FINITE SYMMETRIES IN DIMENSION 4: TEN OPEN QUESTIONS

WEIMIN CHEN

Abstract. Here we list 10 open questions about finite group actions on 4-manifolds. The answers to many of these questions are beyond the reach by the current techniques, so hopefully these questions may stimulate inventions of new ideas or methods. On the other hand, we choose these questions in order to explore the differences between the locally-linear, smooth, symplectic, and holomorphic categories in the study of finite group actions in dimension 4, and to understand the subtleties of some of the issues in group actions when being considered under the different categories.

1. locally-linear vs. smooth: the 4-sphere

Background: The 4-sphere is of the lowest dimension among the spheres which admit nonlinear smooth finite group actions. So the "linearity" question in this dimension naturally takes a weaker form: if a finite group G acts on $\mathbb{S}^4$ locally linearly (resp. smoothly), is G necessarily isomorphic to a subgroup of $O(5)$?

Question 1 *Does there exist a locally linear action on $\mathbb{S}^4$ by a finite group G, where G is not isomorphic to a subgroup of $O(5)$?*

If such a finite group G does exist, then the G-action is necessarily orientation-reversing, cf. [1]. A strategy for constructing such an action by some of the Milnor groups (which are not isomorphic to a subgroup of $O(5)$) was also proposed in [1].

Question 2 *If an orientation-reversing action of a finite group G on $\mathbb{S}^4$ is smooth, must G be isomorphic to a subgroup of $O(5)$?*

Constructing an example of a smooth action on $\mathbb{S}^4$ by a finite group G not isomorphic to a subgroup of $O(5)$ seems to be quite difficult. On the other hand, if the smoothness assumption is necessary for an affirmative answer to this question, and one attempts to use the current techniques of gauge theory to solve it, then one of the immediate obstacles is that the usual set-up of gauge theory requires to fix an orientation of $\mathbb{S}^4$ and the group action needs to be orientation-preserving. Note that since linearity of the actions fails in this dimension, one cannot hope to derive such a conclusion from a "linearity theorem" as one did in dimension 3. Finally, it is not clear whether the role of the standard smooth structure is necessary here.

Remarks: An interesting observation about finite group actions on $\mathbb{S}^4$ is that whatever results that have been established so far for smooth actions also hold true for

Date: March 11, 2016.

locally linear actions. This said, if the answers to both Questions 1 and 2 are affirmative, then this would be the first example to indicate that the locally-linear and smooth categories are different for finite group actions on $\mathbb{S}^4$.

2. Rigidity and smooth structure: the $K3$ surface

Background: Among all the smooth structures supported by the topological 4-manifold of a $K3$ surface, the standard structure seems to be very unique and enjoy a special status. It is the only smooth structure on the 4-manifold known up to date that supports a symplectic or holomorphic structure with trivial canonical bundle; in particular, it is the underlying smooth structure of all $K3$ surfaces. The corresponding smooth 4-manifold is called *the standard $K3$ surface*, or simply *the $K3$ surface.*

Finite groups of holomorphic automorphisms of $K3$ surfaces have been extensively studied and are well-understood. An important feature is the so-called "homological rigidity", i.e., a holomorphic automorphism must be trivial if the induced action on the $K3$ lattice is trivial. As an immediate consequence, any finite group which can be realized as a holomorphic automorphism group of a $K3$ surface, called a *$K3$ group*, must be isomorphic to a subgroup of the automorphism group of the $K3$ lattice. The $K3$ groups are the only finite groups known up to date that can act smoothly on a *homotopy $K3$ surface*, i.e., a smooth 4-manifold homeomorphic to a $K3$ surface.

Question 3 *Does there exist a nontrivial periodic diffeomorphism of the* (*standard*) *$K3$ surface which is homologically trivial?*

This question was originally due to A. Edmonds (see Kirby's Problem List, Problem 4.124 (B)). There exist nontrivial, locally linear, periodic homeomorphisms of the $K3$ surface of any odd prime order which is homologically trivial, so one naturally asks whether this is also possible for some odd prime order in the smooth category. We included this question with an additional motivation, and wanted to emphasize that the underlying smooth structure may play a role here. It is plausible to construct an example of a nontrivial but homologically trivial, periodic diffeomorphism of a homotopy $K3$ surface, say of a relatively small, odd prime order. However, if one insists that the smooth structure be the standard one, then such an example would be much more difficult to obtain. On the other hand, it is very likely, given the special status of the standard smooth structure, that the answer to this question is negative, even though it seems to be a long shot with the current techniques.

Question 4 *Does there exist a smooth action on a homotopy $K3$ surface by a finite group which is not a $K3$ group?*

If one attempts to construct an example by a non-$K3$ group which is cyclic of prime order p, then p must be greater than 19, a fairly large order. Note that a smooth $\mathbb{Z}_p$-action of prime order $p > 19$ on a homotopy $K3$ surface is automatically trivial in homology. Hence an example of smooth action by a prime order cyclic non-$K3$ group would in particular give a nontrivial but homologically trivial periodic diffeomorphism of a homotopy $K3$ surface. Finally, we believe that, for an affirmative answer to this question, the smooth structure has to be non-standard.

Remarks: A negative answer to Question 3 seems to be out of reach by the current methods. However, any progress would shed light on how distinguished the standard smooth structure is from the viewpoint of group actions.

Question 5 *Must a symplectic symmetry of a homotopy $K3$ surface induce a faithful representation on the $K3$ lattice?*

We believe the answer to this question is affirmative, even though the smooth structures here are not required to be standard. The reason is that symplectic symmetries tend to be much more rigid. The special case where the symplectic homotopy $K3$ surface has trivial canonical class (e.g., when the smooth structure is standard) was verified in [2], as one of the first applications of an equivariant version of Taubes' theorem "$SW \Rightarrow Gr$". The key observation here is that the equivariant J-holomorphic curve technique gives nontrivial and useful information about the fixed-point set structure of a symplectic symmetry (cf. [2, 3]), while on the other hand, constraints on the fixed-point set structure of a smooth action that go beyond those satisfied by a locally linear topological action remain largely mysterious.

Question 6 *Does there exist a symplectic exotic $K3$ surface which admits no symplectic $\mathbb{Z}_p$-actions for some prime number $p \leq 19$?*

Symmetries and smooth structures on a homotopy $K3$ surface have interesting correlations, as demonstrated in [4, 5]. In particular, one can arrange to have exotic smooth structures such that some of the $K3$ groups can no longer act smoothly. The arguments require the $K3$ groups to have a somewhat "complicated" group structure, and it has been unsuccessful to extend the arguments to $K3$ groups which have "simpler" structures, in particular, groups of the simplest structure (i.e., cyclic groups of prime order), even if the group actions are assumed to be symplectic.

Remarks: Both Questions 5 and 6 are good testing grounds for developing a general theory of equivariant Gromov-Taubes invariant.

3. Bounding the order: 4-manifolds of "general type"

Background: By a classical theorem of Hurwitz, the order of the automorphism group of a compact Riemann surface of genus $g \geq 2$ is bounded by $84(g-1)$, and the bound is optimal. In a famous work, G. Xiao found the optimal generalization of Hurwitz Theorem to algebraic surfaces of general type. Recently, Xiao's theorem was extended to nonsingular projective varieties of general type of dimensions > 2.

Hurwitz Theorem can be equivalently formulated in terms of topological actions of finite groups. The two questions in this section are concerned with Hurwitz-type bound for smooth or symplectic finite group actions on 4-manifolds of "general type".

Question 7 *Let X be a simply connected, even, and smoothable topological 4-manifold which has non-zero signature. Does there exist a constant $C > 0$, depending only on X, such that there are no smoothable $\mathbb{Z}_p$-actions on X for any $p > C$?*

The very first issue one encounters here is: what is the appropriate notion of "general type" for smooth actions on 4-manifolds? In Hurwitz Theorem, the general-type condition is equivalent to the condition that the Riemann surface admits no circle actions. Such a condition can be used as a starting point, as obviously, it is also a necessary condition for the existence of a Hurwitz-type bound.

Some new phenomenon arises in dimension 4. Hurwitz-type bound exists for holomorphic $\mathbb{Z}_p$-actions which depends only on the integral homology of the 4-manifold, however, even for symplectic $\mathbb{Z}_p$-actions, the Hurwitz-type bound will depend on the underlying smooth structure. A key point explored in constructing such symplectic examples is the fact that existence of smooth circle actions in general depends on the underlying smooth structure, see [3]. Further examples were constructed in [6] which show the dependence of Hurwitz-type bound on the fundamental group.

The "general-type" condition in Question 7 is designed to suppress these issues; indeed, by a theorem of Atiyah and Hirzebruch, the 4-manifold X in Question 7 does not support any smooth circle actions no matter what the underlying smooth structure is. We hope that attempts to answer this question affirmatively may inspire new inventions in equivariant gauge theory.

Question 8 *Let X be a minimal symplectic 4-manifold of symplectic Kodaira dimension 2. Does there exist a universal constant $c > 0$ such that for any finite subgroup G of symplectomorphisms, the order of G satisfies the following bound?*

$$|G| \leq c \cdot c_1^2(TX)$$

For symplectic 4-manifolds, there is a natural notion of general type, thanks to the fundamental work of Taubes on "$SW = Gr$" (for the same reason the symplectic Kodaira dimension is well-defined). This notion of general type depends only on the underlying smooth structure, and is stronger than the condition of non-existence of smooth circle actions. Question 8 is a direct generalization of the aforementioned theorem of G. Xiao, although here we do not specify the value of the constant c (in Xiao's theorem $c = 42^2$).

4. Exoticness: symplectic vs. holomorphic, topological vs. smooth

Background: Primary examples of finite group actions on 4-manifolds are provided by automorphism groups of algebraic surfaces (or more generally, holomorphic actions on Kähler surfaces); in the case of $\mathbb{S}^4$, primary examples are given by the restrictions of linear actions on $\mathbb{R}^5$. A natural question asks whether there are "exotic" smooth actions which deviate from these standard ones. A basic method of producing exotic actions has been to construct actions whose fixed-point set structures are non-standard. However, if one requires the exotic actions to resemble the standard actions in some strong way, then the construction becomes considerably much harder.

Question 9 *Does there exist a symplectic finite group action on a Kähler surface which is not smoothly equivalent to a holomorphic action?*

If a symplectic finite group action has a 2-dimensional fixed-point set, then each component of the fixed-point set must be an embedded symplectic surface. This is the primary reason why all the exotic smooth actions on a Kähler surface known up to date can not be made symplectic. On the other hand, symplectic $\mathbb{Z}_n$-actions on $\mathbb{CP}^2$ or a Hirzebruch surface are smoothly equivalent to a holomorphic action, see [7, 8, 9]. (Of course, proving such a statement for general Kähler surfaces is technically impossible.)

Question 10 *Are there any orientation-preserving, pseudo-free, smooth finite cyclic actions on a simply connected* 4*-manifold, which are topologically equivalent but smoothly non-equivalent?*

The most interesting and relevant cases are when the 4-manifold is either $\mathbb{S}^4$, $\mathbb{CP}^2$, or a Hirzebruch surface, as only on these "small" 4-manifolds the topological classification of orientation-preserving, pseudo-free, smooth finite cyclic actions is practically manageable. In particular, every such an action on $\mathbb{S}^4$ or $\mathbb{CP}^2$ is topologically equivalent to a linear action. In the case of $\mathbb{S}^4$ such an exotic smooth action is related to a certain exotic smooth s-cobordism, whose non-existence was conjectured in [10]. One of the difficulties involved here, besides the construction of such exotic smooth actions, is the lack of an effective invariant (pre-assumably gauge theoretic) that can tell the smooth actions apart. Recently, some progress was made in this regard in the case of Hirzebruch surfaces (cf. [9]).

References

[1] W. Chen, S. Kwasik and R. Schultz, *Finite symmetries of* $\mathbb{S}^4$, Forum Mathematicum **28** no. 2 (2016), 295-310.

[2] W. Chen and S. Kwasik, *Symplectic symmetries of* 4*-manifolds*, Topology **46** no. 2 (2007), 103-128.

[3] W. Chen, *On the orders of periodic diffeomorphisms of 4-manifolds*, Duke Mathematical Journal **156** no. 2 (2011), 273-310.

[4] W. Chen and S. Kwasik, *Symmetries and exotic smooth structures on a* $K3$ *surface*, Journal of Topology **1** (2008), 923-962.

[5] W. Chen and S. Kwasik, *Symmetric symplectic homotopy* $K3$ *surfaces*, Journal of Topology **4** (2011), 406-430.

[6] W. Chen, *Hurwitz-type bound, knot surgery, and smooth* $\mathbb{S}^1$*-four-manifolds*, Mathematische Zeitschrift **276** (2014), 267-279.

[7] W. Chen, *Orbifold adjunction formula and symplectic cobordisms between lens spaces*, Geometry and Topology **8** (2004), 701-734.

[8] W. Chen, *Group actions on 4-manifolds: some recent results and open questions*, Proceedings of the Gökova Geometry-Topology Conference 2009, Akbulut, S. et al ed., pp. 1-21, International Press, 2010.

[9] W. Chen, *G-minimality and invariant negative spheres in G-Hirzebruch surfaces*, Journal of Topology **8** (2015), 621-650.

[10] W. Chen, *Smooth s-cobordisms of elliptic* 3*-manifolds*, Journal of Differential Geometry **73** no.3 (2006), 413-490.