Homework 8—Final Exam

Math 652
Spring 2020

Optional, Due Thursday May 7, if you want it graded

Since most of you have a final project presentation for the Applied Math MS (and since I forgot to post the exam until now), the final exam is strictly extra credit. You do not have to complete it if you are busy with other things. However, I encourage you to have a look, since it will help you to understand the lectures on constrained and convex optimization. If you want me to grade it, please turn it in by May 7.

You have already seen in your study of variational problems related to the finite element method that (strictly) convex minimization problems with objective functions that are bounded below have a unique solution. Something weaker but roughly similar is true for convex programs.

Problem 1. *Consider the general convex program*

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{s. t.} \quad & Ax = b \\ & g_i(x) \le 0 \textit{ for all } i \in \mathscr{I}. \end{aligned}$$

(Here, we assume that f is convex and all the constraint functions g_i are convex. A is a matrix and b is a vector.)

1. *Show that the feasible region Ω is convex.*
2. *Show that any local solution of the convex program is a global solution.*
3. *Show the set of global solutions is convex.*
4. *Would any of this still be true if I changed the inequality constraint to $g_i(x) \ge 0$, where g_i is convex?*

Writing optimization problems in standard forms is an essential skill in convex optimization. I've given you one good exercise below. I suggest Exercises 4.5, 4.6, 4.11, and 4.12 in Boyd and Vandenberghe, as well, if you are interested.

Problem 2. *Let $v : \mathbb{R}^n \to \mathbb{R}^m$. Assume that v is smooth Consider the unconstrained, possibly nonsmooth, problem to*

$$\text{minimize} \quad \|v(x)\|_\infty.$$

Reformulate this problem as a smooth constrained optimization problem. Think about how many constraints are required in order to do this.

Recognizing convex functions is an essential skill in formulating problems as convex programs. Here are a few example forms of convex functions that might not be obvious.

Problem 3.

1. *Suppose that* $\{f_i : i \in I\}$ *are convex functions. Show that* $g(x) = \sup_{i \in I} f_i(x)$ *is convex.* Note: *If you don't know how to handle the supremum, just assume that* I *is finite and make it a maximum. It is important to realize however that the cardinality of* I *doesn't actually matter. It could be uncountable even.*

2. *Show that the "log-sum-exp" function*

$$f(x) = \log\left(\sum_i \exp(x_i)\right)$$

   is convex. Note: *This function arises all the time when you consider logarithms of probabilities. It comes up in machine learning too, for slightly different reasons.*

3. *Show that if* g *is monotone nondecreasing and convex and* f *is convex, then* $g \circ f$ *is convex.*

Finally, you should make sure that you understand the KKT conditions. I propose that you try the following exercise. This isn't well-posed. It's something we could discuss, but there isn't really anything to turn in here.

Problem 4. *Interpret the KKT conditions in the case of* linear *constraints (both equality and inequality). Draw lots of pictures. Use what you know from linear algebra. Pay special attention to the LICQ. The proof of the KKT conditions basically says that the things that are true in the linear case carry over to the nonlinear case using the implicit function theorem. The linear case already contains all of the real insights.*