Homework 1

Math 652
Spring 2020

Due Friday, February 7 2020

Problem 1 (2+5+1+2 points)**.**

1. *Show that Δ_h, an operator on $L(\Omega_h)$, is symmetric. That is, show that for any $v, w \in L(\Omega_h)$ we have*

$$\langle \Delta_h v, w \rangle_h = \langle v, \Delta_h w \rangle_h.$$

   Hint: *Use the summation by parts formula from class. But be careful! Remember that the formula only applies to functions $v, w : \bar{\Omega}_h \to \mathbb{R}$ defined over the entire mesh $\bar{\Omega}_h$ with $v = w = 0$ on Γ_h. On the other hand, the elements of $L(\Omega_h)$ are functions $v : \Omega_h \to \mathbb{R}$ defined over the interior Ω_h of the mesh. This is not a serious difficulty at all, but make sure you understand why not.*

2. *You may be accustomed to thinking of linear operators as matrices. To represent Δ_h as a matrix one would first choose an enumeration of Ω_h. One conventional choice is to begin with $(1,1)$ and count along rows first and then columns. For example, with $N = 4$, the enumeration of Ω_h would be*

1	*(1,1)*
2	*(1,2)*
3	*(1,3)*
4	*(2,1)*
5	*(2,2)*
6	*(2,3)*
7	*(3,1)*
8	*(3,2)*
9	*(3,3)*

   Given an enumeration, each function $v : \Omega_h \to \mathbb{R}$ corresponds to a vector $\tilde{v} \in \mathbb{R}^{(N-1)^2}$. It is probably clear to you what the rule defining $\tilde{v}$ must be, but in case it isn't, here is an example which can serve as a definition: For the enumeration above, we would have $\tilde{v}_6 = v(2h, 3h)$ and $\tilde{v}_2 = v(1h, 2h)$. What is the matrix corresponding to Δ_h under this enumeration? Don't hesitate to ask if this question is not clear to you. It's important.

3. *Is the matrix corresponding to Δ_h symmetric for all enumerations of Ω_h?*

4. *When is it necessary to write Δ_h explicitly as a matrix? Could you code CG or Gauss–Seidel without explicitly choosing an enumeration and construcing a matrix? Could you use the LU and QR decompositions provided in SciPy? Could you prove the theoretical results below?*

You have seen one proof of stability of Δ_h based on summation by parts and the discrete Poincaré inequality. It is also possible to prove stability using Fourier analysis, which gives explicit expressions for all eigenvectors and eigenvalues. You will carry out this approach in the problem below.

Problem 2 (5+3+3 points)**.**

1. *Define $\phi_{k\ell} : \Omega_h \to \mathbb{R}$ by*

$$\phi_{k\ell}(mh, nh) = \sin(k\pi mh)\sin(\ell\pi nh).$$

Show that for any $1 \le k \le N-1$ and $1 \le \ell \le N-1$, $\phi_{k\ell}$ is an eigenvector of Δ_h with eigenvalue

$$\frac{2}{h^2}\left(\cos(k\pi h) + \cos(\ell\pi h) - 2\right).$$

Since Δ_h is an operator on a space $L(\Omega_h)$ of dimension $(N-1)^2$, one can conclude that these are all *of the eigenvectors.* Hint: *Use the angle sum identity for the* sin *function.*

2. *Recall the definition of the h-norm for functions $v : \Omega_h \to \mathbb{R}$:*

$$\|v\|_h = \left\{ h^2 \sum_{m,n=1}^{N-1} \right\}.$$

(If it concerns you that the sums only go up to $N-1$, then you can imagine that $v = 0$ on Γ_h and the sums go all the way to N.) Show that

$$\|\Delta_h^{-1}\|_h = \frac{h^2}{4(1-\cos(\pi h))}$$

and that

$$\lim_{h\to 0} \frac{h^2}{4(1-\cos(\pi h))} = \frac{1}{2\pi^2}.$$

Hint: *By the first problem, you know that Ω_h is symmetric. It follows that $\|\Delta_h^{-1}\|_2 = \max\{|\lambda|^{-1}; \lambda \in \sigma(\Delta_h)\}$. Since the ℓ^2 and h-norms are related by a constant multiple, the ℓ^2 and h operator norms are the same.*

3. *Show that*

$$\kappa_2(\Delta_h) = \frac{1+\cos(\pi h)}{1-\cos(\pi h)}$$

and that

$$\kappa_2(\Delta_h) \sim \frac{\pi^2}{2h^2}$$

in the limit as $h \to 0$. Note: *The "$\sim$" above means show that*

$$\frac{\kappa_2(\Delta_h)}{\frac{\pi^2}{2h^2}} \to 1.$$

Finally, I would like you to prove a discrete version of the Poincaré inequality to complete the proof of stability outlined in class. It's worth noting that this proof of stability works in many cases where a proof by Fourier analysis is not feasible. For example, it will work with only minor modifications in the case of non-constant coefficients.

Problem 3 (5+3 points)**.**

1. *Prove that for any* $v : \bar{\Omega}_h \to \mathbb{R}$ *with* $v = 0$ *on* Γ_h.

   $$\|v\|_h \leq \|D_x^- v\|_h.$$

   Of course, we then have $\|v\|_h \leq \|D_y^- v\|_h$ *as well by symmetry.* Hint: *Mimic the proof of the Poincaré inequality from class. All steps are more or less the same.*

2. *Combine the summation by parts lemma and the discrete Poincaré inequality above to prove the stability result*

   $$\|\Delta_h v\|_h \geq \|v\|_h.$$

   Hint: *Again, mimic the proof of the analogous result for* C^1 *functions from class.*

The above seems like enough for this week. I'll assign some computations next week.