Homework 2

Math 652
Spring 2020

Due Friday, February 14, 2020

Recall that a function $f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is said to be Lipschitz with constant $L > 0$ if

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\|$$

for all $x, y \in \mathbb{R}^n$ and $t \in [0, \infty)$.

Problem 1 (5 points). *Let $f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be Lipschitz with constant L. Suppose that $x : [0, \infty) \to \mathbb{R}^n$ solves*

$$x'(t) = f(x(t), t) \text{ with } x(0) = x_0$$

and that $y : [0, \infty) \to \mathbb{R}^n$ solves

$$y'(t) = f(y(t), t) \text{ with } y(0) = y_0.$$

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$. Show that

$$\|x(t) - y(t)\| \leq \exp(Lt)\|x_0 - y_0\|.$$

Hint: *Use Grönwall's inequality. Mimic the proof of the contraction property of the heat equation.*

I have assigned the above mostly as an exercise in using Grönwall's inequality. Please note that it is very similar to our stability estimates for discretizations of differential equations.

There is a more general version of the Poincaré inequality than the one-dimensional result proved in class.

Theorem 1 (Poincaré Inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and assume that $\partial\Omega$ is smooth. There exists a constant C depending on Ω so that for any continuously differentiable $u : \Omega \to \mathbb{R}$ with $u(x) = 0$ for all $x \in \partial\Omega$, we have*

$$\|u\|_2 \leq C\|\nabla u\|_2.$$

It should be noted that the conditions under which the Poincaré inequality holds can be weakened substantially. In particular, $\partial\Omega$ need only be Lipschitz, not smooth. Also, u need only be *weakly* differentiable. We will ignore these technical details entirely in this class.

The next exercise uses our new, general version of the Poincaré inequality. It is the basis for the finite element method for solving elliptic PDE.

Problem 2 (3+3+3+3 points). *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with piecewise smooth boundary.*[1] *Define $C_0^2(\Omega)$ to be the set of all twice continuously differentiable functions $u : \Omega \to \mathbb{R}$ satisfying the boundary condition $u(x) = 0$ for all $x \in \partial\Omega$. Let $a : \Omega \to \mathbb{R}$ be a continuously differentiable function such that for some $\theta > 0$, $a(x) > \theta$ for all $x \in \Omega$, and let $f : \Omega \to \mathbb{R}$ be continuous. Define the functional $I : C_0^2(\Omega) \to \mathbb{R}$ by*

$$I(u) = \frac{1}{2}\int_\Omega a(x)\nabla u(x) \cdot \nabla u(x)\, dx - \int_\Omega f(x)u(x)\, dx.$$

1. *Show that*

   $$\lim_{\|u\|_2 \to \infty} I[u] = \infty.$$

   Hint: *Use the Poincaré inequality.*

2. *Show that I is convex, i.e. show that for any $\alpha \in [0,1]$, $u, v \in C_0^2(\Omega)$, we have*

   $$I[\alpha u + (1-\alpha)v] \leq \alpha I[u] + (1-\alpha)I[v].$$

   Note: *In fact, something stronger is true. For any $u, v \in C_0^2$ with $v \neq 0$, the function $\alpha \mapsto I[u + \alpha v]$ is a parabola that opens upward. That is, if we restrict I to any line in the space $C_0^2(\Omega)$, the result is a parabola.*

3. *Let $u, v \in C_0^2(\Omega)$ with $v(x) = 0$ for all $x \in \partial\Omega$. Show that*

   $$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} I[u + \varepsilon v] = \int_\Omega (-\operatorname{div}(a(x)\nabla u(x)) - f(x))v(x)\, dx.$$

4. *Explain without giving technical details why the minimizer of I exists, why it is unique, and why it solves the boundary value problem*

   $$\begin{cases} -\operatorname{div}(a(x)\nabla u(x)) = f(x) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

   Hint: *Think about our proof of the variational formulation of $Au = f$ when A is SPD. (If you haven't seen this, visit my office hours and I will explain.) Suppose we were dealing with a function $I : \mathbb{R}^3 \to \mathbb{R}$. What would the graph look like if I were convex and $\lim_{x\to\infty} I(x) = \infty$? It should be noted that any complete proof of the existence of solutions to the boundary value problem will be much more sophisticated than the above. This problem gives only the outline of a proof.*

Finally, we have a computational problem.

[1] Recall that the boundary of a set is smooth if each point on the boundary is surrounded by a neighbhorhood in the boundary that can be parametrized by a continuously differentiable function. Piecewise smooth means that the boundary can be partitioned into a finite number of smooth pieces. For example, the unit cube has a piecewise smooth boundary. One needs some smoothness assumption on the boundary in order to apply the divergence theorem. We are going to ignore this point, for the most part, but you should still be aware.

Problem 3 (10+3+3 points)**.** *Let*

$$f(x,y) = \left(\frac{2}{9\pi^2} - \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right)\sin(3\pi y).$$

We observe that the solution u of the boundary value problem

$$\begin{cases} \Delta u(x,y) = f(x,y) & \text{for } (x,y) \in [0,1]^2 \\ u(x,y) = 0 & \text{for } (x,y) \in \partial[0,1]^2, \end{cases}$$

is

$$u(x,y) = \frac{1}{9\pi^2}\left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right]\sin(3\pi y).$$

1. *Compute the solution of the discrete problem*

$$\begin{cases} \Delta_h u_h(mh, nh) = f(mh, nh) \text{ for } (mh, nh) \in \Omega_h \\ u_h(mh, nh) = 0 \text{ for } (mh, nh) \in \Gamma_h \end{cases}$$

for $N = 16, 32, 64, 128, 256, 512, 1024$. Use your favorite of conjugate gradients or Gauss–Seidel to solve for u_h.[2] *Whichever iteration you choose, terminate at the first step k so that*

$$\|\Delta_h u_h^k - f\|_h \leq 10^{-10}.$$

Plot $\log_{10}\|u_h - u\|_h$ vs. $\log_{10} N$. Does the observed rate of convergence agree with the theory developed in class?

Note: *You* do not *need to choose an enumeration and compute the matrix representing Δ_h to solve for u_h by CG or G–S. If you do not see why, ask me. Generally speaking, it is easier to code iterative methods without explicitly assembling the matrix. However, you can certainly do it using the matrix if you like. Just make sure you use a sparse format for storage!*

2. *Show that if $\|\Delta_h u_h^k - f\|_h \leq 10^{-10}$, then for u_h the exact solution of the discrete problem, we have*

$$\|u_h^k - u_h\|_h \leq \frac{h^2}{4(1 - \cos(\pi h))} 10^{-10}.$$

Hint: *Look over last week's assignment. Note that this justifies the choice of termination criterion for the iterative method.*

[2] See Leveque's book for a review of these methods with tips on how to code them when solving the discrete boundary value problem. There is a link to an online copy of Leveque's book on the syllabus page. Those who took Math 651 with me last semester are strongly encouraged to adopt conjugate gradients as their favorite iterative method.

3. *I chose the tolerance* 10^{-10} *for a reason. When* $N = 1024$*, does it make sense to choose a lower tolerance, terminating for example when*

$$\|\Delta_h u_h^k - f\|_h \leq 10^{-14}?$$

Would you actually expect an improved result with the lower tolerance?

Hint: *Think about floating point errors. What is the condition number of* Δ_h*? How accurate is the matrix product* $\Delta_h u_h^k$ *when computed in floating point? What is machine precision for 64-bit arithmetic? (Recall that the condition number for the matrix product is the same as for the solution of the linear equation. It's* $\kappa = \|\Delta_h\| \|\Delta_h^{-1}\|$*.)*