Homework 4

Math 652
Spring 2020

Due Monday, March 9, 2020

There are many Fourier transforms. They all have similar properties. For example, let $N \in \mathbb{N}$ and $\Delta x = 1/N$, and consider the set of periodic functions defined on the spatial mesh

$$\Omega = \{k\Delta x; k = 0, \dots, N\}.$$

Here, as in class, periodic means that $f(0) = f(N)$. Now define Fourier modes $\psi^k : \Omega \to \mathbb{C}$ by

$$\psi^k(\ell\Delta x) = \exp(2\pi i k\ell\Delta x).$$

The appropriate inner product in this context is the weighted inner product

$$\langle u, v\rangle_{\Delta x} = \Delta x \sum_{\ell=0}^{N-1} u(\ell\Delta x)\overline{v(\ell\Delta x)}$$

introduced for the analysis of the diffusion equation, and the corresponding norm is of course $\|u\|_{\Delta x} = \langle u, u\rangle_{\Delta x}^{\frac{1}{2}}$.

Problem 1 (3+2+2+5 points)**.**

1. *Show that $\{\psi^k; k = 0, \dots, N-1\}$ is an orthonormal set.*

   Hint: *Use the geometric sum formula.*

2. *Consider the matrix*

   $$U = \left(\psi^0 | \dots | \psi^{N-1}\right).$$

   Prove that $\sqrt{\Delta x}U$ is an isometry of $\|\cdot\|_{\Delta x}$. That is, show that for any periodic $v : \Omega_\Delta \to \mathbb{C}$, we have

   $$\|\sqrt{\Delta x}Uv\|_{\Delta x} = \|v\|_{\Delta x}.$$

   Note*: A better way to think of the above result: U is a norm preserving mapping from the space of periodic functions on $N\Omega = \{0, 1, \dots, N\}$ with the $\|\cdot\|_2$-norm to the space of periodic functions on Ω with the $\|\cdot\|_{\Delta x}$-norm.*

   Note: *I made some incorrect statements about this in class. In fact, one should define an operator K to be unitary over a Hermitian product if its* *inverse equals its adjoint K^* over that product. In this case, the conjugate transpose is not the adjoint, since the inner product is weighted. This is inconsequential for our proof of Lax–Richtmeyer stability of Lax–Friedrichs, but nonetheless what I wrote was wrong.*

3. *Prove that $\sqrt{\frac{1}{\Delta x}}U^{-1}$ is an isometry of $\|\cdot\|_{\Delta x}$.*

   Note: *A better way to think of the above result: U^{-1} is a norm preserving mapping from the space of periodic functions on Ω with the $\|\cdot\|_{\Delta x}$-norm to the space of periodic functions on $N\Omega = \{0, 1, \dots, N\}$ with the $\|\cdot\|_2$-norm.*

4. *Prove that $U^{-1} = \Delta x U^\dagger$.*

5. *Recall that the Lax–Friedrichs method is based on the operator*

$$A_\Delta = \frac{\Delta x^2}{2\Delta t} D_\Delta^2 - aD_\Delta^c,$$

   where we define

$$\begin{aligned} M_\Delta u_\Delta(\ell, n) &= \frac{u_\Delta(\ell+1, n) + u_\Delta(\ell-1, n)}{2}, \\ D_\Delta^c u(\ell, n) &= \frac{u_\Delta(\ell+1, m) - u_\Delta(\ell-1, m)}{2\Delta x} \end{aligned}$$

   For each $k = 0, \dots, N-1$, show that ψ^k is an eigenvector of A_Δ and compute the corresponding eigenvalue.

With apologies for all the mistakes, given the above, we have that

$$\begin{aligned} \|L_\Delta^n\|_{\Delta x} &= \|(I + \Delta t A_\Delta)^n\|_{\Delta x} \\ &= \|(I + \Delta t U \Lambda U^{-1})^n\|_{\Delta x} \\ &= \left\| \sqrt{\Delta x} U (I + \Delta t \Lambda)^n \sqrt{\frac{1}{\Delta x}} U^{-1} \right\|_{\Delta x} \\ &= \|(I + \Delta t \Lambda)^n\|_{\Delta x}, \end{aligned}$$

and Lax–Richtmeyer stability follows from an analysis of the spectrum of A_Δ, as discussed in class, when $|a\Delta x/\Delta t| \le 1$.

Problem 2 (5 points)**.** *I claim that the stability condition*

$$\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$$

for the solution of the heat equation by forwards Euler is "unreasonable," i.e. it requires a Δt that is smaller than would otherwise be desireable. But I also claim that the stability condition

$$\left| \frac{a\Delta t}{\Delta x} \right| \le 1$$

for Lax–Friedrichs is very reasonable. Write a couple paragraphs arguing this position. Some points to consider in your answer:

1. *Are the parts of the consistency error corresponding to spatial and time discretization of balanced size when the stability condition holds?*

2. *Should the constant a appear in the stability condition? What happens if we rescale time in the advection equation by a linear factor?*

Finally, I would like for you to prove the easy part of the Lax equivalence theorem.

Problem 3 (10 points)**.** *Let u solve the partial differential equation*

$$\frac{\partial u}{\partial t} = Lu,$$

where L is some linear partial differential operator, with appropriate initial and boundary conditions. For example,

$$L = D\Delta - a \cdot \nabla$$

for the advection-diffusion equation. Let $(\Delta t_m, \Delta x_m)$ be a sequence of discretization parameters with $(\Delta t_m, \Delta x_m) \to (0,0)$ as $m \to \infty$. Let L_Δ be a linear operator arising from a consistent finite difference approximation of L. That is, assume that

$$u(\cdot, (n+1)\Delta t) = L_\Delta u(\cdot, n\Delta t) + H_n(\cdot),$$

where $\|H_n\| \leq C\Delta t(\Delta x^p + \Delta t^q)$ for some constant C independent of Δt, Δx. Assume that L_Δ is Lax–Richtmeyer stable along (x_m, t_m). Show that

$$\max_{n=0,\ldots,\lfloor T/\Delta t \rfloor} \|u_\Delta(\cdot, n\Delta t) - u(\cdot, n\Delta t)\| \leq C(T)(\Delta x_m^p + \Delta t_m^q)$$

for some constant $C(T)$ that may depend on T, but not on u or m.
Hint: *Use the discrete version of variation of parameters as in our study of forwards Euler for the heat equation.*