let $E := \mathbb{RP}^n \setminus \{l(0, 0, \cdots, 1)\}$, with $\pi : E \to \mathbb{RP}^{n-1}$ defined by sending $l(x_1, x_2, \cdots, x_n, x_{n+1})$ to $l(x_1, x_2, \cdots, x_n)$. Note that for any $l \in \mathbb{RP}^{n-1}$, the fiber $\pi^{-1}(l)$ is the set of graphs of linear transformations from the line $l \subset \mathbb{R}^n$ to the x_{n+1}-axis in $\mathbb{R}^{n+1}$, hence is naturally a 1-dimensional vector space over $\mathbb{R}$.

Exercise: Show that $\pi : E \to \mathbb{RP}^{n-1}$ is isomorphic to the dual bundle of the tautological line bundle over $\mathbb{RP}^{n-1}$. (See Example 1.3(2).)

Sub-bundles and quotient bundles: Let $\pi : E \to M$ be a smooth vector bundle of rank n. A subset $E' \subset E$ is called a **sub-bundle** of rank k, if the following holds:

(1) let $\pi' = \pi|_{E'}$, then for any $p \in M$, $E'_p := (\pi')^{-1}(p)$ is a k-dimensional subspace of E_p,

(2) for any $p \in M$, there exists a local frame of E over a neighborhood U of p, denoted by $\sigma_1, \sigma_2, \cdots, \sigma_n$, such that for any $q \in U$, $\sigma_1(q), \sigma_2(q), \cdots, \sigma_k(q)$ is a basis of E'_q.

The **quotient bundle** E/E', which as a set is defined to be $\sqcup_{p \in M} E_p/E'_p$, is a smooth vector bundle of rank $n - k$.

In fact, let G be the Lie subgroup of $GL(n, \mathbb{R})$ which consists of matrices of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where $A \in GL(k, \mathbb{R})$, $C \in GL(n-k, \mathbb{R})$. Then item (2) above implies that the bundle E admits a set of local trivializations such that the associated transition functions $\{\tau_{\beta\alpha}\}$ whose images lie in G. Let $\rho_1 : G \to GL(k, \mathbb{R})$ be the Lie group homomorphism sending $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ to A, and $\rho_2 : G \to GL(n-k, \mathbb{R})$ be the Lie group homomorphism sending $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ to C. Then E', E/E' are the induced bundles of E via ρ_1, ρ_2 respectively.

Example 1.14. Let $S \subset M$ be an embedded submanifold. Then TS is a sub-bundle of $TM|_S$, because for any $p \in S$, there is a slice chart of S containing p, which in particular implies item (2) in the definition of sub-bundles. The quotient bundle $TM|_S/TS$ is called the **normal bundle** of S in M.

Direct sum: Let E, E' be two smooth vector bundles over M. The **direct sum** of E, E', defined to be $E \oplus E' := \sqcup_{p \in M} E_p \oplus E'_p$, is a smooth vector bundle over M, whose bundle structure is determined as follows: we choose an open cover $\{U\}$ of M such that over each U, both E, E' are trivial. Let (σ_i), (σ'_j) be the local frames of E, E'

over U. Then we simply declare that (σ_i, σ'_j) to be a local frame of $E \oplus E'$ over U. It is easy to see that the associated transition functions for $E \oplus E'$ are smooth.

Exercise: Let E be the tautological bundle over $\mathbb{RP}^2$, and let E' be the pull-back bundle of the tautological bundle of the Grassmannian $G_{2,3}$ via the canonical diffeomorphism $\mathbb{RP}^2 \to G_{2,3}$. Show that $E \oplus E'$ is a trivial bundle over $\mathbb{RP}^2$. (Hint: find a global frame of $E \oplus E'$.)

Tensor product: Let E, E' be two smooth vector bundles over M of rank n, n' respectively. The **tensor product** of E, E', defined to be $E \otimes E' := \sqcup_{p \in M} E_p \otimes E'_p$, is a smooth vector bundle over M of rank nn', whose bundle structure is determined as follows: we choose an open cover $\{U\}$ of M such that over each U, both E, E' are trivial. Let (σ_i), (σ'_j) be the local frames of E, E' over U. Then we simply declare that $(\sigma_i \otimes \sigma'_j)$ to be a local frame of $E \otimes E'$ over U. It is easy to see that the associated transition functions for $E \otimes E'$ are smooth.

Exercise: Let E be the (complex) tautological line bundle over $\mathbb{CP}^1$. For any $n > 0$, let E^n be the n-fold tensor product of E, which is a complex line bundle over $\mathbb{CP}^1$. Show that the associated sphere bundle $S(E^n)$, where E^n is regarded as a rank 2 real bundle, is diffeomorphic to the lens space $L(n, 1)$. In particular, for $m \neq n$, E^m and E^n are not isomorphic.

Exercise: (1) Let E be a real line bundle over M. Then $E \otimes E^*$ is also a real line bundle over M. Show that $E \otimes E^*$ is always trivial. (Hint: for any vector spaces V, W, the tensor product $W \otimes V^* = \mathrm{Hom}(V, W)$, the space of linear maps from V to W. Use this fact to find a global frame of $E \otimes E^*$.)

(2) Conversely, let E_1, E_2 be real line bundles over M. Show that if $E_1 \otimes E_2^*$ is trivial, then E_1, E_2 are isomorphic.

2. Vector fields, Lie bracket, and Lie algebras

Definition 2.1. (1) Let M be a smooth manifold. A smooth section $X : M \to TM$ of the tangent bundle is called a **smooth vector field** of M. The set of smooth vector fields is denoted by $\mathcal{X}(M)$. A **local smooth vector field** is one that is only defined over an open subset $U \subset M$. (We will use the following notation: the value of X at $p \in M$ is denoted by $X_p \in T_pM$.)

(2) Let $F : M \to N$ be any smooth map, $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$. We say X, Y are **F-related** if for any $p \in M$, $F_*(X_p) = Y_{F(p)}$.

Remarks: (1) Note that $\mathcal{X}(M)$ is naturally a module over the commutative ring $C^\infty(M)$ of smooth functions on M.

(2) Let $(\frac{\partial}{\partial x^i})$ be a local coordinate frame over U. A section $X : U \to TU$ is smooth iff $X = \sum_i X^i \frac{\partial}{\partial x^i}$ where $X^i \in C^\infty(U)$. Another criterion for smoothness of X is that for any $f \in C^\infty(U)$, $X(f) : p \mapsto X_p(f)$ is a smooth function on U.

Definition 2.2. Let $X, Y \in \mathcal{X}(M)$. We define the **Lie bracket** of X, Y, denoted by $[X, Y]$, to be the $\mathbb{R}$-linear map from $C^\infty(M)$ to itself, by

$$[X, Y](f) := X(Y(f)) - Y(X(f)), \ \ \forall f \in C^\infty(M).$$

For each $p \in M$, let $[X,Y]_p$ be the $\mathbb{R}$-linear map from $C^\infty(M)$ to $\mathbb{R}$, where

$$[X,Y]_p(f) := [X,Y](f)(p) = X_p(Y(f)) - Y_p(X(f)), \ \ \forall f \in C^\infty(M).$$

Lemma 2.3. *For any $p \in M$, $[X,Y]_p \in T_pM$. Consequently, $[X,Y] \in \mathcal{X}(M)$.*

Proof. One only needs to verify

$$[X,Y]_p(fg) = f(p)[X,Y]_p(g) + g(p)[X,Y]_p(f), \ \ \forall f,g \in C^\infty(M),$$

which follows from a direct calculation. □

Local Expression: Let $(\frac{\partial}{\partial x^i})$ be a local coordinate frame over U. Let $X = \sum_i X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$ be two local smooth vector fields over U. Then we can write

$$[X,Y] = \sum_i [X,Y]^i \frac{\partial}{\partial x^i}, \text{ where } [X,Y]^i \in C^\infty(U).$$

Exercise: Show that $[X,Y]^i = X(Y^i) - Y(X^i)$. In particular, $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$.

The following properties of Lie bracket are straightforward, except perhaps the Jacobi Identity.

Proposition 2.4. *(Properties of Lie bracket) Let $X,Y,Z \in \mathcal{X}(M)$.*

(a) $[X,Y] = -[Y,X]$.
(b) $[aX + bY, Z] = a[X,Z] + b[Y,Z]$, $\forall a, b \in \mathbb{R}$.
(c) *Jacobi Identity:* $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.
(d) $[fX,Y] = f[X,Y] - Y(f)X$, *for any* $f \in C^\infty(M)$.
(e) *Let $F : M \to N$ be any smooth map. For $i = 1,2$, if $X_i \in \mathcal{X}(M)$ is F-related to $Y_i \in \mathcal{X}(N)$, then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.*

Definition 2.5. A vector space V is called a **Lie algebra** if it is equipped with a bilinear, antisymmetric map $V \times V \to V$, denoted by $(X,Y) \mapsto [X,Y]$ and called the **bracket**, which satisfies the Jacobi Identity (as in Prop. 2.4). V is called **Abelian** if $[X,Y] = 0$ for any $X,Y \in V$.

So $\mathcal{X}(M)$, with the Lie bracket, is an infinite dimensional Lie algebra. Finite dimensional Lie algebras naturally arise in the study of Lie groups.

The Lie algebra of a Lie group. Let G be a Lie group. For any $g \in G$, let $L_g : G \to G$, $R_g : G \to G$ be the left and right translation by g. A smooth vector field $X \in \mathcal{X}(G)$ is said to be **left-invariant** if for any $g \in G$, $(L_g)_*(X) = X$, i.e., $(L_g)_*(X_h) = X_{gh}$, $\forall h \in G$. (Similarly, one can define **right-invariant** vector fields.) The set of all left-invariant vector fields is denoted by $Lie(G)$. By Prop.2.4(e), for any $X,Y \in Lie(G)$, $[X,Y] \in Lie(G)$. Consequently, $Lie(G)$ is a Lie algebra, which is called the Lie algebra of G.

Theorem 2.6. *The map $r : Lie(G) \to T_eG$ by $X \mapsto X_e$ is an isomorphism.*

Proof. The map $r : Lie(G) \to T_eG$ is clearly linear and injective. It remains to show that it is surjective. To this end, let $A \in T_eG$ be any tangent vector at e. We define a vector filed X with $X_e = A$ as follows: for any $g \in G$, we define $X_g := (L_g)_*(A)$, where $(L_g)_* : T_eG \to T_gG$. By the very definition, X is naturally left-invariant:

$$(L_g)_*(X_h) = (L_g)_*((L_h)_*(A)) = (L_g)_* \circ (L_h)_*(A) = (L_{gh})_*(A) = X_{gh}, \forall g, h \in G.$$

It remains to show that X_g depends smoothly on g, which follows if for any $f \in C^\infty(G)$, $X_g(f)$ is a smooth function g. To this end, we pick a smooth curve $\gamma : (-\epsilon, \epsilon) \to G$ such that $\gamma(0) = e$ and $\gamma'(0) = A$. Then

$$X_g(f) = (L_g)_*(A)(f) = A(f \circ L_g) = \frac{d}{dt}((f \circ L_g) \circ \gamma)|_{t=0} = \frac{\partial}{\partial t}(f(g\gamma(t)))|_{t=0}.$$

The function $\phi(g,t) := f(g\gamma(t))$ is smooth in g, t, as $\phi = f \circ m \circ (Id \times \gamma) : G \times (-\epsilon, \epsilon) \to \mathbb{R}$, where $m : G \times G \to G$, sending (g, h) to gh, is a composition of smooth maps. Hence $X_g(f)$ is smooth in g for any $f \in C^\infty(G)$. This finishes the proof.

□

Corollary 2.7. *Let $F : G \to H$ be a Lie group homomorphism. Then there is an induced Lie algebra homomorphism $F_* : Lie(G) \to Lie(H)$.*

Proof. By Theorem 2.6, for any $X \in Lie(G)$, there is a unique $Y \in Lie(H)$ such that $Y_e = F_*(X_e) \in T_eH$, where $F_* : T_eG \to T_eH$ is induced by the smooth map $F : G \to H$. We let $F_*(X) = Y$, which defines $F_* : Lie(G) \to Lie(H)$ as a linear map between vector spaces. To see it preserves the Lie bracket, we note that $F : G \to H$ is a Lie group homomorphism implies that $F \circ L_g = L_{F(g)} \circ F$ for any $g \in G$. Hence

$$F_*(X_g) = F_* \circ (L_g)_*(X_e) = (L_{F(g)})_* \circ F_*(X_e) = (L_{F(g)})_*(Y_e) = Y_{F(g)}, \forall X \in Lie(G).$$

In other words, $X \in Lie(G)$ and $F_*(X) \in Lie(H)$ are F-related. By Prop.2.4(e), $F_* : Lie(G) \to Lie(H)$ preserves the Lie bracket, hence is a Lie algebra homomorphism.

□

Proposition 2.8. *The Lie algebra of an Abelian Lie group is Abelian.*

Proof. Let G be any Lie group. We first show that for the smooth map $m : G \times G \to G$, the map $m_* : T_{(e,e)}(G \times G) = T_eG \times T_eG \to T_eG$ sends (X, Y) to $X + Y$, i.e., $m_*(X, Y) = X + Y$, $\forall X, Y \in T_eG$.

To see this, for any $X \in T_eG$, let $\gamma(t)$ be a smooth curve in G through e such that $X = \gamma'(0)$. Then $m_*(X, 0)$ is represented by the smooth curve $m(\gamma(t), e) = \gamma(t)$ in G, hence $m_*(X, 0) = X$. Similarly, $m_*(0, Y) = Y$, so that

$$m_*(X, Y) = m_*(X, 0) + m_*(0, Y) = X + Y, \ \ \forall X, Y \in T_eG.$$

With this understood, let $I : G \to G$ be the map where $I(g) = g^{-1}$. Then $I_* : T_eG \to T_eG$ is given by $I_* = -Id$, because for any $g \in G$, $m(g, I(g)) = e$, so that for any $X \in T_eG$, $X + I_*(X) = 0$. Hence the claim.

Now when G is Abelian, $I : g \mapsto g^{-1}$ is a Lie group homomorphism. By Corollary 2.7, for any $X, Y \in Lie(G)$,

$$-[X, Y] = I_*([X, Y]) = [I_*(X), I_*(Y)] = [-X, -Y] = [X, Y],$$

which implies that $[X, Y] = 0$. □

Sometimes it is more convenient to identify $Lie(G)$ with T_eG (as in Theorem 2.6) as in concrete examples, T_eG is a more explicit vector space than $Lie(G)$. With such an identification, T_eG becomes a Lie algebra through $Lie(G)$, with a specific bracket $[-,-]: T_eG\times T_eG\rightarrow T_eG$. More specifically, for any $A,B\in T_eG$, we let $\widetilde{A},\widetilde{B}\in Lie(G)$ whose values at e are A,B respectively. Then we define $[A,B]:=[\widetilde{A},\widetilde{B}]_e\in T_eG$.

Example 2.9. Let $G=GL(n,\mathbb{R})$. Then $T_eG=M(n,\mathbb{R})$, the space of real $n\times n$ matrices. Note that as a vector space over $\mathbb{R}$, $M(n,\mathbb{R})$ is given with the standard smooth structure, and $G\subset M(n,\mathbb{R})$ is an open subset, given with the induced smooth structure. With this understood, note that in fact for any $g\in G$, $T_gG=M(n,\mathbb{R})$. So in what follows, we shall identity T_gG (in particular, T_eG) with $M(n,\mathbb{R})$ throughout.

Let (x_{ij}) be the standard coordinates on $M(n,\mathbb{R})$, with x_{ij} being the (i,j)-entry of a matrix $X\in M(n,\mathbb{R})$. Given any $A=(a_{ij})\in M(n,\mathbb{R})$, let $\widetilde{A}\in Lie(G)$ whose value at e equals A. We observe that for any $g\in G$, the left translation $L_g: G\rightarrow G$ is a linear map so that $(L_g)_*=L_g$. Consequently, for any $X=(x_{ij})\in G$, the value of $\widetilde{A}$ at X is given by $L_X(A)=XA\in M(n,\mathbb{R})$. Equivalently,

$$\widetilde{A}=\sum_{i=1}^n\sum_{j=1}^n(\sum_{k=1}^n x_{ik}a_{kj})\frac{\partial}{\partial x_{ij}}.$$

We let $B=(b_{ij})$. Then $\widetilde{B}=\sum_{i=1}^n\sum_{j=1}^n(\sum_{k=1}^n x_{ik}b_{kj})\frac{\partial}{\partial x_{ij}}$. Using the local expression for Lie bracket, i.e., for $X=\sum_i X^i\frac{\partial}{\partial x^i}$ and $Y=Y^i\frac{\partial}{\partial x^i}$,

$$[X,Y]=\sum_i [X,Y]^i\frac{\partial}{\partial x^i},\ \text{where } [X,Y]^i=X(Y^i)-Y(X^i),$$

we get

$$[\widetilde{A},\widetilde{B}]=\sum_{i=1}^n\sum_{j=1}^n(\tilde{A}(\sum_{k=1}^n x_{ik}b_{kj})-\widetilde{B}(\sum_{k=1}^n x_{ik}a_{kj}))\frac{\partial}{\partial x_{ij}}=\sum_{i=1}^n\sum_{j=1}^n(\sum_{k=1}^n\sum_{l=1}^n x_{ik}a_{kl}b_{lj}-\sum_{k=1}^n\sum_{l=1}^n x_{ik}b_{kl}a_{lj})\frac{\partial}{\partial x_{ij}}.$$

Evaluating the above expression at $X=(x_{ij})=I_n$ (the identity matrix), we get

$$[\widetilde{A},\widetilde{B}]_e=\sum_{i=1}^n\sum_{j=1}^n\sum_{l=1}^n(a_{il}b_{lj}-b_{il}a_{lj})\frac{\partial}{\partial x_{ij}}.$$

It follows easily that $[A,B]=AB-BA$.

Let V be a n-dimensional Lie algebra, $X_1,X_2,\cdots,X_n$ be a basis of the vector space V. Then the bracket on V is determined by a set of constants $\{c_{ij}^k\}$, where

$$[X_i,X_j]=\sum_{k=1}^n c_{ij}^k X_k.$$

Note that the antisymmetry of the bracket implies $c_{ij}^k=-c_{ji}^k$ for any i,j,k, and the Jacobi Identity implies that $\sum_{l=1}^n(c_{jk}^l c_{il}^s+c_{ki}^l c_{jl}^s+c_{ij}^l c_{kl}^s)=0$ for any i,j,k,s.

Exercise: Let V be a 2-dimensional vector space, and X_1,X_2 be a basis of V. We define $[X_1,X_2]=X_2$. Show that this determines a Lie algebra structure on V.

Exercise: Classify all k-dimensional Lie algebras up to isomorphism for $k \leq 3$.

Exercise: Consider the Lie group $\mathbb{S}^3$ of unit quaternions. Note that $T_e\mathbb{S}^3$ is spanned by the quaternions i, j, k. Let $X_1, X_2, X_3 \in Lie(\mathbb{S}^3)$ whose values at $e \in \mathbb{S}^3$ are i, j, k respectively. Show that

$$[X_1, X_2] = 2X_3, \;\; [X_2, X_3] = 2X_1, \;\; [X_3, X_1] = 2X_2.$$

3. Tensor bundles and tensor fields

Tensor bundles: Let V be a n-dimensional vector space over $\mathbb{R}$, let V^* denote its dual space. We shall consider the following tensor spaces associated to V:

(1) $T^k(V) = V^* \otimes V^* \otimes \cdots \otimes V^*$ (k-fold tensor product of V^*), the space of **covariant k-tensors**, which is naturally identified with the space of multilinear maps from $V \times V \times \cdots \times V$ to $\mathbb{R}$.
(2) $T_l(V) = V \otimes V \otimes \cdots \otimes V$ (l-fold tensor product of V), the space of **contravariant l-tensors**.
(3) $T^k_l(V) = T^k(V) \otimes T_l(V)$, the space of **mixed tensors of type** (k, l).

Given any basis $(e_1, e_2, \cdots, e_n)$ of V, we denote by $(\epsilon^1, \epsilon^2, \cdots, \epsilon^n)$ the basis of V^* which is dual to the given basis of V. Then there are bases of the corresponding tensor spaces canonically associated to the given basis $(e_1, e_2, \cdots, e_n)$:

(1) $T^k(V)$: $\{\epsilon^{i_1} \otimes \epsilon^{i_2} \otimes \cdots \otimes \epsilon^{i_k} | 1 \leq i_1, i_2, \cdots, i_k \leq n\}$.
(2) $T_l(V)$: $\{e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_l} | 1 \leq j_1, j_2, \cdots, j_l \leq n\}$.
(3) $T^k_l(V)$: $\{\epsilon^{i_1} \otimes \epsilon^{i_2} \otimes \cdots \otimes \epsilon^{i_k} \otimes e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_l} | 1 \leq i_1, i_2, \cdots, i_k, j_1, j_2, \cdots, j_l \leq n\}$.

Now let $\pi : E \to M$ be a real smooth vector bundle of rank n. Then one has the following associated tensor bundles of E:

$$T^k(E) := \sqcup_{p \in M} T^k(E_p), \;\; T_l(E) := \sqcup_{p \in M} T_l(E_p), \;\; T^k_l(E) := \sqcup_{p \in M} T^k_l(E_p).$$

These are smooth vector bundles over M in a canonical way: for any local frame of E, we declare the corresponding local sections of the associated bases of the tensor bundle to be a local frame of the tensor bundle.

Tensor fields: We shall focus on the case where $E = TM$. The corresponding tensor bundles, denoted by $T^k M$, $T_l M$, and $T^k_l M$ respectively, are called the bundle of **covariant k-tensors**, **contravariant l-tensors**, and **mixed tensors of type** (k, l) on M. The corresponding space of smooth sections is denoted by $\mathcal{T}^k M$, $\mathcal{T}_l M$, and $\mathcal{T}^k_l M$ respectively, and is called the space of **covariant k-tensor fields**, **contravariant l-tensor fields**, and **mixed tensor fields of type** (k, l) on M. Note that for the special case of $k = l = 0$, we have

$$T^0 M = T_0 M = T^0_0 M = M \times \mathbb{R}, \;\; \mathcal{T}^0 M = \mathcal{T}_0 M = \mathcal{T}^0_0 M = C^\infty(M).$$

Finally, note that $\mathcal{T}^k M$, $\mathcal{T}_l M$, and $\mathcal{T}^k_l M$ are naturally modules over the commutative ring $C^\infty(M)$ of smooth functions (via pointwise multiplication).

Local expressions: Let (U, ϕ) be a local coordinate chart on M, with local coordinate functions (x^i). Then we have local coordinate frame $(\frac{\partial}{\partial x^i})$ and local coordinate coframe (dx^i) on U. The corresponding local frames of the tensor bundles T^kM, T_lM, and T_l^kM are

$$(dx^{i_1}\otimes dx^{i_2}\otimes\cdots\otimes dx^{i_k}|1\le i_1,i_2,\cdots,i_k\le n),$$
$$(\frac{\partial}{\partial x^{j_1}}\otimes\frac{\partial}{\partial x^{j_2}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_l}}|1\le j_1,j_2,\cdots,j_l\le n)$$

and

$$(dx^{i_1}\otimes dx^{i_2}\otimes\cdots\otimes dx^{i_k}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\frac{\partial}{\partial x^{j_2}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_l}}|1\le i_1,i_2,\cdots,i_k,j_1,j_2,\cdots,j_l\le n)$$

respectively. A covariant k-tensor field $\sigma\in\mathcal{T}^kM$ can be written locally as

$$\sigma=\sigma_{i_1i_2\cdots i_k}dx^{i_1}\otimes dx^{i_2}\otimes\cdots\otimes dx^{i_k},\quad \sigma_{i_1i_2\cdots i_k}\in C^\infty(U),$$

a contravariant l-tensor field $\sigma\in\mathcal{T}_lM$ can be written locally as

$$\sigma=\sigma^{j_1j_2\cdots j_l}\frac{\partial}{\partial x^{j_1}}\otimes\frac{\partial}{\partial x^{j_2}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_l}},\quad \sigma^{j_1j_2\cdots j_l}\in C^\infty(U),$$

and a mixed tensor field $\sigma\in\mathcal{T}_l^kM$ of type (k,l) can be written as

$$\sigma=\sigma_{i_1i_2\cdots i_k}^{j_1j_2\cdots j_l}dx^{i_1}\otimes dx^{i_2}\otimes\cdots\otimes dx^{i_k}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\frac{\partial}{\partial x^{j_2}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_l}},\quad \sigma_{i_1i_2\cdots i_k}^{j_1j_2\cdots j_l}\in C^\infty(U).$$

Here we adapt the convention that a repeated index indicates a summation over $1,2,\cdots,n$ for that index.

Alternative description: Let $\sigma\in\mathcal{T}^kM$ be a covariant k-tensor field. Then σ defines a multilinear map from $\mathcal{X}(M)\times\mathcal{X}(M)\times\cdots\times\mathcal{X}(M)$ (k-fold product) to $C^\infty(M)$ as follows. For any $X_1,X_2,\cdots,X_k\in\mathcal{X}(M)$, and any $p\in M$,

$$\sigma(X_1,X_2,\cdots,X_k)(p):=\sigma(p)(X_1|_p,X_2|_p,\cdots,X_k|_p).$$

It is clear that σ is linear over the commutative ring $C^\infty(M)$, i.e., for any $f\in C^\infty(M)$, and any index $i=1,2,\cdots,k$, $\sigma(X_1,\cdots,fX_i,\cdots,X_k)=f\sigma(X_1,\cdots,X_i,\cdots,X_k)$.

To see that for any $X_1,X_2,\cdots,X_k\in\mathcal{X}(M)$, $\sigma(X_1,X_2,\cdots,X_k)\in C^\infty(M)$, we check it out in local coordinates. Let $(\frac{\partial}{\partial x^i})$ be a local coordinate frame over U. Then we may write $\sigma=\sigma_{i_1i_2\cdots i_k}dx^{i_1}\otimes dx^{i_2}\otimes\cdots\otimes dx^{i_k}$ and $X_i=X_i^j\frac{\partial}{\partial x^j}$. Consequently,

$$\sigma(X_1,X_2,\cdots,X_k)=\sigma_{i_1i_2\cdots i_k}X_1^{i_1}X_2^{i_2}\cdots X_k^{i_k}\in C^\infty(U),$$

because each $\sigma_{i_1i_2\cdots i_k}$ and $X_i^j\in C^\infty(U)$.

Theorem 3.1. *Each $\sigma\in\mathcal{T}_l^kM$ determines, and vice versa, a $C^\infty(M)$-multilinear map $\psi:\mathcal{X}(M)\times\mathcal{X}(M)\times\cdots\times\mathcal{X}(M)\to\mathcal{T}_lM$, i.e., $\forall X_1,X_2,\cdots,X_k\in\mathcal{X}(M)$, $f\in C^\infty(M)$,*

$$\psi(X_1,\cdots,fX_i,\cdots,X_k)=f\psi(X_1,\cdots,X_i,\cdots,X_k),\ \forall i=1,2,\cdots,k.$$

Proof. It suffices to show that given such a $\psi:\mathcal{X}(M)\times\mathcal{X}(M)\times\cdots\times\mathcal{X}(M)\to\mathcal{T}_lM$, there is a corresponding $\sigma\in\mathcal{T}_l^kM$, such that

$$\psi(X_1,X_2,\cdots,X_k)(p)=\sigma(p)(X_1|_p,X_2|_p,\cdots,X_k|_p)\in T_l(T_pM).$$

The key point of the proof is that for any $p\in M$, $\psi(X_1,X_2,\cdots,X_k)(p)$ depends only on the values of $X_1,X_2,\cdots,X_k$ at p, which by linearity, is equivalent to the statement that $\psi(X_1,\cdots,X_i,\cdots,X_k)(p)=0$ if $X_i|_p=0$, for any i. Assuming this momentarily, we define the corresponding σ by defining, for any $p \in M$ and $Y_1, Y_2, \cdots, Y_k \in T_pM$, the value $\sigma(p)(Y_1, Y_2, \cdots, Y_k) \in T_l(T_pM)$ to be $\psi(X_1, X_2, \cdots, X_k)(p) \in T_l(T_pM)$, where each $X_i \in \mathcal{X}(M)$ such that $X_i|_p = Y_i$. Clearly σ is well-defined and $\sigma \in \mathcal{T}_l^k M$.

It remains to show that $\psi(X_1, \cdots, X_i, \cdots, X_k)(p) = 0$ if $X_i|_p = 0$, for any i. To this end, let $(\frac{\partial}{\partial x^i})$ be a local coordinate frame over U. We choose a smooth partition of unity $\{f_1, f_2\}$ subordinate to the open cover $\{U, M \setminus \{p\}\}$. Then note that supp $f_1 \subset U$ and $f_2(p) = 0$. With this understood, observe that $f_2(p) = 0$ implies

$$\psi(X_1, \cdots, X_i, \cdots, X_k)(p) = \psi(X_1, \cdots, f_1 X_i, \cdots, X_k)(p).$$

Now we write, over U, $X_i = \sum_{j=1}^n a_i^j \frac{\partial}{\partial x^j}$ where each $a_i^j \in C^\infty(U)$. The condition $X_i|_p = 0$ is equivalent to $a_i^j(p) = 0$ for each j. We set $Z_j := \sqrt{f_1} \frac{\partial}{\partial x^j} \in \mathcal{X}(M)$ and $g_i^j := \sqrt{f_1} a_i^j \in C^\infty(M)$. Then note that $g_i^j(p) = 0$ for each j. With this understood,

$$\psi(X_1, \cdots, f_1 X_i, \cdots, X_k)(p) = \sum_{j=1}^n g_i^j(p) \psi(X_1, \cdots, Z_j, \cdots, X_k)(p) = 0.$$

□

Pull-backs of covariant tensor fields: Let $F: M \to N$ be any smooth map. For any $\sigma \in \mathcal{T}^k N$, we define the **pull-back** of σ via F, denoted by $F^*\sigma$, as follows. For any $p \in M$, we define $F^*\sigma(p) \in T^k(T_pM)$ by setting, for any $X_1, X_2, \cdots, X_k \in T_pM$,

$$F^*\sigma(p)(X_1, X_2, \cdots, X_k) := \sigma(F(p))(F_*(X_1), F_*(X_2), \cdots, F_*(X_k)),$$

where $F_*: T_pM \to T_{F(p)}N$. For the special case of $k = 0$, σ is simply a smooth function on N, and $F^*\sigma = \sigma \circ F$.

Computing $F^*\sigma$ in local coordinate charts, we assume $U \subset M$, $V \subset N$ such that $F(U) \subset V$, and let (x^i) and (y^j) be local coordinate functions on U and V respectively. Writing $\sigma = \sigma_{j_1 j_2 \cdots j_k} dy^{j_1} \otimes dy^{j_2} \otimes \cdots \otimes dy^{j_k}$ over V, we have

$$F^*\sigma(p)(\frac{\partial}{\partial x^{i_1}}|_p, \frac{\partial}{\partial x^{i_2}}|_p, \cdots \frac{\partial}{\partial x^{i_k}}|_p) = \sigma_{j_1 j_2 \cdots j_k}(F(p)) dy^{j_1}(F_*(\frac{\partial}{\partial x^{i_1}}|_p)) \cdots dy^{j_k}(F_*(\frac{\partial}{\partial x^{i_k}}|_p)).$$

It follows easily that over U, $F^*\sigma = (F^*\sigma)_{i_1 i_2 \cdots i_k} dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k}$, where

$$(F^*\sigma)_{i_1 i_2 \cdots i_k} = (\sigma_{j_1 j_2 \cdots j_k} \circ F) \frac{\partial (y^{j_1} \circ F)}{\partial x^{i_1}} \cdots \frac{\partial (y^{j_k} \circ F)}{\partial x^{i_k}} \in C^\infty(U).$$

In particular, it follows that $F^*\sigma \in \mathcal{T}^k M$, resulting a $\mathbb{R}$-linear map $F^*: \mathcal{T}^k N \to \mathcal{T}^k M$.

Exercise: Let $F: M \to N$ be any smooth map.

(1) Show that for any $\sigma \in \mathcal{T}^k N$, $\xi \in T^l N$, the tensor product $\sigma \otimes \xi \in T^{k+l} N$ is well-defined, and $F^*(\sigma \otimes \xi) = F^*\sigma \otimes F^*\xi$. In particular, for any $f \in C^\infty(N)$, $F^*(f\sigma) = (f \circ F) F^*\sigma$.

(2) For any $f \in C^\infty(N)$, $df \in \mathcal{T}^1 N$. Show that $F^* df = d(F^* f) = d(f \circ F)$.

Riemannian metrics: A Riemannian metric on a smooth manifold M is a positive definite, symmetric covariant 2-tensor field $g \in \mathcal{T}^2 M$. Here g being symmetric means that for any $p \in M$, $X, Y \in T_pM$, $g(p)(X, Y) = g(p)(Y, X)$ holds true, while g being positive definite means that $g(p)(X, X) \geq 0$ with "=" only when $X = 0$. In a local coordinate system (x^i), we can write $g = g_{ij}dx^i \otimes dx^j$, where the matrix (g_{ij}) is symmetric and positive definite.

Theorem 3.2. *Every smooth manifold possesses a Riemannian metric.*

Proof. We cover M with a smooth atlas $\{(U_\alpha, \phi_\alpha)\}$, with the local coordinate functions on U_α denoted by (x^i_α). We pick a smooth partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$, and set $g_\alpha := \delta_{ij}dx^i_\alpha \otimes dx^j_\alpha$ for each α, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Then $g := \sum_\alpha f_\alpha g_\alpha$ is a Riemannian metric on M.

□

Exercise: Let $F : M \to N$ be an immersion. Show that for any Riemannian metric $g \in \mathcal{T}^2 N$, the pull-back $F^*g \in \mathcal{T}^2 M$ is a Riemannian metric on M. As a consequence, Whitney's embedding theorem implies that every compact smooth manifold admits a Riemannian metric, a proof independent of Theorem 3.2.

Exercise: Show that for any smooth action of a finite group G on M, there is a Riemannian metric g on M which is G-invariant, i.e., for any $h \in G$, let $\theta_h : M \to M$ be the corresponding diffeomorphism, then $\theta_h^* g = g$.

Let g be a Riemannian metric on M. For any smooth curve $\gamma : [a, b] \to M$, we define the length of γ (depending on g) to be

$$L(\gamma) := \int_a^b g(\gamma'(t), \gamma'(t))^{1/2} dt.$$

It is straightforward to generalize the above definition to piecewise smooth curves. Now suppose M is connected. Then for any $p, q \in M$, one can connect p and q by a piecewise smooth curve γ. We define the distance between p and q, denoted by $d(p, q)$, to be the infimum of $L(\gamma)$ among all the piecewise smooth curves γ connecting p to q.

Exercise: Show that $d(p, q)$ is a distance function on M, making M into a metric space. Moreover, show that the topology of M as a metric space is the same as the underlying topology of smooth manifold.

Proposition 3.3. *For any smooth manifold M, TM and T^*M are isomorphic.*

Proof. We pick a Riemannian metric g on M. Then we define a map $\tilde{g} : TM \to T^*M$ as follows. For any $p \in M$, let $\tilde{g}_p : T_pM \to T_p^*M$ be the isomorphism such that for any $X, Y \in T_pM$, $\tilde{g}_p(X)(Y) = g(p)(X, Y)$. To check that $\tilde{g}$ is a smooth map, we compute in local coordinate (x^i), where we assume $g = g_{ij}dx^i \otimes dx^j$. Then if we trivialize TU by the local coordinate frame $(\frac{\partial}{\partial x^i})$ and trivialize T^*U by the local coordinate coframe (dx^i), the map $\tilde{g} : TU \to T^*U$ is given by $Id \times (g_{ij}) : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$, which is smooth. It follows that $\tilde{g} : TM \to T^*M$ is a bundle isomorphism.

□

Let g be a Riemannian metric on M. A local frame (σ_i) of TM over an open subset U is called **orthonormal** if $g(\sigma_i, \sigma_j) = 1$ for $i = j$ and 0 otherwise. Local orthonormal frames always exist by the Gram-Schmidt process.

Definition 3.4. Let E be a smooth real (resp. complex) vector bundle of rank n, and let G be a Lie subgroup of $GL(n,\mathbb{R})$ (resp. $GL(n,\mathbb{C})$). We say E is a G**-bundle** if there is a set of local trivializations of E such that the associated transition functions $\{\tau_{\beta\alpha}\}$ have their images lying in G.

Observe that if E is a $O(n)$-bundle, then the dual bundle E^* must be isomorphic to E because E^* is the induced bundle of E via the Lie group homomorphism $\rho : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ sending A to $(A^T)^{-1}$, and $\rho = Id$ when restricted to $O(n)$.

Exercise: Let M be a smooth manifold of dimension n. Show that TM is a $O(n)$-bundle. As a consequence, TM and T^*M are isomorphic.

Exercise: Let $S \subset M$ be an embedded submanifold, and let g be a Riemannian metric on M. For any $p \in S$, let $N_p \subset T_pM$ be the subspace consisting of tangent vectors which are orthogonal to T_pS with respect to $g(p)$. Let $N(S) := \sqcup_{p\in S} N_p$. Show that $N(S)$ is a sub-bundle of $TM|_S$, $N(S)$ is isomorphic to the normal bundle of S in M, and $TM|_S$ is a direct sum of TS and $N(S)$.

More generally, let E be a smooth real vector bundle of rank n. A metric on E is a smooth section g of the tensor bundle $T^2(E)$, such that g is symmetric and positive definite. The same argument of Theorem 3.2 shows that there exists a metric on any given smooth vector bundle.

Exercise: Let E be a smooth real vector bundle of rank n. Show that the following statements are equivalent.
(1) There is a metric on E.
(2) E is a $O(n)$-bundle.
(3) E^* and E are isomorphic.

Analogously, let E be a smooth complex vector bundle of rank n. One can similarly define the notion of a **Hermitian metric** on E, and the same argument of Theorem 3.2 shows that there exists a Hermitian metric on any given smooth complex vector bundle. On the other hand, the **complex conjugate** of E, denoted by $\overline{E}$, is the complex vector bundle obtained by changing each fiber of E to its complex conjugate. (Let V be a vector space over $\mathbb{C}$. The complex conjugate of V is the complex vector space obtained by changing the complex multiplication of $c \in \mathbb{C}$ on V to the multiplication by its conjugate $\bar{c}$ on V.)

Exercise: Let E be a smooth complex vector bundle of rank n. Show that the following statements are equivalent.
(1) There is a Hermitian metric on E.
(2) E is a $U(n)$-bundle.
(3) E^* and $\overline{E}$ are isomorphic.

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