Consider the initial value problem

\[ x' = f(x) \text{ with } x(0) = y, \]

where \( f : \mathbb{R} \to \mathbb{R} \) is a scalar function. Assume that \( f \) is Lipschitz with constant \( L > 0 \), i.e.

\[ |f(x) - f(y)| \leq L|x - y| \]

for all \( x, y \in \mathbb{R} \). Recall that Euler’s method for the solution of the IVP reads

\[ x_{n+1} = x_n + \Delta t f(x_n) \text{ with } x_0 = y. \]

We have shown that Euler’s method is stable. In particular, for any \( T > 0 \), if \( w_n \) solves the perturbed recurrence relation

\[ w_{n+1} = w_n + \Delta t f(w_n) + G_n \text{ with } z_0 = y, \]

where \( |G_n| \leq \varepsilon \) for all \( n = 0, \ldots, \lfloor T/\Delta t \rfloor \), then

\[ \max_{n=0,\ldots,\lfloor T/\Delta t \rfloor} |w_n - x_n| \leq \frac{C\varepsilon}{L\Delta t} \exp(TL). \]

Moreover, Euler’s method is consistent of order one. That is, the exact solution \( x \) of the IVP solves

\[ x((n + 1)\Delta t) = x(n\Delta t) + \Delta t f(x(n\Delta t)) + H_n, \]

where \( |H_n| \leq C\Delta t^2 \) for some constant \( C > 0 \) independent of \( \Delta t \). The error estimate

\[ \max_{n=0,\ldots,\lfloor T/\Delta t \rfloor} |x(n\Delta t) - x_n| \leq \frac{C}{L} \exp(TL)\Delta t \]

follows. You will now apply the technique of modified equations to the study of Euler’s method.

**Problem 1** (5 + 3 points).
1. Define

\[ g(x) = \frac{f(x)}{1 + \frac{\Delta t}{2} f'(x)}, \]

and consider the modified equation

\[ z'(t) = g(z(t)) \text{ with } z(0) = y. \]

(Note that \( g \) is defined whenever \( \Delta t \) is sufficiently small, if we assume that \( f' \) is bounded.) Show that the Euler recurrence

\[ x_{n+1} = x_n + \Delta t f(x_n) \text{ with } x_0 = y \]

for the original equation has second order consistency error as a numerical method for the modified equation.

2. Prove that if \( z \) solves the modified equation, then

\[ \max_{n=0, \ldots, [T/\Delta t]} |z(n\Delta t) - x_n| \leq C L \exp(TL) \Delta t^2. \]

You may use any results proved in class or stated above.

Recall that the modified equation for the upwind method is the advection-diffusion equation

\[ \frac{\partial v}{\partial t} = -a \frac{\partial v}{\partial x} + \frac{1}{2} a \Delta t \left( 1 - \frac{a \Delta t}{\Delta x} \right) \frac{\partial^2 u}{\partial x^2}. \]

As a consequence, one expects that if upwind is used to solve the advection equation, waves spread out as they propagate. You will now verify that this is the case.

**Problem 2** (5 points). Use the upwind method to solve the advection equation on the domain \([0, 1]\) with \( a = 1 \), with periodic boundary conditions, and with initial condition

\[ u_0(x) = \exp \left( -128 \left| x - \frac{1}{2} \right|^2 \right). \]

For \( i \in \{4, 5, 6, 7\} \), run the upwind method using the parameters \( \Delta x = 2^{-i} \), \( \Delta t = \frac{1}{2} \Delta x \), and \( T = 2 \). For each \( i \), plot the solution at an appropriate sequence of times, say \( t = 0.25, 0.5, 0.75, \ldots, 2 \). Verify that the wave spreads as it propagates; notice that it spreads more slowly when \( \Delta t \) is small.

Finally, I would like you to resolve a few points from the first lecture on Ritz methods.

**Problem 3** (3 points). Let \( \{\phi_1, \ldots, \phi_N\} \) be a basis for a subspace \( V \leq C^1_0 \). Let \( a : \Omega \to \mathbb{R} \) be a continuously differentiable function so that \( \min a > 0 \). Show that the matrix \( M \in \mathbb{R}^{N \times N} \) with entries

\[ M_{k\ell} = \int_\Omega a \nabla \phi_k \cdot \nabla \phi_\ell \, dx \]

is positive definite. Hint: You’re going to have to use the Poincaré inequality.