MATH 703: PART 1: SMOOTH MANIFOLDS

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Contents

1. Smooth manifolds and smooth maps

Definition 1.1. (1) Let M be a topological space. We call M a **smoothable manifold of dimension** n if

(i) M is Hausdorff and second countable (recall that a topology is called second countable if there is a countable basis).

(ii) There is an open cover $\{U_\alpha\}$ of M such that for each α, there is a map $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ which is a homeomorphism onto the open subspace $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$, and for any α, β, $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a smooth (i.e., differentiable) map. Each (U_α, ϕ_α) is called a **local coordinate chart**, and the collection $\{(U_\alpha, \phi_\alpha)\}$ is called a **smooth atlas**.

(2) A maximal smooth atlas of M is called a **smooth structure** on M. A **smooth manifold** is a smoothable manifold together with a given smooth structure.

(3) Suppose $n = 2m$ is even. Let $\{(U_\alpha, \phi_\alpha)\}$ be a smooth atlas, and we canonically identify $\mathbb{R}^n$ in each $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ with $\mathbb{C}^m$. If for any α, β, the map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ between open subsets of $\mathbb{C}^m$ is holomorphic, then $\{(U_\alpha, \phi_\alpha)\}$ is called a **holomorphic atlas**. A maximal holomorphic atlas is called a **complex structure**. In this case, M is called a **complex manifold of dimension** m.

Exercise: (cf. Lemma 1.10 in Lee [1].) Let M be a smoothable manifold. Show that every smooth atlas on M is contained in a unique maximal smooth atlas, hence determines a unique smooth structure on M. Moreover, show that two smooth atlases on M determine the same smooth structure if and only if their union is a smooth atlas.

Example 1.2. (1) Let $M = \mathbb{R}^n$, which is clearly Hausdorff and second countable. Consider the smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ where $\{U_\alpha\}$ consists of a single element $U_\alpha = M$, and the map ϕ_α is simply the identity map $M \to \mathbb{R}^n$. The corresponding smooth structure is called the **standard smooth structure** on $\mathbb{R}^n$. In a similar way, $M = \mathbb{C}^m$ is a complex manifold of dimension m.

In the remaining examples, we recall the fact that the Hausdorff property and second countability are both preserved under subspace topology and product topology.

(2) Let M be a smooth manifold, and N be an open subset of M. Then N, given the subspace topology, is a smooth manifold of the same dimension in a canonical way. To see this, let $\{(U_\alpha, \phi_\alpha)\}$ be a smooth atlas of M. We let $V_\alpha := N \cap U_\alpha$, $\psi_\alpha := \phi_\alpha|_{V_\alpha} : V_\alpha \to \mathbb{R}^n$. Then $\{(V_\alpha, \psi_\alpha)\}$ is a smooth atlas of N.

(3) Let M, N be smooth manifolds, and let $\{(U_\alpha, \phi_\alpha)\}$, $\{(V_\beta, \psi_\beta)\}$ be a smooth atlas of M, N respectively. Then $\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$ is a smooth atlas of $M \times N$, making it into a smooth manifold in a canonical way. The dimension of $M \times N$ is the sum of the dimensions of M and N.

(4) Consider the n-sphere $\mathbb{S}^n$, which is the subspace of $\mathbb{R}^{n+1}$ defined by the equation

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1.$$

Then $\mathbb{S}^n$ is a smooth manifold of dimension n. In what follows, we prove this (i.e., verifying Definition 1.1) for the case of $n = 2$; the general case is the same.

Let $N = (0, 0, 1) \in \mathbb{S}^2$, $S = (0, 0, -1) \in \mathbb{S}^2$ be the north pole and south pole respectively. Let $U_N := \mathbb{S}^2 \setminus \{N\}$, $U_S := \mathbb{S}^2 \setminus \{S\}$ be the complement, which are open subsets and form a cover of $\mathbb{S}^2$. We define $\phi_N : U_N \to \mathbb{R}^2$, $\phi_S : U_S \to \mathbb{R}^2$ by

$$\phi_N(x_1, x_2, x_3) = (\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}),\ \phi_S(x_1, x_2, x_3) = (\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3}).$$

Then an easy calculation shows that

$$\phi_N^{-1}(y_1, y_2) = (\frac{2y_1}{y_1^2 + y_2^2 + 1}, \frac{2y_2}{y_1^2 + y_2^2 + 1}, \frac{y_1^2 + y_2^2 - 1}{y_1^2 + y_2^2 + 1}),$$

and

$$\phi_S^{-1}(y_1, y_2) = (\frac{2y_1}{y_1^2 + y_2^2 + 1}, \frac{2y_2}{y_1^2 + y_2^2 + 1}, \frac{1 - y_1^2 - y_2^2}{1 + y_1^2 + y_2^2}),$$

in particular, $\phi_N : U_N \to \mathbb{R}^2$, $\phi_S : U_S \to \mathbb{R}^2$ are homeomorphisms. Furthermore, one can check that $\phi_S \circ \phi_N^{-1} : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2 \setminus \{(0, 0)\}$ is a smooth map; in fact

$$\phi_S \circ \phi_N^{-1}(y_1, y_2) = (\frac{y_1}{y_1^2 + y_2^2}, \frac{y_2}{y_1^2 + y_2^2}).$$

It follows that $\{(U_N, \phi_N), (U_S, \phi_S)\}$ is a smooth atlas of $\mathbb{S}^2$. The corresponding smooth structure is called the **standard smooth structure on** $\mathbb{S}^2$ (more generally, $\mathbb{S}^n$).

Theorem 1.3. *(cf. Lemma 1.23 of Lee* [1]*) Let M be a set, and suppose there is a collection $\{U_\alpha\}$ of countably many subsets of M such that for each α, there is an injective map $\phi_\alpha : U_\alpha \to \mathbb{R}^n$, with the following properties:*

(i) *$M = \cup_\alpha U_\alpha$.*
(ii) *For each α, $\phi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^n$.*
(iii) *For any α, β, $\phi_\alpha(U_\alpha \cap U_\beta)$, $\phi_\beta(U_\alpha \cap U_\beta)$ are open subsets of $\mathbb{R}^n$.*
(iv) *For any α, β, $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a smooth map.*

Then M can be given a topology such that the collection of all the subsets of the form

$$\phi_\alpha^{-1}(V), \text{ where } V \text{ is an open subset of } \mathbb{R}^n$$

forms a basis of the topology. Furthermore, if M with the above topology is Hausdorff, then M is a smooth manifold of dimension n with $\{(U_\alpha, \phi_\alpha)\}$ being a smooth atlas.

Remarks: If in the above theorem we replace $\mathbb{R}^n$ by $\mathbb{C}^m$ and in condition (iv), we assume $\phi_\beta \circ \phi_\alpha^{-1}$ is holomorphic, then M is a complex manifold of dimension m with $\{(U_\alpha, \phi_\alpha)\}$ being a holomorphic atlas.

Example 1.4. (1) For each $n > 0$, consider the real projective space $\mathbb{RP}^n$, which is the set of lines in $\mathbb{R}^{n+1}$ passing through the origin. As an application of Theorem 1.3, one can show that $\mathbb{RP}^n$ is a smooth manifold of dimension n (in a canonical way). In what follows, we give the details for the case of $n = 2$.

First, we introduce the following notation: for each $(x_1, x_2, x_3) \neq (0, 0, 0)$, denote by $l(x_1, x_2, x_3)$ the line in $\mathbb{R}^3$ which passes through $(0, 0, 0)$ and (x_1, x_2, x_3). Then for each $\alpha = 1, 2, 3$, we let U_α be the subset of $\mathbb{RP}^2$ defined as follows:

$$U_\alpha = \{l(x_1, x_2, x_3) \in \mathbb{RP}^2 | x_\alpha \neq 0\}.$$

For each α, we define a map $\phi_\alpha : U_\alpha \to \mathbb{R}^2$ by

$$\phi_\alpha(l(x_1, x_2, x_3)) = (x_\beta/x_\alpha, x_\gamma/x_\alpha), \text{ where } \beta < \gamma \text{ and } \alpha, \beta, \gamma \text{ are distinct.}$$

It is clear that each ϕ_α is well-defined and injective.

Next we verify the conditions (i)-(iv) in Theorem 1.3. It is clear that $\mathbb{RP}^2 = \cup_{\alpha=1}^3 U_\alpha$, so (i) is true. For (ii), note that for each α, $\phi_\alpha(U_\alpha) = \mathbb{R}^2$. For (iii), note that for any α, β, where $\alpha \neq \beta$, $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$ are a subset of $\mathbb{R}^2$, which is $\mathbb{R}^2$ with a coordinate axis removed. Finally, for (iv), one can easily check that each $\phi_\beta \circ \phi_\alpha^{-1}$ is a smooth map. For example,

$$\phi_2 \circ \phi_1^{-1}(y_1, y_2) = \phi_2(l(1, y_1, y_2)) = (1/y_1, y_2/y_1), \text{ where } y_1 \neq 0.$$

It remains to show that $\mathbb{RP}^2$, with the topology given as in Theorem 1.3, is Hausdorff. To this end, let $l_1, l_2 \in \mathbb{RP}^2$ such that $l_1 \neq l_2$. If l_1, l_2 are both contained in some U_α, then since $\phi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^2$ which is Hausdorff, one can easily verify the Hausdorff property for l_1, l_2 in this case. It remains to consider the case where there is no U_α such that $l_1, l_2 \in U_\alpha$. Then without loss of generality, we may assume that $l_1 \in U_1 \setminus U_2$ and $l_2 \in U_2 \setminus U_1$. This condition implies that there are $u_1, u_2 \in \mathbb{R}$ such that $\phi_1(l_1) = (0, u_1)$ and $\phi_2(l_2) = (0, u_2)$. With this understood, consider

$$V_1 := \{(x_1, x_2 + u_1) | x_1^2 + x_2^2 < \epsilon\}, \;\; V_2 := \{(y_1, y_2 + u_2) | y_1^2 + y_2^2 < \epsilon\}.$$

Then $\phi_1^{-1}(V_1)$, $\phi_2^{-1}(V_2)$ are open neighborhoods of l_1, l_2 respectively. We claim that when $\epsilon > 0$ is sufficiently small, $\phi_1^{-1}(V_1) \cap \phi_2^{-1}(V_2) = \emptyset$.

To see this, suppose to the contrary that there is an $l \in \phi_1^{-1}(V_1) \cap \phi_2^{-1}(V_2)$. Let $\phi_1(l) = (x_1, x_2 + u_1)$, $\phi_2(l) = (y_1, y_2 + u_2)$. Then

$$l(1, x_1, x_2 + u_1) = l = l(y_1, 1, y_2 + u_2),$$

which implies that $x_1 y_1 = 1$. But this is not possible when $\epsilon > 0$ is sufficiently small, because $x_1^2 + x_2^2 < \epsilon$ and $y_1^2 + y_2^2 < \epsilon$.

(2) For each $m > 0$, the complex projective space $\mathbb{CP}^m$ is the set of (complex) lines in $\mathbb{C}^{m+1}$ which pass through the origin. Using the holomorphic version of Theorem 1.3, one can show that $\mathbb{CP}^m$ is a complex manifold of dimension m.

Exercise: Work out the details for the case of $\mathbb{CP}^1$ and $\mathbb{CP}^2$.

(3) For any $0 < k < n$, let $G_{k,n}$ be the set of k-dimensional subspaces of $\mathbb{R}^n$ (called the Grassmannian). Then $G_{k,n}$ is naturally a smooth manifold of dimension $k(n-k)$, which can be proved using Theorem 1.3.

Exercise: Work out the details for the case of $G_{2,4}$ by following the steps below.

(a) For any $i, j = 1, 2, 3, 4$ where $i < j$, let U_{ij} be the set of pairs of vectors (v_1, v_2), where $v_1, v_2 \in \mathbb{R}^4$, such that the i-th coordinate of v_1 equals 1 and the j-coordinate of v_1 equals 0, and i-th coordinate of v_2 equals 0 and the j-coordinate of v_2 equals 1. Note that the map sending (v_1, v_2) to the 2-plane spanned by v_1, v_2 identifies U_{ij} as a subset of $G_{2,4}$. We define an injective map $\phi_{ij} : U_{ij} \to \mathbb{R}^4$ by

$$\phi_{ij}(v_1, v_2) = (x_1, x_2, x_3, x_4),$$

where x_1, x_2 and x_3, x_4 are the remaining (i.e., not i, j) coordinates of v_1, v_2 respectively. With this understood, verify (i)-(iv) of Theorem 1.3 for $\{(U_{ij}, \phi_{ij})\}$.

(b) Prove $G_{2,4}$, with the topology as given in Theorem 1.3, is Hausdorff.

Remarks: The manifolds in Example 1.4 are all examples of homogeneous spaces. After we discuss Lie group actions in Section 4, it should follow from a general theorem that they are smooth manifolds.

Definition 1.5. (1) Let M be a smooth manifold. A continuous function $f : M \to \mathbb{R}$ is called **smooth** if for any local coordinate chart (U, ϕ), $\hat{f} := f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is a smooth function. f is said to be **locally smooth** at a point $p \in M$ if there is an open neighborhood W of p such that $f|_W$ is smooth.

(2) More generally, let M, N be smooth manifolds of dimension m, n respectively. A continuous map $F : M \to N$ is called **smooth** if for any local coordinate charts (U, ϕ) on M and (V, ψ) on N, $\hat{F} := \psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^n$ is a smooth map. In a similar way, one can define **local smoothness** of F.

Remarks: (1) $\hat{f}, \hat{F}$ are called **coordinate representatives** of f, F. In practice, to verify local smoothness at a point p, it suffices to find one local coordinate chart containing p such that the corresponding coordinate representative is smooth. If f or F is locally smooth everywhere, then it is smooth.

(2) Composition of two smooth maps is smooth.

(3) If we replace $\mathbb{R}$ by $\mathbb{C}$, smooth manifolds by complex manifolds, we can define **holomorphic** functions/maps in the same fashion.

Definition 1.6. (1) Let $F : M \to N$ be a smooth map. F is called a **diffeomorphism** if the inverse $F^{-1} : N \to M$ exists and is a smooth map. In this case, M, N are called **diffeomorphic**.

(2) A smooth map $F : M \to N$ is called a **local diffeomorphism** if for any $p \in M$, there exists an open neighborhood U of p such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.

Remarks: A fundamental problem in differential topology is to classify smooth manifolds up to a diffeomorphism.

An important class of local diffeomorphisms is given by **smooth covering maps**, smooth maps which are covering maps between the underlying topological spaces. We observe the following fact, which can be easily proved (cf. Prop. 2.12 in Lee [1]).

Theorem 1.7. *Let M be a connected smooth manifold, and $\tilde{M}$ a connected topological space. Suppose there is a (topological) covering map $\pi : \tilde{M} \to M$. Then $\tilde{M}$ is a smoothable manifold with a unique smooth structure, such that $\pi : \tilde{M} \to M$ is a smooth covering map.*

Example 1.8. (1) If $N \subset M$ is an open subset of a smooth manifold, then the inclusion map $i : N \to M$ is smooth.

(2) Let $M = M_1 \times M_2$, let $\pi_i : M \to M_i$, for $i = 1, 2$, be the projection to the i-th factor, and let $j_1(x) : M_1 \to M$, $j_2(y) : M_2 \to M$ for $x \in M_2$, $y \in M_1$, be defined by $j_1(x)(p) = (p, x)$, $\forall p \in M_1$, and $j_2(y)(q) = (y, q)$, $\forall q \in M_2$. Then $\pi_i, i = 1, 2$, $j_1(x), j_2(y)$ are all smooth maps.

(3) The inclusion map $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a smooth map.

(4) The map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$, defined by sending $(x_1, x_2, \cdots, x_{n+1})$ to the line $l(x_1, x_2, \cdots, x_{n+1})$ passing through $(x_1, x_2, \cdots, x_{n+1})$ and the origin $0 \in \mathbb{R}^{n+1}$, is a smooth map. Similarly, the complex analog $\pi : \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{CP}^m$ is holomorphic.

(5) The composition $F : \mathbb{S}^n \to \mathbb{RP}^n$, i.e., $F : \mathbb{S}^n \to \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$, is smooth.

(6) The composition $F : \mathbb{S}^{2m+1} \to \mathbb{CP}^m$, i.e., $F : \mathbb{S}^{2m+1} \to \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{CP}^m$, which is called **Hopf fibration**, is smooth.

Example 1.9. Here we show that $F : \mathbb{S}^2 \to \mathbb{RP}^2$ is a smooth covering map. The same proof shows that $F : \mathbb{S}^n \to \mathbb{RP}^n$ is a smooth covering map for any $n > 0$.

Let $l \in \mathbb{RP}^2$ be any point, where without loss of generality, we assume $l \in U_1$. We set $\phi_1(l) = (u_1, u_2)$. Then consider the open ϵ-ball in $\mathbb{R}^2$,

$$V := \{(u_1 + y_1, u_2 + y_2) | y_1^2 + y_2^2 < \epsilon^2\},$$

and the open neighborhood $W := \phi_1^{-1}(V)$ of l in $\mathbb{RP}^2$.

Let $\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$ be the map defined in (4) of Example 1.8. Then

$$\pi^{-1}(W) = \{(\lambda, \lambda(u_1 + y_1), \lambda(u_2 + y_2)) | \lambda \in \mathbb{R} \setminus \{0\}, y_1^2 + y_2^2 < \epsilon^2\},$$

which is an open subset of $\mathbb{R}^3 \setminus \{0\}$. Its intersection with $\mathbb{S}^2$ is given by a disjoint union of U_+, U_-, where

$$U_\pm = \{(\lambda, \lambda(u_1 + y_1), \lambda(u_2 + y_2)) | \lambda = \pm \frac{1}{\sqrt{1 + (u_1 + y_1)^2 + (u_2 + y_2)^2}}, y_1^2 + y_2^2 < \epsilon^2\}.$$

Since $\mathbb{S}^2$ is given with the subspace topology, U_+, U_- are open subsets of $\mathbb{S}^2$. Finally, it is easy to see that the restriction of $F : \mathbb{S}^2 \to \mathbb{RP}^2$ to U_+ or U_- is a homeomorphism onto W. This shows that $F : \mathbb{S}^2 \to \mathbb{RP}^2$ is a topological covering map.

It remains to show that $F_{U_+} : U_+ \to W$, $F_{U_-} : U_- \to W$ are diffeomorphisms. Without loss of generality, we only look at F_{U_+}, and need to show that $F_{U_+}^{-1} : W \to U_+$ is smooth. Note that $U_+ \subset U_N$, so we shall check that $\phi_N \circ F_{U_+}^{-1} \circ \phi_1^{-1} : V \to \mathbb{R}^2$ is a smooth map. It is straightforward that

$$\phi_N \circ F_{U_+}^{-1} \circ \phi_1^{-1}(y_1, y_2) = (\frac{\lambda}{1 - \lambda y_2}, \frac{\lambda y_1}{1 - \lambda y_2}), \text{ where } \lambda = \frac{1}{\sqrt{1+y_1^2+y_2^2}},$$

which is differentiable in y_1, y_2. This finishes the proof.

Exercise: Show that $\mathbb{S}^2$ and $\mathbb{CP}^1$ are diffeomorphic.

Exercise: Show that the canonical map $F : \mathbb{RP}^2 \to G_{2,3}$, where for any $l \in \mathbb{RP}^2$, $F(l) \in G_{2,3}$ is the 2-plane in $\mathbb{R}^3$ perpendicular to l, is a diffeomorphism,

We end by mentioning a fundamental tool in smooth manifold theory: smooth partition of unity.

Definition 1.10. Let M be a topological space, and let $\mathcal{U} = \{U_\alpha | \alpha \in \Lambda\}$ be an open cover of M. A **partition of unity subordinate to** $\mathcal{U}$ is a collection of continuous functions $\{f_\alpha : M \to \mathbb{R} | \alpha \in \Lambda\}$, such that

(i) for any $x \in M$, $\alpha \in \Lambda$, $0 \leq f_\alpha(x) \leq 1$;
(ii) $\text{supp} f_\alpha := \overline{\{x \in M | f_\alpha(x) \neq 0\}} \subset U_\alpha$, $\forall \alpha \in \Lambda$;
(iii) the set $\{ \text{supp} f_\alpha | \alpha \in \Lambda\}$ is locally finite, i.e., for any $x \in M$, there is a neighborhood U of x, such that $U \cap \text{supp} f_\alpha \neq \emptyset$ for only finitely many $\alpha \in \Lambda$;
(iv) $\sum_{\alpha \in \Lambda} f_\alpha = 1$ on M.

Theorem 1.11. *(Existence of smooth partition of unity, cf. Lee* [1]*) Let M be any smooth manifold, and let $\mathcal{U} = \{U_\alpha | \alpha \in \Lambda\}$ be an open cover of M. Then there exists a partition of unity $\{f_\alpha | \alpha \in \Lambda\}$ subordinate to $\mathcal{U}$, such that each f_α is smooth.*

2. Tangent vectors and tangent spaces

Let M be a smooth manifold. The set of smooth functions on M, denoted by $C^\infty(M)$, is a naturally a commutative ring, where for any $f, g \in C^\infty(M)$, the sum $f + g$ and the multiplication fg are defined by

$$(f + g)(p) := f(p) + g(p), \ (fg)(p) := f(p)g(p), \forall p \in M.$$

Moreover, under $+$, $C^\infty(M)$ is a vector space over $\mathbb{R}$, where for any $c \in \mathbb{R}$, $f \in C^\infty(M)$, cf is defined by $(cf)(p) := cf(p)$. Finally, observe that for any smooth map $F : M \to N$, there is an induced homomorphism (ring and vector space) $F^* : C^\infty(N) \to C^\infty(M)$ given by $F^*(f) = f \circ F$, $\forall f \in C^\infty(N)$.

Definition 2.1. (1) Let M be a smooth manifold and let $p \in M$ be any given point. A $\mathbb{R}$-linear map $X : C^\infty(M) \to \mathbb{R}$ is called a **tangent vector** at p, if for any $f, g \in C^\infty(M)$,

$$X(fg) = f(p)X(g) + g(p)X(f)$$

holds. The set of all tangent vectors at p is denoted by T_pM, which is naturally a vector space over $\mathbb{R}$, and is called the **tangent space** at p.

(2) Let $F : M \to N$ be any smooth map. Let $p \in M$, $q := F(p) \in N$. Then for any $X \in T_pM$, we define $F_*(X)$ to be the $\mathbb{R}$-linear map from $C^\infty(N)$ to $\mathbb{R}$, by

$$F_*(X)(f) := X(F^*(f)), \ \ \forall f \in C^\infty(N).$$

It is easy to check that $F_*(X) \in T_q(N)$. Moreover, $F_* : T_pM \to T_qN$ is a homomorphism between $\mathbb{R}$-vector spaces. F_*, also denoted by $dF(p)$, is called the **differential of** F **at** p.

It is clear that a diffeomorphism induces an isomorphism between the tangent spaces.

The following localization result is key to understanding tangent spaces in any concrete ways.

Theorem 2.2. *Let M be a smooth manifold, and let $U \subset M$ be any open subset. Then the inclusion map $i_* : U \to M$ induces an isomorphism $i_* : T_pU \to T_pM$ for any point $p \in U$.*

Proof. We begin with the following lemma, which follows from a standard application of partition of unity.

Lemma 2.3. *(1) Let $p \in M$, $f, g \in C^\infty(M)$. If there exists an open subset B containing p, such that $f|_B = g|_B$. Then for any $X \in T_pM$, $X(f) = X(g)$.*

(2) Fixing any closed subset $A \subset U$, there is a $\mathbb{R}$-linear extension map from $C^\infty(U)$ to $C^\infty(M)$, denoted by $f \mapsto \tilde{f}$, such that $\tilde{f}|_A = f|_A$.

Proof. (1). Consider the open cover $\mathcal{U} = \{B, M \setminus \{p\}\}$ of M. Let $\{\psi_1, \psi_2\}$ be a smooth partition of unity subordinate to $\mathcal{U}$, where supp $\psi_1 \subset B$. Then we can write

$$f = \psi_1 f + \psi_2 f, \;\; g = \psi_1 g + \psi_2 g.$$

On the other hand, since $f|_B = g|_B$, we conclude that $\psi_1 f = \psi_1 g$ as supp $\psi_1 \subset B$. With this understood,

$$X(f) = X(\psi_1 f) + X(\psi_2 f) = X(\psi_1 f) + X(\psi_2) f(p) + \psi_2(p) X(f) = X(\psi_1 f) + X(\psi_2) f(p),$$

because supp $\psi_2 \subset M \setminus \{p\}$ so that $\psi_2(p) = 0$. Similarly, $X(g) = X(\psi_1 g) + X(\psi_2) g(p)$. Since $\psi_1 f = \psi_1 g$ and $f(p) = g(p)$, we conclude that $X(f) = X(g)$.

(2) Consider the open cover of M, $\mathcal{U} = \{U, M \setminus A\}$, and pick a smooth partition of unity $\{\psi_1, \psi_2\}$ subordinate to $\mathcal{U}$, where supp $\psi_1 \subset U$. For any $f \in C^\infty(U)$, we define $\tilde{f} := \psi_1 f$. Since supp $\psi_1 \subset U$, $\tilde{f} = \psi_1 f$ can be regarded as in $C^\infty(M)$ by letting it equal zero outside of U. Clearly, the map $f \mapsto \tilde{f}$ is $\mathbb{R}$-linear. Finally, since supp $\psi_2 \subset M \setminus A$ and $\psi_1 + \psi_2 = 1$, it follows that $\psi_1|_A = 1$, so that $\tilde{f}|_A = f|_A$ holds. □

Now we are ready for a proof of Theorem 2.2. To begin, we fix an open neighborhood B of p such that its closure $A := \overline{B} \subset U$. We first prove that $i_* : T_pU \to T_pM$ is injective. Let $X \in T_pU$ such that $i_*(X) = 0 \in T_pM$. We need to show that for any $f \in C^\infty(U)$, $X(f) = 0$. To see this, we consider the extension $\tilde{f} \in C^\infty(M)$ defined in Lemma 2.3(2), using the closed subset $A \subset U$. Then

$$0 = i_*(X)(\tilde{f}) = X(\tilde{f}|_U) = X(f),$$

where the last equality follows from Lemma 2.3(1) because $(\tilde{f}|_U)|_B = f|_B$, which follows from the fact that $(\tilde{f}|_U)|_B = \tilde{f}|_B$ and $\tilde{f}|_A = f|_A$. Hence $i_* : T_pU \to T_pM$ is injective.

To see $i_* : T_pU \to T_pM$ is surjective, for any $Y \in T_pM$, we define a map $X : C^\infty(U) \to \mathbb{R}$ by setting $X(f) = Y(\tilde{f})$ for any $f \in C^\infty(U)$. Since $f \mapsto \tilde{f}$ is $\mathbb{R}$-linear, it follows that X is a $\mathbb{R}$-linear map. To show $X \in T_pU$, it remains to show that for any $f, g \in C^\infty(U)$,

$$X(fg) = f(p)X(g) + g(p)X(f).$$

To see this, we note that $\widetilde{fg}|_B = fg|_B = \tilde{f}|_B\tilde{g}|_B = \tilde{f}\tilde{g}|_B$, so that

$$X(fg) = Y(\widetilde{fg}) = Y(\tilde{f}\tilde{g}) = \tilde{f}(p)Y(\tilde{g}) + \tilde{g}(p)Y(\tilde{f}) = f(p)X(g) + g(p)X(f).$$

Finally, we check $i_*(X) = Y$. For any $f \in C^\infty(M)$,

$$i_*(X)(f) = X(f|_U) = Y(\widetilde{f|_U}) = Y(f).$$

The last equality follows from Lemma 2.3(1) because $\widetilde{f|_U}|_B = (f|_U)_B = f|_B$. Hence $i_*(X) = Y$.

□

As an immediate consequence of Theorem 2.2, a local diffeomorphism induces an isomorphism between tangent spaces.

Now let (U, ϕ) be a local coordinate chart of M, where $\phi : U \to \mathbb{R}^n$, which now is a diffeomorphism onto the open subset $\phi(U) \subset \mathbb{R}^n$. For any $p \in U$, we can identify T_pU with T_pM canonically using Theorem 2.2, while on the other hand, T_pU can be identified with $T_{\hat{p}}\mathbb{R}^n$ via ϕ_*, where $\hat{p} := \phi(p)$ is the image of p in $\mathbb{R}^n$. The next lemma describes the tangent spaces of $\mathbb{R}^n$ in classical terms.

Lemma 2.4. *For any $p \in \mathbb{R}^n$, there is a canonical isomorphism $D : \mathbb{R}^n \to T_p\mathbb{R}^n$, which sends $v \in \mathbb{R}^n$ to the directional derivative D_v, i.e., $\forall f \in C^\infty(\mathbb{R}^n)$,*

$$D_v(f) = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(p), \text{ where } v = (v_1, v_2, \cdots, v_n).$$

Here $x_1, x_2, \cdots, x_n$ are the standard coordinate system on $\mathbb{R}^n$.

Proof. It is easy to check that for each vector $v \in \mathbb{R}^n$, $D_v \in T_p\mathbb{R}^n$, and $v \mapsto D_v$ is linear. To see it is injective, we note that $D_v(x_i) = v_i$ for each $i = 1, 2, \cdots, n$. Finally, to see it is surjective, we claim that for any $X \in T_p\mathbb{R}^n$,

$$X(f) = \sum_{i=1}^n X(x_i)\frac{\partial f}{\partial x_i}(p), \quad \forall f \in C^\infty(\mathbb{R}^n),$$

so that $X = D_v$ where $v = (X(x_1), X(x_2), \cdots, X(x_n))$. Our claim follows easily from the fact that $X(f) = 0$ for any constant function f, and the following fact from multivariable calculus: let $p_1, p_2, \cdots, p_n$ be the coordinates of p, $\forall f \in C^\infty(\mathbb{R}^n)$,

$$f(x) = f(p) + \sum_{i=1}^n (x_i - p_i)\frac{\partial f}{\partial x_i}(p) + \sum_{i=1}^n (x_i - p_i)g_i(x),$$

where $g_i \in C^\infty(\mathbb{R}^n)$ satisfying $g_i(p) = 0$ for any i.

□

As a corollary, note that if M is a smooth manifold of dimension n, then for any $p \in M$, the tangent space T_pM is a n-dimensional vector space over $\mathbb{R}$. In fact, there is more to say. Note that by Lemma 2.4, $T_p\mathbb{R}^n$ has a canonical basis, i.e., the partial derivatives $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}$. Now for any local coordinate chart (U, ϕ), $p \in U$, there is a set of coordinate functions $x^i := x_i \circ \phi \in C^\infty(U)$, $i = 1, 2, \cdots, n$, where $x_1, x_2, \cdots, x_n$ are the standard coordinate system on $\mathbb{R}^n$. Through ϕ_*, we obtain a basis of T_pM,

$$\frac{\partial}{\partial x^i}|_p := \phi_*^{-1}(\frac{\partial}{\partial x_i}),\ i = 1, 2, \cdots, n.$$

Proposition 2.5. *Let (U, ϕ), (V, ψ) be two local coordinate charts both containing p. Let (x^i), (y^i) denote the corresponding coordinate functions associated to (U, ϕ), (V, ψ) respectively. Then*

$$\frac{\partial}{\partial x^j}|_p = \sum_{i=1}^n \frac{\partial}{\partial x^j}|_p(y^i)\frac{\partial}{\partial y^i}|_p,\ j = 1, 2, \cdots, n.$$

Moreover, the matrix $(\frac{\partial}{\partial x^j}|_p(y^i))$ is simply $D(\psi \circ \phi^{-1})(\phi(p))$, where D is the Jacobian of a smooth map from an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$.

Exercise: Prove Proposition 2.5.

Proposition 2.6. *Let $F : M \to N$ be any smooth map, $p \in M$, $q := F(p) \in N$. Let (U, ϕ) be a local coordinate chart containing p, (V, ψ) be a local coordinate chart containing q. Then with respect to the bases of T_pM, T_qN associated to (U, ϕ), (V, ψ) respectively, $F_* : T_pM \to T_qN$ is given by the matrix $D(\psi \circ F \circ \phi^{-1})(\phi(p))$, where D is the Jacobian of a smooth map between open subsets of Euclidean spaces.*

Exercise: Prove Proposition 2.6.

Two Special Cases: (1) Let M be a smooth manifold, $p \in M$. For any $f \in C^\infty(M)$, the differential $df(p) : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$ is linear, so $df(p)$ is an element of the dual space of T_pM. We call the dual space of T_pM, denoted by T_p^*M, the **cotangent space** of M at p. For any local coordinate chart (U, ϕ) containing p, let (x^i), $x^i \in C^\infty(U)$, be the corresponding coordinate functions. Then $dx^i(p) \in T_p^*M$ and $(dx^i(p))$ is the dual basis of the basis $(\frac{\partial}{\partial x^i}|_p)$ of T_pM.

(2) Let M be a smooth manifold, $p \in M$. A (parametrized) **smooth curve in M through** p is a smooth map $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$. For any such γ, the map γ_* at 0 is uniquely determined by $\gamma_*(\frac{\partial}{\partial t}) \in T_pM$, where t is the coordinate on $(-\epsilon, \epsilon)$. We will denote $\gamma_*(\frac{\partial}{\partial t})$ by $\gamma'(0)$ or $\gamma'|_p$, called the **tangent vector** of γ at p. We note that when $M = \mathbb{R}^n$, after identifying $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ in the canonical way (as in Lemma 2.4), $\gamma'|_p \in \mathbb{R}^n$ is simply the tangent vector of the smooth curve γ at p in the usual sense.

An Alternative Approach: Let M be a smooth manifold, $p \in M$. We consider the set of all smooth curves through p, denoted by $\mathcal{C}_p(M)$, and introduce an equivalence relation $\sim$ on $\mathcal{C}_p(M)$ as follows. For any $\gamma_1, \gamma_2 \in \mathcal{C}_p(M)$, $\gamma_1 \sim \gamma_2$ if for any $f \in C^\infty(M)$, $\frac{d}{dt}(f \circ \gamma_1)(0) = \frac{d}{dt}(f \circ \gamma_2)(0)$. We denote by $\mathcal{V}_p(M)$ the set of equivalence classes $[\gamma]$,

$\gamma \in \mathcal{C}_p(M)$. Note that if $F : M \to N$ is a smooth map, $q = F(p)$, then there is an induced mapping $F_* : \mathcal{V}_p(M) \to \mathcal{V}_q(N)$, sending $[\gamma]$ to $[F \circ \gamma]$.

Now we observe that $[\gamma] \mapsto \gamma'|_p$ defines a $1:1$ correspondence between $\mathcal{V}_p(M)$ and T_pM. Under this correspondence, $F_* : \mathcal{V}_p(M) \to \mathcal{V}_q(N)$ is the same as $F_* : T_pM \to T_qN$. So an alternative approach to tangent vectors, which is more intuitive, is to regard a tangent vector at $p \in M$ as an equivalence class of smooth curves in M through p, or to represent a tangent vector at p by a smooth curve in M through p.

Example 2.7. Consider the inclusion map $i : \mathbb{S}^2 \to \mathbb{R}^3$. The differential $i_* : T_p\mathbb{S}^2 \to T_p\mathbb{R}^3$ sends the tangent space $T_p\mathbb{S}^2$ to a subspace of $\mathbb{R}^3$ after identifying $T_p\mathbb{R}^3$ canonically with $\mathbb{R}^3$. On the other hand, $\mathbb{S}^2$ is a hypersurface in $\mathbb{R}^3$, defined by

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

For any $p \in \mathbb{S}^2$, $\mathbb{S}^2$ has a tangent plane at p, which consists of all vectors in $\mathbb{R}^3$ perpendicular to the vector $p \in \mathbb{S}^2 \subset \mathbb{R}^3$. One would naturally guess that $i_*(T_p\mathbb{S}^2)$ is the tangent plane of $\mathbb{S}^2$ at p. There are two different ways to verify this.

(1) The first approach is to pick a local coordinate chart, either (U_N, ϕ_N) or (U_S, ϕ_S), which contains $p \in \mathbb{S}^2$. Then $i_* : T_p\mathbb{S}^2 \to T_p\mathbb{R}^3$ is represented by the corresponding Jacobian (cf. Prop. 2.6), a 3×2 matrix which can be explicitly computed. One can verify that this matrix has rank 2 and the two column vectors of the matrix are both perpendicular to the vector $p \in \mathbb{R}^3$.

(2) The second approach is to represent a tangent vector $v \in T_p\mathbb{S}^2$ by a smooth curve $\gamma \in \mathcal{C}_p(\mathbb{S}^2)$. Then under the inclusion map $i : \mathbb{S}^2 \to \mathbb{R}^3$, γ becomes a smooth curve in $\mathbb{R}^3$ through p. The image $i_*(v)$ is represented by γ as a smooth curve in $\mathbb{R}^3$ through p, so $i_*(v)$ must be the tangent vector (in the usual sense) of $\gamma \in \mathcal{C}_p(\mathbb{R}^3)$ at p. From this, it follows immediately that $i_*(T_p\mathbb{S}^2)$ is the tangent plane of $\mathbb{S}^2$ at $p \in \mathbb{R}^3$.

Exercise: Let $F : \mathbb{S}^3 \to \mathbb{CP}^1$ be the Hopf fibration. Show that for any $p \in \mathbb{S}^3$, $F_* : T_p\mathbb{S}^3 \to T_{F(p)}\mathbb{CP}^1$ is surjective.

3. Inverse function theorem and maps of constant rank

Let $F : U \to V$ be a smooth map between open subsets of Euclidean spaces. The **rank** of F at $p \in U$ is defined to be the rank of the Jacobian $DF(p)$. More generally, the rank of a smooth map $F : M \to N$ between smooth manifolds at a point $p \in M$ is simply the rank of $F_* : T_pM \to T_{F(p)}N$.

The results collected in this section are based on the following theorem from multivariable calculus.

Theorem 3.1. *(Inverse Function Theorem) Let U, V be open subsets of $\mathbb{R}^n$, $F : U \to V$ a smooth map. For any $p \in U$, if the Jacobian $DF(p)$ is nonsingular, then there exist connected neighborhoods $U_0 \subset U$ of p, $V_0 \subset V$ of $F(p)$, such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

First application: local coordinate functions

Let M be a smooth manifold of dimension n. Let $y^1, y^2, \cdots, y^n$ be a set of n locally smooth functions near a point $p \in M$, such that the differentials $dy^i(p) \in T_p^*M$, $i = 1, 2, \cdots, n$, form a basis of the cotangent space at p. Then there is an open neighborhood U of p, such that (U, ϕ), where $\phi : U \to \mathbb{R}^n$ is defined by $\phi(q) = (y^1(q), y^2(q), \cdots, y^n(q))$, $\forall q \in U$, is a local coordinate chart.

The proof is a straightforward application of Theorem 3.1. We choose a local coordinate chart (V, ψ) containing p over which y^i's are defined. Let (x^i) be the associate coordinate functions on V. Then the Jacobian of $\phi \circ \psi^{-1} : \psi(V) \to \mathbb{R}^n$ at $\psi(p)$ equals the matrix $(dy^i(p)(\frac{\partial}{\partial x^j}|_p))$, which is nonsingular because $(dy^i(p))$ is a basis of T_p^*M. By Theorem 3.1, $\phi \circ \psi^{-1}$ is a diffeomorphism from an open neighborhood W_1 of $\psi(p)$ onto an open neighborhood W_2 of $\phi(p)$. We simply let $U := \phi^{-1}(W_2)$. One can easily check that ϕ is a diffeomorphism from U onto $W_2 \subset \mathbb{R}^n$.

The following theorem concerning maps of constant rank is the most relevant application of Inverse Function Theorem (see Lee [1], Theorems 7.8 and 7.13).

Theorem 3.2. *(Maps of Constant Rank) Let M, N be smooth manifolds of dimension m and n respectively. Let $F : M \to N$ be a smooth map of constant rank k. Then for any $p \in M$, there are local coordinate charts (U, ϕ) containing p, (V, ψ) containing $F(p)$, such that the coordinate representative $\hat{F} := \psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ takes the following standard form*

$$(x_1, x_2, \cdots, x_k, x_{k+1}, \cdots, x_m) \mapsto (x_1, x_2, \cdots, x_k, 0, \cdots, 0) \in \mathbb{R}^n.$$

Definition 3.3. (1) A smooth map $F : M \to N$ is called an **immersion** if for any point $p \in M$, $F_* : T_pM \to T_{F(p)}N$ is injective. An immersion which is also a topological embedding is called a **smooth embedding**.

(2) A smooth map $F : M \to N$ is called a **submersion** if for any point $p \in M$, $F_* : T_pM \to T_{F(p)}N$ is surjective.

Remarks: (1) One can show that if an immersion is one to one, and the map is a proper map, then it must be a smooth embedding.

(2) Both immersions and submersions are maps of constant rank, so their local coordinate representatives can be chosen to have a canonical form as described in Theorem 3.2.

Example 3.4. (1) The inclusion map $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a smooth embedding.

(2) The Hopf fibration $F : \mathbb{S}^{2m+1} \to \mathbb{CP}^m$ is a submersion.

Definition 3.5. Let M be a smooth manifold of dimension n. A subset $S \subset M$ is called an **embedded submanifold** of dimension k, for $k < n$, if for any $p \in S$, there exists a local coordinate chart (U, ϕ) containing p, such that $S \cap U$ is given by $x^{k+1} = c_1, x^{k+2} = c_2, \cdots, x^n = c_{n-k}$ for some constants $c_1, c_2 \cdots, c_{n-k}$, where $x^1, x^2, \cdots, x^n$ are the coordinate functions associated to (U, ϕ). We shall call (U, ϕ) a **slice chart**, $x^1, x^2, \cdots, x^n$ **slice coordinates**, and $n - k$ the **codimension** of S.

Exercise: Prove, using Theorem 3.2, that the image of a smooth embedding is an embedded submanifold.

Exercise: Let $F : M \to N$ be a smooth map. The graph of F is the subset $\Gamma(F) := \{(p, q) \in M \times N | q = F(p)\}$ of $M \times N$. Show that $\Gamma(F)$ is the image of a smooth embedding, hence an embedded submanifold.

The converse is given by the following theorem.

Theorem 3.6. *Let $S \subset M$ be an embedded submanifold of dimension k. Given the subspace topology, S is a smooth manifold of dimension k, with a unique smooth structure, such that the inclusion map $i : S \to M$ is a smooth embedding.*

Proof. With subspace topology, S is Hausdorff and second countable. To show S is a smooth manifold, we need to construct a smooth atlas on S. For any $p \in S$, there is a slice chart (U, ϕ) containing p. We set $V := S \cap U$, which is an open subset of S. The collection of all such open subsets V forms an open cover of S. For each V, we define $\psi = \pi \circ \phi|_V : V \to \mathbb{R}^k$, where $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is the projection onto the first k coordinates. Since $S \cap U$ is given by $x^{k+1} = c_1, x^{k+2} = c_2, \cdots, x^n = c_{n-k}$ for some constants $c_1, c_2 \cdots, c_{n-k}$, it follows immediately that $\psi : V \to \mathbb{R}^k$ is a homeomorphism onto its image. Moreover, note that its inverse $\psi^{-1} : \psi(V) \to V$ is given by $\phi^{-1} \circ j$, where $j : \mathbb{R}^k \to \mathbb{R}^n$ is the embedding sending $(x_1, x_2, \cdots, x_k)$ to $(x_1, x_2, \cdots, x_k, c_1, c_2, \cdots, c_{n-k})$. Finally, if (V', ψ') is another such local coordinate chart obtained from a slice chart (U', ϕ'), then

$$\psi' \circ \psi^{-1} = (\pi \circ \phi') \circ (\phi^{-1} \circ j) = \pi \circ (\phi' \circ \phi^{-1}) \circ j,$$

which is smooth because $\phi' \circ \phi^{-1}$ is smooth. This shows that $\{(V, \psi)\}$ is a smooth atlas on S, and S is a smooth manifold of dimension k. It is clear from the construction that with this smooth structure, $i : S \to M$ is a smooth embedding.

It remains to show that the smooth structure is unique. To this end, we need to show that for any local coordinate chart (W, θ) of a smooth structure on S with respect to which $i : S \to M$ is a smooth embedding, (W, θ) must be smoothly compatible with (V, ψ) constructed from a slice chart, i.e., $\psi \circ \theta^{-1} : \theta(W \cap V) \to \psi(W \cap V)$ is a diffeomorphism between open subsets of $\mathbb{R}^k$. This follows because 1) $\psi \circ \theta^{-1} = (\pi \circ \phi) \circ \theta^{-1} = \pi \circ (\phi \circ \theta^{-1})$ is smooth and has nonsingular Jacobian, 2) $\psi \circ \theta^{-1} : \theta(W \cap V) \to \psi(W \cap V)$ is a homeomorphism.

□

The following useful observation follows easily from the proof above.

Proposition 3.7. *Let $S \subset N$ be an embedded submanifold, and $F : M \to N$ be a smooth map such that $F(M) \subset S$. Then the map $F : M \to S$ is also a smooth map.*

Exercise: Prove Proposition 3.7.

Regular value of a smooth map: Let $F : M \to N$ be a smooth map. A point $q \in N$ is called a **regular value** of F, if for any $p \in F^{-1}(q)$, $F_* : T_pM \to T_qN$ is surjective.

Proposition 3.8. *Let M, N be smooth manifold of dimension m and n respectively. Suppose $q \in N$ is a regular value of a smooth map $F : M \to N$. Then $F^{-1}(q)$ is an embedded submanifold of M of dimension $m - n$. Moreover, for any $p \in F^{-1}(q)$, the tangent space of $F^{-1}(q)$ at p is given by the kernel of $F_* : T_pM \to T_qN$.*

Proof. For any $p \in F^{-1}(q)$, there is an open neighborhood W of p, such that for any $x \in W$, $F_* : T_xM \to T_{F(x)}N$ is surjective. In other words, $F|_W : W \to N$ is a smooth map of constant rank n. By Theorem 3.2, there exist local coordinate charts (U, ϕ) containing p, (V, ψ) containing q, such that $\psi \circ F \circ \phi^{-1}$ takes the form

$(x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_m) \mapsto (x_1, x_2, \cdots, x_n)$. Let $\psi(q) = (c_1, c_2, \cdots, c_n)$. Then $F^{-1}(q) \cap U$ is given by $x^1 = c_1, x^2 = c_2, \cdots, x^n = c_n$. This proves that $F^{-1}(q)$ is an embedded submanifold of M of dimension $m - n$. The last statement follows from the fact that the tangent space of $F^{-1}(q)$ at p is spanned by $\frac{\partial}{\partial x^{n+1}}|_p, \frac{\partial}{\partial x^{n+2}}|_p, \cdots, \frac{\partial}{\partial x^m}|_p$. □

Example 3.9. (1) Consider the smooth function $F : \mathbb{R}^{n+1} \to \mathbb{R}$, where

$$F(x_1, x_2, \cdots, x_{n+1}) = x_1^2 + x_2^2 + \cdots + x_{n+1}^2.$$

Then $1 \in \mathbb{R}$ is a regular value of F, and $\mathbb{S}^n = F^{-1}(1)$ is an embedded submanifold of $\mathbb{R}^{n+1}$. By the uniqueness in Theorem 3.6, the smooth structure on $\mathbb{S}^n$ is the standard smooth structure as in Example 1.2(4).

(2) The Hopf fibration $F : \mathbb{S}^{2m+1} \to \mathbb{CP}^m$ is a submersion, so every point $q \in \mathbb{CP}^m$ is a regular value of F. The fiber $F^{-1}(q)$ is an embedded submanifold of $\mathbb{S}^{2m+1}$ of dimension 1, an embedded circle. In fact, each $q \in \mathbb{CP}^m$ represents a complex line $L(q)$ in $\mathbb{C}^{m+1}$, and $F^{-1}(q)$ is simply the intersection of $\mathbb{S}^{2m+1}$ with $L(q)$, which is the circle in $L(q)$ of distance 1 to the origin.

Transversal maps: Let $S \subset N$ be an embedded submanifold. A smooth map $F : M \to N$ is said to be **transversal to** S if for any $p \in F^{-1}(S)$, the tangent space $T_{F(p)}N$ is the sum of $F_*(T_pM)$ and $T_{F(p)}S$.

Proposition 3.10. *Let M, N be smooth manifolds of dimension m and n respectively, and $S \subset N$ be an embedded submanifold of dimension k. Suppose a smooth map $F : M \to N$ is transversal to S. Then $F^{-1}(S)$ is an embedded submanifold of M of dimension $m + k - n$.*

Exercise: Prove Proposition 3.10. More generally, let $F_i : M_i \to N$, where $i = 1, 2$, be smooth maps. We say F_1, F_2 are **transversal to each other** if for any $p_1 \in M_1$, $p_2 \in M_2$ such that $F_1(p_1) = F_2(p_2) = q \in N$, the tangent space T_qN is the sum of $(F_1)_*(T_{p_1}M_1)$ and $(F_2)_*(T_{p_2}M_2)$. Show that if F_1, F_2 are transversal to each other, then the subset $\{(p_1, p_2) \in M_1 \times M_2 | F_1(p_1) = F_2(p_2)\}$ is an embedded submanifold of $M_1 \times M_2$ of dimension $\dim M_1 + \dim M_2 - \dim N$.

Whitney Embedding Theorem: The following theorem shows that every compact smooth manifold is an embedded submanifold of a Euclidean space.

Theorem 3.11. *Let M be a compact smooth manifold. Then for large enough N, there is a smooth embedding $F : M \to \mathbb{R}^N$.*

Proof. For any $p \in M$, there exists a local coordinate chart (W, ϕ) containing p. For each such (W, ϕ), we choose an open neighborhood U of p such that the closure $\overline{U} \subset W$. Now there is a smooth partition of unity subordinate to the open cover $\{W, M \setminus \overline{U}\}$. We let λ be the smooth function from the partition of unity such that supp $\lambda \subset W$. Note that $\lambda \equiv 1$ on $\overline{U}$, and the collection of subsets U is an open cover of M.

Since M is compact, there are finitely many U_i, where $i = 1, 2, \cdots, m$, such that $\{U_i\}$ is a cover of M. Let $\{(W_i, \phi_i)\}$ be the corresponding local coordinate charts, and $\lambda_i \in C^\infty(M)$ the smooth functions. For each i, we define $F_i : M \to \mathbb{R}^{n+1}$, where $\phi_i : W_i \to \mathbb{R}^n$ and $F_i = (\lambda_i \phi_i, \lambda_i)$, and define $F : M \to \mathbb{R}^{(n+1)m}$ by $F = (F_1, F_2, \cdots, F_m)$.

First, we shall that F is an immersion. To see this, for any $p \in M$, there is a U_i such that $p \in U_i$. Then observe that since $\lambda_i \equiv 1$ on U_i, $F_i = (\phi_i, 1)$ in a small neighborhood of p. Since $\phi_i : W_i \to \mathbb{R}^n$ is a local diffeomorphism, it follows easily that F_i, hence F, must be an immersion near p.

It remains to show that F is one to one. To this end, let $p, q \in M$ such that $p \neq q$. We choose a U_i such that $p \in U_i$. In particular, $\lambda_i(p) = 1$. If $\lambda_i(q) = 1$, then $F_i(q) = (\phi(q), 1) \neq (\phi_i(p), 1) = F_i(p)$. If $\lambda_i(q) < 1$, then $F_i(q) \neq F_i(p)$ as well. This shows that F is one to one. Since M is compact, $F : M \to \mathbb{R}^{(n+1)m}$ must be a topological embedding, hence a smooth embedding.

□

A natural question asks what is the minimal value of N in Theorem 3.11. One possible way to reduce the dimension of $\mathbb{R}^N$ in the smooth embedding theorem is to compose $F : M \to \mathbb{R}^N$ with a map $\pi_l : \mathbb{R}^N \to \mathbb{R}^{N-1}$, where π_l is the projection of $\mathbb{R}^N$ onto $\mathbb{R}^{N-1}$ along a line l in $\mathbb{R}^N$. It turns out that, based on the so-called Sard's Theorem (see below), one can show that, as long as $N > 2n+1$ (here n is the dimension of M), for a generic choice of line l in $\mathbb{R}^N$, the map $\pi_l \circ F : M \to \mathbb{R}^{N-1}$ continues to be a smooth embedding. Repeating this argument, one can show that there exists a smooth embedding of M into $\mathbb{R}^{2n+1}$, which is called the *Whitney Embedding Theorem.*

Theorem 3.12. *(Sard's Theorem) Let $U \subset \mathbb{R}^m$ be any open subset. Then for any smooth map $F : U \to \mathbb{R}^n$, the complement of regular values (i.e., the set of critical values) of F in $\mathbb{R}^n$ has measure zero.*

There is a whole package of transversality theory in differential topology that is based on Sard's Theorem, see [2].

Regarding Whitney Embedding Theorem, an interesting question asks for a given individual smooth manifold M, what is the minimal value of N such that there is a smooth embedding of M in $\mathbb{R}^N$?

Exercise: Consider $F : \mathbb{R}^3 \to \mathbb{R}^4$, where $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Show that $F(\mathbb{S}^2) \subset \mathbb{R}^4$ is an embedded submanifold which is diffeomorphic to $\mathbb{RP}^2$.

However, there is no smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^3$.

4. Lie groups and Lie group actions

Definition 4.1. Let G be a group. If G is a smooth manifold such that the multiplication map $G \times G \to G$, $(g, h) \mapsto gh$, and the inverse map $G \to G$, $g \mapsto g^{-1}$, are both smooth maps, then G is called a **Lie group**. A homomorphism between Lie groups which is also a smooth map is called a **Lie group homomorphism**. A subgroup of a Lie group which is also an embedded submanifold is called a **Lie subgroup**. (A Lie subgroup is naturally a Lie group.)

Example 4.2. (1) General linear groups $GL(n, \mathbb{R})$, $GL(m, \mathbb{C})$ are naturally Lie groups.

(2) Orthogonal groups $O(n)$, $SO(n)$, unitary groups $U(m)$, $SU(m)$, are Lie subgroups of $GL(n, \mathbb{R})$, $GL(m, \mathbb{C})$ respectively.

(3) $\mathbb{S}^1 \subset \mathbb{C}$ under complex multiplication, $\mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{H}$ under quaternion multiplication, are Lie groups. It is known that $\mathbb{S}^3$ is isomorphic to $SU(2)$.

(4) (Spin groups) For any $n > 2$, $\pi_1(SO(n)) = \mathbb{Z}_2$. The universal cover of $SO(n)$, denoted by $Spin(n)$, is called a **spin group**. It is known that $Spin(3) = \mathbb{S}^3$, and $Spin(4) = \mathbb{S}^3 \times \mathbb{S}^3$.

(5) Let G be a finite or countably infinite group, given with the discrete topology. Then G is a 0-dimensional Lie group, called a **discrete group**.

Definition 4.3. Let M be a smooth manifold, G a Lie group, with $e \in G$ being the identity element. A **smooth left-action** of G on M is a smooth map $\theta : G \times M \to M$, with $\theta(g, p)$ denoted by $g \cdot p$, which satisfies the following conditions:

$$g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p \text{ and } e \cdot p = p, \ \ \forall g_1, g_2 \in G, p \in M.$$

A **smooth right-action** of G on M is a smooth map $\theta : M \times G \to M$, with $\theta(p, g)$ denoted by $p \cdot g$, which satisfies the following conditions:

$$(p \cdot g_1) \cdot g_2 = p \cdot (g_1 g_2) \text{ and } p \cdot e = p, \ \ \forall g_1, g_2 \in G, p \in M.$$

Remarks: If $p \cdot g$ is a given right-action, then $(g, p) \mapsto p \cdot g^{-1}$ is a left-action. Hence without loss of generality, we shall only consider left-actions here. A smooth manifold equipped with a smooth left-action (or right-action) of G is called a **smooth G-manifold**. Note that for any $g \in G$, $\theta_g : M \to M$ defined by $\theta_g(p) = g \cdot p$ is a diffeomorphism, with inverse $\theta_g^{-1} = \theta_{g^{-1}}$.

Definition 4.4. (1) For any $p \in M$, the subset $G \cdot p := \{q \in M | q = g \cdot p \text{ for some } g \in G\}$ is called the **orbit of** p under the G-action. The set of orbits $M/G := \{G \cdot p | p \in M\}$ is called the **quotient space**, which comes with a natural map $\pi : M \to M/G$ sending p to its orbit $G \cdot p$. We give M/G the quotient topology. The G-action is called **transitive** if $M = G \cdot p$ for some $p \in M$.

(2) For any $p \in M$, the **isotropy subgroup at** p is the subgroup

$$G_p := \{g \in G | g \cdot p = p\}.$$

The G-action is called **free** if $G_p = \{e\}$ (i.e., is trivial), $\forall p \in M$, and the G-action is called **effective** if $g \in G_p$ for all $p \in M$ implies that $g = e$.

Example 4.5. (1) (Trivial actions) The map $\theta : G \times M \to M$ such that $\theta(g, p) = p$ for any $g \in G$, $p \in M$.

(2) For $G = GL(n, \mathbb{R})$, $M = \mathbb{R}^n$, the smooth left-action $\theta : G \times M \to M$ given by $\theta(A, v) = Av$, $\forall A \in G, v \in \mathbb{R}^n$.

(3) Given any smooth left-action $G \times M \to M$, and a Lie group homomorphism $\rho : H \to G$, there is a canonically induced smooth left-action $H \times M \to M$, defined by $h \cdot p = \rho(h) \cdot p$, $\forall h \in H, p \in M$.

(4) Let $G = O(n) \subset GL(n, \mathbb{R})$, $M = \mathbb{S}^{n-1} \subset \mathbb{R}^n$, the smooth left-action in (2) induces a smooth left-action of $O(n)$ on $\mathbb{S}^{n-1}$ (cf. Proposition 3.7).

(5) For any Lie group G, there is a canonical smooth left-action of G on G itself, given by $L_g : G \to G$ sending h to gh for any $g, h \in G$. L_g is called a **left translation**. There is also a canonical right-action, defined by $R_g : G \to G$, where $R_g(h) = hg$, for any $g, h \in G$. R_g is called a **right translation**.

(6) (Linear representations). Let V be a finite dimensional real vector space, $GL(V)$ be the group of automorphisms of V. (Note that V can be identified with $\mathbb{R}^n$ for some

n, hence a smooth manifold, and $GL(V)$ with $GL(n.\mathbb{R})$ hence a Lie group.) Given any Lie group homomorphism $\rho : G \to GL(V)$, there is an induced smooth left-action of G on V via ρ, i.e., $(g, v) \mapsto \rho(g)(v)$, $\forall g \in G$, $v \in V$.

(7) (The adjoint representation). Let G be any Lie group. Consider the following smooth left-action of G on G itself, defined by $\theta(g, h) = ghg^{-1}$. Note that for any $g \in G$, $\theta_g : G \to G$ leaves e fixed, i.e., $\theta_g(e) = e$. Thus $\forall g \in G$, $(\theta_g)_* \in GL(T_eG)$. The representation $Ad : G \to GL(T_eG)$, where $Ad(g) = (\theta_g)_*$, is called the **adjoint representation**.

Definition 4.6. Let M, N be smooth G-manifolds. A smooth map $F : M \to N$ is called **equivariant** if $F(g \cdot p) = g \cdot F(p)$ for any $g \in G$, $p \in M$.

Theorem 4.7. *Let $F : M \to N$ be an equivariant map between two smooth G-manifolds. Suppose the G-action on M is transitive. Then F has constant rank. As a consequence, for any $q \in N$, the subset $F^{-1}(q) \subset M$ is an embedded submanifold.*

Proof. Let $\theta : G \times M \to M$, $\varphi : G \times N \to N$ be the G-actions on M, N respectively. Fix a point $p_0 \in M$, then since the G-action on M is transitive, for any $p \in M$, there is a $g \in G$ such that $g \cdot p_0 = p$. Now observe that $F_* : T_pM \to T_{F(p)}N$ equals the composition $(\theta_g)_*^{-1} : T_pM \to T_{p_0}M$, $F_* : T_{p_0}M \to T_{F(p_0)}N$, and $(\varphi_g)_* : T_{F(p_0)}N \to T_{F(p)}N$, hence has the same rank as $F_* : T_{p_0}M \to T_{F(p_0)}N$. This shows that F has constant rank. It follows easily from Theorem 3.2 that for any $q \in N$, the subset $F^{-1}(q) \subset M$ is an embedded submanifold.

□

Example 4.8. Let $F : G \to H$ be a Lie group homomorphism. Consider the G-action on G by left translations, and the G-action on H by the left translations of G via F. Then it is easy to check that F is equivariant. Since the left translations on G are transitive, it follows from Theorem 4.7 that for any $h \in H$, $F^{-1}(h)$ is an embedded submanifold of G. In particular, the kernel $K := F^{-1}(e)$ is a Lie subgroup of G. For example, let $SL(n, \mathbb{R})$ be the set of real $n \times n$ matrices with determinant 1. Then $SL(n, \mathbb{R})$ is a Lie subgroup of $GL(n, \mathbb{R})$, because it is the kernel of the determinant homomorphism.

Definition 4.9. A smooth G-action on M is called **proper** if the map $\Theta : G \times M \to M \times M$, sending (g, p) to $(g \cdot p, p)$, is a proper map.

Remarks: (1) For a proper action, G_p is compact for any $p \in M$.

(2) Compact Lie group actions are proper.

(3) For a proper, discrete Lie group action, G_p is finite. Hence for any $p \in M$, there is a G_p-invariant neighborhood of p: we simply pick any neighborhood U of p, and let $V := \cap_{g \in G_p}(g \cdot U)$, which is G_p-invariant.

Theorem 4.10. *(Quotient Manifold Theorem) Let M be a smooth manifold, equipped with a smooth G-action by a Lie group G, which is free and proper. Then the quotient space M/G is a smooth manifold of dimension $\dim M - \dim G$, with a unique smooth structure such that the quotient map $\pi : M \to M/G$ sending p to its orbit $G \cdot p$ is a submersion.*

Example 4.11. (1) Consider $\mathbb{S}^{2m+1} \subset \mathbb{C}^m$ and the smooth action of $\mathbb{S}^1$ on it via complex multiplication. The action is clearly free and proper, and it is easy to see that the quotient map is the Hopf fibration $\mathbb{S}^{2m+1} \to \mathbb{CP}^m$.

(2) Consider the action of $\mathbb{Z}_2 = \{\pm 1\}$ on $\mathbb{S}^n$, given by the antipodal map

$$\tau : (x_1, x_2, \cdots, x_{n+1}) \mapsto (-x_1, -x_2, \cdots, -x_{n+1}).$$

The quotient map is the $2:1$ covering $\mathbb{S}^n \to \mathbb{RP}^n$ in Example 1.9.

(3) For any $0 < q < p$ where $gcd(p, q) = 1$, we consider the $\mathbb{Z}_p$-action on $\mathbb{S}^3$, generated by

$$(z_1, z_2) \mapsto (\exp(2\pi i/p) z_1, \exp(2\pi i q/p) z_2).$$

It is clearly free and proper. The quotient manifold $\mathbb{S}^3/\mathbb{Z}_p$ is a 3-dimensional manifold, called a **lens space**, and is denoted by $L(p, q)$. Note that $L(2, 1) = \mathbb{RP}^3$.

(4) Consider a smooth, free and proper action of a discrete Lie group G on M. In this case, the quotient manifold M/G has the same dimension as M, and the quotient map $\pi : M \to M/G$ is a smooth covering map. Conversely, for any regular smooth covering map, the action of the group of deck transformations on the covering manifold is a smooth, free and proper action of a discrete Lie group.

We begin a proof of the Quotient Manifold Theorem with the following

Lemma 4.12. *For each $p \in M$, the orbit $G \cdot p$ is an embedded submanifold of M.*

Proof. Consider the map $\theta_p : G \to M$ by $\theta_p(g) = g \cdot p$, which is equivariant with respect to the left translations on G and the given G-action on M. Since the left translations on G are transitive, θ_p has constant rank by Theorem 4.7. On the other hand, θ_p is a one to one map, as the G-action is free. Being one to one in turn implies that θ_p must be an immersion, because if otherwise, θ_p being of constant rank will violate Theorem 3.2. Consequently, in order to show $G \cdot p$ is an embedded submanifold, it remains to show that θ_p is a proper map. But this follows from the assumption that the G-action on M is proper.

□

As a consequence, we have

Lemma 4.13. *Let $k = \dim G$, $n = \dim M - \dim G$. Then for any $p \in M$, there is a local coordinate chart (U, ϕ) centered at p (i.e., $\phi(p) = 0$), such that*

(i) *$\phi(U) = U_1 \times U_2 \subset \mathbb{R}^k \times \mathbb{R}^n$, with coordinates $(x_1, \cdots, x_k, y_1, \cdots, y_n)$;*

(ii) *each orbit of the G-action intersects U either in empty or in a single slice of the form: $y_1 \equiv c_1$, $y_2 \equiv c_2, \cdots, y_n \equiv c_n$.*

Proof. Let (W, ψ) be a slice chart of $G \cdot p$ centered at p, with local coordinate functions $u^1, u^2, \cdots, u^k, v^1, v^2, \cdots, v^n$ such that $W \cap G \cdot p$ is given by $v^1 = v^2 = \cdots = v^n = 0$. We consider the subset $S \subset W$ defined by $u^1 = u^2 = \cdots = u^k = 0$, which is an embedded submanifold of dimension n in W. Let $\theta : G \times S \to M$ be the smooth map, sending (g, q) to $g \cdot q$. Then at (e, p), $\theta_* : T_{(e,p)} G \times S \to T_p M$ is an isomorphism, implying that θ is a local diffeomorphism near (e, p). With this understood, let (X, α) be a local coordinate chart of G centered at e, (Y, β) be a local coordinate chart of S centered at p, such that θ maps $X \times Y$ diffeomorphically onto an open subset $U := \theta(X \times Y) \subset W$.

Let $\phi : U \to \mathbb{R}^k \times \mathbb{R}^n$, where $\phi = (\alpha, \beta) \circ \theta^{-1}$. Then (U, ϕ) is a local coordinate chart of M centered at p. Note that $\phi(U) = U_1 \times U_2$, where $U_1 = \alpha(X) \subset \mathbb{R}^k$, $U_2 = \beta(Y) \subset \mathbb{R}^n$.

Let $(x_1, \cdots, x_k, y_1, \cdots, y_n)$ be the coordinates on $\mathbb{R}^k \times \mathbb{R}^n$. From the construction of (U, ϕ), it is clear that each slice $y_1 \equiv c_1$, $y_2 \equiv c_2, \cdots, y_n \equiv c_n$ lies in the same orbit of the G-action. It remains to show that when choosing $Y \subset S$ sufficiently small, different slices lie in different orbits of the G-action. Suppose to the contrary that this is not true. Then there exists a sequence $q_i \in Y$, $q_i' \in Y$, where $q_i \neq q_i'$ and both $\{q_i\}$, $\{q_i'\}$ converge to p, such that for each i, q_i and q_i' are in the same orbit of the G-action, which means that there is a $g_i \in G$, such that $q_i' = g_i \cdot q_i$ for all i.

Exercise: Show that the properness of the G-action on M is equivalent to the following statement: for any convergent sequence $p_i \in M$, and any sequence $g_i \in G$, if the sequence $\{g_i \cdot p_i\}$ contains a convergent subsequence, then $\{g_i\}$ must also contains a convergent subsequence.

Hence the assumption that the G-action is proper implies that $\{g_i\}$ contains a convergent subsequence, which is still denoted by $\{g_i\}$, and let the limit be $g \in G$. Since both $\{q_i\}$, $\{q_i'\}$ converge to p, we have $p = g \cdot p$, which implies that $g = e$ as the G-action is free. Hence for sufficiently large i, $g_i \in X$. But this contradicts the fact that θ is injective on $X \times Y$, as $\theta(g_i, q_i) = q_i' = \theta(e, q_i')$, but $(g_i, q_i) \neq (e, q_i')$ in $X \times Y$. $\square$

Exercise: Show that the map $\theta : G \times S \to M$ is a diffeomorphism onto its image in M when S is chosen sufficiently small. Such a S is called a **local slice**.

We now complete the proof of the theorem. First, we show that $\pi : M \to M/G$ is an open map. To see this, let U be any open subset of M. To see $\pi(U)$ is open in M/G, we note that $\pi^{-1}(\pi(U)) = \cup_{g \in G} \theta_g(U)$. Since θ_g is a diffeomorphism, each $\theta_g(U)$ is open, which implies that $\pi^{-1}(\pi(U))$ is open in M. With the quotient topology on M/G, this shows $\pi(U)$ is open. As a consequence, M/G is second countable.

Now for each (U, ϕ) in Lemma 4.13, we set $V := \pi(U)$, which is an open subset of M/G. We define a map $\psi : V \to \mathbb{R}^n$ as follows. Identify U_2 with $\{0\} \times U_2$ in $\phi(U)$, we observe that $\pi : \phi^{-1}(U_2) \to V$ is a homeomorphism. We simply let ψ be the inverse of this map followed by ϕ. It is clear that ψ sends V homeomorphically onto $U_2 \subset \mathbb{R}^n$. We declare each (V, ψ) to be a local coordinate chart of M/G. One can easily check that if $(\tilde{U}, \tilde{\phi})$ is another chart from Lemma 4.13, with the corresponding chart $(\tilde{V}, \tilde{\psi})$ for M/G, then the transition map $\tilde{\psi} \circ \psi^{-1}$ is smooth. Finally, we point out that M/G is Hausdorff is a consequence of the assumption that the G-action is proper. We leave the details as an exercise.

Homogeneous Spaces: Let G be a Lie group, H be a Lie subgroup of G. We consider the smooth right-action of H on G, $\theta : G \times H \to G$, given by $(g, h) \mapsto gh$, which is clearly free. We claim it is also proper. To see this, suppose $g_i \in G$ is a convergent sequence, $h_i \in H$ is a sequence such that $g_i h_i$ is convergent in G. We need to show h_i is convergent in H. Note that h_i is convergent in G, so this boils down to the following lemma.

Lemma 4.14. *A Lie subgroup H of a Lie group G is a closed subset in G.*

Proof. Suppose $h_i \in H$ converges to $g \in G$. We choose a slice chart (U, ϕ) of H centered at $e \in G$, and choose an open subset W such that $\overline{W} \subset U$. Since the map $m : G \times G \to G$, sending (g_1, g_2) to $g_1 g_2^{-1}$, is smooth, there is an neighborhood V of $e \in G$, such that $m(V \times V) \subset W$. Now for large enough i, $h_i g^{-1} \in V$, so that for large enough i, j, $h_i h_j^{-1} = (h_i g^{-1})(h_j g^{-1})^{-1} \in W$. Let j goes to infinity, we have $h_i g^{-1} \in \overline{W} \subset U$. Since $H \cap U$ is a slice in U, it is closed, which implies that $h_i g^{-1} \in H$. Hence $g \in H$, and H is closed. □

By Theorem 4.10, the quotient space $G/H := \{g \cdot H | g \in G\}$, which is the set of coset gH, is a smooth manifold of dimension $\dim G - \dim H$. We observe that there is a natural smooth left-action of G on G/H, by $(g', gH) \mapsto g'gH$. The isotropy subgroup at the coset $eH = H$ is exactly the Lie subgroup H, i.e., $G_H = H$. Moreover, it is clear that the action of G on G/H is transitive.

Definition 4.15. A smooth manifold M is called a **homogeneous space** if there is a smooth, transitive Lie group action on it.

Theorem 4.16. *Let M be a homogeneous space, with a smooth, transitive left-action of G on it. For any point $p \in M$, the map $G/G_p \to M$ sending gG_p to $g \cdot p$ is an equivariant diffeomorphism.*

Proof. Consider the smooth map $F : G \to M$ by $F(g) = g \cdot p$, which is clearly equivariant with respect to the left translations of G on G and the given G-action on M. The G-action on G is transitive, so F has constant rank. On the other hand, since the G-action on M is transitive, F is onto, which implies that F must be a submersion given it is of constant rank. In particular, this also implies that $G_p = F^{-1}(p)$ is an embedded submanifold of G, therefore must be a Lie subgroup of G.

The map $F : G \to M$ factors through G/G_p, which induces a one to one and onto map from G/G_p to M. From the proof of the Quotient Manifold Theorem, it follows easily that the map $G/G_p \to M$ is smooth. Note that it is equivariant, and since the G-action on G/G_p is transitive, it must be a diffeomorphism. □

Remarks: If M is just a set with a transitive Lie group action of G, and if for a point $p \in M$ the isotropy subgroup G_p is a Lie subgroup of G, then the above proof shows that there is a one to one and onto map from G/G_p to M. We can use this map to give M a smooth manifold structure, so that it is diffeomorphic to G/G_p.

Example 4.17. (1) Let $M = \mathbb{S}^n$, and consider the $O(n+1)$-action on $\mathbb{S}^n$ which is transitive. The isotropy subgroup at $(0, 0, \cdots, 1) \in \mathbb{S}^n$ is $O(n)$, so $\mathbb{S}^n$ is diffeomorphic to $O(n+1)/O(n)$.

(2) Let $M = \mathbb{RP}^n$, and consider the $SO(n+1)$-action on it which is transitive. The isotropy subgroup at $l(0, 0, \cdots, 1) \in \mathbb{RP}^n$ is $O(n)$, so $\mathbb{RP}^n$ is diffeomorphic to $SO(n+1)/O(n)$.

(3) The orthogonal group $O(n)$ acts transitively on the Grassmannian $G_{k,n}$. One can check that the isotropy subgroup at the k-plane spanned by the first k coordinates is a Lie subgroup of $O(n)$. Hence the Grassmannians are homogeneous spaces.

Exercise: Let M be the set of oriented 2-planes in $\mathbb{R}^4$.

(1) Show that there is a natural left action of $SO(4)$ on M which is transitive.

(2) Let $P_0 \in M$ be the 2-plane of the first two coordinates given with the standard orientation. Determine the isotropy subgroup at P_0 and show that it is a Lie subgroup of $SO(4)$.

(3) Combining (1) and (2), show that M is a compact, connected smooth manifold of dimension 4.

(4) Let $G := \mathbb{S}^3 \times \mathbb{S}^3$, where $\mathbb{S}^3$ is the Lie group of unit quaternions. Consider the homomorphism $\rho : G \to GL(4, \mathbb{R})$ obtained as follows: for any $(p, q) \in G$, $x \in \mathbb{H} = \mathbb{R}^4$, $\rho(p, q)(x) := pxq^{-1}$. Show that ρ induces a homomorphism from G onto $SO(4)$, with kernel $\mathbb{Z}_2$.

(5) Define a smooth left action of G on M via $\rho : G \to SO(4)$. Then the G-action on M is also transitive. Determine the isotropy subgroup G_{P_0}.

(6) Show that $G/G_{P_0} = \mathbb{S}^2 \times \mathbb{S}^2$, which implies that M is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$.

(7) There is a natural $2:1$ covering map from M to $G_{2,4}$ by forgetting the orientation of the 2-planes in M. With M being identified with $\mathbb{S}^2 \times \mathbb{S}^2$, describe the $\mathbb{Z}_2$-action on $\mathbb{S}^2 \times \mathbb{S}^2$, and use it to determine $G_{2,4}$.

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