First, we review a few facts about the (semi-discrete) Fourier transform defined in class on February 26. For any 1-periodic function $u : \mathbb{R} \to \mathbb{C}$, we define $\hat{u} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{u}(k) = \langle \psi_k, u \rangle = \int_{x=0}^{1} u(x) \exp(-2\pi ikx) \, dx.$$ 

Here, the functions $\{\psi_k, k \in \mathbb{Z}\}$ defined by $\psi_k(x) = \exp(2\pi ikx)$ are called the characters. When we used the Fourier transform to solve the advection-diffusion equation, we proved that for any 1-periodic $u \in C^n$,

$$\frac{d^n u}{dx^n}(k) = (2\pi ik)^n \hat{u}(k).$$

We also recall that

$$\|u\|_{L^2} = \|\hat{u}\|_{\ell^2} = \left\{ \sum_{k \in \mathbb{Z}} \hat{u}(k)^2 \right\}^{\frac{1}{2}}.$$

You have recently verified something similar for the finite Fourier transform. The general result is sometimes called Plancherel’s theorem. It is pretty easy to prove once you remember that the characters are orthonormal and that by the Fourier inversion formula

$$u = \sum_{k \in \mathbb{Z}} \hat{u}(k) \psi_k,$$

where the sum converges in $L^2$.

The Fourier transform provides a means of characterizing Sobolev spaces. We say that a 1-periodic function $u$ is weakly differentiable if there exists a 1-periodic $Du$ so that for any 1-periodic $v \in C^\infty$,

$$\int_{x=0}^{1} Du(x)v(x) \, dx = -\int_{x=0}^{1} u(x)v'(x) \, dx.$$
Let $H^n(\mathbb{R}/\mathbb{Z})$ be the set of all 1-periodic functions with $n$ weak derivatives such that each weak derivative is in $L^2$. One can show that $u \in H^n(\mathbb{R}/\mathbb{Z})$ if and only if
\[
\sum_{k \in \mathbb{Z}} (1 + k^2)^n |\hat{u}(k)|^2 < \infty.
\]

**Problem 1** (2+1 points).

1. Show that if $u \in H^1(\mathbb{R}/\mathbb{Z})$, then
\[
\frac{d\hat{u}}{dx}(k) = 2\pi ik\hat{u}(k).
\]

Hint: You can use the argument from class. You just have to check that all the steps work when $u$ is only weakly differentiable.

2. Show that if $u \in H^1(\mathbb{R}/\mathbb{Z})$, then
\[
\sum_{k \in \mathbb{Z}} (1 + k^2)|\hat{u}(k)|^2 < \infty.
\]

I won’t make you prove that $\sum_{k \in \mathbb{Z}} (1 + k^2)^n |\hat{u}(k)|^2 < \infty$ implies $u \in H^n(\mathbb{R}/\mathbb{Z})$, but I will say that you now know enough analysis to do it, at least mostly.

We now consider the accuracy of a Galerkin method based on the Fourier characters. This is an example of a spectral method. Let
\[
V_m = \text{span}\{\psi_{-m}, \ldots, \psi_m\}.
\]

Our goal is to estimate the best approximation error
\[
\inf_{v \in V_m} \|u - v\|_2,
\]
where $u$ is the solution of some boundary value problem (or other similar variational problem).

**Problem 2** (5+3 points). Let $u \in H^n(\mathbb{R}/\mathbb{Z})$. Define
\[
S_m u = \sum_{k = -m}^{m} \hat{u}(k) \psi_k.
\]

1. Show that
\[
\|u - S_m u\|_2 = O\left(\frac{1}{m^{(n-1)}}\right)
\]
in the limit as $n \to \infty$.

Hint: Observe that if $\sum_{k \in \mathbb{Z}} \hat{u}(k)^2 k^{2n} < \infty$, then we can only have $|\hat{u}(k)|^2 < 1/k^{2n}$ for at most finitely many $k$, hence $|\hat{u}(k)|^2 \leq C/k^{2n}$ for some constant $C$.  

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2. How would the above result change if you wanted to measure the error in the $H^1(\mathbb{R}/\mathbb{Z})$-norm instead of in $L^2$?

Observe that if $u \in C^\infty$, then

$$
\|u - S_m u\|_2 = O \left( \frac{1}{m^{\ell}} \right)
$$

for all $\ell > 0$! In this case, one says that the approximations converge with infinite order or at spectral speed. By comparison, convergence of finite elements is quite slow. For example, if we took $V_m$ to be the set of all periodic, piecewise affine functions with respect to the mesh $\{0, 1/N, 2/N, \ldots, 1\}$, then by the standard analysis based on Bramble–Hilbert, we have only the estimate

$$
\|u - S_m u\|_2 = O \left( \frac{1}{N^2} \right).
$$

Finite element methods do not converge at spectral speed. However, this does not mean that finite element methods are useless. Spectral methods lead to system matrices that are large and dense, so it can be very hard to solve the minimization problem. Also, for BVPs on domains of complicated shapes it can be difficult or impossible to choose an appropriate basis for a spectral method.

There is a more refined (but also much more complicated) explanation of the convergence properties of spectral methods in the book by Arieh Iserles, see the course website for a link to the online version. There are some things that can go wrong with spectral approximations, so the book is a worthwhile read.