Extension of Twisted Pluricanonical Sections with Plurisubharmonic Weight and Invariance of Semipositively Twisted Plurigenera for Manifolds Not Necessarily of General Type

Yum-Tong Siu*

Department of Mathematics, Harvard University, Cambridge, MA, USA
E-mail address: siu@math.harvard.edu

Abstract Let X be a holomorphic family of compact complex projective algebraic manifolds with fibers X_t over the open unit 1-disk Δ. Let K_{X_t} and K_X be respectively the canonical line bundles of X_t and X. We prove that, if L is a holomorphic line bundle over X with a (possibly singular) metric $e^{-\varphi}$ of semipositive curvature current on X such that $e^{-\varphi}|_{X_0}$ is locally integrable on X_0, then for any positive integer m, any $s \in \Gamma(m K_{X_0} + L)$ with $|s|^2 e^{-\varphi}$ locally bounded on X_0 can be extended to an element of $\Gamma(X, m K_X + L)$. In particular, $\dim \Gamma(X_t, m K_{X_t} + L)$ is independent of t for φ smooth. The case of trivial L gives the deformational invariance of the plurigenera. The method of proof uses an appropriately formulated effective version, with estimates, of the argument in the author's earlier paper on the invariance of plurigenera for general type. A delicate point of the estimates involves the use of metrics as singular as possible for $p K_{X_0} + a_p L$ on X_0 to make the dimension of the space of L^2 holomorphic sections over X_0 bounded independently of p, where a_p is the smallest integer $\geq \frac{p-1}{m}$. These metrics are constructed from s. More conventional metrics, independent of s, such as generalized Bergman kernels are not singular enough for the estimates.

Table of Contents

Introduction 224
1 Review of Existing Argument for Invariance of Plurigenera . . 228
2 Global Generation of Multiplier Ideal Sheaves with Estimates . 234
3 Extension Theorems of Ohsawa-Takegoshi Type from Usual Basic Estimates with Two Weight Functions 241
4 Induction Argument with Estimates 248
5 Effective Version of the Process of Taking Powers and Roots of Sections 256
6 Remarks on the Approach of Generalized Bergman Kernels . 264
References 276

* Partially supported by a grant from the National Science Foundation.
2000 *Mathematics Subject Classification*: 32G05, 32C35.

0 Introduction

For a holomorphic family of compact complex projective algebraic manifolds, the plurigenera of a fiber are conjectured to be independent of the fiber. The case when the fibers are of general type was proved in [Siu98]. Generalizations were made by Kawamata [Kaw99] and Nakayama [Nak98] and, in addition, they recast the transcendentally formulated methods in [Siu98] into a completely algebraic geometric setting. Recently Tsuji put on the web a preprint on the deformational invariance of the plurigenera for manifolds not necessarily of general type [Tsu01], in which, in addition to the techniques of [Siu98], he uses his theory of analytic Zariski decomposition and generalized Bergman kernels. Tsuji's approach of generalized Bergman kernels naturally reduces the problem of the deformational invariance of the plurigenera to a growth estimate on the generalized Bergman kernels. This crucial estimate is still lacking. We will explain it briefly in (1.5.2) and discuss it in more details in §6 at the end of this paper.

In this paper we use an appropriately formulated effective version, with estimates, of the argument in [Siu98] to prove the following extension theorem (Theorem 0.1) which implies the invariance of semipositively twisted plurigenera (Corollary 0.2). The results in this paper can be regarded as generalizations of the deformational invariance of plurigenera.

Theorem 0.1. *Let $\pi : X \to \Delta$ be a holomorphic family of compact complex projective algebraic manifolds over the open unit 1-disk $\Delta = \{z \in \mathbf{C} \,\big|\, |z| < 1\}$. For $t \in \Delta$, let $X_t = \pi^{-1}(t)$ and K_t be the canonical line bundle of X_t. Let L be a holomorphic line bundle over X with a (possibly singular) metric $e^{-\varphi}$ whose curvature current $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ is semi-positive on X such that $e^{-\varphi}|_{X_0}$ is locally integrable on X_0. Let m be any positive integer. Then any element $s \in \Gamma(X_0, m K_{X_0} + L)$ with $|s|^2 e^{-\varphi}$ locally bounded on X_0 can be extended to an element $\tilde{s} \in \Gamma(X, m K_X + L)$ in the sense that $\tilde{s}|_{X_0} = s \wedge \pi^*(dt)$, where K_X is the canonical line bundle of X.*

Corollary 0.2. *Let $\pi : X \to \Delta$ be a holomorphic family of compact complex projective algebraic manifolds over the open unit 1-disk Δ with fiber X_t. Let L be a holomorphic line bundle over X with a smooth metric $e^{-\varphi}$ whose curvature form $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ is semi-positive on X. Let m be any positive integer. Then the complex dimension of $\Gamma(X_t, m K_{X_t} + L)$ is independent of t for $t \in \Delta$.*

Corollary 0.3. *Let $\pi : X \to \Delta$ be a holomorphic family of compact complex projective algebraic manifolds over the open unit 1-disk Δ with fiber X_t. Let m be any positive integer. Then the complex dimension of $\Gamma(X_t, m K_{X_t})$ is independent of t for $t \in \Delta$.*

So far as the logical framework is concerned, the method in this paper simply follows that of [Siu98], the only difference being the monitoring of estimates in this paper. Some of the estimates are quite delicate. The estimates depend on a choice, at the beginning, of singular metrics for the twisted pluricanonical line bundles of the initial fiber. In contrast to the metrics chosen in [Siu98], for the effective argument in this paper metrics as singular as possible have to be chosen for the twisted pluricanonical bundles on X_0, as long as the relevant sections on X_0 remain L^2. The use of usual abstractly-defined general metrics for the twisted pluricanonical line bundle of the initial fiber, such as generalized Bergman kernels on X_0, would contribute an uncontrollable factor in the final estimate (see (1.5.2) below). If one uses generalized Bergman kernels on X as for example in [Tsu01], there are difficulties with norm changes similar to the norm-change problems encountered in the papers of Nash [Nas54], Moser [Mos61], and Grauert [Gra60]. The norm changes in our case mean the need to shrink the domain for supremum estimates in each of an infinite number of steps. Our situation here is different from those which could be handled by the norm-change techniques of Nash [Nas54], Moser [Mos61], and Grauert [Gra60] (see §6, in particular (6.4)). The difficulty here with the use of generalized Bergman kernels cannot be overcome.

The method of this paper should be applicable to give the deformational invariance of the plurigenera twisted by a numerically effective line bundle, i.e., Corollary 0.2 in which L is assumed to be numerically effective instead of having a smooth metric with semi-positive curvature. In this paper we give the easier semi-positive case to avoid one more layer of complication in our estimates. For the numerically effective case of Corollary 0.2 we have to use a sufficiently ample line bundle A so that $mL + A$ is very ample for any positive integer m and to keep track of the limiting behavior of the m-th root of some canonically defined smooth metric of $mL + A$. In order not to be distracted from the main arguments of this paper by another lengthy peripheral limiting process, we will leave the numerically effective case of Corollary 0.2 to another occasion.

Unlike the case of general type (see (1.5.1)), for the proof in this paper a genuine limiting process is being used. It is not clear whether this proof can be translated into a completely algebraic geometric setting. Of course, instead of L being semi-positive, the algebraic geometric formulation of Corollary 0.2 would have to assume that L is generated by global holomorphic sections on X. The difficulty of translating the limiting process into an algebraic geometric setting occurs already for the case of trivial L (Corollary 0.3).

A more general formulation of the deformational invariance of plurigenera is for the setting of a holomorphic family of compact Kähler manifolds. Such a setting is completely beyond the reach of the methods of [Siu98] and this paper. For that setting, the only known approach is that of Levine [Lev83] in which he uses Hodge theory to extend a pluricanonical section from the initial fiber to its finite neighborhood of second order over a double point of the base. The conjecture for the Kähler case is the following.

Conjecture 0.4. (Conjecture on Deformational Invariance of Plurigenera for the Kähler Case) Let $\pi : X \to \Delta$ be a holomorphic family of compact Kähler manifolds over the open unit 1-disk Δ with fiber X_t. Then for any positive integer m the complex dimension of $\Gamma\left(X_t, m K_{X_t}\right)$ is independent of t for $t \in \Delta$.

The case of twisting by a numerically effective line bundle can be formulated for the Kähler case in the form of a conjecture as follows.

Conjecture 0.5. (Conjecture on Deformational Invariance of Plurigenera for the Kaehler Case with Twisting by Numerically Effective Line Bundles) Let $\pi : X \to \Delta$ be a holomorphic family of compact Kähler manifolds over the open unit 1-disk Δ with fiber X_t. Let L be a holomorphic line bundle on X which is numerically effective in the sense that, for any strictly negative $(1,1)$-form ω on X and any compact subset W of X, there exists a smooth metric for L whose curvature form is $\geq \omega$ on W. Then for any positive integer m the complex dimension of $\Gamma\left(X_t, (L|X_t) + m K_{X_t}\right)$ is independent of t for $t \in \Delta$.

Since the complex dimension of $\Gamma\left(X_t, (L|X_t) + m K_{X_t}\right)$ is always upper semi-continuous as a function of t, to prove its independence on t it suffices to show that every element s of $\Gamma\left(X_{t_0}, (L|X_{t_0}) + m K_{X_{t_0}}\right)$ can be extended to an element $\tilde{s}$ of $\Gamma\left(X, L + m K_X\right)$ in the sense that $s \wedge \pi^*(dt) = \tilde{s}|X_{t_0}$. Thus Corollaries 0.2 and 0.3 follow readily from Theorem 0.1. Most of the rest of this paper is devoted to the proof of Theorem 0.1. For notational simplicity, in referring to the extension of a twisted pluricanonical section on X_0 to X, we identify K_{X_0} with $K_X|X_0$ by the map which is defined by the wedge product with $\pi^*(dt)$.

In the proof of the deformational invariance of the plurigenera in [Siu98], there are the following four major ingredients.

(i) Global generation of the multiplier ideal sheaf after twisting by a sufficiently ample line bundle (by the method of Skoda [Sko72]).

(ii) Extension of sections from the initial fiber which are L^2 with respect to a singular metric on the family with semi-positive curvature current (by the method of Ohsawa-Takegoshi [OT87])).

(iii) An induction argument on m, which uses the two preceding ingredients and regards a section of the m-canonical bundle as a top-degree form with values in the $(m-1)$-canonical bundle, in order to construct singular metrics on the $(m-1)$-canonical bundle on the family and extend sections of the m-canonical bundle from the initial fiber.

(iv) The process of raising a section on X_0 to a high power ℓ and later taking the ℓ-th root after extending to X its product with the canonical section s_D of some fixed effective line bundle D on X. For the case of general type, when $m K_X$ is written as the sum of D and a sufficiently ample line bundle A for some large m, multiplication by s_D provides us with the twisting by A which is needed for the global generation of the multiplier ideal sheaf in the first ingredient. This technique of taking powers, extending the product with s_D, and taking roots is to eliminate the effect of multiplication by s_D, or equivalently to eliminate the effect of multiplication of a section of A.

The rough and naive motivation underlying the idea of the proof in [Siu98] is that, if one could write an element $s^{(m)}$ of $\Gamma(X_0, m K_{X_0})$ as a sum of terms, each of which is the product of an element $s^{(1)}$ of $\Gamma(X_0, K_{X_0})$ and an element $s^{(m-1)}$ of $\Gamma(X_0, (m-1) K_{X_0})$, then one can extend $s^{(m)}$ to an element of $\Gamma(X, m K_X)$ by induction on m. Of course, in general it is clearly impossible to so express $s^{(m)}$ as a sum of such products. However, one could successfully implement a very much modified form of this rough and naive motivation by using the above four ingredients in the case of general type. The actual proof in [Siu98] by induction on m, which uses the modified form of the argument, appears very different from this rough and naive motivation and is not recognizable as related in any way to it, but it was in fact from such a rough and naive motivation that the actual proof in [Siu98] evolved. The modification is to require only $s^{(1)}$ to be just a local holomorphic function and twist each $m K_{X_0}$ by the same sufficiently ample line bundle A on X so that locally the absolute value of $s^{(m)}$ can be estimated by the sum of absolute values of elements of $\Gamma(X_0, (m-1) K_{X_0} + A)$. Such an estimate enables us to inductively get the extension and then to use the technique of taking powers and roots to get rid of the twisting by A.

The assumption of general type is used in the fourth ingredient listed above. For manifolds not necessarily of general type, in the fourth ingredient multiplication by s_D is replaced by multiplication by a section s_A of A and one has to pass to limit as the integer, which is both the power and the root order, goes to infinity, in order to eliminate the effect of multiplication by s_A. To carry out the limiting argument, one needs to have a good control of the estimates, which necessitates the use of an effective version of the argument of [Siu98].

Besides proving Theorem 0.1, the paper includes in §3 a simple approach to extension theorems which uses the usual basic estimates and two weight functions (see §3) and explains in §6 the delicate point of the estimates, especially why there are difficulties with the simple natural approach of using generalized Bergman kernels.

Every metric of holomorphic line bundles in this paper is allowed to be singular, but, so far as regularity is concerned, its curvature current is assumed to be no worse than the sum of a smooth form and a semi-positive current. For a metric $e^{-\varphi}$ of a holomorphic line bundle, we use the notation $\mathcal{I}_\varphi$ to denote the *multiplier ideal sheaf* which consists of all holomorphic function germs f such that $|f|^2 e^{-\varphi}$ is locally integrable. Multiplier ideal sheaves were introduced by Nadel [Nad89] for his vanishing theorem. Nadel's vanishing theorem, in the algebraic setting and in the special case of algebraically-definable singular metrics, is reducible to the vanishing theorem of Kawamata-Viehweg [Kaw82], [Vie82].

The structure of this paper is as follows. In §1 we review the argument of [Siu98] for the deformational invariance of plurigenera for the case of general type. The purpose of the review is to first present the logical framework of the argument without the estimates so that it can serve as a guide for following the later complicated details of the estimates in the effective version. Sections 2, 3, 4, and 5 with detailed estimates correspond respectively to the first, second, third, and fourth ingredients listed above. Section 6 contains some remarks on the difficulties of using generalized Bergman kernels for the effective arguments for the problem of the deformational invariance of plurigenera for manifolds not necessarily of general type.

1 Review of Existing Argument for Invariance of Plurigenera

The first two ingredients of the argument of [Siu98] for the deformational invariance of plurigenera for general type uses the global generation of multiplier ideal sheaves (by the method of Skoda [Sko72]) and an extension theorem of Ohsawa-Takegoshi type [OT87]. Let us first recall the precise statements of these two results.

1.1 *Global Generation of Multiplier Ideal Sheaves* [Siu98, p.664, Prop. 1]

Let L be a holomorphic line bundle over an n-dimensional compact complex manifold Y with a Hermitian metric which is locally of the form $e^{-\xi}$ with ξ plurisubharmonic. Let $\mathcal{I}_\xi$ be the multiplier ideal sheaf of the Hermitian metric $e^{-\xi}$. Let E be an ample holomorphic line bundle over Y such that for every point P of Y there are a finite number of elements of $\Gamma(Y, E)$ which all vanish to order at least $n+1$ at P and which do not simultaneously vanish outside P. Then $\Gamma(Y, \mathcal{I}_\xi \otimes (L+E+K_Y))$ generates $\mathcal{I}_\xi \otimes (L+E+K_Y)$ at every point of Y.

Theorem 1.2. *(Extension Theorem of Ohsawa-Takegoshi Type)* [Siu98, p.666, Prop. 2]. *Let $\gamma : Y \to \Delta$ be a projective algebraic family of compact complex manifolds parametrized by the open unit 1-disk Δ. Let $Y_0 = \gamma^{-1}(0)$ and let n be the complex dimension of Y_0. Let L be a holomorphic line bundle with a Hermitian metric which locally is represented by $e^{-\chi}$ such that $\sqrt{-1}\partial\bar\partial\chi \geq \omega$ in the sense of currents for some smooth positive $(1,1)$-form ω on Y. Let $0 < r < 1$ and $\Delta_r = \{ t \in \Delta \mid |t| < r \}$. Then there exists a positive constant A_r with the following property. For any holomorphic L-valued n-form f on Y_0 with*

$$\int_{Y_0} |f|^2 e^{-\chi} < \infty ,$$

there exists a holomorphic L-valued $(n+1)$*-form* $\tilde{f}$ *on* $\gamma^{-1}(\Delta_r)$ *such that* $\tilde{f}|_{Y_0} = f \wedge \gamma^*(dt)$ *at points of* Y_0 *and*

$$\int_Y |\tilde{f}|^2 e^{-\chi} \leq A_r \int_{Y_0} |f|^2 e^{-\chi}\,.$$

Note that Theorem 2.1 (respectively Theorem 3.1) below is an effective version of Theorem 1.1 (respectively Theorem 1.2).

1.3 *Induction Argument in Axiomatic Formulation*

To prepare for later adaptation to the effective version, we now formulate axiomatically the induction argument of [Siu98]. The induction argument is the precise formulation of the rough and naive motivation explained in the Introduction. The main idea of the induction argument is to start with suitable metrics $e^{-\varphi_p}$ for $p\,K_{X_0} + D$ on X_0 with semi-positive curvature current on X_0. The induction step is to use a sufficiently ample line bundle A to extend L^2 sections of $m\,K_{X_0} + D + A$ with respect to $e^{-\varphi_{m-1}}$ from X_0 to X and to produce a metric $e^{-\chi_m}$ of $m\,K_X + D + A$ on X with semi-positive curvature current so that L^2 sections of $(m+1)\,K_{X_0} + D + A$ on X_0 with respect to $e^{-\varphi_m}$ are $\tilde{L}^2$ with respect to $e^{-\chi_m}|_{X_0}$.

Fix a positive integer m_0 and a holomorphic line bundle D over X. Assume that A is a sufficiently ample line bundle on X so that for any point P of X_0 there are a finite number of elements of $\Gamma(X_0, A)$ which all vanish to order at least $n+1$ at P and which do not simultaneously vanish outside P. Thus the theorem 1.1 of global generation of multiplier ideal sheaf on X_0 holds with twisting by $A|X_0$.

Suppose we have

(i) a metric $e^{-\varphi_p}$ for $p\,K_{X_0} + D$ over X_0 with semi-positive curvature current on X_0 for $0 \leq p < m_0$, and
(ii) a metric $e^{-\varphi_D}$ of D over X, with semi-positive curvature current on X,

such that

(a) $\mathcal{I}_{\varphi_p} \subset \mathcal{I}_{\varphi_{p-1}}$ on X_0 for $0 < p < m_0$,
(b) $\mathcal{I}_{\varphi_D|X_0}$ agrees with $\mathcal{I}_{\varphi_0}$ on X_0.

Proposition 1.4. *If a holomorphic family* $\pi : X \to \Delta$ *of compact complex projective algebraic manifolds satisfies Assumptions (i), (ii) and Conditions (a), (b) of (1.3), then any element* $f \in \Gamma\,(X_0, m_0\,K_{X_0} + D + A)$ *which locally belongs to* $\mathcal{I}_{\varphi_{m_0-1}}$ *can be extended to an element* $\tilde{f} \in \Gamma\,(X, m_0\,K_X + D + A)$.

Proof. By (1.1) and Condition (a) of (1.3), we can cover X_0 by a finite number of open subsets U_λ $(1 \leq \lambda \leq \Lambda)$ so that, for some nowhere zero element $\xi_\lambda \in \Gamma\,(U_\lambda, -K_{X_0})$, we can write

$$\xi_\lambda\, f|U_\lambda = \sum_{k=1}^{N_{m_0-1}} b_k^{(m_0-1,\lambda)} s_k^{(m_0-1)}\,, \qquad (1.4.1)$$

where $b_k^{(m_0-1,\lambda)}$ is a holomorphic function on U_λ and

$$s_k^{(m_0-1)} \in \Gamma\left(X_0, \mathcal{I}_{\varphi_{m_0-2}}\left((m_0-1)\,K_{X_0}+D+A\right)\right) .$$

Moreover, we can assume that the open subsets U_λ are chosen small enough so that, for $1 \le p \le m_0-2$, inductively by (1.1) we can write

$$\xi_\lambda\, s_j^{(p+1)}\big|U_\lambda = \sum_{k=1}^{N_p} b_{j,k}^{(p,\lambda)} s_k^{(p)} , \tag{1.4.2}$$

where $b_{j,k}^{(p,\lambda)}$ is a holomorphic function on U_λ and

$$s_k^{(p)} \in \Gamma\left(X_0, \mathcal{I}_{\varphi_{p-1}}\left(p\,K_{X_0}+D+A\right)\right) .$$

For this we have used already Condition (a) and have also used Assumption (i) of (1.3) in order to apply (1.1) on the global generation of multiplier ideal sheaves.

We are going to show, by induction on $1 \le p < m_0$, the following claim.

(1.4.3) *Claim*

$$s_j^{(p)} \in \Gamma\left(X_0, p\,K_{X_0}+D+A\right)$$

can be extended to

$$\tilde{s}_j^{(p)} \in \Gamma\left(X, p\,K_X+D+A\right)$$

for $1 \le j \le N_p$ and $1 \le p < m_0$.

The case $p=1$ of Claim (1.4.3) clearly follows from Assumption (ii) and Condition (b) of (1.3) and the extension theorem (Theorem 1.2). Suppose Claim (1.4.3) has been proved for Step p and we are going to prove Step $p+1$, where p is replaced by $p+1$.

Let

$$\chi_p = \log \sum_{k=1}^{N_p} \left|\tilde{s}_k^{(p)}\right|^2$$

so that $e^{-\chi_p}$ is a metric of $p\,K_X+D+A$. Since $\tilde{s}_j^{(p)}|X_0 = s_j^{(p)}$ for $1 \le p \le N_p$ by Step p of Claim 1.4.3, it follows from (1.4.2) that the germ of $s_j^{(p+1)}$ at any point of X_0 belongs to $\mathcal{I}_{\chi_p|X_0}$ for $1 \le j \le N_{p+1}$. By the extension theorem 1.2,

$$s_j^{(p+1)} \in \Gamma\left(X_0, (p+1)\,K_{X_0}+D+A\right)$$

can be extended to

$$\tilde{s}_j^{(p+1)} \in \Gamma\left(X, (p+1)\,K_X+D+A\right) ,$$

finishing the verification of Step $p+1$ of Claim (1.4.3).

Since $\tilde{s}_j^{(m_0-1)}|X_0 = s_j^{(m_0-1)}$ for $1 \leq p \leq N_{m_0-1}$ from Claim (1.4.1), by (1.4.3) the germ of f at any point of X_0 belongs to $\mathcal{I}_{X_{m_0-1}|X_0}$. By the extension theorem 1.2

$$f \in \Gamma\left(X_0, m_0\, K_{X_0} + D + A\right)$$

can be extended to

$$\tilde{f} \in \Gamma\left(X, (m_0\, K_X + D + A)\right) .$$

□

1.5 *Metrics for the Induction Argument for the Case of General Type*

We now discuss how the metrics $e^{-\varphi_p}$ in the axiomatic formulation of the induction argument can be chosen for the case of general type.

In [Siu98], a sufficiently large positive number a is chosen so that $aK_X = D + A$ for some effective divisor D. Let

$$s_1^{(m)}, \cdots, s_{q_m}^{(m)} \in \Gamma\left(X_0, m\, K_{X_0}\right)$$

be a basis over C. Let s_D be the canonical section of D whose divisor is D and h_A be a smooth metric of A with positive curvature form.

For the case of general type, one can simply use

$$\varphi_p = \log \sum_{j=1}^{q_{m_0}} \left| s_j^{(m_0)} \right|^{\frac{2p}{m_0}} + \log |s_D|^2$$

for $0 \leq p \leq m_0 - 1$.

1.5.1 In the proof of the invariance of plurigenera in [Siu98] the infinite series

$$\varphi = \log \sum_{m=1}^{\infty} \varepsilon_m \sum_{j=1}^{q_m} \left| s_j^{(m)} \right|^{\frac{2}{m}}$$

is used to define

$$\varphi_p = p\, \varphi + \log |s_D|^2 \ ,$$

where a sequence of positive numbers ε_m $(1 \leq m < \infty)$ is chosen which decreases sufficiently rapidly, as m increases, to guarantee local convergence of the infinite series φ. The infinite series φ was introduced in [Siu98] more for notational expediency than for absolute necessity. In effective arguments where estimates have to be controlled, clearly we should avoid the complications of unnecessary infinite processes. So the metric given in (1.5) is more to our advantage than the one given in [Siu98] in the form of an infinite series.

1.5.2 *Reason for Metrics for Pluricanonical Bundles on Initial Fiber as Singular as Possible.* As a matter of fact, in order to guarantee convergence in the effective arguments where D is set to be 0, the more singular the metric $e^{-\varphi_p}$ of $p\,K_{X_0}$ is, the easier it is to control the estimates. For the extension $s^{(m_0)} \in \Gamma\left(X_0, m_0\,K_{X_0}\right)$ to X in the effective argument, we will use the metric

$$e^{-\varphi_p} = \frac{1}{\left|s^{(m_0)}\right|^{\frac{2p}{m_0}}}\,.$$

It is introduced in (4.1) as $e^{-p\psi}$. (The metric $e^{-p\psi}$ in (4.1) contains also the contribution from the semi-positive line bundle L and would be equal to $e^{-\varphi_p}$ when L is the trivial line bundle. Here for simplicity we assume $L = 0$ in our discussion here in (1.5.2).)

The reason why we want more singularity for $e^{-\varphi_p} = e^{-p\psi}$ is that, for the final estimates, a uniform bound, independent of p, is needed for the complex dimension of $\Gamma\left(X_0, \mathcal{I}_{\varphi_p}\left(p\,K_{X_0} + A\right)\right)$. If some other less singular metrics are used for $p\,K_{X_0}$, one may run into the difficulty that the complex dimension of $\Gamma\left(X_0, \mathcal{I}_{\varphi_p}\left(p\,K_{X_0} + A\right)\right)$ grows as a positive power of p. When we take the ℓ-th root of the extension of $\left(s^{(m_0)}\right)^{\ell} s_A$ from X_0 to X, where $s_A \in \Gamma\left(X, A\right)$, and let $\ell \to \infty$, in the estimate there is a contribution of the factor which is the ℓ-th root of the product of the complex dimension of $\Gamma\left(X_0, \mathcal{I}_{\varphi_p}\left(p\,K_{X_0} + A\right)\right)$ for $0 \leq p < \ell\, m_0$. If the complex dimension of $\Gamma\left(X_0, \mathcal{I}_{\varphi_p}\left(p\,K_{X_0} + A\right)\right)$ grows at least like a positive power of p, a positive power of the factor $(\ell!)^{\frac{1}{\ell}}$ occurs in the final estimate and becomes unbounded as ℓ goes to infinity. The occurrence of such a factor is due to the estimate in (5.3.4). So the use of usual abstractly-defined general metrics for $p\,K_{X_0}$, such as generalized Bergman kernels, would not work in the effective argument.

1.6 *Taking Powers and Roots of Sections for the Case of General Type.*

We take $s \in \Gamma\left(X_0, m_0\,K_{X_0}\right)$. To finish the last step of the proof of the deformational invariance of the plurigenera for general type, we need to extend s to an element of $\Gamma\left(X, m_0\,K_{X_0}\right)$.

Take a positive integer ℓ and let $m = m_0\,\ell$. We use the formula (1.5) to define φ_p for the larger range $0 \leq p \leq m-1$. Then the germ of $s^{\ell}\,s_D$ at any point of X_0 belongs to $\mathcal{I}_{\varphi_{m-1}}$. By Theorem 1.1 on the global generation of multiplier ideal sheaves, we can write

$$s^{\ell}\,s_D = \sum_{k=1}^{q_m} b_k\,s_k^{(m)} \tag{1.6.1}$$

locally on X_0 for some local holomorphic functions b_k on X_0 and for

$$s_k^{(m)} \in \Gamma\left(X_0, \mathcal{I}_{\varphi_{m-1}}\left(m\,K_{X_0} + D + A\right)\right)\,.$$

Then, by Proposition 1.4 with m_0 replaced by m, $s_k^{(m)}$ can be extended to

$$\tilde{s}_k^{(m)} \in \Gamma\left(X, m\,K_X + D + A\right) = \Gamma\left(X_0, (m+a)\,K_X\right)$$

for $1 \le k \le q_m$. Let

$$\chi_m = \log \sum_{k=1}^{q_m} \left|\tilde{s}_k^{(m)}\right|^2 .$$

From (1.6.1) the germ of of $s^\ell s_D$ at any point of X_0 belongs to the multiplier ideal sheaf $\mathcal{I}_{\chi_m|X_0}$.

We now apply Hölder's inequality. Let h_D be a smooth metric for D without any curvature condition. Let $h_{K_{X_0}}$ be a smooth metric for K_{X_0} without any curvature condition. Then $dV_{X_0} := \left(K_{X_0}\right)^{-1}$ is a smooth volume form of X_0. Let $\frac{1}{\ell} + \frac{1}{\ell'} = 1$ for ℓ sufficiently large. Since $\frac{1}{\ell'} = 1 - \frac{1}{\ell}$ and $\ell' = \frac{\ell}{\ell - 1}$, it follows that Hölder's inequality gives

$$\begin{aligned}
&\int_{X_0} |s|^2 e^{-\left(\frac{m_0}{m+a}\right)\chi_m}\, dV_{X_0} \\
&= \int_{X_0} \left(\left|s\,(s_D)^{\frac{1}{\ell}}\right|^2 h_D\, e^{-\left(\frac{m_0}{m+a}\right)\chi_m}\right) \left(\frac{1}{\left|(s_D)^{\frac{1}{\ell}}\right|^2 h_D}\right) dV_{X_0} \\
&\le \left(\int_{X_0} \left|s^\ell s_D\right|^2\, h_D\, e^{-\ell\left(\frac{m_0}{m+a}\right)\chi_m} dV_{X_0}\right)^{\frac{1}{\ell}} \left(\int_{X_0} \frac{1}{\left(h_D\,|s_D|^2\right)^{\frac{1}{\ell-1}}} dV_{X_0}\right)^{1-\frac{1}{\ell}} \\
&\le \left(\sup_{X_0} h_{K_X}\, e^{\left(\frac{1}{m+a}\right)\chi_m}\right)^{\frac{a}{\ell}} \left(\int_{X_0} \frac{\left|s^\ell s_D\right|^2 h_D\, e^{-\chi_m} dV_{X_0}}{\left(h_{K_{X_0}}\right)^a}\right)^{\frac{1}{\ell}} \cdot \\
&\qquad \cdot \left(\int_{X_0} \frac{dV_{X_0}}{\left(h_D\,|s_D|^2\right)^{\frac{1}{\ell-1}}}\right)^{1-\frac{1}{\ell}}
\end{aligned}$$

which is finite for ℓ sufficiently large. Hence

$$\int_{X_0} |s|^2 e^{-\left(\frac{m_0-1}{m+a}\right)\chi_m} \le \left(\sup_{X_0} h_{K_X}\, e^{\left(\frac{1}{m+a}\right)\chi_m}\right) \int_{X_0} |s|^2 e^{-\left(\frac{m_0}{m+a}\right)\chi_m}\, dV_{X_0} < \infty .$$

By the extension theorem 1.2, s can be extended to an element

$$\tilde{s} \in \Gamma\left(X, m_0\,K_X\right) .$$

This finishes the review of the argument of [Siu98].

2 Global Generation of Multiplier Ideal Sheaves with Estimates

Now we give the effective version, with estimates, of the global generation of multiplier ideal sheaves.

Theorem 2.1. *(Effective Version of Global Generation of Multiplier Ideal Sheaves). Assume that for every point P_0 of X_0 one has a coordinate chart $\tilde{U}_{P_0} = \{|z^{(P_0)}| < 2\}$ of X_0 with coordinates*

$$z^{(P_0)} = \left(z_1^{(P_0)}, \cdots, z_n^{(P_0)}\right)$$

centered at P_0 such that the set U_{P_0} of points of $\tilde{U}_{P_0}$ where $\left|z^{(P_0)}\right| < 1$ is relatively compact in $\tilde{U}_{P_0}$. Let ω_0 be a Kähler form of X_0. Let C_{X_0} be a positive number such that the supremum norm of $\bar{\partial} z_j^{(P_0)}$ with respect to ω_0 is $\leq C_{X_0}$ on U_{P_0} for $1 \leq j \leq n$. Let $0 < r_1 < r_2 \leq 1$. Let A be an ample line bundle over X_0 with a smooth metric h_A of positive curvature. Assume that, for every point P_0 of X_0, there exists a singular metric h_{A,P_0} of A, whose curvature current dominates $c_A \omega_0$ for some positive constant c_A, such that

$$\frac{h_A}{|z^{(P_0)}|^{2(n+1)}} \leq h_{A,P_0}$$

on U_{P_0} and

$$\sup_{r_1 \leq |z^{(P_0)}| \leq r_2} \frac{h_{A,P_0}(z^{(P_0)})}{h_A(z^{(P_0)})} \leq C_{r_1,r_2}$$

and

$$\sup_{X_0} \frac{h_A}{h_{A,P_0}} \leq C^\sharp$$

for some constants C_{r_1,r_2} and $C^\sharp \geq 1$ independent of P_0. Let m be a positive integer. Assume that there is a (possibly singular) metric $e^{-\varphi_{m-1}}$ for $(m-1)K_{X_0}$ with semi-positive curvature current. Let

$$C^\flat = 2n\left(\frac{1}{r_1^{2(n+1)}} + 1 + C^\sharp \frac{1}{c_A} C_{r_1,r_2} \left(\frac{2\,r_2\,C_{X_0}}{r_2^2 - r_1^2}\right)^2\right).$$

Let $0 < r < 1$ and let

$$\hat{U}_{P_0,r} = U_{P_0} \cap \left\{\left|z^{(P_0)}\right| < \frac{r}{n\sqrt{C^\flat}}\right\}.$$

Let N_m be the complex dimension of the subspace of all elements

$$s \in \Gamma\left(X_0, m\,K_{X_0} + A\right)$$

such that

$$\int_{X_0} |s|^2 \, e^{-\varphi_{m-1}} \, h_A < \infty \, .$$

Then there exist

$$\sigma_1^{(m)}, \cdots, \sigma_{N_m}^{(m)} \in \Gamma\left(X_0, m\,K_{X_0} + A\right)$$

with

$$\int_{X_0} \left|\sigma_k^{(m)}\right|^2 e^{-\varphi_{m-1}} h_A \leq 1$$

($1 \leq k \leq N_m$) such that, for any $P_0 \in X_0$ and for any holomorphic section s of $m\,K_{X_0} + A$ over U_{P_0} with

$$\int_{U_{P_0}} |s|^2 e^{-\varphi_{m-1}} h_A = C_s < \infty \, ,$$

one can find holomorphic functions $b_{P_0,m,k}$ on $\hat{U}_{P_0,r}$ such that

$$s = \sum_{k=1}^{N_m} b_{P_0,m,k} \, \sigma_k^{(m)}$$

on $\hat{U}_{P_0,r}$ and

$$\sup_{\hat{U}_{P_0,r}} \sum_{k=1}^{N_m} \left|b_{P_0,m,k}\right|^2 \leq \frac{1}{(1-r)^2} \, C^\flat \, C_s \, .$$

The proof of Theorem 2.1 will depend on the following lemma.

Lemma 2.2. *Under the assumption of Theorem 2.1, given any holomorphic section s of $m\,K_{X_0} + A$ over U_{P_0} with*

$$\int_{U_{P_0}} |s|^2 e^{-\varphi_{m-1}} h_A = C_s < \infty$$

there exist $\sigma \in \Gamma\left(X_0, m\,K_{X_0} + A\right)$ and $v_j \in \Gamma\left(U_{P_0}, m\,K_{X_0} + A\right)$ for $1 \leq j \leq n$ such that $s - \sigma = \sum_{j=1}^n z_j^{(P_0)} \, v_j$ with

$$\int_{X_0} |\sigma|^2 e^{-\varphi_{m-1}} h_A \leq C^\flat \, C_s$$

and

$$\int_{U_{P_0}} |v_j|^2 \, e^{-\varphi_{m-1}} h_A \leq C^\flat \, C_s \, .$$

Before we prove Lemma 2.2, we recall the following result of Skoda [Sko72, Th. 1, pp.555-556].

Theorem 2.3. *(Skoda). Let Ω be a pseudoconvex domain in $\mathbf{C}^n$ and ψ be a plurisubharmonic function on Ω. Let $g_1, \ldots, g_p$ be holomorphic functions on Ω. Let $\alpha > 1$ and $q = \inf(n, p-1)$. Then for every holomorphic function f on Ω such that*

$$\int_\Omega |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda < \infty \, ,$$

there exist holomorphic functions $h_1, \ldots, h_p$ on Ω such that

$$f = \sum_{j=1}^p g_j h_j$$

and

$$\int_\Omega |h|^2 |g|^{-2\alpha q} e^{-\psi} d\lambda \leq \frac{\alpha}{\alpha - 1} \int_\Omega |f|^2 |g|^{-2\alpha q - 2} e^{-\psi} d\lambda \, ,$$

where

$$|g| = \left(\sum_{j=1}^p |g_j|^2 \right)^{\frac{1}{2}}, \qquad |h| = \left(\sum_{j=1}^p |h_j|^2 \right)^{\frac{1}{2}},$$

and $d\lambda$ is the Euclidean volume element of $\mathbf{C}^n$.

2.4 *Proof of Lemma 2.2.*

We take a smooth cut-off function $0 \leq \varrho(x) \leq 1$ of a single real variable x so that $\varrho(x) = 1$ for $x \leq r_1^2$ and $\varrho(x) = 0$ for $x > r_2^2$ and

$$\left| \frac{d\varrho}{dx}(x) \right| \leq \frac{2}{r_2^2 - r_1^2} \, .$$

Let $\varrho_{P_0} = \varrho\left(\left| z^{(P_0)} \right|^2 \right)$. Then the supremum norm of $\bar\partial \varrho_{P_0}$ with respect to ω_0 is no more than

$$\frac{2 \, r_2 \, C_{X_0}}{r_2^2 - r_1^2} \, .$$

Consider $s \bar\partial \varrho_{P_0}$. Then

$$\begin{aligned}
&\int_{X_0} \left| s \, \bar\partial \varrho_{P_0} \right|^2 e^{-\varphi_{m-1}} h_{A,P_0} \\
&= \int_{X_0} \left| s \, \bar\partial \varrho_{P_0} \right|^2 e^{-\varphi_{m-1}} h_A \left(\frac{h_{A,P_0}}{h_A} \right) \\
&\leq \left(\sup_{r_1 \leq \left| z^{(P_0)} \right| \leq r_2} \frac{h_{A,P_0}}{h_A} \right) \int_{X_0} \left| s \, \bar\partial \varrho_{P_0} \right|^2 e^{-\varphi_{m-1}} h_A \leq C_{r_1, r_2} \left(\frac{2 \, r_2 C_{X_0}}{r_2^2 - r_1^2} \right)^2 C_s \, .
\end{aligned}$$

We now solve the equation $\bar\partial u = s\,\bar\partial \varrho_{P_0}$ with the estimate

$$\int_{X_0} |u|^2\, e^{-\varphi_m} h_{A,P_0} \le \frac{1}{c_A}\, C_{r_1,r_2} \left(\frac{2\,r_2\,C_{X_0}}{r_2^2 - r_1^2}\right)^2 C_s\,.$$

Let $\sigma = s\varrho_{P_0} - u$. Then

$$\begin{aligned}\int_{X_0} |\sigma|^2 e^{-\varphi_{m-1}} h_A &= \int_{X_0} |s\varrho_{P_0} - u|^2 e^{-\varphi_{m-1}} h_A \\ &\le 2\int_{X_0} \left(|s|^2 + |u|^2\right) e^{-\varphi_{m-1}} h_A \\ &\le 2\,C_s + 2\int_{X_0} |u|^2\, e^{-\varphi_{m-1}} h_{A,P_0} \left(\frac{h_A}{h_{A,P_0}}\right) \\ &\le 2\,C_s + 2\left(\sup_{X_0} \frac{h_A}{h_{A,P_0}}\right) \int_{X_0} |u|^2\, e^{-\varphi_{m-1}} h_{A,P_0} \\ &\le 2\left(1 + C^\sharp\, \frac{1}{c_A}\, C_{r_1,r_2} \left(\frac{2\,r_2\,C_{X_0}}{r_2^2 - r_1^2}\right)^2\right) C_s\end{aligned}$$

which is $\le C^\flat\, C_s$. Since $s - \sigma = (1 - \varrho_{P_0})\,s + u$, it follows that

$$\begin{aligned}\int_{U_{P_0}} \frac{|s-\sigma|^2}{\left|z^{(P_0)}\right|^{2(n+1)}}\, e^{-\varphi_m} h_A &\le 2\int_{U_{P_0}} \frac{\left|(1-\varrho_{P_0})\,s\right|^2 + |u|^2}{\left|z^{(P_0)}\right|^{2(n+1)}} e^{-\varphi_m} h_A \\ &\le 2\left(\frac{1}{r_1^{2(n+1)}} + \frac{1}{c_A}\, C_{r_1,r_2} \left(\frac{2\,r_2\,C_{X_0}}{r_2^2 - r_1^2}\right)^2\right) C_s\,,\end{aligned}$$

because $\varrho_{P_0} = 1$ on $\left\{\left|z^{(P_0)}\right| < r_1\right\}$ and

$$\frac{h_A}{\left|z^{(P_0)}\right|^{2(n+1)}} \le h_{A,P_0}$$

on U_{P_0}. By Skoda's theorem 2.3 with $g_j = z_j$, $p = n$, and $\alpha = \frac{n}{n-1}$, we can write

$$s - \sigma = \sum_{j=1}^{n} z_j^{(P_0)}\, v_j\,,$$

where v_j is a holomorphic section of $m\,K_{X_0} + A$ on U_{P_0} with

$$\begin{aligned}\int_{U_{P_0}} \frac{|v_j|^2}{\left|z^{(P_0)}\right|^{2n}} e^{-\varphi_{m-1}} h_A &\le n\int_{U_{P_0}} \frac{|s-\sigma|^2}{\left|z^{(P_0)}\right|^{2(n+1)}}\, e^{-\varphi_{m-1}} h_A \\ &\le 2n\left(\frac{1}{r_1^{2(n+2)}} + \frac{1}{c_A}\, C_{r_1,r_2} \left(\frac{2\,r_2\,C_{X_0}}{r_2^2 - r_1^2}\right)^2\right) C_s \le C^\flat\, C_s\end{aligned}$$

for $1 \le j \le n$. Since $U_{P_0} = \left\{\left|z^{(P_0)}\right| < 1\right\}$, it follows that

$$\int_{U_{P_0}} |v_j|^2 \, e^{-\varphi_{m-1}} h_A \leq C^\flat \, C_s \, . \qquad \square$$

2.5 *Proof of Theorem 2.1.*

Let

$$\sigma_1^{(m)}, \cdots, \sigma_{N_m}^{(m)} \in \Gamma \left(X_0, m \, K_{X_0} + A\right)$$

be an orthonormal basis, with respect to $e^{-\varphi_{m-1}} h_A$ on X_0, of the subspace of all elements $\sigma^{(m)} \in \Gamma \left(X_0, m \, K_{X_0} + A\right)$ such that

$$\int_{X_0} \left|\sigma^{(m)}\right|^2 \, e^{-\varphi_{m-1}} \, h_A < \infty \, .$$

By Lemma 2.2 there exist

$$\sigma \in \Gamma \left(X_0, m \, K_{X_0} + A\right)$$

and

$$v_j \in \Gamma \left(U_{P_0}, m \, K_{X_0} + A\right)$$

for $1 \leq j \leq n$ such that $s - \sigma = \sum_{j=1}^n z_j^{(P_0)} \, v_j$ with

$$\int_{X_0} |\sigma|^2 e^{-\varphi_{m-1}} h_A \leq C^\flat \, C_s$$

and

$$\int_{U_{P_0}} |v_j|^2 \, e^{-\varphi_{m-1}} h_A \leq C^\flat \, C_s \, .$$

We can uniquely write $\sigma = \sum_{k=1}^{N_m} c_k \, \sigma_k^{(m)}$ with $c_k \in \mathbf{C}$ for $1 \leq k \leq N_m$. Then $s = \sum_{k=1}^{N_m} c_k \, \sigma_k^{(m)} + \sum_{j=1}^n z_j^{(P_0)} \, v_j$ on U_{P_0} and

$$\sum_{k=1}^{N_m} |c_k|^2 \leq C^\flat \, C_s \, .$$

We are going to iterate this process. We do it with s replaced by v_j and use induction. This iteration process is simply an effective version, with estimates, of Nakayama's lemma in algebraic geometry. For notational convenience in the iteration and induction process, we rewrite the index $1 \leq j \leq n$ as $1 \leq j_1 \leq n$. We now replace s by v_{j_1} in the preceding argument and get

$$v_{j_1} = \sum_{k=1}^{N_m} c_{k,j_1} \sigma_k^{(m)} + \sum_{j_2=1}^{n} z_{j_2}^{(P_0)} \, v_{j_1,j_2}$$

with $c_{k,j_1} \in \mathbf{C}$ and $v_{j_1,j_2} \in \Gamma \left(U_{P_0}, m \, K_{X_0} + A\right)$ such that

$$\sum_{k=1}^{N_m} |c_{k,j_1}|^2 \leq C^\flat \int_{U_{P_0}} |v_{j_1}|^2 e^{-\varphi_{m-1}} h_A \leq \left(C^\flat\right)^2 \, C_s$$

and

$$\int_{U_{P_0}} |v_{j_1,j_2}|^2 \, e^{-\varphi_{m-1}} h_A \leq C^\flat \int_{U_{P_0}} |v_{j_1}|^2 e^{-\varphi_{m-1}} h_A \leq \left(C^\flat\right)^2 C_s \,.$$

By induction on ℓ and applying Lemma 2.2 with s replaced by $v_{j_1,\cdots,j_{\ell-1}}$, we get

$$v_{j_1,\cdots,j_{\ell-1}} = \sum_{k=1}^{N_m} c_{k,j_1,\cdots,j_{\ell-1}} \sigma_k^{(m)} + \sum_{j_\ell=1}^{n} z_{j_2}^{(P_0)} v_{j_1,\cdots,j_{\ell-1},j_\ell}$$

with $c_{k,j_1,\cdots,j_{\ell-1}} \in \mathbf{C}$ and $v_{j_1,\cdots,j_\ell} \in \Gamma\left(U_{P_0}, m\, K_{X_0} + A\right)$ such that

$$\sum_{k=1}^{N_m} |c_{k,j_1,\cdots,j_\nu}|^2 \leq \left(C^\flat\right)^{\nu+1} C_s$$

and

$$\int_{U_{P_0}} |v_{j_1,\cdots,j_\ell}|^2 \, e^{-\varphi_{m-1}} h_A \leq \left(C^\flat\right)^\ell C_s \,.$$

Thus

$$\begin{aligned} s = & \sum_{j_1,\cdots,j_\ell=1}^{n} v_{j_1,\cdots,j_\ell} z_{j_1} \cdots z_{j_\ell} \\ & + \sum_{k=1}^{N_m} \left(c_k + \sum_{\nu=1}^{\ell-1} \sum_{j_1,\cdots,j_\nu=1}^{n} c_{k,j_1,\cdots,j_\nu} z_{j_1}^{(P_0)} \cdots z_{j_\nu}^{(P_0)} \right) \sigma_k^{(m)} \,. \end{aligned}$$

Let

$$b_{P_0,m,k} = c_k + \sum_{\nu=1}^{\infty} \sum_{j_1,\cdots,j_\nu=1}^{n} c_{k,j_1,\cdots,j_\nu} z_{j_1}^{(P_0)} \cdots z_{j_\nu}^{(P_0)} \,.$$

We are going to verify that the series defining $b_{P_0,m,k}$ converges to a holomorphic function on $\hat{U}_{P_0,r}$ and

$$\sup_{\hat{U}_{P_0,r}} \sum_{k=1}^{N_m} |b_{P_0,m,k}|^2 \leq \frac{1}{(1-r)^2} C^\flat C_s \,.$$

Since

$$\sup_{\hat{U}_{P_0,r}} \left| z_{j_1}^{(P_0)} \cdots z_{j_\nu}^{(P_0)} \right|^2 \leq \left(\frac{r^2}{n^2 C^\flat} \right)^\nu$$

it follows that on $\hat{U}_{P_0,r}$ and for $1 \leq \nu < \infty$ one has the estimate

$$\begin{aligned}
&\sum_{k=1}^{N_m} \left| \sum_{j_1,\cdots,j_\nu=1}^{n} c_{k,j_1,\cdots,j_\nu} z_{j_1}^{(P_0)} \cdots z_{j_\nu}^{(P_0)} \right|^2 \\
&\leq \left(\frac{r^2}{n^2 C^\flat} \right)^\nu \sum_{k=1}^{N_m} \left| \sum_{j_1,\cdots,j_\nu=1}^{n} c_{k,j_1,\cdots,j_\nu} \right|^2 \\
&\leq \left(\frac{r^2}{n^2 C^\flat} \right)^\nu n^\nu \sum_{k=1}^{N_m} \sum_{j_1,\cdots,j_\nu=1}^{n} |c_{k,j_1,\cdots,j_\nu}|^2 \\
&= \left(\frac{r^2}{n^2 C^\flat} \right)^\nu n^\nu \sum_{j_1,\cdots,j_\nu=1}^{n} \sum_{k=1}^{N_m} |c_{k,j_1,\cdots,j_\nu}|^2 \\
&\leq \left(\frac{r^2}{n^2 C^\flat} \right)^\nu n^\nu \sum_{j_1,\cdots,j_\nu=1}^{n} \left(C^\flat \right)^{\nu+1} C_s = r^{2\nu} C^\flat C_s . \qquad (2.5.1)
\end{aligned}$$

Let F_ν denotes the N_m-tuple

$$F_\nu = (F_{\nu,1}, \cdots, F_{\nu,N_m})$$

of holomorphic functions on $\hat{U}_{P_0,r}$ with

$$F_{\nu,k} = \sum_{j_1,\cdots,j_\nu=1}^{n} c_{k,j_1,\cdots,j_\nu} z_{j_1}^{(P_0)} \cdots z_{j_\nu}^{(P_0)}$$

for $\nu \geq 1$ and $F_{0,k} = c_k$. Then the pointwise L^2 norm $|F_\nu|$ of F_ν is the function

$$\left(|F_{\nu,1}|^2 + \cdots + |F_{\nu,N_m}|^2 \right)^{\frac{1}{2}} .$$

By (2.5.1),

$$|F_\nu| \leq r^\nu \sqrt{C^\flat C_s}$$

on $\hat{U}_{P_0,r}$. Hence

$$\left| \sum_{\nu=0}^{\infty} F_\nu \right| \leq \left(\sum_{\nu=0}^{\infty} r^\nu \right) \sqrt{C^\flat C_s} = \frac{1}{1-r} \sqrt{C^\flat C_s} .$$

Since

$$b_k = \sum_{\nu=0}^{\infty} F_{\nu,k} ,$$

it follows that b_k is holomorphic on $\hat{U}_{P_0,r}$ and

$$\sup_{\hat{U}_{P_0,r}} \sum_{k=1}^{N_m} |b_{P_0,m,k}|^2 \leq \frac{1}{(1-r)^2} C^\flat C_s . \qquad \square$$

3 Extension Theorems of Ohsawa-Takegoshi Type from Usual Basic Estimates with Two Weight Functions

In this section we are going to state and derive the extension theorem of Ohsawa-Takegoshi type with estimates which we need for the effective version of the arguments of [Siu98]. Such extension theorems originated in a paper of Ohsawa-Takegoshi [OT87] and generalizations were made by Manivel [Man93] and a series of papers of Ohsawa [Ohs88], [Ohs94], [Ohs95], [Ohs01]. Ohsawa's series of papers [Ohs88], [Ohs94], [Ohs95], [Ohs01] contain more general results, which were proved from identities in Kähler geometry and specially constructed complete metrics. Here we use the simple approach of the usual basic estimates with two weight functions. We choose to derive here the extension theorem we need instead of just quoting from more general results, because the simple approach given here gives a clearer picture what and why additional techniques of solving the $\bar{\partial}$ equation other than the standard ones are required for the proof of the extension result. The derivation given here is essentially the same as the one given in [Siu96] with the modifications needed for the present case of no strictly positive lower bound for the curvature current. The only modifications consist of the use of $|\langle u, dw\rangle|$ instead of $|u|$ in some inequalities between (3.5.2) and (3.6.1). The modification simply replaces the strictly positive lower bound of the curvature current in all directions by the strictly positive lower bound of the curvature just for the direction normal to the hypersuface from which the holomorphic section is extended. The precise statement which we need is the following.

Theorem 3.1. *Let Y be a complex manifold of complex dimension n. Let w be a bounded holomorphic function on Y with nonsingular zero-set Z so that dw is nonzero at any point of Z. Let L be a holomorphic line bundle over Y with a (possibly singular) metric $e^{-\kappa}$ whose curvature current is semipositive. Assume that there exists a hypersurface V in Y such that $V \cap Z$ is a subvariety of codimension at least 1 in Z and $Y - V$ is the union of a sequence of Stein subdomains Ω_ν of smooth boundary and Ω_ν is relatively compact in $\Omega_{\nu+1}$. If f is an L-valued holomorphic $(n-1)$-form on Z with*

$$\int_Z |f|^2 \, e^{-\kappa} < \infty \,,$$

then $f dw$ can be extended to an L-valued holomorphic n-form F on Y such that

$$\int_Y |F|^2 \, e^{-\kappa} \leq 8\pi\, e \sqrt{2 + \frac{1}{e}} \left(\sup_Y |w|^2\right) \int_Z |f|^2 \, e^{-\kappa} \,.$$

We will devote most of this section to the proof of Theorem 3.1. We will fix ν and solve the problem on Ω_ν, instead of on Y, and we will do it with the estimate on L^2 norms which is independent of ν and then we will take limit as $\nu \to \infty$. For notational simplicity, in the presentation of our argument, we will drop the index ν in Ω_ν and simply denote Ω_ν by Ω. After dividing w by the supremum of $|w|$ on Y, we can assume without loss of generality that the supremum norm of w on Y is no more than 1. Moreover, since $\Omega_{\nu+1}$ is Stein there exists a holomorphic L-valued holomorphic $(n-1)$-form $\tilde{f}$ on $\Omega_{\nu+1}$ such that $(f \wedge dw)\,|(\Omega_{\nu+1} \cap Z)$ is the restriction of $\tilde{f}$ to $\Omega_{\nu+1} \cap Z$. Of course, when such an extension $\tilde{f}$ is obtained simply by the Stein property of $\Omega_{\nu+1}$, we do not have any L^2 norm estimate on $\tilde{f}$ which is independent of ν.

3.2 *Functional Analysis Preliminaries.*

We recall the standard technique of using functional analysis and Hilbert spaces to solve the $\bar{\partial}$ equation. Consider an operator T which later will be an operator modified from $\bar{\partial}$. Let S be an operator such that $ST = 0$. The operator S later will be an operator modified from the $\bar{\partial}$ operator of the next step in the Dolbeault complex. Given g with $Sg = 0$ we would like to solve the equation $Tu = g$. The equation $Tu = g$ is equivalent to $(v, Tu) = (v, g)$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$, which means $(T^*v, u) = (v, g)$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$. To get a solution u it suffices to prove that the map $T^*v \to (v, g)$ can be extended to a bounded linear functional, which means that there exists a positive constant C such that $|(v,g)| \leq C\|T^*v\|$ for all $v \in \operatorname{Ker} S \cap \operatorname{Dom} T^*$. In that case we can solve the equation $Tu = g$ with $\|u\| \leq C$. We could also use the equivalent inequality

$$|(v,g)|^2 \leq C^2 \left(\|T^*v\|^2 + \|Sv\|^2\right)$$

for all $v \in \operatorname{Dom} S \cap \operatorname{Dom} T^*$.

3.3 *Bochner-Kodaira Formula with Two Weights.*

The crucial point of the argument is the use of two different weights. One weight is for the norm and the other is for the definition of the adjoint of $\bar{\partial}$. We now derive the formula for the Bochner-Kodaira formula in which the weight for the norm is different from the weight used to define the adjoint of $\bar{\partial}$. Formulas, of such a kind, for different weight functions were already given in the literature in the nineteen sixties by authors such as Hörmander (for example, [Hör66]). There is nothing particular new here, except that we need the statement in the form precisely stated below for our case at hand.

We start with a weight $e^{-\varphi}$ and use the usual Bochner-Kodaira formula for this particular weight. Let η be a positive-valued function and let $e^{-\psi} = \frac{e^{-\varphi}}{\eta}$. We will use the weight $e^{-\psi}$ for the definition of the adjoint of $\bar{\partial}$. We use $\bar{\partial}^*_\varphi$ (respectively $\bar{\partial}^*_\psi$) to denote the formal adjoint of $\bar{\partial}$ with respect to the weight function $e^{-\varphi}$ (respectively $e^{-\psi}$). We agree to use the summation convention that, when a lower-case Greek index appears twice in a term, once with a bar and once without a bar, we mean the contraction of the two indices by the Kähler metric tensor. An index without (respectively with) a bar inside the complex conjugation of a factor is counted as an index with (respectively without) a bar. We use $\langle\cdot,\cdot\rangle$ to denote the pointwise inner product. Let $\bar{\nabla}$ be the covariant differentiation in the $(0,1)$-direction. The formula we seek is the following.

Proposition 3.4. *Let Ω be defined by $r<0$ so that $|dr|$ with respect to the Kähler metric is identically 1 on the boundary $\partial\Omega$ of Ω. Let u be an $(n,1)$-form in the domain of the actual adjoint of $\bar\partial$ on Ω. Then*

$$\begin{aligned}\int_\Omega \left\langle \bar\partial^*_\psi u, \bar\partial^*_\psi u\right\rangle e^{-\varphi} + \int_\Omega \left\langle \bar\partial u, \bar\partial u\right\rangle e^{-\varphi} &= \int_{\partial\Omega} \overline{u_{\bar\beta}} u_{\bar\alpha} (\partial_{\bar\beta}\partial_\alpha r) e^{-\varphi} + \int_\Omega \left\langle \bar\nabla u, \bar\nabla u\right\rangle e^{-\varphi} \\ &\quad + \int_\Omega \overline{u_{\bar\beta}} u_{\bar\alpha} \left(\partial_{\bar\beta}\partial_\alpha \psi\right) e^{-\varphi} - \int_\Omega \left(\frac{\partial_{\bar\alpha}\partial_\beta \eta}{\eta}\right) u_\alpha \overline{u_\beta} e^{-\varphi} \\ &\quad + 2\,Re \int_\Omega \left(\frac{\partial_\alpha \eta}{\eta}\right) u_{\bar\alpha}\, \overline{\left(\bar\partial^*_\psi u\right)} e^{-\varphi}.\end{aligned}$$

Proof. The usual Bochner-Kodaira formula for a domain with smooth boundary for the same weight (also known as the basic estimate) gives

$$\begin{aligned}\int_\Omega \left\langle \bar\partial^*_\varphi u, \bar\partial^*_\varphi u\right\rangle e^{-\varphi} + \int_\Omega \left\langle \bar\partial u, \bar\partial u\right\rangle e^{-\varphi} &= \int_{\partial\Omega} \overline{u_{\bar\beta}} u_{\bar\alpha} (\partial_{\bar\beta}\partial_\alpha r) e^{-\varphi} + \int_\Omega \left\langle \bar\nabla u, \bar\nabla u\right\rangle e^{-\varphi} \\ &\quad + \int_\Omega \overline{u_{\bar\beta}} u_{\bar\alpha} \left(\partial_{\bar\beta}\partial_\alpha \varphi\right) e^{-\varphi}. \qquad (3.4.1)\end{aligned}$$

The relation between the formal adjoints of $\bar\partial$ for different weights is as follows:

$$\bar\partial^*_\varphi u = -e^\varphi \partial_\alpha \left(e^{-\varphi} u_{\bar\alpha}\right) = -\frac{e^\psi}{\eta} \partial_\alpha \left(\eta\, e^{-\psi} u_{\bar\alpha}\right) = -\frac{\partial_\alpha \eta}{\eta} u_{\bar\alpha} + \bar\partial^*_\psi u\,.$$

Thus

$$\begin{aligned}\left|\bar\partial^*_\varphi u\right|^2 e^{-\varphi} &= \left| -\frac{\partial_\alpha \eta}{\eta} u_{\bar\alpha} + \bar\partial^*_\psi u\right|^2 e^{-\varphi} \\ &= \left|\frac{\partial_\alpha \eta}{\eta} u_{\bar\alpha}\right|^2 e^{-\varphi} - 2\,\mathrm{Re}\left(\frac{\partial_\alpha \eta}{\eta} u_{\bar\alpha}\, \overline{\left(\bar\partial^*_\psi u\right)}\right) e^{-\varphi} + \left|\bar\partial^*_\psi u\right|^2 e^{-\varphi}\,.\end{aligned}$$

We now rewrite (3.4.1) as

$$\begin{aligned}\int_\Omega \left\langle \bar\partial^*_\psi u, \bar\partial^*_\psi u\right\rangle e^{-\varphi} + \int_\Omega \left\langle \bar\partial u, \bar\partial u\right\rangle e^{-\varphi} &= \int_{\partial\Omega} \overline{u_{\bar\beta}} u_{\bar\alpha} (\partial_{\bar\beta}\partial_\alpha r) e^{-\varphi} + \int_\Omega \left\langle \bar\nabla u, \bar\nabla u\right\rangle e^{-\varphi} \\ &\quad + \int_\Omega \overline{u_{\bar\beta}} u_{\bar\alpha} \left(\partial_{\bar\beta}\partial_\alpha \varphi\right) e^{-\varphi} - \int_\Omega \left|\frac{\partial_\alpha \eta}{\eta} u_{\bar\alpha}\right|^2 e^{-\varphi}\end{aligned}$$

$$+\, 2\,\mathrm{Re}\int_\Omega \frac{\partial_\alpha\eta}{\eta}\, u_{\bar\alpha}\,\overline{\left(\bar\partial^*_\psi u\right)} e^{-\varphi}\,. \tag{3.4.2}$$

From $\varphi = \psi - \log\eta$ it follows that

$$\partial\bar\partial\varphi = \partial\bar\partial\psi - \frac{\partial\bar\partial\eta}{\eta} + \frac{\partial\eta\wedge\bar\partial\eta}{\eta^2}\,.$$

Hence we can rewrite (3.4.2) as

$$\begin{aligned}
&\int_\Omega \left\langle \overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u\right\rangle e^{-\varphi} + \int_\Omega \left\langle \overline{\partial} u, \overline{\partial} u\right\rangle e^{-\varphi}\\
&\quad = \int_{\partial\Omega} \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} (\partial_{\overline{\beta}}\partial_\alpha r) e^{-\varphi} + \int_\Omega \left\langle \overline{\nabla} u, \overline{\nabla} u\right\rangle e^{-\varphi}\\
&\qquad + \int_\Omega \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} \left(\partial_{\overline{\beta}}\partial_\alpha\psi\right) e^{-\varphi} - \int_\Omega \left(\frac{\partial_{\bar\alpha}\partial_\beta\eta}{\eta}\right) u_\alpha \overline{u_\beta} e^{-\varphi}\\
&\qquad + 2\,\mathrm{Re}\int_\Omega \left(\frac{\partial_\alpha\eta}{\eta}\right) u_{\bar\alpha}\,\overline{\left(\bar\partial^*_\psi u\right)} e^{-\varphi}\,.
\end{aligned}$$

□

3.5 *Choice of Two Different Weights.*

Since Ω is weakly pseudoconvex, the Levi form of r is semi-positive at every point of the boundary $\partial\Omega$ of Ω. The inequality in Proposition 3.4 becomes

$$\begin{aligned}
&\int_\Omega \left\langle \overline{\partial}^*_\psi u, \overline{\partial}^*_\psi u\right\rangle e^{-\varphi} + \int_\Omega \left\langle \overline{\partial} u, \overline{\partial} u\right\rangle e^{-\varphi} \geq \int_\Omega \overline{u_{\overline{\beta}}} u_{\overline{\alpha}} \left(\partial_{\overline{\beta}}\partial_\alpha\psi\right) e^{-\varphi}\\
&\quad - \int_\Omega \left(\frac{\partial_{\bar\alpha}\partial_\beta\eta}{\eta}\right) u_\alpha \overline{u_\beta} e^{-\varphi} + 2\,\mathrm{Re}\int_\Omega \left(\frac{\partial_\alpha\eta}{\eta}\right) u_{\bar\alpha}\,\overline{\left(\bar\partial^*_\psi u\right)} e^{-\varphi}\,.
\end{aligned} \tag{3.5.1}$$

Take any positive number $A > e$, where e is the base of the natural logarithm. Let

$$\varepsilon_0 = \sqrt{\frac{A}{e} - 1}\,.$$

For any positive $\varepsilon < \varepsilon_0$, we let

$$\eta = \log\frac{A}{|w|^2 + \varepsilon^2}\,,$$

$$\gamma = \frac{1}{|w|^2 + \varepsilon^2}\,.$$

Then $\eta > 1$ on Ω, because the supremum norm of w is no more than 1 on Ω.

$$-\partial_w\partial_{\bar w}\eta = \frac{\varepsilon^2}{(|w|^2+\varepsilon^2)^2}\,,$$

$$\partial_w\eta = -\frac{\bar w}{|w|^2+\varepsilon^2}\,,$$

$$\partial_{\bar w}\eta = -\frac{w}{|w|^2+\varepsilon^2}\,.$$

We have the estimate

$$\left|2\mathrm{Re}\int_\Omega \frac{\partial_\alpha\eta}{\eta}\,u_{\bar\alpha}(\overline{\partial}^*_\psi u)e^{-\varphi}\right|$$
$$\leq\; 2\int_\Omega \frac{|w|}{|w|^2+\varepsilon^2}\,|\langle u,dw\rangle|\;\left|\bar\partial^*_\psi u\right|\;e^{-\psi}$$
$$\leq\;\int_\Omega \frac{|w|^2}{|w|^2+\varepsilon^2}\;|\langle u,dw\rangle|^2\,e^{-\psi}+\int_\Omega \frac{1}{|w|^2+\varepsilon^2}\left|\bar\partial^*_\psi u\right|^2 e^{-\psi}$$
$$=\;\int_\Omega \frac{|w|^2}{|w|^2+\varepsilon^2}\;|\langle u,dw\rangle|^2\,e^{-\psi}+\int_\Omega \gamma\,|\overline{\partial}^*_\psi u|^2 e^{-\psi}\,. \quad (3.5.2)$$

Choose $\psi=|w|^2+\kappa$. Since

$$\eta(\partial_\alpha\partial_{\overline{\beta}}\psi)u_{\overline{\alpha}}\overline{u_{\overline{\beta}}}\geq|\langle u,dw\rangle|^2\geq\frac{|w|^2}{|w|^2+\varepsilon^2}\,|\langle u,dw\rangle|^2\;,$$

it follows that

$$\int_\Omega \overline{u_{\overline{\beta}}}u_{\overline{\alpha}}\left(\partial_{\overline{\beta}}\partial_\alpha\psi\right)e^{-\varphi}+\int_\Omega \gamma\,|\overline{\partial}^*_\psi u|^2\,e^{-\psi}$$
$$=\;\int_\Omega \eta(\partial_\alpha\partial_{\overline{\beta}}\psi)u_{\overline{\alpha}}\overline{u_{\overline{\beta}}}\,e^{-\psi}+\int_\Omega \gamma\,|\overline{\partial}^*_\psi u|^2\,e^{-\psi}$$
$$\geq\;\int_\Omega \frac{|w|^2}{|w|^2+\varepsilon^2}\;|\langle u,dw\rangle|^2\,e^{-\psi}+\int_\Omega \gamma\;\left|\bar\partial^*_\psi u\right|^2 e^{-\psi}$$
$$\geq\;\left|2\mathrm{Re}\int_\Omega \frac{\partial_\alpha\eta}{\eta}\,u_{\bar\alpha}(\overline{\partial}^*_\psi u)e^{-\varphi}\right|\;,$$

where the last inequality is from (3.5.2). Adding $\int_\Omega \gamma\,|\overline{\partial}^*_\psi u|^2\,e^{-\psi}$ to both sides of (3.5.1), we obtain

$$\int_\Omega\left\langle(\eta+\gamma)\overline{\partial}^*_\psi u,\overline{\partial}^*_\psi u\right\rangle e^{-\psi}+\int_\Omega\langle\eta\,\overline{\partial}u,\bar\partial u\rangle\,e^{-\psi}$$
$$\geq\int_\Omega \overline{u_{\overline{\beta}}}u_{\overline{\alpha}}\left(\partial_{\overline{\beta}}\partial_\alpha\psi\right)e^{-\varphi}+\int_\Omega \gamma\,|\overline{\partial}^*_\psi u|^2\,e^{-\psi}$$
$$-\int_\Omega\left(\frac{\partial_{\bar\alpha}\partial_\beta\eta}{\eta}\right)u_\alpha\overline{u_\beta}e^{-\varphi}+2\,\mathrm{Re}\int_\Omega\left(\frac{\partial_\alpha\eta}{\eta}\right)u_{\bar\alpha}\,\overline{\left(\bar\partial^*_\psi u\right)}e^{-\varphi}$$
$$\geq-\int_\Omega\left(\frac{\partial_{\bar\alpha}\partial_\beta\eta}{\eta}\right)u_\alpha\overline{u_\beta}e^{-\varphi}$$
$$=\;\int_\Omega\frac{\varepsilon^2}{(|w|^2+\varepsilon^2)^2}\,|\langle u,dw\rangle|^2\,e^{-\psi}\,. \quad (3.5.3)$$

We now consider the operator T defined by $Tu = \bar\partial(\sqrt{\eta+\gamma}\,u)$ and the operator S defined by $Su = \sqrt{\eta}\,\bar\partial u$. Then $ST = 0$ and we can rewrite (3.5.3) as

$$\|T^* u\|^2_{\Omega,\psi} + \|Su\|^2_{\Omega,\psi} \geq \int_\Omega \frac{\varepsilon^2}{(|w|^2+\varepsilon^2)^2} \left|\langle u, dw\rangle\right|^2 e^{-\psi} . \tag{3.5.4}$$

Here $\|\cdot\|_{\Omega,\psi}$ means the L^2 norm over Ω with respect to the weight function $e^{-\psi}$.

3.6 *Choice of Cut-Off Function.*

Choose any positive number $\delta < 1$. Choose a C^∞ function $0 \leq \varrho(x) \leq 1$ of a single real variable x on $[0,\infty)$ so that the support of ϱ is in $[0,1]$ and $\varrho(x)$ is identically 1 on $[0,\frac{\delta}{2}]$ and the supremum norm of $\frac{\partial}{\partial x}\varrho(x)$ on $[0,1]$ is no more than $1+\delta$.

Let $\varrho_\varepsilon(w) = \varrho\left(\frac{|w|^2}{\varepsilon^2}\right)$ and let

$$g_\varepsilon = \frac{\left(\tilde f \wedge dw\right)\bar\partial\varrho_\varepsilon}{w} = \frac{\tilde f \wedge dw}{\varepsilon^2}\varrho'\left(\frac{|w|^2}{\varepsilon^2}\right) d\bar w .$$

We would like to solve the equation $T h_\varepsilon = g_\varepsilon$ for some $(n,0)$-form h_ε on Ω. For that we would like to verify the inequality

$$\left|(u, g_\varepsilon)_{\Omega,\psi}\right|^2 \leq C^2\left(\|T^* u\|^2_{\Omega,\psi} + \|S\,u\|^2_{\Omega,\psi}\right)$$

for some positive constant C and for all $u \in \operatorname{Dom} S \cap \operatorname{Dom} T^*$. Here $(\cdot,\cdot)_{\Omega,\psi}$ means the inner product on Ω with respect to the weight function $e^{-\psi}$. We have

$$\begin{aligned}
\left|(u, g_\varepsilon)_{\Omega,\psi}\right|^2 &= \int_\Omega \left|\langle u, g_\varepsilon\rangle\right| e^{-\psi} \\
&= \int_\Omega \left|\left\langle u, \frac{\tilde f \wedge dw}{\varepsilon^2}\varrho'\left(\frac{|w|^2}{\varepsilon^2}\right) d\bar w\right\rangle\right| e^{-\psi} \\
&\leq \left(\int_\Omega \left|\frac{\tilde f \wedge dw}{\varepsilon^2}\varrho'\left(\frac{|w|^2}{\varepsilon^2}\right)\right|^2 \frac{(|w|^2+\varepsilon^2)^2}{\varepsilon^2} e^{-\psi}\right) \\
&\quad \cdot \left(\int_\Omega \left|\langle u, d\bar w\rangle\right|^2 \frac{\varepsilon^2}{(|w|^2+\varepsilon^2)^2} e^{-\psi}\right) \\
&\leq C_\varepsilon(\|T^* u\|^2_{\Omega,\psi} + \|S\,u\|^2_{\Omega,\psi}) ,
\end{aligned}$$

where

$$C_\varepsilon = \int_\Omega \left|\frac{\tilde f \wedge dw}{\varepsilon^2}\varrho'\left(\frac{|w|^2}{\varepsilon^2}\right)\right|^2 \frac{(|w|^2+\varepsilon^2)^2}{\varepsilon^2} e^{-\psi}$$

and the last inequality is from (3.5.4). We can solve $\bar{\partial}(\sqrt{\eta+\gamma}\, h_\varepsilon) = g_\varepsilon$ with

$$\int_\Omega |h_\varepsilon|^2 e^{-\psi} \le C_\varepsilon \,. \tag{3.6.1}$$

As $\varepsilon \to 0$, we have the following bound for the limit of C_ε.

$$\limsup_{\varepsilon\to 0} C_\varepsilon \le \left(\int_{\Omega\cap Y} |f|^2 e^{-\kappa}\right)\left(\limsup_{\varepsilon\to 0}\,(1+\delta)^2 \int_{|w|\le\varepsilon} \frac{(|w|^2+\varepsilon^2)^2}{\varepsilon^6}|dw|^2\right)$$

$$\le 8\pi\,(1+\delta)^2 \int_{\Omega\cap Y} |f|^2 e^{-\kappa} \,. \tag{3.6.2}$$

3.7 *Final Step in the Proof of Theorem 3.1.*

We now set

$$F_\varepsilon = \varrho_\varepsilon \tilde{f} \wedge dw - w\sqrt{\eta+\gamma}\, h_\varepsilon \,.$$

Then

$$\lim_{\varepsilon\to 0} \int_\Omega \left|\varrho_\varepsilon \tilde{f} \wedge dw\right|^2 e^{-\kappa} = 0 \,,$$

because $\tilde{f} \wedge dw$ is smooth in a relatively compact open neighborhood of $\bar{\Omega}$ in $Y - V$ and the support of $\varrho_\varepsilon \tilde{f} \wedge dw$ approaches a set of measure zero in Ω as $\varepsilon \to 0$. The supremum norm of $w\sqrt{\eta+\gamma}$ on $\Omega \subset \{|w| < 1\}$ is no more than the square root of

$$\sup_{0<x\le 1} x^2 \left(\log A + \log\frac{1}{x^2+\varepsilon^2} + \frac{1}{x^2+\varepsilon^2}\right) \le \log A + \frac{1}{e} + 1 \,,$$

because the maximum of $y \log \frac{1}{y}$ on $0 < y \le 1$ occurs at $y = \frac{1}{e}$ where its value is $\frac{1}{e}$ as one can easily verify by checking the critical points of $y \log \frac{1}{y}$. Since A is any number greater than the base of natural logarithm e and δ is any positive number, when we take limit as $A \to e$ and $\delta \to 0$ and $\nu \to \infty$ and we use (3.6.2) and

$$\int_\Omega |h_\varepsilon|^2 e^{-\kappa} \le e\, C_\varepsilon$$

from (3.6.1) and $\sup_\Omega |w| \le 1$, it follows that the limit of F is an L-valued holomorphic n-form on Y whose restriction to Z is $f \wedge dw$ with the following estimate on its norm.

$$\int_Y |F|^2 e^{-\kappa} \le 8\pi e \sqrt{2 + \frac{1}{e}} \int_Z |f|^2 e^{-\kappa} \,.$$

This finishes the proof of Theorem 3.1. The following version of extension from submanifolds of higher codimension follows from successive applications of Theorem 3.1.

Theorem 3.8. *Let Y be a complex manifold of complex dimension n. Let $1 \leq k \leq n$ be an integer and $w_1, \cdots, w_k$ be bounded holomorphic functions on Y whose common zero-set is a complex submanifold Z of complex codimension k in Y (with multiplicity 1 at every point of it). Let L be a holomorphic line bundle over Y with a (possibly singular) metric $e^{-\kappa}$ whose curvature current is semipositive. Assume that there exists a hypersurface V in Y such that $V \cap Z$ is a subvariety of dimension $\leq n-k-1$ in Z and $Y - V$ is the union of a sequence of Stein subdomains Ω_ν of smooth boundary and Ω_ν is relatively compact in $\Omega_{\nu+1}$. If f is an L-valued holomorphic $(n-k)$-form on Z with*

$$\int_Z |f|^2 \, e^{-\kappa} < \infty \, ,$$

then $f dw_1 \wedge \cdots \wedge dw_k$ can be extended to an L-valued holomorphic n-form F on Y such that

$$\int_Y |F|^2 \, e^{-\kappa} \leq \left(8\pi \, e \sqrt{2 + \frac{1}{e}} \right)^k \left(\sup_Y |w_1 \cdots w_k|^2 \right) \int_Z |f|^2 \, e^{-\kappa} \, .$$

Proof. We can find a hypersurface V_ν in Ω_ν such that $V_\nu \cap Z$ is of complex dimension $\leq n-k-1$ in Z and $dw_1 \wedge \cdots \wedge dw_k$ is nowhere zero on $\Omega_\nu - V_\nu$. We can now apply Theorem 3.1 to the case with Y replaced

$$(\Omega_\nu - V_\nu) \cap \{w_\ell = \cdots = w_k = 0\}$$

and Z replaced by

$$(\Omega_\nu - V_\nu) \cap \{w_{\ell+1} = \cdots = w_k = 0\}$$

and w replaced by $w_{\ell+1}$ for $0 \leq \ell < k$ and use descending induction on ℓ. The theorem now follows by removable singularity for L^2 holomorphic functions and the independence of the constants on $\Omega_\nu - V_\nu$ so that one can pass to limit as $\nu \to \infty$. □

4 Induction Argument with Estimates

In this section we carefully keep track of the estimates in the induction argument of [Siu98]. We follow the logical framework set forth in the axiomatic formulation of the induction argument in (1.3). The effective version of the induction argument estimates the L^2 norm of the quotient of the absolute value of a twisted $(p+1)$-canonical section by the maximum of the absolute value of twisted p-canonical sections. Again, this is the implementation of the rough and naive motivation explained in the Introduction. The effective versions of the global generation of multiplier ideal sheaves (2.1) and the extension theorem of Ohsawa-Takegoshi type (3.1) will be used in the process. As explained in the Introduction, metrics for the relevant bundles on the initial fibers have to be as singular as possible (see (5.2)). Given an element of $s^{(m_0)} \in \Gamma(X_0, m_0 K_{X_0} + L)$, we use the $\left|s^{(m_0)}\right|^{\frac{-2p}{m_0}}$ for the construction of metrics for a properly twisted $p K_{X_0}$. To avoid fractional multiples of L, for coefficients of L in (4.1) below we have to introduce integers closest to the quotient $\frac{p}{m_0}$.

Proposition 4.1. *Let A, U_{P_0}, and $C^\flat$ be as in Proposition 2.1. Let U_λ ($1 \leq \lambda \leq \Lambda$) be a covering of X_0 such that each U_λ is of the form*

$$U_{P_\lambda} \cap \left\{ \left| z^{(P_\lambda)} \right| < \frac{1}{2n\sqrt{C^\flat}} \right\}$$

for some point P_λ of X_0 (i.e., $U_\lambda = \hat{U}_{P_\lambda, r}$ with $r = \frac{1}{2}$). Let $C^\heartsuit$ be a positive constant such that, for any holomorphic line bundle E over X with a (possibly singular) metric h_E of semi-positive curvature current and for any element

$$s \in \Gamma(X_0, E + K_{X_0})$$

with

$$\int_{X_0} |s|^2 \, h_E < \infty \, ,$$

there exists an extension

$$\tilde{s} \in \Gamma(X, E + K_X)$$

such that

$$\int_X |\tilde{s}|^2 \, h_E \leq C^\heartsuit \int_{X_0} |s|^2 \, h_E \, .$$

(According to Theorem 3.1 the constant $C^\heartsuit$ can be taken to be $8\pi e \sqrt{2 + \frac{1}{e}}$.) Let $\{\varrho_\lambda\}_{1 \leq \lambda \leq \Lambda}$ be a partition of unity subordinate to the covering $\{U_\lambda\}_{1 \leq \lambda \leq \Lambda}$ of X_0. For $1 \leq \lambda \leq \Lambda$, let

$$\begin{aligned} \tau_{\lambda, A} &\in \Gamma(U_\lambda, A) \, , \\ \tau_{\lambda, L} &\in \Gamma(U_\lambda, L) \, , \\ \xi_\lambda &\in \Gamma(U_\lambda, -K_{X_0}) \end{aligned}$$

be all nowhere zero. Let L be a holomorphic line bundle over X with a (possibly singular) metric $e^{-\varphi}$ whose curvature current is semi-positive on X. Let m_0 be an integer ≥ 2 and let

$$s^{(m_0)} \in \Gamma(X_0, m_0 K_{X_0} + L)$$

be non identically zero such that $\left|s^{(m_0)}\right|^2 e^{-\varphi}$ is locally bounded on X_0. Let

$$C^{(\%)} = \int_{X_0} \left| s^{(m_0)} \right|^{\frac{2}{m_0}} e^{-\frac{1}{m_0}\varphi} \, .$$

Let $\psi = \frac{1}{m_0}\log\left|s^{(m_0)}\right|^2$ *so that* $e^{-m_0\psi}$ *is a metric for* $m_0 K_{X_0} + L$ *and* $e^{\psi - \frac{1}{m_0}\varphi}$ *is locally bounded on* X_0. *Let* ℓ *be a positive number* ≥ 2 *and let* $m_1 = \ell\, m_0$. *For* $1 \leq p \leq m_1$, *let* a_p *denote the smallest integer which is no less than* $\frac{p-1}{m_0}$. *Let* $d_p = a_p - \frac{p-1}{m_0}$ *for* $1 \leq p \leq m_1$ *and* $c_p = a_p - a_{p-1}$ *for* $2 \leq p \leq m_1$. *Let* $C^{\natural}$ *be the maximum of*

$$\begin{cases} 4\, C^{\flat} C^{(\%)} \sup_{U_{P_\lambda}} \left(\left| \xi_\lambda \tau_{\lambda,L}^{-c_p} \right|^2 e^{\psi + \left(c_p - \frac{1}{m_0}\right)\varphi} \right) \text{ for } 1 \leq \lambda \leq \Lambda \text{ and } 2 \leq p < m_1, \\ 4\, C^{\flat} C^{(\%)} \sup_{U_{P_\lambda}} \left(\left| \xi_\lambda \tau_{\lambda,L}^{-c_{m_1}} \tau_{\lambda,A} \right|^2 e^{\psi + \left(c_{m_1} - \frac{1}{m_0}\right)\varphi} h_A \right) \text{ for } 1 \leq \lambda \leq \Lambda\,. \end{cases}$$

For $1 \leq p < m_1$ *let* N_p *be the complex dimension of the subspace of all elements* $s \in \Gamma(X_0, p\, K_{X_0} + a_p\, L + A)$ *such that*

$$\int_{X_0} |s|^2\, e^{-(p-1)\psi - d_p\varphi}\, h_A < \infty\,.$$

Then there exist

$$s_1^{(p)}, \cdots, s_{N_p}^{(p)} \in \Gamma(X_0, p\, K_{X_0} + a_p\, L + A)$$

for $1 \leq p < m_1$ *with*

$$\int_{X_0} \left| s_j^{(p)} \right|^2 e^{-(p-1)\psi - d_p\varphi}\, h_A \leq 1$$

for $1 \leq j \leq N_p$ *such that*

(i) $s_j^{(p)}$ *can be extended to*

$$\tilde{s}_j^{(p)} \in \Gamma(X, p\, K_X + a_p\, L + A)$$

for $1 \leq p < m_1$ *and* $1 \leq j \leq N_p$,

(ii)

$$\int_X \frac{\left|\tilde{s}_j^{(p+1)}\right|^2 e^{-c_{p+1}\varphi}}{\max_{1\leq k\leq N_p} \left|\tilde{s}_k^{(p)}\right|^2} \leq C^{\heartsuit}\, C^{\natural} \int_{X_0} \sum_{\lambda=1}^{\Lambda} \frac{\varrho_\lambda \left|\tau_{\lambda,L}^{c_{p+1}}\right|^2 e^{-c_{p+1}\varphi}}{|\xi_\lambda|^2}$$

for $1 \leq p \leq m_1 - 2$ *and* $1 \leq j \leq N_{p+1}$,

(iii)

$$\int_X \left|\tilde{s}_j^{(1)}\right|^2 h_A \leq C^{\heartsuit}$$

for $1 \leq j \leq N_1$, *and*

(iv)

$$\int_{X_0} \frac{\left|\left(s^{(m_0)}\right)^{\ell} s_A\right|^2 e^{-c_{m_1}\varphi}}{\max_{1\le k\le N_{m_1-1}} \left|\tilde{s}_k^{(m_1-1)}\right|^2} \le C^{\natural} \int_{X_0} \sum_{\lambda=1}^{\Lambda} \frac{\varrho_\lambda \left|s_A \tau_{\lambda,L}^{c_{m_1}}\right|^2 e^{-c_{m_1}\varphi}}{\left|\xi_\lambda \tau_{\lambda,A}\right|^2},$$

where s_A is any holomorphic section of A over X.

In particular, $\left(s^{(m_0)}\right)^{\ell} s_A$ can be extended to

$$\hat{s}^{(m_1)} \in \Gamma\left(X, m_1 K_X + \ell L + A\right)$$

such that

$$\int_{X} \frac{\left|\hat{s}^{(m_1)}\right|^2 e^{-c_{m_1}\varphi}}{\max_{1\le k\le N_{m_0-1}} \left|\tilde{s}_k^{(m_1-1)}\right|^2} \le C^{\heartsuit} C^{\natural} \int_{X_0} \sum_{\lambda=1}^{\Lambda} \frac{\varrho_\lambda \left|s_A \tau_{\lambda,L}^{c_{m_1}}\right|^2 e^{-c_{m_1}\varphi}}{\left|\xi_\lambda \tau_{\lambda,A}\right|^2}.$$

Proof of Proposition 4.1. The nonnegative integers a_p and the numbers $0 \le d_p < 1$ are introduced so that $e^{-(p-1)\psi - d_p\varphi}$ is a metric for the line bundle $(p-1)\,K_{X_0} + a_p L$ on X_0. The integers c_p are introduced so that multiplication by $\xi_\lambda \tau_{\lambda,L}^{-c_p}$ defines a map from the sheaf of germs of holomorphic sections of $p\,K_{X_0} + a_p L$ on U_λ to the sheaf of germs of holomorphic sections of $(p-1)\,K_{X_0} + a_{p-1}L$ on U_λ. We note that

$$a_1 = d_1 = 0\,, \quad a_{m_1} = \ell, \quad d_{m_1} = \frac{1}{m_0}\,, \quad 0 \le c_p \le 1 \text{ for } 2 \le p \le m_1. \tag{4.1.1}$$

For $1 \le p \le m_1$ the local function

$$(p-1)\,\psi + d_p\,\varphi = \frac{p-1}{m_0} \log\left|s^{(m_0)}\right|^2 + d_p\,\varphi \ \text{ is plurisubharmonic.} \tag{4.1.2}$$

By the definition of $C^{(\%)}$ and the value of d_{m_1} from (4.1.1) we have

$$\int_{X_0} \left|\left(s^{(m_0)}\right)^{\ell}\right|^2 e^{-(m_1-1)\psi - d_{m_1}\varphi} = C^{(\%)}\,. \tag{4.1.3}$$

Now we are going to apply Theorem 2.1 on the global generation of multiplier ideal sheaves with estimates to prove the following claim by descending induction on $1 \le p \le m_1 - 1$. This claim involving estimates corresponds to the step containing (1.4.1) and (1.4.2) in the proof of Proposition 1.4.

(4.1.4) *Claim.* There exist

(i)
$$s_1^{(p)}, \cdots, s_{N_p}^{(p)} \in \Gamma\left(X_0, p\,K_{X_0} + a_p\,L + A\right)$$
for $1 \leq p \leq m_1 - 1$,

(ii) holomorphic functions $b_{j,k}^{(p,\lambda)}$ on U_λ, for $1 \leq p \leq m_1 - 2$, $1 \leq j \leq N_{p+1}$, $1 \leq k \leq N_p$, and $1 \leq \lambda \leq \Lambda$, and

(iii) holomorphic functions $b_k^{(m_1-1,\lambda)}$ on U_λ for $1 \leq k \leq N_{m_1-1}$ and $1 \leq \lambda \leq \Lambda$ such that, for $1 \leq \lambda \leq \Lambda$,

(a)
$$\xi_\lambda\, \tau_{\lambda,A}\, \tau_{\lambda,L}^{-c_{m_1}} \left(s^{(m_0)}\right)^\ell = \sum_{k=1}^{N_{m_1-1}} b_k^{(m_1-1,\lambda)}\, s_k^{(m_1-1)}$$
on U_λ,

(b)
$$\xi_\lambda\, \tau_{\lambda,L}^{-c_{p+1}}\, s_j^{(p+1)} = \sum_{k=1}^{N_p} b_{j,k}^{(p,\lambda)}\, s_k^{(p)}$$
on U_λ for $1 \leq p \leq m_1 - 2$ and $1 \leq j \leq N_{p+1}$,

(c)
$$\sup_{U_\lambda} \sum_{1 \leq k \leq N_{m_1-1}} \left| b_k^{(m_1-1,\lambda)} \right|^2 \leq C^{\natural},$$

(d)
$$\sup_{U_\lambda} \sum_{k=1}^{N_p} \left| b_{j,k}^{(p,\lambda)} \right|^2 \leq C^{\natural}$$
for $1 \leq j \leq N_{p+1}$ and $1 \leq k \leq N_p$ and $1 \leq p \leq m_1 - 2$, and

(e)
$$\int_{X_0} \left| s_k^{(p)} \right|^2 e^{-(p-1)\psi - d_p\,\varphi}\, h_A \leq 1$$
for $1 \leq k \leq N_p$ and $1 \leq p \leq m_1 - 1$.

To verify Claim (4.1.4) by descending induction on $1 \leq p \leq m_1 - 1$, we do first the step $p = m_1 - 1$ and start out with

$$\xi_\lambda\, \tau_{\lambda,A}\, \tau_{\lambda,L}^{-c_{m_1}} \left(s^{(m_0)}\right)^\ell \in \Gamma\left(U_\lambda, (m_1 - 1)\, K_{X_0} + a_{m_1-1} L + A\right).$$

By (4.1.3) we have

$$\int_{U_{P_\lambda}} \left| \xi_\lambda\, \tau_{\lambda,A}\, \tau_{\lambda,L}^{-c_{m_1}} \left(s^{(m_0)}\right)^\ell \right|^2 e^{-(m_1-2)\psi - d_{m_1-1}\varphi}\, h_A$$
$$\leq C^{(\%)} \sup_{U_{P_\lambda}} \left(\left| \xi_\lambda\, \tau_{\lambda,A}\, \tau_{\lambda,L}^{-c_{m_1}} \right|^2 e^{\psi + \left(c_{m_1} - \frac{1}{m_0}\right)\varphi}\, h_A \right).$$

By applying Theorem 2.1 with $r = \frac{1}{2}$ to the line bundle $(m_1 - 2) K_{X_0} + a_{m_1 - 1} L$ with the metric $e^{-(m_1-2)\psi - d_{m_1-1}\varphi}$, from (4.1.2) we obtain

$$s_1^{(m_1-1)}, \cdots, s_{N_{m_1-1}}^{(m_1-1)} \in \Gamma\left(X_0, (m_1 - 1) K_{X_0} + a_{m_1-1} L + A\right)$$

and holomorphic functions $b_k^{(m_1-1,\lambda)}$ on U_λ, for $1 \leq k \leq N_{m_1-1}$, such that

$$\xi_\lambda \, \tau_{\lambda,A} \, \tau_{\lambda,L}^{-c_{m_1}} \left(s^{(m_0)}\right)^{\ell} = \sum_{k=1}^{N_{m_1-1}} b_k^{(m_1-1,\lambda)} \, s_k^{(m_1-1)}$$

and

$$\begin{aligned}
&\sup_{U_\lambda} \sum_{k=1}^{N_{m_1-1}} \left| b_{j,k}^{(m_1-1,\lambda)} \right|^2 \\
&\qquad \leq 4 C^{\flat} \int_{U_{P_\lambda}} \left| \xi_\lambda \, \tau_{\lambda,L}^{-c_{m_1}} \, \tau_{\lambda,A} \left(s^{(m_0)}\right)^{\ell} \right|^2 e^{-(m_1-2)\psi - d_{m_1-1}\varphi} \\
&\qquad \leq 4 C^{\flat} \sup_{U_{P_\lambda}} \left(\left| \xi_\lambda \, \tau_{\lambda,L}^{-c_{m_1}} \, \tau_{\lambda,A} \right|^2 , e^{\psi + \left(c_{m_1} - \frac{1}{m_0}\right)\varphi} \, h_A \right) \\
&\qquad\quad \cdot \left(\int_{U_{P_\lambda}} \left| \left(s^{(m_0)}\right)^{\ell} \right|^2 e^{-(m_1-1)\psi - d_{m_1}\varphi} \right) \\
&\qquad \leq 4 C^{\flat} C^{(\%)} \sup_{U_{P_\lambda}} \left(\left| \xi_\lambda \, \tau_{\lambda,L}^{-c_{m_1}} \, \tau_{\lambda,A} \right|^2 e^{\psi + \left(c_{m_1} - \frac{1}{m_0}\right)\varphi} \, h_A \right) \leq C^{\natural}
\end{aligned}$$

and

$$\int_{X_0} \left| s_k^{(m_1-1)} \right|^2 e^{-(m_1-2)\psi - d_{m_1-1}\varphi} \, h_A \leq 1 \, .$$

Thus we have (4.1.4)(a) and (4.1.4)(c) and the case $p = m_1 - 1$ of (4.1.4)(e). This finishes the initial step $p = m_1 - 1$.

Suppose we have done this for Step p and we would like to do it for the next step which is Step $p - 1$. Since

$$\xi_\lambda \, \tau_{\lambda,L}^{-c_p} \, s_k^{(p)} \in \Gamma\left(U_\lambda, (p-1) K_{X_0} + a_{p-1} L + A\right)$$

with

$$\begin{aligned}
&\int_{U_{P_\lambda}} \left| \xi_\lambda \, \tau_{\lambda,L}^{-c_p} \, s_k^{(p)} \right|^2 e^{-(p-2)\psi - d_{p-1}\varphi} \, h_A \\
&\qquad \leq \sup_{U_{P_\lambda}} \left(\left| \xi_\lambda \, \tau_{\lambda,L}^{-c_p} \right|^2 e^{\psi + \left(c_p - \frac{1}{m_0}\right)\varphi} \right) \int_{U_{P_\lambda}} \left| s_k^{(p)} \right|^2 e^{-(p-1)\psi - d_p \varphi} \, h_A \\
&\qquad \leq \sup_{U_{P_\lambda}} \left(\left| \xi_\lambda \, \tau_{\lambda,L}^{-c_p} \right|^2 e^{\psi + \left(c_p - \frac{1}{m_0}\right)\varphi} \right)
\end{aligned}$$

from (4.1.4)(e) of Step p, it follows from Theorem 2.1 with $r = \frac{1}{2}$ applied to the line bundle $(p-1)\,K_{X_0} + a_{p-1}L$ with the metric $e^{-(p-2)\psi - d_{p-1}\varphi}$ and from (4.1.2) that we obtain

$$s_k^{(p-1)} \in \Gamma\left(X_0, (p-1)K_{X_0} + a_{p-1}\,L + A\right)$$

for $1 \leq k \leq N_{p-1}$ and holomorphic functions $b_{j,k}^{(p-1,\lambda)}$ on U_λ, for $1 \leq j \leq N_p$ and $1 \leq k \leq N_{p-1}$ such that

$$\xi_\lambda\, \tau_{\lambda,L}^{-c_p}\, s_j^{(p)} = \sum_{k=1}^{N_{p-1}} b_{j,k}^{(p-1,\lambda)}\, s_k^{(p-1)}$$

and

$$\sup_{U_\lambda} \sum_{k=1}^{N_{p-1}} \left| b_{j,k}^{(p-1,\lambda)} \right|^2 \leq 4\, C^\flat \sup_{U_{P_\lambda}} \left(\left| \xi_\lambda\, \tau_{\lambda,L}^{-c_p} \right|^2 e^{\psi + \left(c_p - \frac{1}{m_0}\right)\varphi} \right) \leq C^\natural .$$

Thus we have (4.1.4)(b) and (4.1.4)(d) and (4.1.4)(e). This finishes the the verification of Claim (4.1.4) by induction.

Next we use induction on $1 \leq p < m_1$ to verify the following claim. This claim involving estimates corresponds to Claim (1.4.3) in the proof of Proposition 1.4.

(4.1.5) *Claim.* One can extend $s_j^{(p)}$ to

$$\tilde{s}_j^{(p)} \in \Gamma\left(X, p\,K_X + a_p\,L + A\right)$$

for $1 \leq j \leq N_p$ and $1 \leq p < m_1$ such that the extension

$$\tilde{s}_j^{(p)} \in \Gamma\left(X, p\,K_X + a_p\,L + A\right)$$

satisfies the inequalities (ii) and (iii) in the statement of Proposition 4.1.

To start the induction process when $p = 1$, we simply observe that, since the curvature form of the metric h_A of A is positive on X and since

$$\int_{X_0} \left| s_k^{(1)} \right|^2 h_A \leq 1 ,$$

it follows from the assumption on $C^\heartsuit$ that $s_j^{(1)}$ can be extended to

$$\tilde{s}_j^{(1)} \in \Gamma\left(X, K_X + A\right)$$

with

$$\int_X \left| \tilde{s}_j^{(1)} \right|^2 h_A \leq C^\heartsuit ,$$

which is inequality (iii) in the statement of Proposition 4.1.

Suppose Step p has been proved and we go to the next step which is Step $p+1 < m_1$. Since

$$\xi_\lambda \, \tau_{\lambda,L}^{-c_{p+1}} \, s_j^{(p+1)} = \sum_{k=1}^{N_p} b_{j,k}^{(p,\lambda)} \, s_k^{(p)}$$

by (4.1.4)(b) and

$$\sup_{U_\lambda} \sum_{k=1}^{N_p} \left| b_{j,k}^{(p,\lambda)} \right|^2 \leq C^{\natural}$$

by (4.1.4)(d), it follows that

$$\int_{X_0} \frac{\left| s_j^{(p+1)} \right|^2 e^{-c_{p+1}\varphi}}{\max_{1 \leq k \leq N_p} \left| \tilde{s}_k^{(p)} \right|^2} = \int_{X_0} \frac{\sum_{\lambda=1}^{\Lambda} \varrho_\lambda \left| \left(\frac{\tau_{\lambda,L}^{c_{p+1}}}{\xi_\lambda} \right) \tau_{\lambda,L}^{-c_{p+1}} \, \xi_\lambda \, s_j^{(p+1)} \right|^2 e^{-c_{p+1}\varphi}}{\max_{1 \leq k \leq N_p} \left| s_k^{(p)} \right|^2}$$

$$\leq \left(\sup_{U_\lambda} \sum_{k=1}^{N_p} \left| b_{j,k}^{(p,\lambda)} \right|^2 \right) \int_{X_0} \sum_{\lambda=1}^{\Lambda} \frac{\varrho_\lambda \left| \tau_{\lambda,L}^{c_{p+1}} \right|^2 e^{-c_{p+1}\varphi}}{\left| \xi_\lambda \right|^2}$$

$$\leq C^{\natural} \int_{X_0} \sum_{\lambda=1}^{\Lambda} \frac{\varrho_\lambda \left| \tau_{\lambda,L}^{c_{p+1}} \right|^2 e^{-c_{p+1}\varphi}}{\left| \xi_\lambda \right|^2} .$$

By the assumption on $C^{\heartsuit}$, one can extend $s_j^{(p+1)}$ to

$$\tilde{s}_j^{(p+1)} \in \Gamma\left(X, (p+1)K_X + a_{p+1}\, L + A\right)$$

with

$$\int_X \frac{\left| \tilde{s}_j^{(p+1)} \right|^2 e^{-c_{p+1}\varphi}}{\max_{j=1}^{N_p} \left| \tilde{s}_j^{(p)} \right|^2} \leq C^{\heartsuit} \, C^{\natural} \int_{X_0} \sum_{\lambda=1}^{\Lambda} \frac{\varrho_\lambda \left| \tau_{\lambda,L}^{c_{p+1}} \right|^2 e^{-c_{p+1}\varphi}}{\left| \xi_\lambda \right|^2} ,$$

which is inequality (ii) in the statement of Proposition 4.1. This finishes the verification of Claim (4.1.5) by induction.

Finally, since

$$\xi_\lambda \, \tau_{\lambda,A} \, \tau_{\lambda,L}^{-c_{m_1}} \left(s^{(m_0)} \right)^{\ell} = \sum_{k=1}^{N_{m_1-1}} b_k^{(m_1-1,\lambda)} \, s_k^{(m_1-1)}$$

by (4.1.4)(a) and

$$\sup_{U_\lambda} \sum_{k=1}^{N_{m_1-1}} \left| b_k^{(m_1-1,\lambda)} \right|^2 \leq C^{\natural}$$

by (4.1.4)(c), it follows that

$$\int_{X_0} \frac{\left|\left(s^{(m_0)}\right)^{\ell} s_A\right|^2 e^{-c_{m_1}\varphi}}{\max_{1\leq k\leq N_p}\left|\tilde{s}_k^{(m_1-1)}\right|^2}$$

$$= \int_{X_0} \frac{\sum_{\lambda=1}^{A} \varrho_\lambda \left|\left(\frac{s_A \tau_{\lambda,L}^{c_{m_1}}}{\xi_\lambda}\right) \tau_{\lambda,L}^{-c_{m_1}} \xi_\lambda \left(s^{(m_0)}\right)^{\ell}\right|^2 e^{-c_{m_1}\varphi}}{\max_{1\leq k\leq N_p}\left|s_k^{(p)}\right|^2}$$

$$\leq \left(\sum_{k=1}^{N_{m_1-1}} \sup_{U_\lambda}\left|b_k^{(m_1-1,\lambda)}\right|^2\right) \int_{X_0} \sum_{\lambda=1}^{A} \frac{\varrho_\lambda \left|s_A \tau_{\lambda,L}^{c_{m_1}}\right|^2 e^{-c_{m_1}\varphi}}{\left|\xi_\lambda \tau_{\lambda,A}\right|^2}$$

$$\leq C^{\natural} \int_{X_0} \sum_{\lambda=1}^{A} \frac{\varrho_\lambda \left|s_A \tau_{\lambda,L}^{c_{m_1}}\right|^2 e^{-c_{m_1}\varphi}}{\left|\xi_\lambda \tau_{\lambda,A}\right|^2},$$

which is inequality (iv) in the statement of Proposition 4.1. By the assumption on $C^{\heartsuit}$, $\left(s^{(m_0)}\right)^{\ell} s_A$ can be extended to

$$\hat{s}^{(m_1)} \in \Gamma\left(X, m_1 K_X + \ell L + A\right)$$

such that

$$\int_X \frac{\left|\hat{s}^{(m_1)}\right|^2 e^{-c_{m_1}\varphi}}{\max_{1\leq k\leq N_{m_1-1}}\left|\tilde{s}_k^{(m_1-1)}\right|^2} \leq C^{\heartsuit} C^{\natural} \int_{X_0} \sum_{\lambda=1}^{A} \frac{\varrho_\lambda \left|s_A \tau_{\lambda,L}^{c_{m_1}}\right|^2 e^{-c_{m_1}\varphi}}{\left|\xi_\lambda \tau_{\lambda,A}\right|^2}.$$

□

5 Effective Version of the Process of Taking Powers and Roots of Sections

This section corresponds to the effective version of the fourth ingredient listed in the Introduction, which is the process of raising a section on the initial fiber to a high power and later taking the root of the same order after extending its product with a holomorphic section of some fixed line bundle on X.

5.1 To apply Theorem 2.1 on the global generation of the multiplier ideal sheaf with estimates, we have to introduce an auxiliary sufficiently ample line bundle A. To eliminate the undesirable effects from A at the end, we take the ℓ-th power of an m_0-canonical section on X_0 to be extended and then we multiply the power by some section of A on X to make the extension of the product to X possible by Proposition 4.1 and then we take the ℓ-th root of the extension and let $\ell \to \infty$.

The goal of the limiting process is to produce a metric for $(m_0 - 1) K_X$ on X so that the m_0-canonical section on X_0 has finite L^2 norm on X_0 with respect to it. For the limiting process, we have to control the estimates to guarantee the convergence of the limit. We do this by using the concavity of the logarithmic function and the sub-mean-value property of the logarithm of the absolute value of a holomorphic function. A delicate point is that we have to get the bound on the dimension of certain spaces of sections independent of ℓ. It is because of this delicate point that, as explained in the Introduction and (1.5.2), the metric chosen for the pluricanonical line bundles of the initial fiber at the beginning of the effective argument has to be as singular as possible. For that purpose, one cannot use usual abstractly-defined general metrics for the pluricanonical line bundle of the initial fiber such as generalized Bergman kernels. The bound for the dimension in question will be given below in (5.2).

Fix a positive number m_0 and take a non identically zero

$$s^{(m_0)} \in \Gamma\left(X_0, m_0 K_{X_0} + L\right) .$$

Our goal is to extend it to an element of $\Gamma\left(X, m_0 K_X + L\right)$. Since the case of $m_0 = 1$ is an immediate consequence of the extension theorem 3.1, we can assume without loss of generality that $m_0 \geq 2$.

Fix an element $s_A \in \Gamma\left(X, A\right)$ such that its zero-set in X_0 is of complex codimension at least 1 in X_0. We are going to apply Proposition 4.1 and let $\ell \to \infty$. We need the following lemma concerning the bound on the dimension of certain spaces of sections independent of ℓ.

Lemma 5.2. *In the notations of the statement of Proposition 4.1, let $\tilde{N}_{m_0}$ be the maximum of the complex dimension of $\Gamma\left(X_0, k\, K_{X_0} + a\, L + A\right)$ for integers $1 \leq k \leq m_0$ and $a = 0, 1$. Then $N_p \leq \tilde{N}_{m_0}$ for $1 \leq p < m_0 \ell$ for any integer $\ell \geq 2$.*

Proof. We follow the notations of Proposition 4.1. For $1 \leq p < m_1$ let Γ_p be the subspace of all elements $s \in \Gamma\left(X_0, p\, K_{X_0} + a_p\, L + A\right)$ such that

$$\int_{X_0} |s|^2\, e^{-(p-1)\psi - d_p \varphi}\, h_A < \infty .$$

Let b_p be the largest integer not exceeding $\frac{p-1}{m_0}$. For any $s \in \Gamma_p$, the definition of Γ_p implies that

$$\frac{s}{\left(s^{(m_0)}\right)^{b_p}}$$

is a holomorphic section of $(p - b_p m_0)\, K_{X_0} + (a_p - b_p)\, L + A$ over X_0. Thus the map

$$\Xi_p : \Gamma_p \to \Gamma\left(X_0, (p - b_p m_0)\, K_{X_0} + (a_p - b_p)\, L + A\right)$$

defined by

$$\Xi_p(s) = \frac{s}{\left(s^{(m_0)}\right)^{b_p}}$$

is injective. We conclude that

$$\dim_{\mathbb{C}} \Gamma_p \leq \dim_{\mathbb{C}} \Gamma\left(X_0, (p - b_p m_0) K_{X_0} + (a_p - b_p) L + A\right)$$

Since

$$b_p \leq \frac{p-1}{m_0}, \quad b_p + 1 > \frac{p-1}{m_0}, \quad a_p \geq \frac{p-1}{m_0}, \quad a_p - 1 < \frac{p-1}{m_0},$$

it follows that $1 \leq p - b_p m_0 \leq m_0$ and $0 \leq a_p - b_p \leq 1$. The definitions of $\tilde{N}_{m_0}$ and N_p imply that $N_p \leq \tilde{N}_{m_0}$ for $1 \leq p < \ell m_0$ for any integer $\ell \geq 2$.

The reason for imposing the condition $1 \leq p < \ell m_0$ in the conclusion, instead of just getting the conclusion for all positive integers p, is that technically in the statement of Proposition 4.1 the number N_p is defined only for $p < m_1$ and $m_1 = \ell m_0$. □

5.3 *Application of Induction Argument.*

Take an arbitrary integer $\ell \geq 2$. We now apply Proposition 4.1 to

$$s^{(m_0)} \in \Gamma\left(X_0, m_0 K_{X_0} + L\right)$$

so that we can extend

$$\left(s^{(m_0)}\right)^{\ell} s_A \in \Gamma\left(X_0, \ell m_0 K_{X_0} + \ell L + A\right)$$

to

$$\hat{s}^{(\ell m_0, A)} \in \Gamma\left(X, \ell m_0 K_X + \ell L + A\right).$$

This extension from Proposition 4.1 comes with estimates. We use the notations $N_{p,A}$, ϱ_λ, ξ_λ, $\tau_{\lambda,A}$, $C^{\natural}$, and $C^{\heartsuit}$ of Proposition 4.1. By Lemma 5.2, $N_{p,A} \leq N_{m_0}$ for all $1 \leq p \leq \ell m_0 - 1$.

Let $C^{\diamondsuit}$ be the maximum of the following positive numbers

$$C^{\heartsuit}, \quad C^{\heartsuit} C^{\natural} \int_{X_0} \sum_{\lambda=1}^{\Lambda} \frac{\varrho_\lambda \left| s_A \, \tau_{\lambda,L}^{c_{m_1}} \right|^2 e^{-c_{m_1}\varphi}}{\left|\xi_\lambda \, \tau_{\lambda,A}\right|^2},$$

$$C^{\heartsuit} C^{\natural} \int_{X_0} \sum_{\lambda=1}^{\Lambda} \frac{\varrho_\lambda \left| \tau_{\lambda,L}^{c_{p+1}} \right|^2 e^{-c_{p+1}\varphi}}{\left|\xi_\lambda\right|^2} \quad \text{for } 1 \leq p \leq m_1 - 2.$$

The finiteness of $C^{\diamondsuit}$ follows from $0 \leq c_p \leq 1$ in 4.1.1 and from the local integrability of $e^{-\varphi}|_{X_0}$ on X_0. According to Proposition 4.1 we have

$$s_1^{(p)}, \cdots, s_{N_p}^{(p)} \in \Gamma\left(X_0, p K_{X_0} + a_p L + A\right)$$

for $1 \leq p < m_1$ such that

(i) the section $s_j^{(p)}$ can be extended to

$$\tilde{s}_j^{(p)} \in \Gamma\left(X, p\,K_X + a_p\,L + A\right)$$

for $1 \leq p < m_1$ and $1 \leq p \leq N_p$

(ii)

$$\int_X \frac{\left|\tilde{s}_j^{(p+1)}\right|^2 e^{-c_{p+1}\,\varphi}}{\max_{1 \leq k \leq N_p} \left|\tilde{s}_k^{(p)}\right|^2} \leq C^{\diamond} \tag{5.3.1}$$

for $1 \leq p \leq m_1 - 2$ and $1 \leq j \leq N_{p+1}$,

(iii)

$$\int_X \left|\tilde{s}_j^{(1)}\right|^2 h_A \leq C^{\diamond} \tag{5.3.2}$$

for $1 \leq j \leq N_1$, and

(iv)

$$\int_X \frac{\left|\hat{s}^{(m_1)}\right|^2 e^{-c_{m_1}\,\varphi}}{\max_{1 \leq k \leq N_{m_1-1}} \left|\tilde{s}_k^{(m_1-1)}\right|^2} \leq C^{\diamond}. \tag{5.3.3}$$

From (5.3.1) we have

$$\begin{aligned} \int_X \frac{\max_{1 \leq k \leq N_{p+1}} \left|\tilde{s}_j^{(p+1)}\right|^2 e^{-c_{p+1}\,\varphi}}{\max_{1 \leq k \leq N_p} \left|\tilde{s}_k^{(p)}\right|^2} &\leq \sum_{j=1}^{N_{p+1}} \int_X \frac{\left|\tilde{s}_j^{(p+1)}\right|^2 e^{-c_{p+1}\,\varphi}}{\max_{1 \leq k \leq N_p} \left|\tilde{s}_k^{(p)}\right|^2} \\ &\leq N_{p+1}\,C^{\diamond} \leq \tilde{N}_{m_0}\,C^{\diamond} \end{aligned} \tag{5.3.4}$$

for $1 \leq p \leq m_1 - 2$ and $1 \leq j \leq N_{p+1}$. It is for this step that we need the bound on N_{p+1} given in Lemma 5.2 which is independent of ℓ.

5.4 *Supremum Estimates by the Concavity of Logarithm and the Sub-Mean-Value Property of the Logarithm of the Absolute Value of a Holomorphic Function.*

To continue with our estimates, we have to switch at this point to the use of supremum norms. The reason is that we are estimating the ℓ-th root of the product of $\ell\,m_0$ factors, for each of which we have only an L^2 estimate and we cannot continue with L^2 estimates by using the Hölder inequality. We switch to supremum estimates by using the concavity of logarithm and the sub-mean-value property of the logarithm of the absolute value of a holomorphic function. The use of the sub-mean-value property for switching to supremum estimates necessitates the shrinking of the domain on which estimates are made. For notational convenience we assume that the family X can be extended to a larger one over a larger disk in $\mathbf{C}$.

More precisely, without loss of generality we can assume the following. The given family $\pi : X \to \Delta$ can be extended to a holomorphic family $\tilde{\pi} : \tilde{X} \to \tilde{\Delta}$ of compact complex projective algebraic manifolds so that $\tilde{\Delta}$ is an open disk of radius > 1 in $\mathbf{C}$ centered at 0. We can assume that the coordinate charts U_λ $(1 \leq \lambda \leq \Lambda)$ from Proposition 4.1 are restrictions of coordinate charts in $\tilde{X}$ in the following way. We have coordinate charts W_λ in $\tilde{X}$ $(1 \leq \lambda \leq \Lambda)$ with coordinates

$$\left(z_1^{(\lambda)}, \cdots, z_n^{(\lambda)}, t\right)$$

so that

$$\bigcup_{\lambda=1}^{\Lambda} W_\lambda = \tilde{X} .$$

Moreover, we assume that for each $1 \leq \lambda \leq \Lambda$ we have a relative compact open subset $\tilde{U}_\lambda$ of W_λ such that

$$\tilde{U}_\lambda = W_\lambda \bigcap \left\{ \left| z_1^{(\lambda)} \right| < 1, \cdots, \left| z_n^{(\lambda)} \right| < 1, |t| < 1 \right\}$$

and

$$\bigcup_{\lambda=1}^{\Lambda} \tilde{U}_\lambda = X$$

and

$$U_\lambda = X_0 \cap \tilde{U}_\lambda .$$

We can also assume that there exists nowhere zero elements

$$\hat{\xi}_\lambda \in \Gamma(W_\lambda, -K_{\tilde{X}}), \quad \hat{\tau}_{\lambda,A} \in \Gamma(W_\lambda, A), \quad \hat{\tau}_{\lambda,L} \in \Gamma(W_\lambda, L)$$

for $1 \leq \lambda \leq \Lambda$ such that

$$\xi_\lambda = \hat{\xi}_\lambda \,|\, U_\lambda ,$$
$$\tau_{\lambda,A} = \hat{\tau}_{\lambda,A} \,|\, U_\lambda ,$$
$$\tau_{\lambda,L} = \hat{\tau}_{\lambda,L} \,|\, U_\lambda$$

for $1 \leq \lambda \leq \Lambda$.

There exists some $0 < r_0 < 1$ such that, if we let $X' = \pi^{-1}(\Delta_{r_0})$ and

$$U'_\lambda = \tilde{U}_\lambda \bigcap \left\{ \left| z_1^{(\lambda)} \right| < r_0, \cdots, \left| z_n^{(\lambda)} \right| < r_0, |t| < r_0 \right\},$$

for $1 \leq \lambda \leq \Lambda$, then $X' = \bigcup_{\lambda=1}^{\Lambda} U'_\lambda$. Let $dV_{z^{(\lambda)},t}$ be the Euclidean volume form in the coordinates system

$$\left(z^{(\lambda)}, t\right) = \left(z_1^{(\lambda)}, \cdots, z_n^{(\lambda)}, t\right).$$

From (5.3.4) we have

$$\begin{aligned}
&\frac{1}{\pi^{n+1}}\int_{\tilde{U}_\lambda}\left(\log \max_{1\le j\le N_{p+1}}\left|\tau_{\lambda,A}^{-1}\,\tau_{\lambda,L}^{-a_{p+1}}\,\xi_\lambda^{p+1}\,\tilde{s}_j^{(p+1)}\right|^2\right)dV_{z(\lambda),t}\\
&\quad-\frac{1}{\pi^{n+1}}\int_{\tilde{U}_\lambda}\left(\log \max_{1\le k\le N_p}\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_p}\,\xi_\lambda^{p}\tilde{s}_k^{(p)}\right|^2\right)dV_{z(\lambda),t}\\
&=\frac{1}{\pi^{n+1}}\int_{\tilde{U}_\lambda}\left(\log\frac{\max_{1\le j\le N_{p+1}}\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_{p+1}}\xi_\lambda^{p+1}\tilde{s}_j^{(p+1)}\right|^2}{\max_{1\le k\le N_p}\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_p}\xi_\lambda^{p}\tilde{s}_k^{(p)}\right|^2}\right)dV_{z(\lambda),t}\\
&\le\log\left(\frac{1}{\pi^{n+1}}\int_{\tilde{U}_\lambda}\frac{\max_{1\le j\le N_{p+1}}\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_{p+1}}\xi_\lambda^{p+1}\tilde{s}_j^{(p+1)}\right|^2}{\max_{1\le k\le N_p}\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_p}\xi_\lambda^{p}\tilde{s}_k^{(p)}\right|^2}dV_{z(\lambda),t}\right)\\
&\le\log\left(\sup_{\tilde{U}_\lambda}\left(\frac{1}{\pi^{n+1}}\left|\tau_{\lambda,L}^{-c_{p+1}}\,\xi_\lambda\right|^2 e^{c_{p+1}\varphi}\,dV_{z(\lambda),t}\right)\right.\\
&\qquad\left.\cdot\int_{\tilde{U}_\lambda}\frac{\max_{1\le j\le N_{p+1}}\left|\tilde{s}_j^{(p+1)}\right|^2 e^{-c_{p+1}\varphi}}{\max_{1\le k\le N_p}\left|\tilde{s}_k^{(p)}\right|^2}\right)\\
&\le\log\left(\tilde{N}_{m_0}\,C^{\diamondsuit}\sup_{\tilde{U}_\lambda}\left(\frac{1}{\pi^{n+1}}\left|\tau_{\lambda,L}^{-c_{p+1}}\,\xi_\lambda\right|^2 e^{c_{p+1}\varphi}\,dV_{z(\lambda),t}\right)\right)
\end{aligned}\tag{5.4.1}$$

for $1\le p\le m_1-2$. Here for the first inequality we have used the concavity of the logarithm. Summing (5.4.1) from $p=1$ to $p=m_1-2$, we get

$$\begin{aligned}
&\frac{1}{\pi^{n+1}}\int_{\tilde{U}_\lambda}\left(\log \max_{1\le j\le N_{m_1-1}}\left|\tau_{\lambda,A}^{-1}\,\tau_{\lambda,L}^{-a_{m_1-1}}\,\xi_\lambda^{m_1-1}\,,\tilde{s}_j^{(m_1-1)}\right|^2\right)dV_{z(\lambda),t}\\
&\le\frac{1}{\pi^{n+1}}\int_{\tilde{U}_\lambda}\left(\log \max_{1\le k\le N_1}\left|\tau_{\lambda,A}^{-1}\,\xi_\lambda\,\tilde{s}_k^{(1)}\right|^2\right)dV_{z(\lambda),t}\\
&\quad+\sum_{p=1}^{m_1-2}\log\left(\tilde{N}_{m_0}C^{\diamondsuit}\sup_{\tilde{U}_\lambda}\left(\frac{1}{\pi^{n+1}}\left|\tau_{\lambda,L}^{-c_{p+1}}\xi_\lambda\right|^2 e^{c_{p+1}\varphi}dV_{z(\lambda),t}\right)\right).
\end{aligned}\tag{5.4.2}$$

Likewise, from (5.3.3)

$$\frac{1}{\pi^{n+1}}\int_{\tilde U_\lambda}\left(\log\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_{m_1}}\xi_\lambda^{m_1}\hat s^{(m_1)}\right|^2\right)dV_{z(\lambda),t}$$

$$-\frac{1}{\pi^{n+1}}\int_{\tilde U_\lambda}\left(\log\max_{1\le k\le N_{m_1-1}}\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_{m_1-1}}\xi_\lambda^{m_1-1}\tilde s_k^{(m_1-1)}\right|^2\right)dV_{z(\lambda),t}$$

$$=\frac{1}{\pi^{n+1}}\int_{\tilde U_\lambda}\left(\log\frac{\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_{m_1}}\xi_\lambda^{m_1}\hat s^{(m_1)}\right|^2}{\max_{1\le k\le N_{m_1-1}}\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_{m_1-1}}\xi_\lambda^{m_1-1}\tilde s_k^{(m_1-1)}\right|^2}\right)dV_{z(\lambda),t}$$

$$\le\log\left(\frac{1}{\pi^{n+1}}\int_{\tilde U_\lambda}\frac{\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_{m_1}}\xi_\lambda^{m_1}\hat s^{(m_1)}\right|^2}{\max_{1\le k\le N_{m_1-1}}\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_{m_1-1}}\xi_\lambda^{m_1-1}\tilde s_k^{(m_1-1)}\right|^2}dV_{z(\lambda),t}\right)$$

$$\le\log\left(\sup_{\tilde U_\lambda}\left(\frac{1}{\pi^{n+1}}\left|\tau_{\lambda,L}^{-c_{m_1}}\xi_\lambda\right|^2e^{c_{m_1}\varphi}dV_{z(\lambda),t}\right)\cdot\int_{\tilde U_\lambda}\frac{\left|\hat s^{(m_1)}\right|^2e^{-c_{m_1}\varphi}}{\max_{1\le k\le N_{m_1-1}}\left|\tilde s_k^{(m_1-1)}\right|^2}\right)$$

$$\le\log\left(C^\diamond\sup_{\tilde U_\lambda}\left(\frac{1}{\pi^{n+1}}\left|\tau_{\lambda,L}^{-c_{m_1}}\xi_\lambda\right|^2e^{c_{m_1}\varphi}dV_{z(\lambda),t}\right)\right).\qquad(5.4.3)$$

Again here for the first inequality we have used the concavity of the logarithm. Adding up (5.4.2) and (5.4.3), we get

$$\frac{1}{\pi^{n+1}}\int_{\tilde U_\lambda}\left(\log\left|\tau_{\lambda,L}^{-a_{m_1}}\tau_{\lambda,A}^{-1}\xi_\lambda^{m_1}\hat s^{(m_1)}\right|^2\right)dV_{z(\lambda),t}$$

$$\le\frac{1}{\pi^{n+1}}\int_{\tilde U_\lambda}\left(\log\max_{1\le k\le N_1}\left|\tau_{\lambda,A}^{-1}\xi_\lambda\tilde s_k^{(1)}\right|^2\right)dV_{z(\lambda),t}$$

$$+\sum_{p=1}^{m_1-1}\log\left(\tilde N_{m_0}C^\diamond\sup_{\tilde U_\lambda}\left(\frac{1}{\pi^{n+1}}\left|\tau_{\lambda,L}^{-c_{p+1}}\xi_\lambda\right|^2e^{c_{p+1}\varphi}dV_{z(\lambda),t}\right)\right).\quad(5.4.4)$$

By the sub-mean-value property of plurisubharmonic functions

$$\sup_{U'_\lambda}\log\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_{m_1}}\xi_\lambda^{m_1}\hat s^{(m_1)}\right|^2$$

$$\leq \frac{1}{\left(\pi\left(1-r_0\right)^2\right)^{n+1}} \int_{\tilde{U}_\lambda}\left(\log\left|\tau_{\lambda,A}^{-1}\tau_{\lambda,L}^{-a_{m_1}}\xi_\lambda^{m_1}\hat{s}^{(m_1)}\right|^2\right) dV_{z^{(\lambda)},t}. \quad (5.4.5)$$

Let $C^\spadesuit$ be defined as the number such that $\frac{1}{m_0}\log C^\spadesuit$ is the maximum of

$$\frac{1}{\left(1-r_0\right)^{2(n+1)}} \log\left(\tilde{N}_{m_0}\, C^\diamondsuit \sup_{\tilde{U}_\lambda}\left(\frac{1}{\pi^{n+1}}\left|\tau_{\lambda,L}^{-c_{p+1}}\,\xi_\lambda\right|^2 e^{c_{p+1}\varphi}\, dV_{z^{(\lambda)},t}\right)\right)$$

for $1 \leq \lambda \leq \Lambda$ and $1 \leq p \leq m_1 - 1$. Since c_{p+1} only takes the value 0 and 1 according to (4.1.1), the number $C^\spadesuit$ is independent of ℓ.

Let $\hat{C}$ be defined by

$$\log \hat{C} = \frac{1}{\left(\pi\left(1-r_0\right)^2\right)^{n+1}} \sup_{1\leq\lambda\leq\Lambda}\int_{\tilde{U}_\lambda}\left(\log \max_{1\leq k\leq N_1}\left|\tau_{\lambda,A}^{-1}\,\xi_\lambda\,\tilde{s}_k^{(1)}\right|^2\right) dV_{z^{(\lambda)},t}\,.$$

Since $m_1 = \ell\, m_0$ and $a_{m_1} = \ell$, from (5.4.4) and (5.4.5) we obtain

$$\sup_{1\leq\lambda\leq\Lambda}\ \sup_{U_\lambda'}\left|\tau_{\lambda,A}^{-1}\,\tau_{\lambda,L}^{-\ell}\,\xi_\lambda^{\ell\, m_0}\,\hat{s}^{(\ell\, m_0)}\right|^2 \leq \hat{C}\,\left(C^\spadesuit\right)^{\ell\, m_0-1}. \quad (5.4.6)$$

5.5 *Construction of Metric as Limit.*

For $1 \leq \lambda \leq \Lambda$ let χ_λ be the function on U_λ' which is the upper semi-continuous envelope of

$$\limsup_{\ell\to\infty}\log\left|\tau_{\lambda,L}^{-\ell}\,\xi_\lambda^{\ell\, m_0}\,\hat{s}^{(\ell\, m_0)}\right|^{\frac{2}{\ell}}.$$

From the definition of χ_λ, we have

$$\sup_{X_0\cap U_\lambda'}\left(\left|\tau_{\lambda,L}^{-1}\,\xi_\lambda^{m_0}\, s^{(m_0)}\right|^2 e^{-\chi_\lambda}\right) \leq 1 \quad (5.5.1)$$

for $1 \leq \lambda \leq \Lambda$. By (5.4.6), we have

$$\sup_{U_\lambda'}\chi_\lambda \leq m_0 \log C^\spadesuit \quad (5.5.2)$$

for $1 \leq \lambda \leq \Lambda$. Moreover, from the definition of χ_λ we have the transformation rule

$$e^{-\chi_\lambda} = \left|\frac{\tau_{\lambda,L}^{-1}\xi_\lambda^{m_0}}{\tau_{\mu,L}^{-1}\xi_\mu^{m_0}}\right|^2 e^{-\chi_\mu}$$

on $U_\lambda' \cap U_\mu'$ for $1 \leq \lambda, \mu \leq \Lambda$. Hence the collection $\{e^{-\chi_\mu}\}_{1\leq\lambda\leq\Lambda}$ defines a metric $e^{-\chi}$ for $(m_0 K_X + L)\,|X'$ with respect to the local trivializations

$$m_0 K_X + L \to \mathcal{O}_X$$

on U'_λ defined by multiplication by $\tau_{\lambda,L}^{-1}\xi_\lambda^{m_0}$ on U'_λ for $1 \leq \lambda \leq \Lambda$. More precisely,

$$e^{-\chi} = \frac{\left|\xi_\lambda^{m_0}\right|^2 e^{-\chi_\lambda}}{\left|\tau_{\lambda,L}\right|^2}$$

on U'_λ for $1 \leq \lambda \leq \Lambda$. The curvature current of $e^{-\chi}$ is semi-positive and

$$\begin{aligned}
&\int_{X_0} \left|s^{(m_0)}\right|^2 e^{-\left(\frac{m_0-1}{m_0}\right)\chi - \left(\frac{1}{m_0}\right)\varphi} \\
&= \sum_{1\leq\lambda\leq\Lambda} \int_{X_0\cap U'_\lambda} \left(\left|\tau_{\lambda,L}^{-1}\xi_\lambda^{m_0} s^{(m_0)}\right|^2 e^{-\chi_\lambda}\right) e^{\frac{1}{m_0}\chi_\lambda} \left(\left|\tau_{\lambda,L}\right|^2 e^{-\varphi}\right)^{\frac{1}{m_0}} \left(\frac{\varrho_\lambda}{\left|\xi_\lambda\right|^2}\right) \\
&\leq C^\spadesuit \sum_{1\leq\lambda\leq\Lambda} \int_{X_0\cap U'_\lambda} \left(\left|\tau_{\lambda,L}\right|^2 e^{-\varphi}\right)^{\frac{1}{m_0}} \frac{\varrho_\lambda}{\left|\xi_\lambda\right|^2} < \infty\,,
\end{aligned}$$

where for the last inequality (5.5.1) and (5.5.2) are used. The finiteness of

$$\int_{X_0\cap U'_\lambda} \left(\left|\tau_{\lambda,L}\right|^2 e^{-\varphi}\right)^{\frac{1}{m_0}} \frac{\varrho_\lambda}{\left|\xi_\lambda\right|^2}$$

follows from the local integrability of $e^{-\varphi}|_{X_0}$ on X_0 and $m_0 \geq 1$.

By Theorem 3.1, we can extend $s^{(m_0)}$ to an element of $\Gamma(X', m_0K_X + L)$. Extension to $\Gamma(X, m_0K_X + L)$ follows from Stein theory and Grauert's coherence of the proper direct image of coherent sheaves [Gra60] as follows.

Let $\mathcal{F}$ be the coherent sheaf on Δ which is the zeroth direct image $R^0\pi_*(\mathcal{O}_X(m_0K_X + L))$ of the zeroth direct image, under π, of the sheaf $\mathcal{O}_X(m_0K_X + L)$ of germs of holomorphic sections of $m_0K_X + L$ on X. The Stein property of Δ and the coherence of $\mathcal{F}$ imply the surjectivity of the map

$$\Gamma(\Delta, \mathcal{F}) \to \mathcal{F}_0/\mathbf{m}_{\Delta,0}\mathcal{F}_0\,,$$

where $\mathcal{F}_0$ is the stalk of $\mathcal{F}$ at 0 and $\mathbf{m}_{\Delta,0}$ is the maximum ideal of Δ at 0. This finishes the proof of Theorem 0.1.

6 Remarks on the Approach of Generalized Bergman Kernels

As discussed in (1.5.2), (5.1), (5.2), and (5.3.4), the metric for the twisted pluricanonical bundles on X_0 should be chosen as singular as possible to avoid the undesirable unbounded growth of the dimension of the space of L^2 holomorphic sections. Usual naturally-defined metrics such as generalized Bergman kernels would not be singular enough to give the uniform bound of the dimension of the space of L^2 holomorphic sections. The uniform bound of the dimensions of section spaces is needed when we use the concavity of the logarithm for the estimates which leads to (5.3.4). There are a number of other natural alternative ways of estimation which do not need such a uniform bound of dimensions of section spaces. Here we make some remarks on the difficulties of such natural alternatives in the case of generalized Bergman kernels on X as used, for example, in [Tsu01].

Generalized Bergman kernels use square integrable (possibly twisted) pluricanonical sections for definition, instead of just square integrable canonical sections used for the usual Bergman kernels. To use square integrable (possibly twisted) m-canonical sections, we need a metric for the (possibly twisted) $(m-1)$-canonical bundle. There are more than one way of constructing a metric for the (possibly twisted) $(m-1)$-canonical bundle in order to define the generalized Bergman kernel. In order to make the definition applicable to the problem of the deformational invariance of the plurigenera, one has to use the inductive definition, which inductively constructs a metric for the (possibly twisted) $(m-1)$-canonical bundle. Such an inductive definition is defined as follows.

6.1 *Inductively Defined Generalized Bergman Kernels.*

Let Y be a complex manifold of complex dimension n and E be a holomorphic line bundle over Y. Let $e^{-\kappa}$ be a (possibly singular) metric for $K_Y + E$. We inductively define a generalized Bergman kernel Φ_m as follows, so that $\frac{1}{\Phi_m}$ is a metric for $m\,K_Y + E$. Let $\Phi_1 = e^{\kappa}$ and

$$\Phi_m(P) = \sup\left\{\left|\sigma^{(m)}(P)\right|^2 \,\middle|\, \sigma^{(m)} \in \Gamma\left(Y, m\,K_Y + E\right),\ \int_Y \frac{\left|\sigma^{(m)}\right|^2}{\Phi_{m-1}} \leq 1\right\}$$

for $m \geq 2$. Equivalently, we can define

$$\Phi_m = \sum_j \left|\sigma_j^{(m)}\right|^2$$

for an orthonormal basis $\left\{\sigma_j^{(m)}\right\}_j$ of the Hilbert space Γ_m of all $\sigma^{(m)} \in \Gamma\left(Y, m\,K_Y + E\right)$ with

$$\int_Y \frac{\left|\sigma^{(m)}\right|^2}{\Phi_{m-1}} < \infty\,.$$

The reason for the equivalence is that any $\sigma^{(m)} \in \Gamma_m$ with

$$\int_Y \frac{\left|\sigma^{(m)}\right|^2}{\Phi_{m-1}} = 1$$

can be written as $\sum_j c_j \sigma_j^{(m)}$ with $\sum_j |c_j|^2 = 1$ and that for any $P \in Y$ one can first choose $\left\{\sigma_j^{(m)}\right\}_{j \geq 2}$ as the orthonormal basis for the subspace of Γ_m consisting of all elements vanishing at P and then add $\sigma_1^{(m)}$.

6.2 *How Generalized Bergman Kernels Are Used.*

The use of generalized Bergman kernels is meant for the fourth ingredient listed in the Introduction. Consider the case of L being trivial. Use X as Y. Use as E a sufficiently ample line bundle A over X so that we have a metric $e^{-\kappa}$ for $K_X + E$ with positive curvature on X. Let $s^{m_0} \in \Gamma(X_0, m_0 K_{X_0})$ be given which is to be extended to an element of $\Gamma(X, m_0 K_X)$. Let s_A be an element of $\Gamma(X, A)$ whose restriction to X_0 is not identically zero. The setting is the following situation after (5.3). For any integer $\ell \geq 2$, $\left(s^{(m_0)}\right)^{\ell} s_A$ can be extended to $\hat{s}^{(\ell m_0)} \in \Gamma(X, \ell m_0 K_X + A)$ such that

$$\int_X \frac{\left|\hat{s}^{(\ell m_0)}\right|^2}{\Phi_{\ell m_0 - 1}} \leq \left(C^{\diamondsuit}\right)^{\ell m_0}$$

for some positive constant $C^{\diamondsuit}$ independent of ℓ. Hence

$$\left|\hat{s}^{(\ell m_0)}\right|^2 \leq \left(C^{\diamondsuit}\right)^{\ell m_0} \Phi_{\ell m_0}.$$

Suppose on X_0 one were able to prove that $(\Phi_m)^{\frac{1}{m}}$ is locally bounded on X_0 as $m \to \infty$. If we denote by Θ the upper semi-continuous envelope of the limit superior of $(\Phi_m)^{\frac{1}{m}}$ as $m \to \infty$, then $\frac{1}{\Theta}$ is a metric of K_X with semi-positive curvature current and

$$\int_{X_0} \frac{\left|s^{(m_0)}\right|^2}{\Theta^{m_0}} \leq \left(C^{\diamondsuit}\right)^{\ell}.$$

Thus $s^{(m_0)}$ can be extended to an element of $\Gamma(X, m_0 K_X)$. In the rest of §6 we will discuss the difficulties of various approaches to get the local bound of $(\Phi_m)^{\frac{1}{m}}$ as $m \to \infty$.

6.3 *Bounds of Roots of Generalized Bergman Kernels.*

We now return to the general setting of Y in (6.2). Assume that on Y we have a finite number of coordinate charts $\tilde{G}_\lambda$ with coordinates

$$\left(z_1^{(\lambda)}, \cdots, z_n^{(\lambda)}\right).$$

We assume that there exists nowhere zero elements

$$\xi_\lambda \in \Gamma\left(\tilde{G}_\lambda, -K_Y\right),$$
$$\tau_{\lambda,E} \in \Gamma\left(\tilde{G}_\lambda, E\right).$$

Let dV_λ be the Euclidean volume of the coordinate chart $\tilde{G}_\lambda$. For $0 < r \leq 1$ let

$$G_{\lambda,r} = \tilde{G}_\lambda \cap \left\{\left|z_1^{(\lambda)}\right| < r, \cdots, \left|z_n^{(\lambda)}\right| < r\right\}.$$

Assume that $G_{\lambda,r}$ is relatively compact in $\tilde{G}_\lambda$ for $1 \leq \lambda \leq \Lambda$ and $0 < r \leq 1$. Let $G_\lambda = G_{\lambda,1}$. Let

$$\Theta_{m,r} = \sup_{1 \leq \lambda \leq \Lambda} \sup_{G_{\lambda,r}} \Phi_m \left| \xi_\lambda^m \tau_{\lambda,E}^{-1} \right|^2 ,$$

$$C_1^{\clubsuit} = \sup_{1 \leq \lambda \leq \Lambda} \sup_{G_\lambda} \left| \xi_\lambda \tau_{\lambda,E}^{-1} \right|^2 e^{\kappa},$$

$$C_2^{\clubsuit} = \sup_{1 \leq \lambda, \mu \leq \Lambda} \sup_{G_\lambda \cap G_\mu} \left| \frac{\xi_\lambda}{\xi_\mu} \right|^2 ,$$

$$C_3^{\clubsuit} = \sup_{1 \leq \lambda, \mu \leq \Lambda} \sup_{G_\lambda \cap G_\mu} \left| \frac{\tau_{\lambda,E}}{\tau_{\mu,E}} \right|^2 .$$

As explained in (6.2), the kind of estimates required for the invariance of plurigenera is a bound for $(\Theta_{m,r})^{\frac{1}{m}}$ as $m \to \infty$. We are going to inductively estimate $\Theta_{m,r}$ in terms of $C_1^{\clubsuit}$ and $C_2^{\clubsuit}$ and show where the difficulty lies in getting a finite local bound for $\limsup_{m \to \infty} (\Theta_{m,r})^{\frac{1}{m}}$. For the estimates we will first use the plurisubharmonicity of the absolute value squares of holomorphic sections. Later we will discuss the use of the concavity of the logarithm.

6.4 *Difficulty of Shrinking Domain from Sub-Mean-Value Property.*

Take $0 < r_0 < 1$. Fix an arbitrary point $P_0 \in G_{\lambda,r_0}$ for some $1 \leq \lambda \leq \Lambda$. For $0 < r < 1 - r_0$ let

$$\Delta_r^{(P_0,\lambda)} = G_\lambda \cap \left\{ \left| z_1^{(\lambda)} - z_1^{(\lambda)}(P_0) \right| < r, \cdots, \left| z_n^{(\lambda)} - z_n^{(\lambda)}(P_0) \right| < r \right\}.$$

By the definition of Φ_m there exists some

$$\sigma^{(m)} \in \Gamma\left(Y, m K_Y + E\right)$$

such that

$$\int_Y \frac{\left|\sigma^{(m)}\right|^2}{\Phi_{m-1}} \leq 1$$

and

$$\Phi_m(P_0) = \left| \sigma^{(m)}(P_0) \right|^2 .$$

For $m \geq 2$ and $0 < r \leq 1 - r_0$, by the sub-mean-value property of the absolute value square of a holomorphic function, we have

$$\begin{aligned}
&\left(\Phi_m \left|\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2\right)(P_0) = \left|\sigma^{(m)}\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2 (P_0) \\
&\leq \frac{1}{(\pi r^2)^n} \int_{\Delta_r^{(P_0,\lambda)}} \left|\sigma^{(m)}\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2 dV_\lambda \\
&= \frac{1}{(\pi r^2)^n} \int_{\Delta_r^{(P_0,\lambda)}} \frac{\left|\sigma^{(m)}\right|^2}{\Phi_{m-1}} \left(\Phi_{m-1}\left|\xi_\lambda^{m-1}\tau_{\lambda,E}^{-1}\right|^2\right) |\xi_\lambda|^2 \, dV_\lambda \\
&\leq \left(\sup_{\Delta_r^{(P_0,\lambda)}} \Phi_{m-1}\left|\xi_\lambda^{m-1}\tau_{\lambda,E}^{-1}\right|^2\right) \left(\sup_{\Delta_r^{(P_0,\lambda)}} |\xi_\lambda|^2 \, dV_\lambda\right) \frac{1}{(\pi r^2)^n} \int_{\Delta_r^{(P_0,\lambda)}} \frac{\left|\sigma^{(m)}\right|^2}{\Phi_{m-1}} \\
&\leq \left(\sup_{\Delta_r^{(P_0,\lambda)}} \Phi_{m-1}\left|\xi_\lambda^{m-1}\tau_{\lambda,E}^{-1}\right|^2\right) \left(\sup_{\Delta_r^{(P_0,\lambda)}} |\xi_\lambda|^2 \, dV_\lambda\right) \frac{1}{(\pi r^2)^n}. \qquad (6.4.1)
\end{aligned}$$

Choose positive numbers $r_1, \cdots, r_{m-1}$ such that $\sum_{j=1}^{m-1} r_j \leq 1 - r_0$. Let $\delta_j = 1 - (r_1 + \cdots + r_{j-1})$. From (6.4.1) and by induction on m we have

$$\left(\Phi_m \left|\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2\right)(P_0) \leq \left(\sup_{\Delta_{\delta_1}^{(P_0,\lambda)}} \left|\xi_\lambda \tau_{\lambda,E}^{-1}\right|^2 e^\kappa\right) \prod_{j=1}^{m-1} \sup_{\Delta_{\delta_j}^{(P_0,\lambda)}} \frac{|\xi_\lambda|^2 \, dV_\lambda}{\left(\pi r_j^2\right)^n} \qquad (6.4.2)$$

for $m \geq 2$. The estimate is for $P_0 \in G_{\lambda, r_0}$ when $r_0 < 1 - \sum_{j=1}^{m-1} r_j$. When we increase m, we have to add more and more r_j $(1 \leq j < m)$. Their sum have to remain less than 1 and the m-th root of their product have to remain bounded in m.

We have to decide between two choices. The first choice is to use smaller and smaller r_j to maintain a fixed positive lower bound for r_0. This choice results in the lack of a finite bound for the m-th root of $\prod_{j=1}^{m-1} \frac{1}{r_j^2}$ as m goes to infinity. The second choice is to keep the m-th root of $\prod_{j=1}^{m-1} \frac{1}{r_j^2}$ bounded in m. This choice forces r_0 to become eventually negative so that the domain of supremum estimates becomes empty.

This kind of difficulties with norm change, involving supremum norms on different domains, is similar to the norm-change problems encountered in the papers of Nash [Nas54], Moser [Mos61], and Grauert [Gra60]. However, the situation here is different from those which could be handled by the norm-change techniques of Nash [Nas54], Moser [Mos61], and Grauert [Gra60].

6.5 *The Compact Case and Difficulty from Transition Functions.*

For the resolution of the difficulty explained in (6.4), let us consider the case of a compact Y. When Y is compact, a shrunken cover of Y is still a cover of Y and the difficulty of the shrinking domain for the supremum estimate seems not to exist. However, if we use this approach to resolve the problem in the compact case, the transition functions of the m-canonical bundle would give us trouble.

When we consider a point of an unshrunken member of the cover as being in another shrunken member, for the estimation of the bound in question we have to use the transition function of the m-canonical bundle whose bound grows as the m-th power of a positive constant. Let us more precisely describe the contribution from the transition functions of the m-canonical bundle.

From (6.4.1) we obtain

$$\sup_{1\leq\lambda\leq\Lambda}\sup_{G_{\lambda,r_0}}\Phi_m\left|\xi_\lambda^m\tau_{\lambda,E}^{-1}\right|^2\leq\frac{C_1^\clubsuit}{\left(\pi\left(1-r_0\right)^2\right)^n}\,\Theta_{m-1,1}\tag{6.5.1}$$

for $m\geq 2$. Since Y is compact, we can assume $Y=\bigcup_{1\leq\lambda\leq\Lambda}G_{\lambda,r_0}$. Any $P_0\in G_\lambda$ belongs to some G_{μ,r_0} and

$$\begin{aligned}\left(\Phi_m\left|\xi_\lambda^m\tau_{\lambda,E}^{-1}\right|^2\right)(P_0)&=\left|\frac{\xi_\lambda^m\tau_{\lambda,E}^{-1}}{\xi_\nu^m\tau_{\lambda,E}^{-1}}\right|^2\left(\Phi_m\left|\xi_\mu^m\tau_{\lambda,E}^{-1}\right|^2\right)(P_0)\\&\leq\left(C_2^\clubsuit\right)^m C_3^\clubsuit\left(\Phi_m\left|\xi_\mu^m\tau_{\mu,E}^{-1}\right|^2\right)(P_0)\\&\leq\left(C_2^\clubsuit\right)^m\frac{C_1^\clubsuit C_3^\clubsuit}{\left(\pi\left(1-r_0\right)^2\right)^n}\,\Theta_{m-1,1}\end{aligned}$$

for $m\geq 2$ by (6.5.1). Hence

$$\Theta_{m,1}\leq\left(C_2^\clubsuit\right)^m\frac{C_1^\clubsuit C_3^\clubsuit}{\left(\pi\left(1-r_0\right)^2\right)^n}\,\Theta_{m-1,1}\tag{6.5.2}$$

for $m\geq 2$. By induction on $m\geq 2$, we conclude from (6.5.2) that

$$\Theta_{m,1}\leq\left(\frac{C_1^\clubsuit C_3^\clubsuit}{\left(\pi\left(1-r_0\right)^2\right)^n}\right)^{m-1}\left(C_2^\clubsuit\right)^{\frac{m(m+1)}{2}-1}\Theta_{1,1}\tag{6.5.3}$$

for $m\geq 2$. When we take the m-th root of both sides of (6.5.3), we get

$$\left(\Theta_{m,1}\right)^{\frac{1}{m}}\leq\left(\frac{C_1^\clubsuit C_3^\clubsuit}{\left(\pi\left(1-r_0\right)^2\right)^n}\right)^{1-\frac{1}{m}}\left(C_2^\clubsuit\right)^{\frac{m+1}{2}-\frac{1}{m}}\left(\Theta_{1,1}\right)^{\frac{1}{m}}$$

for $m \geq 2$ and we would have trouble bounding the factor $\left(C_2^\clubsuit\right)^{\frac{m+1}{2}}$ as $m \to \infty$. So even in the compact case, the difficulty cannot be completely circumvented, because of the undesirable contribution from the transition functions of the m-canonical bundle.

6.6 *Difficulty with Bounds for Nonempty Limit of Shrinking Domain.*

Let us return to the general case in which Y is noncompact. We look more closely at the difficulty of the shrinking domain. In our case of $\pi : X \to \Delta$, we use the notations of (5.4) and let

$$\tilde{U}_{\lambda,r} = W_\lambda \bigcap \left\{ \left| z_1^{(\lambda)} \right| < r, \cdots, \left| z_n^{(\lambda)} \right| < r, |t| < r \right\}.$$

We apply the above argument to $Y = X$ and $G_{\lambda,r} = \tilde{U}_{\lambda,r}$ and with n replaced by $n+1$. Let $r_1, \cdots, r_{m-1}$ be positive numbers such that $r_1 + \cdots + r_{m-1} < 1$. With $\delta_j = 1 - (r_1 + \cdots + r_{j-1})$, from (6.4.2) we have

$$\begin{aligned}
\sup_{\tilde{U}_{\lambda,\delta_m}} \Phi_m \left| \xi_\lambda^m \tau_{\lambda,E}^{-1} \right|^2 &\leq \left(\sup_{\tilde{U}_{\lambda,\delta_1}} \left| \xi_\lambda \tau_{\lambda,E}^{-1} \right|^2 e^\kappa \right) \prod_{j=1}^{m-1} \sup_{\tilde{U}_{\lambda,\delta_j}} \frac{\left|\xi_\lambda\right|^2 dV_\lambda}{\left(\pi r_j^2\right)^n} \\
&= \frac{1}{\pi^{n(m-1)} \prod_{j=1}^{m-1} r_j^2} \left(\sup_{\tilde{U}_{\lambda,\delta_1}} \left| \xi_\lambda \tau_{\lambda,E}^{-1} \right|^2 e^\kappa \right) \prod_{j=1}^{m-1} \sup_{\tilde{U}_{\lambda,\delta_j}} \left(\left|\xi_\lambda\right|^2 dV_\lambda \right)
\end{aligned}$$

for $m \geq 2$. After we take the m-th of both sides, we end up with a factor $\frac{1}{\left(\prod_{j=1}^m r_j^2\right)^{\frac{1}{m}}}$ which is not bounded in m if $\sum_{j=1}^\infty r_j < 1$. For example, if the sequence r_j is non-increasing and if there exists some positive number C such that $\frac{1}{\left(\prod_{j=1}^m r_j^2\right)^{\frac{1}{m}}} \leq C$ for all $m \geq 1$, then

$$\begin{aligned}
&\frac{1}{r_\ell^2} \leq \left(\prod_{j=\ell}^{2\ell} \frac{1}{r_j^2} \right)^{\frac{1}{\ell}} \leq C^2 \quad \text{for } \ell \geq 1 \\
\Longrightarrow \quad & r_\ell \geq \frac{1}{C} \quad \text{for } \ell \geq 1 \\
\Longrightarrow \quad & \sum_{\ell=1}^m r_\ell \geq \frac{m}{C} \to \infty \quad \text{as } m \to \infty.
\end{aligned}$$

It shows that the difficulty of the shrinking domain for the supremum estimate is an essential one when the generalized Bergman kernels are inductively defined.

6.7 *Non-Inductively Defined Generalized Bergman Kernels.*

We would like to point out that, if the metric for the $(m-1)$-canonical bundle is not constructed inductively in order to define the generalized Bergman kernel, then we would be able to avoid the difficulty with the bound of

$$\limsup_{m\to\infty} \left(\Theta_{m,r}\right)^{\frac{1}{m}}.$$

This non-inductive definition works only when E is the trivial line bundle, which we assume only for the discussion here in (6.7). Let $e^{-\kappa_0}$ be a metric for K_Y. The non-inductive definition defines $\tilde{\Phi}_1 = e^{\kappa_0}$ and

$$\tilde{\Phi}_m(P) = \sup\left\{\left|\sigma^{(m)}(P)\right|^2 \,\middle|\, \sigma^{(m)} \in \Gamma\left(Y, m\,K_Y\right) \int_Y \left|\sigma^{(m)}\right|^2 e^{-(m-1)\kappa_0} \leq 1\right\}$$

for $m \geq 2$. We let

$$\tilde{\Theta}_{m,r} = \sup_{1\leq\lambda\leq\Lambda}\sup_{G_{\lambda,r}} \tilde{\Phi}_m \left|\xi_\lambda^m\right|^2.$$

Again take $0 < r_0 < 1$ and fix an arbitrary point $P_0 \in G_{\lambda,r_0}$ for some $1 \leq \lambda \leq \Lambda$. By the definition of $\tilde{\Phi}_m$ there exists some

$$\sigma^{(m)} \in \Gamma\left(Y, m\,K_Y\right)$$

such that

$$\int_Y \left|\sigma^{(m)}\right|^2 e^{-(m-1)\kappa_0} \leq 1 \tag{6.7.1}$$

and

$$\tilde{\Phi}_m\left(P_0\right) = \left|\sigma^{(m)}\left(P_0\right)\right|^2.$$

Let $0 < r \leq 1 - r_0$. For $m \geq 2$, by the sub-mean-value property of the absolute value square of a holomorphic function, from (6.7.1) we have

$$\begin{aligned}\left(\tilde{\Phi}_m \left|\xi_\lambda^m\right|^2\right)\left(P_0\right) &= \left|\sigma^{(m)}\xi_\lambda^m\right|^2\left(P_0\right)\\ &\leq \frac{1}{\left(\pi\, r^2\right)^n}\int_{\Delta_r^{(P_0,\lambda)}} \left|\sigma^{(m)}\xi_\lambda^m\right|^2 dV_\lambda\\ &\leq \frac{1}{\left(\pi\, r^2\right)^n}\left(\sup_{\Delta_r^{(P_0,\lambda)}} \left|\xi_\lambda^m\right|^2 e^{(m-1)\kappa_0}\, dV_\lambda\right).\end{aligned}$$

Hence

$$\left(\tilde{\Theta}_{m,r_0}\right)^{\frac{1}{m}} \leq \left(\frac{1}{\left(\pi\left(1-r_0\right)^2\right)^n}\left(\sup_{1\leq\lambda\leq\Lambda}\sup_{G_\lambda}\left|\xi_\lambda^m\right|^2 e^{(m-1)\kappa_0}\, dV_\lambda\right)\right)^{\frac{1}{m}}$$

is bounded as $m \to \infty$. However, this kind of non-inductively defined generalized Bergman kernel is not useful to the problem of the deformational invariance of plurigenera. The induction argument of [Siu98] for the problem of the deformational invariance of plurigenera requires the inductive definition if generalized Bergman kernels are used. Moreover, the non-inductive definition works only when E is trivial, which is not useful for the case of manifolds not necessarily of general type.

6.8 *Hypothetical Situation of Sub-Mean-Value Property of Quotients.*

Now we return to the general case where E may not be trivial and $e^{-\kappa}$ is a metric for $K_Y + E$. We would like to remark that, if

$$\frac{\left|\sigma^{(m)}\xi_\lambda\right|^2}{\Phi_{m-1}}$$

were to have the sub-mean-value property for $\sigma^{(m)} \in \Gamma\left(Y, m\,K_Y\right)$, the difficulty of the shrinking of the domain of supremum estimate for the inductively defined generalized Bergman kernel would disappear. The reason is as follows.

Again fix an arbitrary point $P_0 \in G_{\lambda,r_0}$ for some $1 \leq \lambda \leq \Lambda$. By the definition of Φ_m there exists some

$$\sigma^{(m)} \in \Gamma\left(Y, m\,K_Y + E\right)$$

such that

$$\int_Y \frac{\left|\sigma^{(m)}\right|^2}{\Phi_{m-1}} \leq 1 \tag{6.8.1}$$

and

$$\Phi_m\left(P_0\right) = \left|\sigma^{(m)}\left(P_0\right)\right|^2.$$

Let $0 < r \leq 1 - r_0$. For $m \geq 2$, by the hypothetical sub-mean-value property of

$$\frac{\left|\sigma^{(m)}\xi_\lambda\right|^2}{\Phi_{m-1}},$$

we would have

$$\begin{aligned}\left(\frac{\Phi_m\left|\xi_\lambda\right|^2}{\Phi_{m-1}}\right)\left(P_0\right) &= \left(\frac{\left|\sigma^{(m)}\xi_\lambda\right|^2}{\Phi_{m-1}}\right)\left(P_0\right) \\ &\leq \frac{1}{\left(\pi\, r^2\right)^n}\int_{\Delta_r^{(P_0,\lambda)}} \frac{\left|\sigma^{(m)}\xi_\lambda\right|^2}{\Phi_{m-1}}\, dV_\lambda \\ &\leq \frac{1}{\left(\pi\, r^2\right)^n}\left(\sup_{\Delta_r^{(P_0,\lambda)}} \left|\xi_\lambda\right|^2 dV_\lambda\right) \end{aligned} \tag{6.8.2}$$

by (6.8.1). Hence by (6.8.2) and by induction on m, we have

$$\left(\frac{\Phi_m \left|\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2}{\Phi_1 \left|\xi_\lambda \tau_{\lambda,E}^{-1}\right|^2}\right)(P_0) \leq \left(\frac{1}{(\pi r^2)^n}\left(\sup_{\Delta_r^{(P_0,\lambda)}} |\xi_\lambda|^2 \, dV_\lambda\right)\right)^{m-1}$$

and

$$\left(\Theta_{m,r_0}\right)^{\frac{1}{m}} \leq \left(\frac{1}{\left(\pi\left(1-r_0\right)^2\right)^n}\left(\sup_{1\leq\lambda\leq\Lambda} \sup_{\Delta_r^{(P_0,\lambda)}} |\xi_\lambda|^2 \, dV_\lambda\right)\right)^{1-\frac{1}{m}} \left(\Theta_{1,r_0}\right)^{\frac{1}{m}}$$

which is bounded as $m \to \infty$. Of course, unfortunately in general

$$\frac{\left|\sigma^{(m)}\xi_\lambda\right|^2}{\Phi_{m-1}}$$

does not have the sub-mean-value property for $\sigma^{(m)} \in \Gamma(Y, m K_Y + E)$.

6.9 *Growth of Dimension of Section Space When Concavity of Logarithm is Used.*

In the case of a compact Y, for $0 < r_0 < 1$ we can follow the method of (5.4) and use the concavity of the logarithm for the supremum estimate of $(\Theta_{m,r_0})^{\frac{1}{m}}$ as $m \to \infty$. Let N_m be the complex dimension of the space Γ_m of all $\sigma^{(m)} \in \Gamma(Y, m K_Y + E)$ such that

$$\int_Y \frac{\left|\sigma^{(m)}\right|^2}{\Phi_{m-1}} < \infty .$$

As in (5.4), for this method of estimation the bound of $(\Theta_{m,r_0})^{\frac{1}{m}}$ cannot be independent of m if the growth order of N_m is at least that of some positive power of m. More precisely, the estimation analogous to that of (5.4) is given as follows.

From the definition of m we have

$$\int_Y \frac{\Phi_m}{\Phi_{m-1}} \leq N_m$$

for $m \geq 2$. By the concavity of the logarithm,

$$\begin{aligned}
&\frac{1}{\pi^n}\int_{G_\lambda}\left(\log \Phi_m \left|\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2\right) dV_\lambda - \frac{1}{\pi^n}\int_{G_\lambda}\left(\log \Phi_{m-1}\left|\xi_\lambda^{m-1}\tau_{\lambda,E}^{-1}\right|^2\right) dV_\lambda \\
&\quad = \frac{1}{\pi^n}\int_{G_\lambda}\left(\log \frac{\Phi_m |\xi_\lambda|^2}{\Phi_{m-1}}\right) dV_\lambda \\
&\quad \leq \log\left(\frac{1}{\pi^n}\int_{G_\lambda} \frac{\Phi_m |\xi_\lambda|^2}{\Phi_{m-1}} dV_\lambda\right)
\end{aligned}$$

$$\leq \log\left(\frac{N_m}{\pi^n}\sup_{G_\lambda}|\xi_\lambda|^2\,dV_\lambda\right) \tag{6.9.1}$$

for $m \geq 2$. Adding up (6.9.1) from $m = 2$ to m, we obtain

$$\begin{aligned}
&\frac{1}{\pi^n}\int_{G_\lambda}\left(\log \Phi_m \left|\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2\right) dV_\lambda \\
&\leq \frac{1}{\pi^n}\int_{G_\lambda}\left(\log\left|\xi_\lambda \tau_{\lambda,E}^{-1}\right|^2 e^{-\kappa}\right) dV_\lambda + \textstyle\sum_{j=2}^m \log\left(\frac{N_m}{\pi^n}\sup_{G_\lambda}|\xi_\lambda|^2\,dV_\lambda\right) \\
&\leq \log\left(\frac{1}{\pi^n}\int_{G_\lambda}\left|\xi_\lambda \tau_{\lambda,E}^{-1}\right|^2 e^{-\kappa} dV_\lambda\right) + \sum_{j=2}^m \log\left(\frac{N_m}{\pi^n}\sup_{G_\lambda}|\xi_\lambda|^2\,dV_\lambda\right), \quad (6.9.2)
\end{aligned}$$

where for the last inequality the concavity of the logarithm is used. Using the sub-mean-value property of $\log \Phi_m \left|\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2$, we obtain

$$\sup_{G_{\lambda,r_0}} \log \Phi_m \left|\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2 \leq \frac{1}{\left(\pi\,(1-r_0)^2\right)^n}\int_{G_\lambda}\left(\log \Phi_m \left|\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2\right) dV_\lambda \tag{6.9.3}$$

for $0 < r_0 < 1$. Let

$$\begin{aligned}
\gamma &= \frac{1}{\left(\pi\,(1-r_0)^2\right)^n}, \\
C_1 &= \frac{1}{\pi^n}\sup_{1\leq\lambda\leq\Lambda}\int_{G_\lambda}\left|\xi_\lambda \tau_{\lambda,E}^{-1}\right|^2 e^{-\kappa} dV_\lambda, \\
C_2 &= \frac{1}{\pi^n}\sup_{G_\lambda}|\xi_\lambda|^2\,dV_\lambda .
\end{aligned}$$

By (6.9.2) and (6.9.3) we have

$$\left(\Theta_{m,r_0}\right)^{\frac{1}{m}} \leq \left(C_1^{\frac{1}{m}} C_2^{1-\frac{1}{m}}\left(\prod_{j=2}^m N_m\right)^{\frac{1}{m}}\right)^\gamma$$

and whether $\left(\Theta_{m,r_0}\right)^{\frac{1}{m}}$ is bounded in m depends on whether $\left(\prod_{j=2}^m N_m\right)^{\frac{1}{m}}$ is bounded in m.

If the order of growth of N_m is at least that of some positive power of m (i.e., $N_m \geq \alpha\, m^\beta - C$ for some positive α and β and C), then

$$\left(\prod_{j=2}^m N_m\right)^{\frac{1}{m}} \geq \hat{C}\,(m!)^{\frac{\beta}{m}}$$

for some positive $\hat{C}$ and for m sufficiently large and is therefore unbounded in m.

Even if we start with some very singular $e^{-\kappa}$, the definition of Φ_m may make $\frac{1}{\Phi_m}$ gain some regularity in each step as m increases. As a result, the growth order of N_m may become comparable to that of the dimension of $\Gamma(Y, m K_Y + E)$ as $m \to \infty$.

Besides the difficulty of the growth order of N_m, this method of the concavity of the logarithm does not apply to the case of noncompact Y, which is what the use of generalized Bergman kernels for the deformation invariance of plurigenera would require.

6.10 *Modification of Generalized Bergman Kernels to Make the Method of Concavity of Logarithm Applicable to the Case of Open Manifolds.*

To make the method of the concavity of the logarithm applicable to the noncompact case of $\pi : X \to \Delta$ at hand, we can do the following. Let E be a holomorphic line bundle over X, $e^{-\kappa}$ be a metric of $K_X + E$, and $C^{\heartsuit} > 0$. Let Φ_m be the generalized Bergman metric for $m K_{X_0} + E|_{X_0}$ and $e^{-\kappa}$, as defined in (6.1) with Y equal to X_0. Let $\sigma_j^{(m)}$ $(1 \leq j \leq N_m)$ form an orthonormal basis of the space Γ_m of all $\sigma^{(m)} \in \Gamma(X_0, m K_{X_0} + E|_{X_0})$ with

$$\int_{X_0} \frac{\left|\sigma^{(m)}\right|^2}{\Phi_{m-1}} < \infty .$$

Then inductively let $\hat{\sigma}_j^{(m)} \in \Gamma(X, m K_X + E)$ be an extension of $\sigma_j^{(m)}$ with

$$\int_X \frac{\left|\hat{\sigma}^{(m)}\right|^2}{\hat{\Phi}_{m-1}} < \infty$$

and

$$\hat{\Phi}_m = \max_{1 \leq j \leq N_m} \left|\hat{\sigma}_j^{(m)}\right|^2 .$$

Define

$$\hat{\Theta}_{m,r} = \sup_{1 \leq \lambda \leq \Lambda} \sup_{G_{\lambda,r}} \hat{\Phi}_m \left|\xi_\lambda^m \tau_{\lambda,E}^{-1}\right|^2 ,$$

where $G_{\lambda,r}, \xi_\lambda^m, \tau_{\lambda,E}$ are as defined above when Y is replaced by X. We can then apply the method of the concavity of the logarithm to $\hat{\Phi}_m$ on X for the supremum estimate of $\left(\hat{\Theta}_{m,r}\right)^{\frac{1}{m}}$ as $m \to \infty$. Of course, $\hat{\Phi}_m$ is no longer the generalized Bergman kernel. We would still have the difficulty with the growth order of N_m when it is at least of the order of some positive power of m.

6.11 The conclusion to these remarks in §6 is that, in the various approaches discussed above, the generalized Bergman kernels pose insurmountable difficulties when they are used for estimates in the problem of the deformational invariance of plurigenera for manifolds not necessarily of general type.

References

[Gra60] H. Grauert, Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, *Inst. Hautes Études Sci. Publ. Math.* No. 5 (1960), 64 pp; Berichtigung zu der Arbeit "Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen", *Inst. Hautes Études Sci. Publ. Math.* No. 16 (1963), 35–36.

[Hör66] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Van Nostrand, Princeton, New Jersey, 1966.

[Kaw82] Y. Kawamata, A generalization of Kodaira-Ramanujam's vanishing theorem, *Math. Ann.* **261** (1982), 43–46.

[Kaw99] Y. Kawamata, Deformations of canonical singularities, *J. Amer. Math. Soc.* **12** (1999), 85–92.

[Lev83] M. Levine, Pluri-canonical divisors on Kähler manifolds, *Invent. Math.* **74** (1983), 293–903.

[Man93] L. Manivel, Un thórème de prolongement L^2 de sections holomorphes d'un fibré hermitien, *Math. Zeitschr.* **212** (1993), 107–122.

[Mos61] J. Moser, On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.* **14** (1961), 577–591.

[Nad89] A. Nadel, Multiplier ideal sheaves and the existence of Kähler-Einstein metrics of positive scalar curvature, *Proc. Natl. Acad. Sci. USA* **86** (1989), 7299–7300; *Ann. of Math.* **132** (1989), 549–596.

[Nak98] N. Nakayama, Invariance of the Plurigenera of Algebraic Varieties, preprint, *Research Inst. for Math. Sci.*, RIMS-1191, March 1998.

[Nas54] J. Nash, C^1 isometric imbeddings, *Ann. of Math.* **60** (1954), 383–396.

[Ohs88] T. Ohsawa, On the extension of L^2 holomorphic functions, II, *Publ. Res. Inst. Math. Sci.* **24** (1988), 265–275.

[Ohs94] T. Ohsawa, On the extension of L^2 holomorphic functions, IV, A new density concept, in *Geometry and Analysis on Complex Manifolds*, 157–170, World Sci. Publishing, River Edge, NJ, 1994.

[Ohs95] T. Ohsawa, On the extension of L^2 holomorphic functions, III. Negligible weights, *Math. Z.* **219** (1995), 215–225.

[Ohs01] T. Ohsawa, On the extension of L^2 holomorphic functions, V. Effects of generalization, *Nagoya Math. J.* **161** (2001), 1–21.

[OT87] T. Ohsawa and K. Takegoshi, On the extension of L^2 holomorphic functions, *Math. Z.* **195** (1987), 197–204.

[Siu96] Y.-T. Siu, The Fujita conjecture and the extension theorem of Ohsawa-Takegoshi, in *Geometric Complex Analysis* ed. Junjiro Noguchi *et al*, World Scientific, Singapore, New Jersey, London, Hong Kong, 1996, pp. 577–592.

[Siu98] Y.-T. Siu, Invariance of plurigenera, *Invent. Math.* **134** (1998), 661–673.

[Sko72] H. Skoda, Application des techniques L^2 à la théorie des ideaux d'un algèbre de fonctions holomorphes avec poids, *Ann. Sci. Ec. Norm. Sup.* **5** (1972), 548–580.

[Tsu92] H. Tsuji, Analytic Zariski decomposition, *Proc. Japan Acad.* **61** (1992), 161–163.

[Tsu99] H. Tsuji, Existence and applications of analytic Zariski decompositions, in *Analysis and Geometry in Several Complex Variables*, ed. Komatsu and Kuranishi, *Trends in Math.*, Birkhäuser, 1999, pp. 253–272.

[Tsu01] H. Tsuji, Deformational invariance of plurigenera, Preprint 2001, math. AG/0011257; (earlier version: Invariance of plurigenera of varieties with nonnegative Kodaira dimensions, math.AG/0012225).

[Vie82] E. Viehweg, Vanishing theorems, *J. reine und angew. Math.* **335** (1982), 1–8.