# Conformal Mapping of Basic Algebraic Functions:

## Visualizing Complex Functions

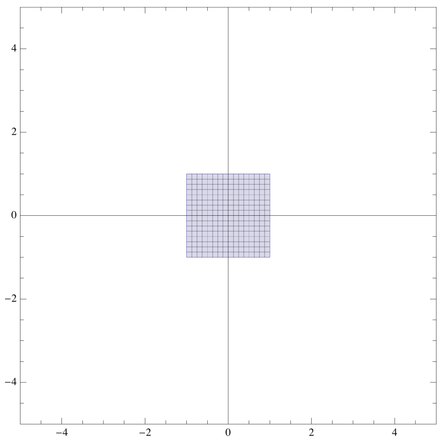
# Simple Algebraic Functions I

*nealy bowden*

In this section we will look at examples of simple algebraic functions and examine how they transform a given area in the complex plane.

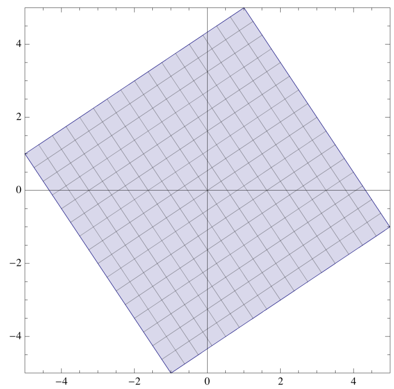
*We begin by considering a square centered at the origin with side length 2 units.*

```
f[z_] := z;
ParametricPlot[{Re[f[x + I * y]], Im[f[x + I * y]]}, {x, -1, 1}, {y, -1, 1}, PlotRange → {{-5, 5}, {-5, 5}}]
```



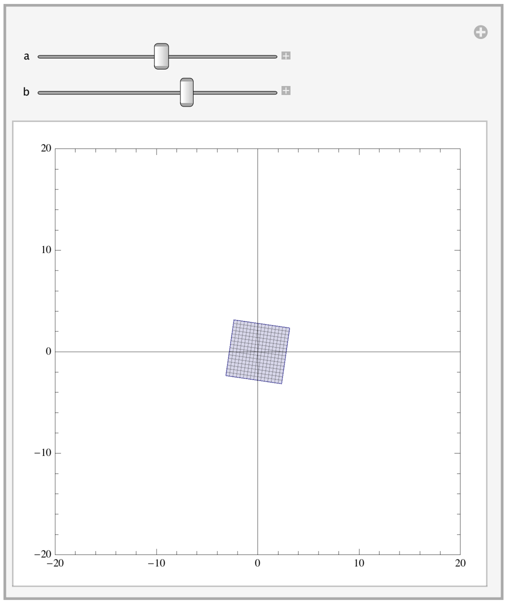
*Let's see what happens when we scale this area by a complex constant, w=3+2i.*

```
f[z_] :=  (3 + 2 I) z ;
ParametricPlot[{Re[f[x + I * y]], Im[f[x + I * y]]}, {x, -1, 1}, {y, -1, 1}, PlotRange → {{-5, 5}, {-5, 5}}]
```



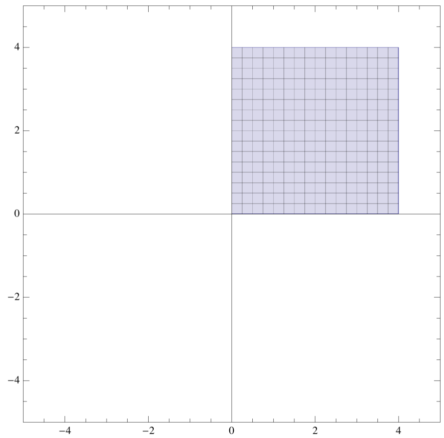
*Now, consider a more general case: w=a+bi. Using the manipulate function we can see the effects of the real and imaginary parts a and b on the square we described above.*

```
Manipulate[ParametricPlot[{Re[(x + I * y) * (a + I * b)], Im[(x + I * y) * (a + I * b)]},
  {x, -1, 1}, {y, -1, 1}, PlotRange → {{-20, 20}, {-20, 20}}], {a, -10, 10}, {b, -10, 10}]
```



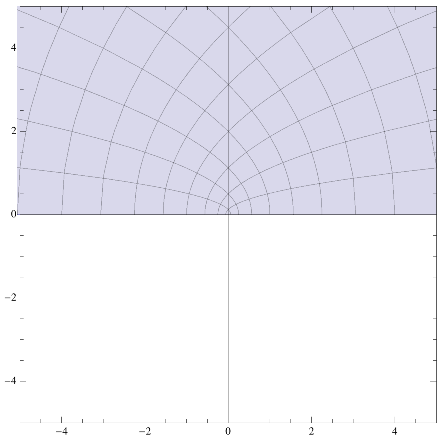
*Of course, we can also look at squares not centered at the origin.*

```
f[z_] := z;
ParametricPlot[{Re[f[x + I * y]], Im[f[x + I * y]]}, {x, 0, 4}, {y, 0, 4}, PlotRange → {{-5, 5}, {-5, 5}}]
```



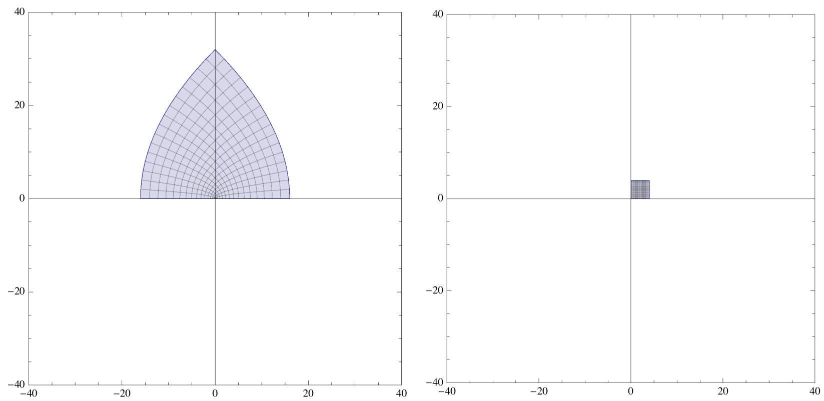
*Let's consider a familiar function,* $f(z) = z^2$*, over this interval.*

```
f[z_] := z^2;
ParametricPlot[{Re[f[x + I * y]], Im[f[x + I * y]]}, {x, 0, 4}, {y, 0, 4}, PlotRange → {{-5, 5}, {-5, 5}}]
```



*Zooming out, we see the whole picture.*

```
f[z_] := z^2;
ParametricPlot[{Re[f[x+I*y]], Im[f[x+I*y]]}, {x, 0, 4}, {y, 0, 4}, PlotRange → {{-40, 40}, {-40, 40}}]
```

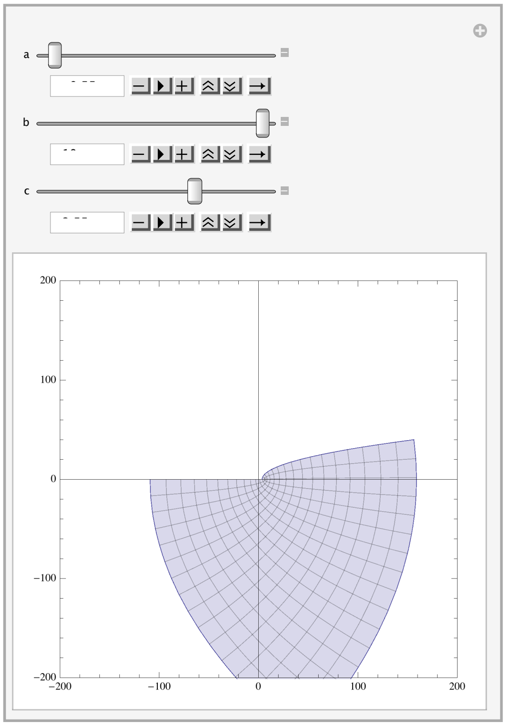



*Considering our domain in this new frame of reference, we get a better understanding of the magnitude of $z^2$.*

```
f[z_] := z;
ParametricPlot[{Re[f[x+I*y]], Im[f[x+I*y]]}, {x, 0, 4}, {y, 0, 4}, PlotRange → {{-40, 40}, {-40, 40}}]
```

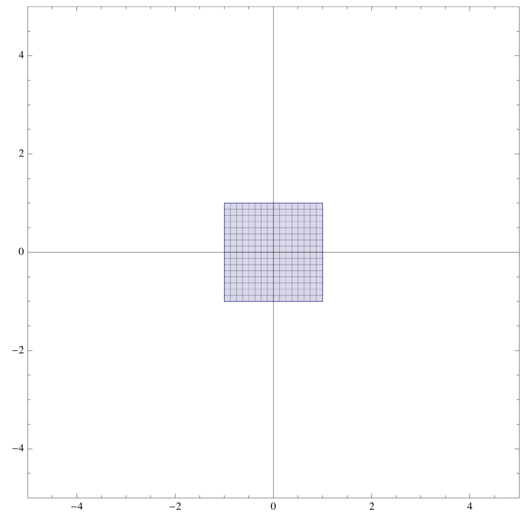
*Let's look at the the domain (the square in quadrant I) when transformed by the function* $f(z) = \text{a}z^2+\text{b}z+\text{c}$.

```
Manipulate[ParametricPlot[{Re[a * ((x + I * y) ^ 2) + b * (x + I * y) + c], Im[a * ((x + I * y) ^ 2) + b * (x + I * y) + c]},
  {x, 0, 4}, {y, 0, 4}, PlotRange → {{-200, 200}, {-200, 200}}], {a, -10, 10}, {b, -10, 10}, {c, -10, 10}]
```



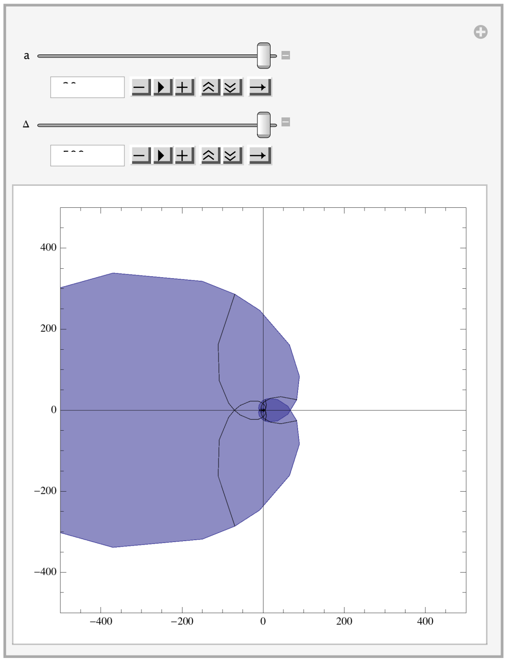
*Now, let's take a gander at some higher order polynomials. We return our focus to a square centered at the origin.*

```
f[z_] := z;
ParametricPlot[{Re[f[x + I * y]], Im[f[x + I * y]]}, {x, -1, 1}, {y, -1, 1}, PlotRange → {{-5, 5}, {-5, 5}}]
```



*Consider $f(z) = z^a$, for some a in [1, 20]. We can use the manipulate function to see what higher order terms do to the space in question.*

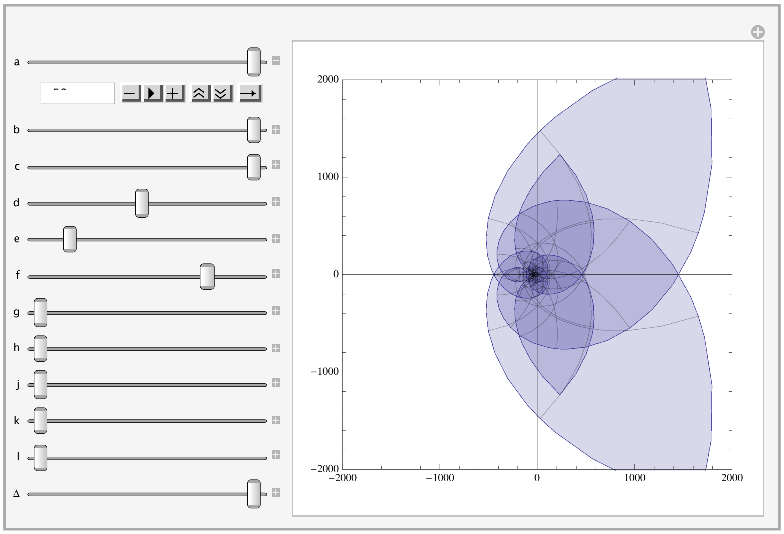
```
In[56]:= Manipulate[ParametricPlot[{Re[(x + I * y) ^ a], Im[(x + I * y) ^ a]}, {x, -1, 1}, {y, -1, 1}, PlotRange → {{-Δ, Δ}, {-Δ, Δ}}], {a, 1, 20}, {Δ, 1, 500}]
```



*Now, let's get a better understanding of the effects that individual terms have on the graphs of higher order polynomials. Let's consider the function:*

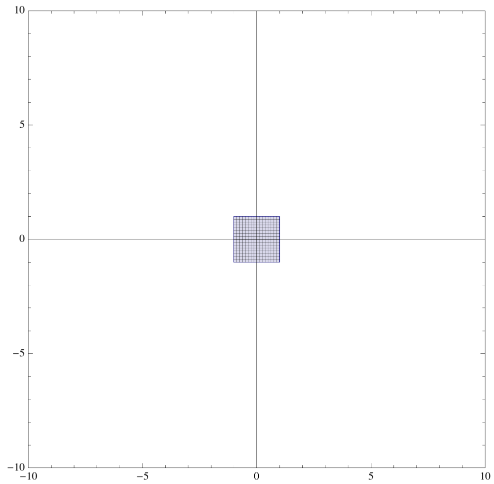
$f(z) = az^{10} + bz^9 + cz^8 + dz^7 + ez^6 + fz^5 + g z^4 + hz^3 + jz^2 + kz + l$ *Applying a manipulate function to f(z), we can investigate the effects of the coefficients on the graph. (Note: The below graph displays real-valued coefficients).*

```
In[57]:= Manipulate[
  ParametricPlot[{Re[a * ((x + I * y)^10) + b * ((x + I * y)^9) + c * ((x + I * y)^8) + d * ((x + I * y)^7) + e * ((x + I * y)^6) + f * ((x + I * y)^5) + g * ((x + I * y)^4) + h * ((x + I * y)^3) + j * ((x + I * y)^2) + k * (x + I * y) + l],
   Im[a * ((x + I * y)^10) + b * ((x + I * y)^9) + c * ((x + I * y)^8) + d * ((x + I * y)^7) + e * ((x + I * y)^6) + f * ((x + I * y)^5) + g * ((x + I * y)^4) + h * ((x + I * y)^3) + j * ((x + I * y)^2) + k * (x + I * y) + l]}, {x, -1, 1},
  {y, -1, 1}, PlotRange → {{-Δ, Δ}, {-Δ, Δ}}], {a, -50, 50}, {b, -50, 50}, {c, -50, 50}, {d, -50, 50}, {e, -50, 50}, {f, -50, 50}, {g, -50, 50}, {h, -50, 50}, {j, -50, 50}, {k, -50, 50}, {l, -50, 50}, {Δ, 0, 2000}]
```



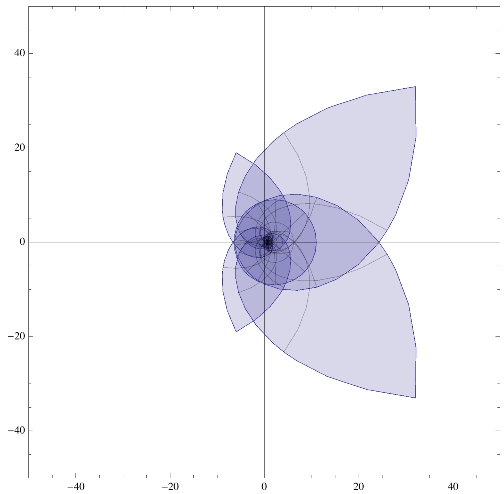
*Now, let's look at some specific cases. First, we will let all values be 0 excluding only k, to which we assign the value 1, the result is our initial square.*

```
DynamicModule[{a = 0, b = 0, c = 0, d = 0, e = 0, f = 0, g = 0, h = 0, j = 0, k = 1, l = 0},
 ParametricPlot[{Re[a (x + i y)^10 + b (x + i y)^9 + c (x + i y)^8 + d (x + i y)^7 + e (x + i y)^6 + f (x + i y)^5 + g (x + i y)^4 + h (x + i y)^3 + j (x + i y)^2 + k (x + i y) + l],
   Im[a (x + i y)^10 + b (x + i y)^9 + c (x + i y)^8 + d (x + i y)^7 + e (x + i y)^6 + f (x + i y)^5 + g (x + i y)^4 + h (x + i y)^3 + j (x + i y)^2 + k (x + i y) + l]},
  {x, -1, 1}, {y, -1, 1}, PlotRange → {{-10, 10}, {-10, 10}}]]
```



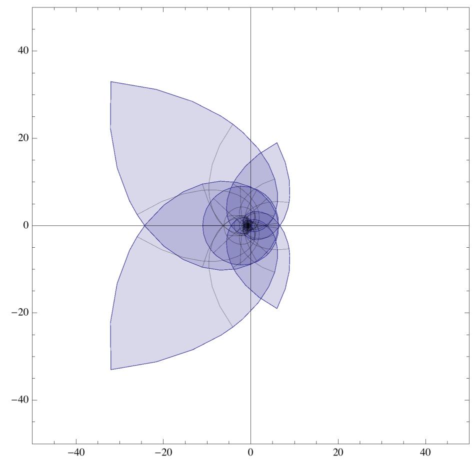
## *Here’s what we get when all coefficients are 1.*

```
DynamicModule[{a = 1, b = 1, c = 1, d = 1, e = 1, f = 1, g = 1, h = 1, j = 1, k = 1, l = 1},
 ParametricPlot[{Re[a (x + i y)^10 + b (x + i y)^9 + c (x + i y)^8 + d (x + i y)^7 + e (x + i y)^6 + f (x + i y)^5 + g (x + i y)^4 + h (x + i y)^3 + j (x + i y)^2 + k (x + i y) + l],
   Im[a (x + i y)^10 + b (x + i y)^9 + c (x + i y)^8 + d (x + i y)^7 + e (x + i y)^6 + f (x + i y)^5 + g (x + i y)^4 + h (x + i y)^3 + j (x + i y)^2 + k (x + i y) + l]},
  {x, -1, 1}, {y, -1, 1}, PlotRange → {{-50, 50}, {-50, 50}}]]
```



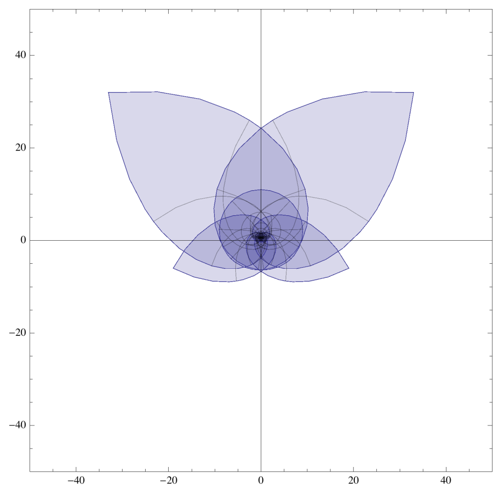
## *Here's what we get when all coefficients are -1.*

```
DynamicModule[{a = -1, b = -1, c = -1, d = -1, e = -1, f = -1, g = -1, h = -1, j = -1, k = -1, l = -1},
 ParametricPlot[{Re[a (x + i y)^10 + b (x + i y)^9 + c (x + i y)^8 + d (x + i y)^7 + e (x + i y)^6 + f (x + i y)^5 + g (x + i y)^4 + h (x + i y)^3 + j (x + i y)^2 + k (x + i y) + l],
   Im[a (x + i y)^10 + b (x + i y)^9 + c (x + i y)^8 + d (x + i y)^7 + e (x + i y)^6 + f (x + i y)^5 + g (x + i y)^4 + h (x + i y)^3 + j (x + i y)^2 + k (x + i y) + l]},
  {x, -1, 1}, {y, -1, 1}, PlotRange → {{-50, 50}, {-50, 50}}]]
```



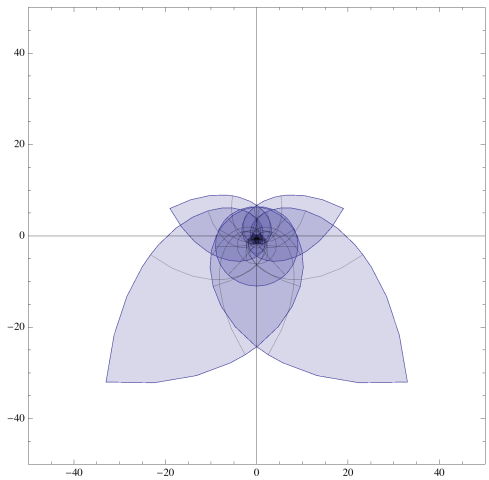
*Here's what we get when all coefficients are i.*

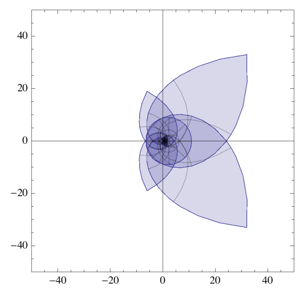
```
DynamicModule[{a = I, b = I, c = I, d = I, e = I, f = I, g = I, h = I, j = I, k = I, l = I},
 ParametricPlot[{Re[a (x + i y)^10 + b (x + i y)^9 + c (x + i y)^8 + d (x + i y)^7 + e (x + i y)^6 + f (x + i y)^5 + g (x + i y)^4 + h (x + i y)^3 + j (x + i y)^2 + k (x + i y) + l],
   Im[a (x + i y)^10 + b (x + i y)^9 + c (x + i y)^8 + d (x + i y)^7 + e (x + i y)^6 + f (x + i y)^5 + g (x + i y)^4 + h (x + i y)^3 + j (x + i y)^2 + k (x + i y) + l]},
  {x, -1, 1}, {y, -1, 1}, PlotRange → {{-50, 50}, {-50, 50}}]]
```

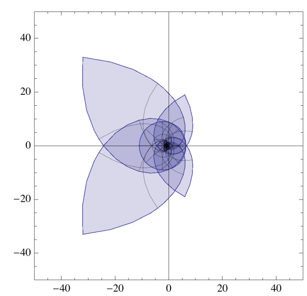


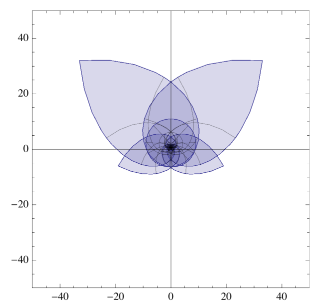
## *Here's what we get when all coefficients are -i*

```
DynamicModule[{a = -I, b = -I, c = -I, d = -I, e = -I, f = -I, g = -I, h = -I, j = -I, k = -I, l = -I},
 ParametricPlot[{Re[a (x + i y)^10 + b (x + i y)^9 + c (x + i y)^8 + d (x + i y)^7 + e (x + i y)^6 + f (x + i y)^5 + g (x + i y)^4 + h (x + i y)^3 + j (x + i y)^2 + k (x + i y) + l],
   Im[a (x + i y)^10 + b (x + i y)^9 + c (x + i y)^8 + d (x + i y)^7 + e (x + i y)^6 + f (x + i y)^5 + g (x + i y)^4 + h (x + i y)^3 + j (x + i y)^2 + k (x + i y) + l]},
  {x, -1, 1}, {y, -1, 1}, PlotRange → {{-50, 50}, {-50, 50}}]]
```




40
20
0
−20
−40
−40 −20 0 20 40


40
20
0
−20
−40
−40 −20 0 20 40


40
20
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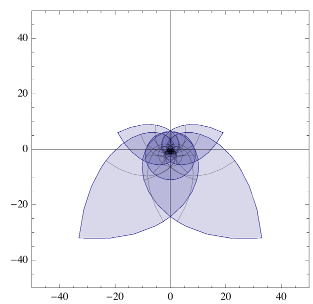
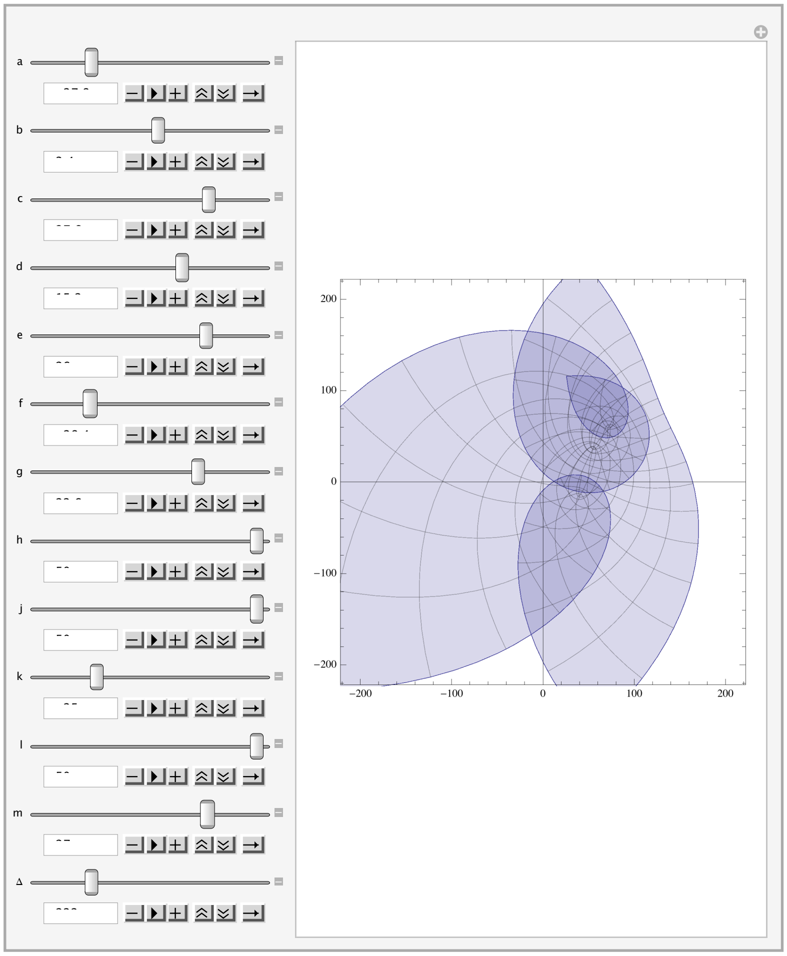
−20
−40
−40 −20 0 20 40


40
20
0
−20
−40
−40 −20 0 20 40

*Now, consider* $f(z) = (a + bi)z^5 + (c + di)z^4 + (e + fi)z^3 + (g + hi)z^2 + (j + ki)z + (l + mi)$ *in the image below we can examine the effects of both the real and imaginary parts of each coefficint on the square centered at the origin with side length 2.*

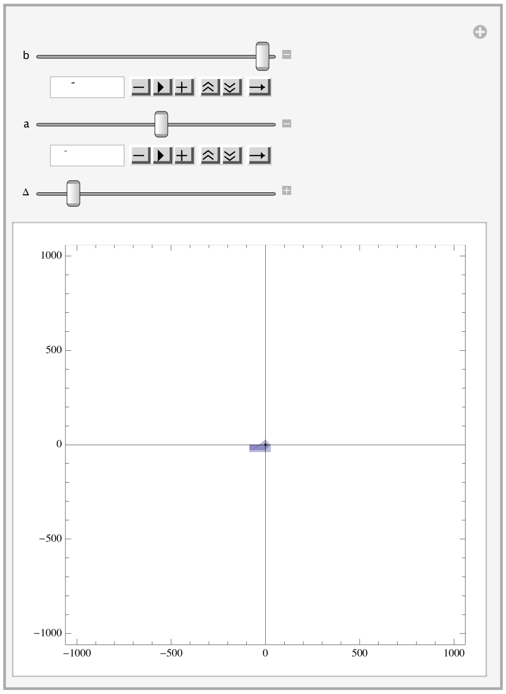
```
Manipulate[ParametricPlot[{Re[(a + b * I) * ((x + I * y)^5) + (c + d * I) * ((x + I * y)^4) + (e + f * I) * ((x + I * y)^3) + (g + h * I) * ((x + I * y)^2) + (j + k * I) * (x + I * y) + (l + m * I)],
   Im[(a + b * I) * ((x + I * y)^5) + (c + d * I) * ((x + I * y)^4) + (e + f * I) * ((x + I * y)^3) + (g + h * I) * ((x + I * y)^2) + (j + k * I) * (x + I * y) + (l + m * I)]}, {x, -1, 1}, {y, -1, 1}, PlotRange → {{-Δ, Δ}, {-Δ, Δ}}],
 {a, -50, 50}, {b, -50, 50}, {c, -50, 50}, {d, -50, 50}, {e, -50, 50}, {f, -50, 50}, {g, -50, 50}, {h, -50, 50}, {j, -50, 50}, {k, -50, 50}, {l, -50, 50}, {m, -50, 50}, {Δ, 0, 1000}]
```



*We can also consider a function of the form $f(z) = az^b$, where b is in [-15, -1] and a is in [-25, 25].*

*Let's begin by looking at $f(z)=1/z$.*

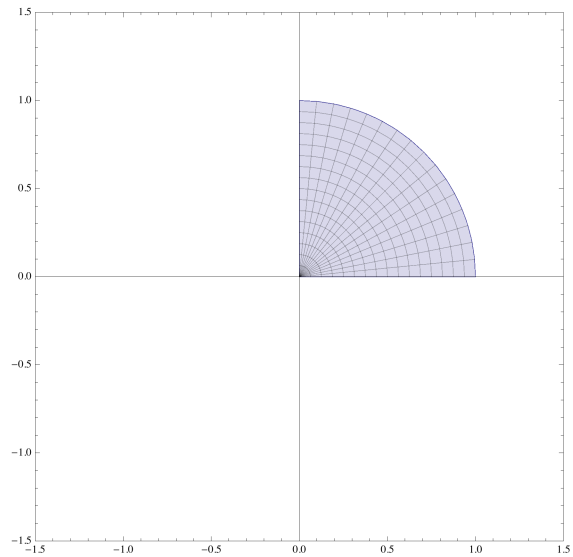
In[58]:=
```
Manipulate[ParametricPlot[{Re[a * ((x + I * y) ^ b)], Im[a * ((x + I * y) ^ b)]}, {x, -1, 1}, {y, -1, 1}, PlotRange → {{-Δ, Δ}, {-Δ, Δ}}],
  {b, -15, -1}, {a, -25, 25}, {Δ, 0, 10 000}]
```



*Now, we may want to consider a function over a disk, or even a fraction of a disk.*

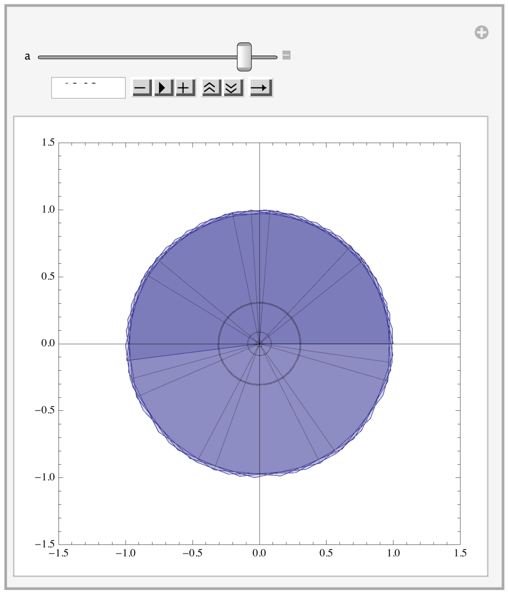
*Below, we have one quarter of the unit disk.*

```
ParametricPlot[{r * Cos[t], r * Sin[t]}, {r, 0, 1}, {t, 0, Pi / 2}, PlotRange → {-1.5, 1.5}]
```



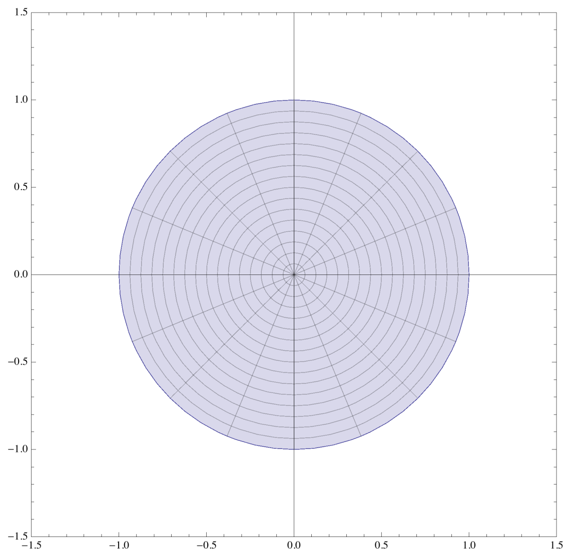
*Let's raise this quarter-disk to the ath power, with a in (0, 20].*

```
Manipulate[ParametricPlot[{Re[(r^a) * Exp[a * I * t]], Im[(r^a) * Exp[a * I * t]]}, {r, 0, 1
  }, {t, 0, Pi / 2}, PlotRange → {-1.5, 1.5}], {a, .0001, 20}]
```



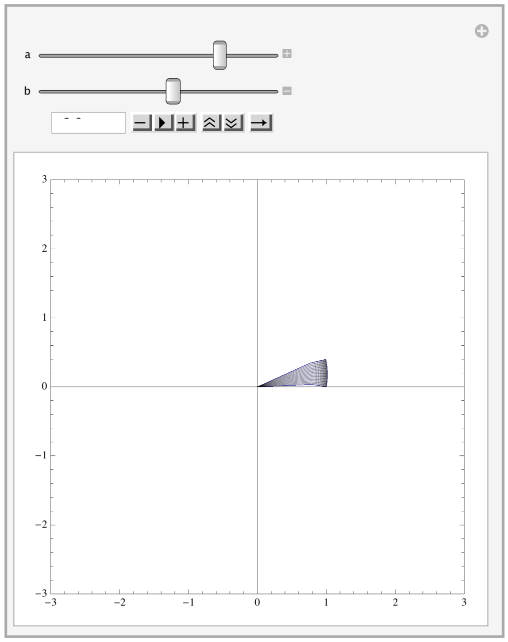
*Now, let's consider the entire unit disk.*

```
ParametricPlot[{r * Cos[t], r * Sin[t]}, {r, 0, 1}, {t, 0, 2 Pi}, PlotRange → {-1.5, 1.5}]
```



*Below is a graph of the a +bi^th root of z.*

```
Manipulate[ParametricPlot[{Re[(r^(1/(a+b*I)))*Exp[(1/(a+b*I))*I*t]], Im[(r^(1/(a+b*I)))*Exp[(1/(a+b*I))*I*t]]}, {r, 0, 1
  }, {t, 0, 2 Pi}, PlotRange → {-3, 3}], {a, 1, 20}, {b, -20, 20}]
```



# Simple Algebraic Functions II

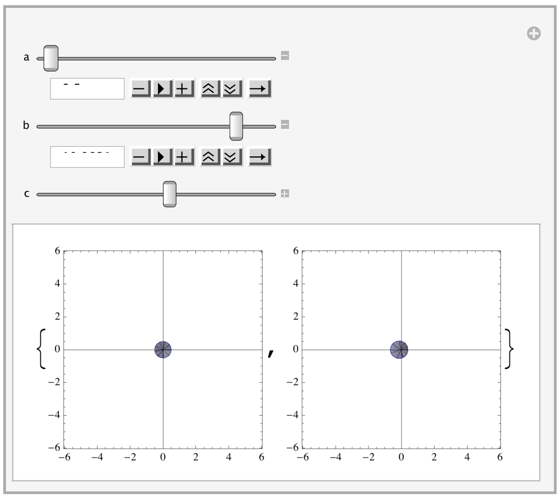
## Mobius Transformations of Circles and Lines

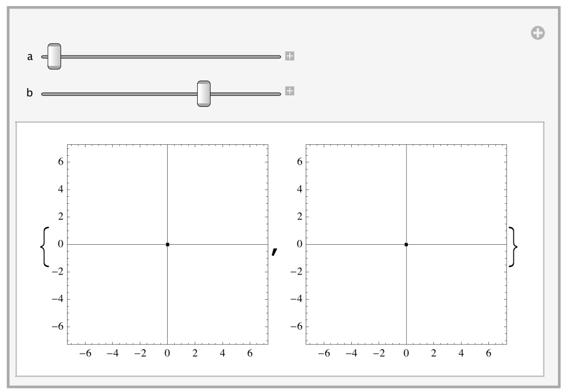
*josh hussey*

*Consider the function* $g(z)=(2z)/(z+2)$.

```
g[z_Complex] := (2 z) / (z + 2);
```

```
In[37]:= Manipulate[{ParametricPlot[{r * Cos[t], r * Sin[t]}, {r, 0, a}, {t, 0, b}, PlotRange → {{-c, c}, {-c, c}}],
           ParametricPlot[{Re[g[r * Exp[I * t]]], Im[g[r * Exp[I * t]]]}, {r, 0, a}, {t, 0, b}, PlotRange → {{-c, c}, {-c, c}}]}, {a, 0.5, 10}, {b, 0.001, 4 * Pi}, {c, 1, 10}]
         Manipulate[{ParametricPlot[{Re[x + I * y], Im[x + I * y]}, {x, -a, a}, {y, -a, a}, PlotRange → {{-b, b}, {-b, b}}],
           ParametricPlot[{Re[g[x + I * y]], Im[g[x + I * y]]}, {x, -a, a}, {y, -a, a}, PlotRange → {{-b, b}, {-b, b}}]}, {a, 0.1, 4}, {b, 1, 10}]
```



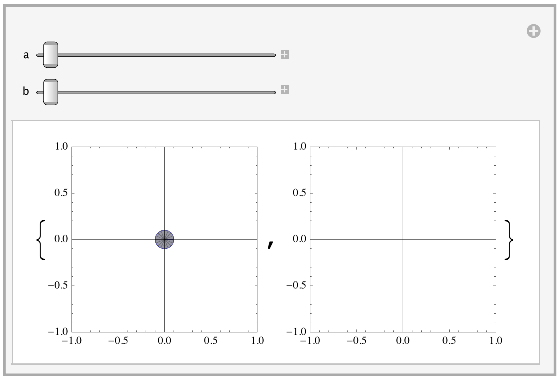



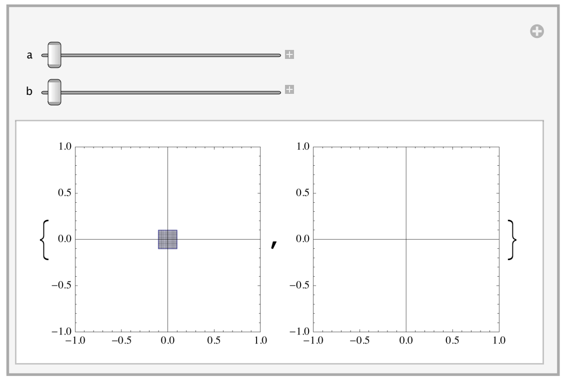
*Consider the function* $h(z)=(2z+3)/(3z+2)$.

```
In[27]:= h[z_Complex] := (2 z + 3) / (3 z + 2);

Manipulate[{ParametricPlot[{r*Cos[t], r*Sin[t]}, {r, 0, a}, {t, 0, 2*Pi}, PlotRange → {{-b, b}, {-b, b}}],
    ParametricPlot[{Re[h[r*Exp[I*t]]], Im[h[r*Exp[I*t]]]}, {r, 0, a}, {t, 0, 2 Pi}, PlotRange → {{-b, b}, {-b, b}}]}, {a, 0.1, 4},
  {b, 1, 10}] Manipulate[{ParametricPlot[{Re[x+I*y], Im[x+I*y]}, {x, -a, a}, {y, -a, a}, PlotRange → {{-b, b}, {-b, b}}],
    ParametricPlot[{Re[h[x+I*y]], Im[h[x+I*y]]}, {x, -a, a}, {y, -a, a}, PlotRange → {{-b, b}, {-b, b}}]}, {a, 0.1, 4}, {b, 1, 10}]
```

Out[28]=



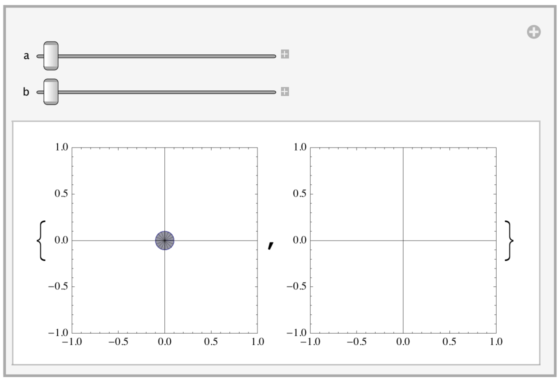



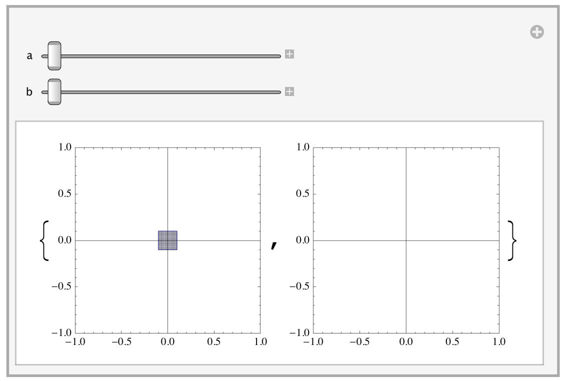
## *Consider the function* $k(z) = (3z+2)/(2z-1)$.

In[21]:=
```
k[z_Complex] := (3 z + 2) / (2 z - 1);
```

In[29]:=
```
Manipulate[{ParametricPlot[{r*Cos[t], r*Sin[t]}, {r, 0, a}, {t, 0, 2*Pi}, PlotRange → {{-b, b}, {-b, b}}],
   ParametricPlot[{Re[k[r*Exp[I*t]]], Im[k[r*Exp[I*t]]]}, {r, 0, a}, {t, 0, 2 Pi}, PlotRange → {{-b, b}, {-b, b}}]}, {a, 0.1, 4}, {b, 1, 10}]
 Manipulate[{ParametricPlot[{Re[x+I*y], Im[x+I*y]}, {x, -a, a}, {y, -a, a}, PlotRange → {{-b, b}, {-b, b}}],
   ParametricPlot[{Re[k[x+I*y]], Im[k[x+I*y]]}, {x, -a, a}, {y, -a, a}, PlotRange → {{-b, b}, {-b, b}}]}, {a, 0.1, 4}, {b, 1, 10}]
```

Out[29]=



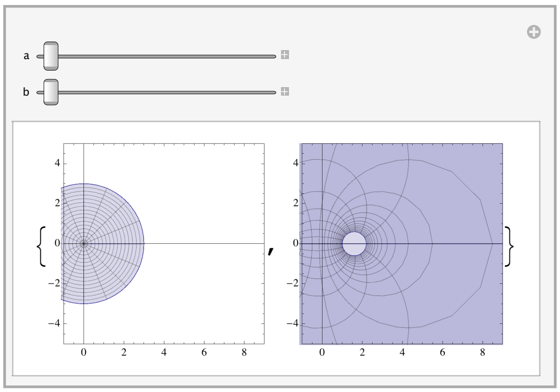



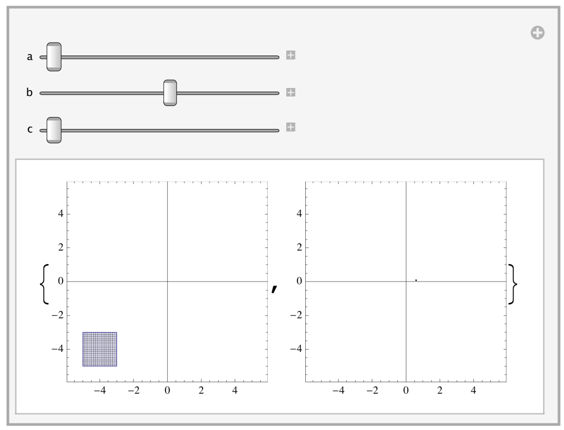
# Mapping away from the origin

*Consider the function* $h(z) = (2z+3)/(3z+2)$.

```
In[30]:= h[z_Complex] := (2 z + 3) / (3 z + 2);

In[39]:= Manipulate[{ParametricPlot[{r*Cos[t] + a, r*Sin[t] + b}, {r, 0, 3}, {t, 0, 2*Pi}, PlotRange → {{-1, 9}, {-5, 5}}], ParametricPlot[
    {Re[k[(a + b*I) + r*Exp[I*t]]], Im[k[(a + b*I) + r*Exp[I*t]]]}, {r, 0, 3}, {t, 0, 2*Pi}, PlotRange → {{-1, 9}, {-5, 5}}]}, {a, 0, 5}, {b, 0, 5}]
Manipulate[{ParametricPlot[{Re[x + I*y], Im[x + I*y]}, {x, a - 1, a + 1}, {y, c - 1, c + 1}, PlotRange → {{-b, b}, {-b, b}}],
  ParametricPlot[{Re[h[x + I*y]], Im[h[x + I*y]]}, {x, a - 1, a + 1}, {y, c - 1, c + 1}, PlotRange → {{-b, b}, {-b, b}}]}, {a, -4, 4}, {b, 1, 10}, {c, -4, 4}]
```



## *Consider the function* $m(z)=(2z)/(z-4)$.

In[33]:=
```
m[z_Complex] := (2 z) / (z - 4);
```

In[35]:=
```
Manipulate[{ParametricPlot[{r * Cos[t] + a, r * Sin[t] + b}, {r, 0, 3}, {t, 0, 2 * Pi}, PlotRange → {{-1, 9}, {-5, 5}}], ParametricPlot[
    {Re[m[(a + b * I) + r * Exp[I * t]]], Im[m[(a + b * I) + r * Exp[I * t]]]}, {r, 0, 3}, {t, 0, 2 * Pi}, PlotRange → {{-1, 9}, {-5, 5}}]}, {a, 0, 5}, {b, 0, 5}]
Manipulate[{ParametricPlot[{Re[x + I * y], Im[x + I * y]}, {x, a - 1, a + 1}, {y, c - 1, c + 1}, PlotRange → {{-b, b}, {-b, b}}],
  ParametricPlot[{Re[m[x + I * y]], Im[m[x + I * y]]}, {x, a - 1, a + 1}, {y, c - 1, c + 1}, PlotRange → {{-b, b}, {-b, b}}]}, {a, -4, 4}, {b, 1, 10}, {c, -4, 4}]
```