MODULI SPACES OF LINE CONFIGURATIONS AND VECTOR BUNDLES

An Honors Thesis Presented

By

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ABSTRACT

The study of moduli spaces is a cornerstone of algebraic geometry. In this thesis, we investigate moduli spaces of two kinds. First, we study realization spaces of 10_3 line configurations. We prove that these spaces all have dense rational points (in both the Zariski and analytic topologies). We find that for exactly four out of the ten 10_3 configurations, the realization space admits a compactification by a K3 surface of maximal Picard rank, which is itself a moduli space of GIT-stable "weak" realizations. Next, we turn to the moduli space $SU_C(2)$ of semistable rank 2 vector bundles with trivial determinant on a smooth projective curve C. Generalizing Tevelev-Torres' proof of the BGMN conjecture, we construct a noncommutative resolution of singularities of $SU_C(2)$ (crepant if the genus of C is even) with a semiorthogonal decomposition into blocks equivalent to even symmetric powers of C. This provides evidence (via the Kuznetsov rationality proposal) toward the longstanding expectation that $SU_C(2)$ is a rational variety.

Contents

Chapter 1

Introduction

This thesis concerns moduli spaces in algebraic geometry. Broadly speaking, given a class $\mathcal{C}$ of geometric objects, a *moduli space* for $\mathcal{C}$ is a geometric space whose points are in one-to-one correspondence with objects in $\mathcal{C}$ up to a suitable notion of isomorphism. Moreover, the geometry of the moduli space should encode the ways these objects can vary in "nice" families. As a basic but critical example, take $\mathcal{C}$ to be the set of lines through the origin in $\mathbb{R}^{n+1}$. The moduli space for $\mathcal{C}$ is real projective space $\mathbb{RP}^n$, discovered by Renaissance artists studying perspective. Other important examples include moduli spaces of (smooth or stable) algebraic curves, Hilbert schemes, and Quot schemes (see Subsection 1.1.1 for more).

In algebraic geometry, the geometric spaces we consider are algebraic varieties or, more generally, schemes. In modern categorical language, the data of a moduli problem is a contravariant functor F from the category of schemes to the category of sets which associates to each scheme T the set $F(T)$ of appropiate families over T of objects in $\mathcal{C}$, and to each morphism $f : T' \to T$ a pullback map $F(f)$ which takes families over T to families over T'. A (fine) moduli space for this functor is a scheme M which represents F, in the sense that F is isomorphic to the functor $\mathrm{Hom}(-, M)$. That is, a morphism to M is exactly a family of objects in $\mathcal{C}$. In particular, one has a universal family on M itself (corresponding to the identity morphism $M \to M$), such that any other family is a pullback of the universal one.

Unfortunately, many functors of interest are not representable by schemes. There is no fine moduli space of elliptic curves, for instance. There are two main workarounds. One option is to relax our notion of moduli space such that we only require a (universal) natural transformation

$F \to \mathrm{Hom}(-, M)$, rather than an isomorphism; such M are called *coarse* moduli spaces, which are easier to come by. Alternatively, we can enlarge our category of geometric spaces, which leads to the theory of algebraic stacks. Moduli stacks uphold our ideal of what a moduli space should be, but are significantly more difficult to work with than varieties or schemes. Studying the geometric properties of moduli spaces and stacks is a central pursuit in modern algebraic geometry.

In this work, we focus on two kinds of moduli space: realization spaces of line configurations and moduli spaces of vector bundles on curves. A *configuration* is a collection $\mathcal{L}$ of subsets of $\mathcal{P} = \{1, \ldots, n\}$ such that $|\ell \cap \ell'| \leq 1$ for all $\ell, \ell' \in \mathcal{L}$. We think of the elements of $\mathcal{P}$ as points in the projective plane, and elements of $\mathcal{L}$ as lines containing those points. An n_3 *configuration* satisfies the additional conditions that $|\mathcal{L}| = n$, each line contains three of the points, and each point is contained in exactly three of the lines. A *realization* of $\mathcal{L}$ over a field k is a map $\mathcal{P} \to \mathbb{P}^2(k)$ such that the images of distinct $p, q, r \in \mathcal{P}$ are collinear if and only if there is an $\ell \in \mathcal{L}$ with $p, q, r \in \ell$. The realizations of $\mathcal{L}$ over k modulo projective transformations of $\mathbb{P}^2$ correspond to the k-points of a quasiprojective variety called the *realization space* of $\mathcal{L}$. Our key question is: what is the geometry of the realization spaces of 10_3 configurations? For example, what are their dimensions? Do they have singularities? If they are surfaces (as we would expect from a naïve dimension count), where do they fall in the classification of algebraic surfaces? Finally, do they have dense rational points with respect to either the Zariski topology or the classical analytic topology?

For a smooth projective curve C of genus $g \geq 2$ and line bundle $\Lambda \in \mathrm{Pic}^d(C)$, one can construct the moduli space $\mathcal{M}_C(2, \Lambda)$ of semistable rank 2 vector bundles on C of determinant Λ. (Here, a vector bundle E is semistable if every line subbundle $L \subset E$ has degree at most $d/2$.) It's easy to see that up to isomorphism, $\mathcal{M}_C(2, \Lambda)$ depends only on the parity of the degree d of Λ. When d is even, this moduli space is conventionally denoted $\mathcal{SU}_C(2)$. Tevelev and Torres [TT21, Tev23] recently proved the BGMN conjecture (named for Belmans–Galkin–Mukhopadhyay and Narasimhan), which states that when d is odd, the bounded derived category $D^b(\mathcal{M}_C(2, \Lambda))$ of coherent sheaves on $\mathcal{M}_C(2, \Lambda)$ admits a semiorthogonal decomposition with blocks equivalent to $D^b(\mathrm{Sym}^k C)$ for $0 \leq k \leq g - 1$. We ask if their methods can be generalized to study the even degree case. While it is known that a direct analog of the BGMN conjecture cannot be true due to its singularities, we instead seek a noncommutative resolution of singularities of $\mathcal{SU}_C(2)$ (see Subsection 1.1.4).

1.1 Background

1.1.1 Moduli spaces and GIT

The theory of moduli spaces began in [Rie57] with Riemann's count of the $3g - 3$ parameters (or *moduli*) on which the complex structure of a compact Riemann surface of genus $g \geq 2$ depends. We refer to [AJP16] for the early history of the subject, including the contributions of Klein, Poincaré, Hilbert, Teichmüller, and many others. Noteworthy examples of classical moduli problems in algebraic geometry are linear systems of divisors, the Jacobian variety of a curve, and Chow varieties (see [Kol23, Sec. 1.1]). The most important foundational developments for the present thesis are those of Grothendieck and Mumford. In a series of ten talks at Henri Cartan's seminar in 1960–1961 [Gro62], Grothendieck introduced the modern perspective on moduli problems in terms of representable functors. That is, when trying to define a moduli space for some class of objects $\mathcal{C}$, one should first introduce a functor $Sch^{op} \to Set$ which assigns to each scheme T the set of "nice" families over T of objects in $\mathcal{C}$, and then construct a scheme which represents this functor. A fundamental example is the Hilbert scheme $\mathcal{H}ilb(X)$ of a projective variety X, constructed by Grothendieck in [Gro95]. This represents the functor $T \mapsto \{\text{Subschemes of } T \times X \text{ flat over } T\}$.

An essential tool in the construction of such moduli spaces is Mumford's geometric invariant theory (GIT), introduced in [Mum65]. Building on Hilbert's classical theory of invariants [Hil93], GIT is a method for constructing quotients of a quasiprojective variety X by the action of a reductive algebraic group G. An ordinary quotient space often does not exist in the category of varieties; this is exemplified by the obvious action of the multiplicative group $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$, where the quotient fails to be separated due to the origin lying in the closure of every other orbit. Mumford's solution is to remove a closed subset of *unstable* points before taking the quotient. More precisely, one first chooses an ample line bundle L on X together with a fiberwise linear action of G extending the one on X (a *linearization*). The *semistable locus* $X_{ss}(L) \subseteq X$ is defined to be the union $\bigcup_s U_s$ over G-invariant global sections $s \in H^0(X, L^{\otimes n})^G$ with $n > 0$, where $U_s = \{x \in X \mid s(x) \neq 0\}$. The *GIT quotient* $X/\!\!/_L G$ is then given by gluing together affine schemes of the form $\operatorname{Spec} A_s^G$, where $U_s = \operatorname{Spec} A_s$.

There is a natural surjection $\pi : X_{ss}(L) \to X/\!\!/_L G$ induced by $A_s^G \hookrightarrow A_s$ which is a categorical quotient, meaning that any G-invariant map from X_{ss} factors through π. Moreover, there is an open subset $X_s(L) \subseteq X_{ss}(L)$ called the *stable locus* such that the restriction of π to $X_s(L)$ is a geometric quotient, in the sense that its fibers are closed G-orbits. Returning to our example of $\mathbb{G}_m$ acting on $\mathbb{A}^{n+1}$, we we can take $L = \mathcal{O}(1)$, the trivial line bundle $\mathbb{A}^{n+1} \times \mathbb{A}^1$ with the linearization $\lambda \cdot (x, v) = (\lambda x, \lambda v)$. The semistable locus (which here equals the stable locus) is then $\mathbb{A}^{n+1} \smallsetminus \{0\}$, and the GIT quotient $\mathbb{A}^{n+1} /\!\!/_L \mathbb{G}_m$ is projective space $\mathbb{P}^n$, as one would expect. This construction is used extensively in moduli theory.

A pertinent question is how the quotient $X /\!\!/_L G$ depends on the choice of G-linearized ample line bundle L. This has been studied in detail by Dolgachev and Hu in [DH98]. The set of G-linearized ample line bundles span the *G-ample cone* $C^G(X)$ in $\mathrm{NS}^G(X) \otimes \mathbb{R}$, where $\mathrm{NS}^G(X)$ is the group of all G-linearlized line bundles on X modulo homological equivalence. This cone is then divided into finitely many *walls* and *chambers*, where the class of L lies on a wall exactly when $X_s(L) \neq X_{ss}(L)$. Each chamber is an equivalence class under the relation $L \sim L' \iff X_{ss}(L) = X_{ss}(L')$; the other equivalence classes are unions of finitely many *cells*, which are connected subsets of walls. This analysis shows that there are finitely many possible GIT quotients of X by G. The authors also show that under certain conditions, the GIT quotients on opposite sides of a wall are related by a birational transformation similar to a Mori flip. These "wall crossing" transformations are an essential component of Tevelev's proof of the BGMN conjecture [Tev23].

1.1.2 Line configurations

Configurations were defined by Reye in 1876 and studied by Kantor and Schroeter, among others; see [Gro97] and references therein for a historical treatment. Gropp notes a lack of interest in line configurations throughout most of the 20th century, but the subject was renewed in the 80s and 90s. This was partly due to the advent of powerful symbolic algebra software, as exemplified by Sturmfels and White's computer-assisted proof that all 11_3 and 12_3 configurations are realizable over $\mathbb{Q}$ [SW90], which completed the classification of n_3 configurations for $n \leq 12$. Other noteworthy results include Mnëv's universality theorem [Mnë88] and Vakil's so-called "Murphy's law" [Vak06].

The starting point for our study of 10_3 configuration is Sturmfels' article [Stu91]. There, Sturmfels offers a strategy for describing realization spaces using geometric construction sequences and Caley algebra. He also introduces the problem of determining whether the rational realizations of a configuration are dense, and notes that the answer is unknown for a particular 10_3 configuration.

1.1.3 Derived categories

The derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ was introduced by Verdier in his thesis [Ver96]. The objects of this category are cochain complexes with terms in $\mathcal{A}$, and the morphisms are (homotopy classes of) complex morphisms together with formal inverses for quasi-isomorphisms. (A quasi-isomorphism is a complex morphism which induces isomorphisms in all cohomology groups.) These categories are the natural context for derived functors such as Tor and sheaf cohomology. Morally, the framework of derived functors allows one to keep track of chain complexes up to quasi-isomoprhism, rather than just the cohomology groups of such complexes. Moreover, derived functors can now be applied to chain complexes, rather than just objects, which in particular allows for the composition of derived functors. When $\mathcal{A} = \mathrm{Coh}(X)$ is the category of coherent sheaves on a smooth projective variety X, we typically restrict ourselves to the bounded derived category $D^b(X) := D^b(\mathrm{Coh}(X))$ whose objects have only finitely many nonzero cohomology groups.

While Verdier viewed derived categories as a mere technical device, $D^b(X)$ was later found to be a powerful and subtle geometric invariant in its own right. The first instance of this viewpoint is Mukai's paper [Muk81], wherein the author constructs a very nontrivial equivalence between the derived category $D^b(X)$ of an abelian variety X and the derived category of its dual variety $\hat{X}$. This is done by introducing *Fourier-Mukai functors*. Given an object $K \in D^b(X \times Y)$ (called the *kernel* in analogy with integral transforms), the Fourier-Mukai functor $\mathcal{P}_K : D^b(X) \to D^b(Y)$ is given by $F \mapsto Rq_*(Lp^*F \overset{L}{\otimes} K)$ where p and q are projections from $X \times Y$ onto X and Y, respectively. The equivalence $D^b(X) \cong D^b(\hat{X})$ is given by $\mathcal{P}_{\mathcal{F}}$ where $\mathcal{F}$ is the normalized Poincaré bundle on $X \times \hat{X}$.

Further fundamental results on derived categories in algebraic geometry are due to Beĭlinson, Kapranov, Bondal, Orlov, and others (e.g., [Beĭ83, BK89, Orl92, BO95]). Of special importance is the notion of *semiorthogonal decomposition*, where a derived category is broken down into smaller subcategories with simple relations among them; namely, a sequence of (admissible) subcategories $\langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ is called *semiorthogonal* if $\mathrm{Hom}(A, B) = 0$ for all $A \in \mathcal{D}_i$, $B \in \mathcal{D}_j$ with $i > j$ (see [Huy06, Ch. 1] for details). Beĭlinson showed that $D^b(\mathbb{P}^n)$ admits a semiorthogonal decomposition $\langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n) \rangle$ (where $\mathcal{O}(k)$ is shorthand for the image of the Fourier-Mukai functor $\mathcal{P}_{\mathcal{O}(k)} : D^b(pt) \to D^b(\mathbb{P}^n)$). In [Orl92], Orlov proved his blowup formula, which states that the derived category of the blowup of X along a smooth center Z admits a semiorthogonal decomposition of the form $\langle D^b(X), D^b(Z), \ldots, D^b(Z)\rangle$, where $D^b(Z)$ appears $\operatorname{codim}(Z)-1$ times. This was generalized in [BO95] to handle flips over projective bundles. In the same groundbreaking paper, Bondal and Orlov prove an essential criterion for fully-faithfulness of Fourier-Mukai functors, give a semiorthogonal decomposition for the intersection of two even-dimensional quadraics, and show that a variety is determined by its derived category if its cannonical bundle is ample or anti-ample.

Essential to the present endeavor is the theory of windows as developed by Teleman, Halpern-Leistner, and Ballard–Favero–Katzarkov [Tel00, HL15, BFK19]. Halpern–Leistner showed that the derived category of a GIT quotient (or rather the quotient stack $[X_{ss}/G]$) is equivalent to a "window" subcategory $\mathbf{G}_w$ of the equivariant derived category $D^b([X/G])$. $\mathbf{G}_w$ consists of objects whose restriction to each Kempf–Ness stratum of the unstable locus have weights lying in a prescribed range, or window. (Here, *weight* refers to the action of the one-parameter subgroup associated to the stratum; see [HL15, Theorem 2.10] for a precise statement.) This technique be used to describe the effect of Dolgachev–Hu's wall-crossing transformations on the derived categories of GIT quotients: under certain weight conditions, the derived category on one side of the wall embeds into the one on the other side. Some applications of this technique, including to one of the problems at hand, can be found in Torres' thesis [Tor21].

To end this subsection, we mention Kuznetsov's article [Kuz16] on rationality problems and derived categories. Kuznetsov suggests that irrationality of a smooth projective variety X should be detectable by the presence of "Griffiths components", meaning semiorthogonal components of $D^b(X)$ with geometric dimension at least $\dim X - 1$. On the other hand, if a variety is expected to be rational, the "geometric" blocks in its derived category should hint at how to construct the birational map to projective space. This is still largely conjectural, as the set of Griffiths components is not even well defined. Nevertheless, this proposal is considered an important heuristic in the study of derived categories and rationality.

1.1.4 Noncommutative resolutions of singularities

In [Kuz08], Kuznetsov defines a *noncommutative* (or *categorical*) *resolution of singularities* of a variety X as a smooth triangulated category $\mathcal{D}$ with an adjoint pair of functors $f_* : \mathcal{D} \to D^b(X)$ and $f^* : \mathrm{Perf}(X) \to \mathcal{D}$ such that $f_* \circ f^* \cong \mathrm{Id}$. (This definition is motivated by the relationship between $D^b(X)$ and the derived category of an ordinary ("commutative") resolution of singularities when X has rational singularities.) A noncommutative resolution is said to be (*strongly*) *crepant* if the identity functor is a relative Serre functor (see [Kuz08, Section 3]). By the Bondal–Orlov conjecture [BO02], it is expected that a noncommutative crepant resolution should be minimal in the sense that it embeds as an admissible subcategory of any other resolution.

Based on work of Špenko and Van den Bergh [ŠVdB21, ŠVdB23], Pădurariu [Păd21] constructs a noncommutative resolution of singularities of the good moduli space of a symmetric stack satisfying various technical assumptions. He shows that this resolution always embeds into the derived category of the Kirwan resolution.

1.1.5 Moduli of vector bundles

The early development of moduli spaces of vector bundles was carried out in the 1960s by Mumford, Narasimhan, Seshadri, Ramanan, Newstead, and others (e.g., [Mum65, NS64, New68, NR69]). We refer to [TT21] for an account of the history of and prior results toward the BGMN conjecture. In that paper, Tevelev and Torres prove this conjecture up to the possibility of a "phantom" block; the same result was obtained independently in [XY21] using different methods. In [Tev23], Tevelev eliminates this possibility, completing the proof.

Tevelev's proof proceeds by instead studying the derived categories of Thaddeus' moduli spaces of stable pairs M_i [Tha94]. These are smooth projective varieties that parameterize pairs (E, ϕ) of a rank 2 vector bundle E on C of determinant Λ with a nonzero global section $\phi \in H^0(C, E)$, subject to a GIT-stability condition depending on the integer parameter $i \in [0, g-1]$. M_0 is a projective space, whose derived category is well understood via the Beilinson exceptional collection. As shown in [TT21], $D^b(M_{i-1})$ embeds into $D^b(M_i)$ as a window subcategory. M_{i-1} and M_i are related by a standard flip of projective bundles over $\mathrm{Sym}^i C$, so the orthogonal complement of $D^b(M_{i-1})$ in $D^b(M_i)$ consists of blocks equivalent to $D^b(\mathrm{Sym}^i C)$.

Iterating this wall-crossing procedure gives a semiorthogonal decomposition of $D^b(M_{g-1})$ with blocks given by $D^b(\mathrm{Sym}^k C)$ for $0 \leq k \leq g-1$, as in the conjecture. On the other hand, there is a "forgetful" morphism $\zeta : M_{g-1} \to N := \mathcal{M}_C(2, \Lambda)$ given by $(E, \phi) \mapsto E$; this morphism is birational, so $L\zeta^*$ gives a fully faithful embedding of $D^b(N)$ into $D^b(M_{g-1})$. This yields a different semiorthogonal decomposition of $D^b(M_{g-1})$ than the one above. In [Tev23], Tevelev devised a long series of mutations relating these two decompositions, which breaks up into a series of simpler steps dubbed “weaving patterns.” The blocks that wind up in $L\zeta^*(D^b(N))$ are exactly those conjectured by BGMN.

In the even degree setting, it is known that $\mathcal{SU}_C(2)$ has rational singularities along the strictly semistable locus. There are three desingularizations studied in the literature: one due to Seshadri [Ses77] based on moduli of parabolic bundles, one due to Narasimhan–Ramanan [NR78] based on Hecke correspondences, and one due to Kirwan [Kir86] in the more general setting of GIT quotients. The geometric relationship between these desingularizations is worked out in [KL04, CCK05]. The Hodge numbers of Seshadri's model are computed in [DB02] by motivic methods. This computation led to Belmans' conjecture [Bel21] on the blocks that should appear in a noncommutative crepant resolution of $\mathcal{SU}_C(2)$.

In contrast to odd degree, where N is known to be rational [KS99], whether $\mathcal{SU}_C(2)$ is a rational variety is a longstanding open question (except if $g = 2$, in which case $\mathcal{SU}_C(2)$ is known to be a projective space [NR69]). Early attempts to solve this problem were unsuccessful [New75, New80, Tyu64, Tyu65].

1.2 Goals and methodology

As mentioned, our goal in studying 10_3 line configurations is to understand the geometry of their realization spaces. Up to isomorphism, there are exactly ten such configurations, so we can study them one by one. Following Sturmfels' suggestion [Stu91], we address the question of whether the rational points of these realization spaces are dense in the real points. We approach this problem from the perspective of elliptic surfaces; see [SS10] for a comprehensive review. It is clear from Sturmfels' construction of one such realization space ($\mathcal{R}(\mathcal{L}_{\mathrm{V}})$ below) that it is an elliptic surface; we study the Mordell–Weil group of this surface to generate many rational points. We apply similar techniques to construct and analyze the other surfaces. We also study the algebro-geometric properties of these spaces, such as their canonical bundles, Hodge numbers, Picard ranks, etc. using standard techniques. Finally, we wish to compactify these realization spaces, and give a modular interpretation of these compactifications. We make essential use of the software packages Magma [BCP97] and Macaulay2 [GS06] for computations.

Our objective in studying the moduli space $\mathcal{SU}_C(2)$ is to generalize the techniques of Tevelev and Torres to explicitly construct its noncommutative resolution of singularities. As described in Subsection 1.1.5, this involves a "two-ray game" where the Fano varities $\mathbb{P}^{3g-2}$ and $\mathcal{SU}_C(2)$ are realized as extremal contractions of another Fano variety $M = M_{g-1}$. We endeavor to use the geometry of M mutate the usual Beĭlinson semiorthogonal decompostion of $D^b(\mathbb{P}^{3g-2})$ into a semiorthogonal decomposition of $D^b(M)$ compatible with the contraction to $\mathcal{SU}_C(2)$; the result should give the desired noncommutative resolution. The Thaddeus space M is qualitatively insensitive to degree, so many of Tevelev–Torres' methods and results will still apply. On the other hand, there are a number of technical complications stemming from the existence of strictly semistable bundles, and some new ideas and methods will be required. In particular, one needs to be much more careful in distinguishing various moduli stacks when the degree is even.

1.3 Results

Our results on line configurations are detailed in Chapter 2. We show in Theorem 2.1.1 that the realization space of each of the ten 10_3 configuration has analytically dense rational points. In Theorem 2.1.2, we find that four of these realization spaces admit compactifications by K3 surfaces, and compute enough invariants of these surfaces to determine them up to isomorphism. We also give a modular interpretation of these K3 surfaces in terms of "weak" realizations.

In Chapter 3, we construct the desired noncommutative resolution $\mathcal{D}$ of $\mathcal{SU}_C(2)$ as a subcategory of $D^b(M)$. It has a semiorthogonal decomposition with blocks equivalent to $D^b(\mathrm{Sym}^{2k}C)$ for $0 \leq 2k \leq g-1$. We find that it agrees with Pădurariu's construction [Păd21] and that it is strongly crepant when g is even, in which case our result confirms Belmans' prediction [Bel21]. This decomposition provides evidence (via Kuznetsov's proposal above) that $\mathcal{SU}_C(2)$ is rational.

1.4 Further research

The most natural followup to our study of line configurations would be to study realization spaces of n_3 configurations for $n > 10$. By the arguments in Section 2.1, $n = 11$ should give Calabi–Yau threefolds (of interest in physics). The number of configurations grows fairly quickly with n, and the computational techniques used in our treatment are unlikely to scale well to larger dimensions (though more specialized software packages, such as the matroid facilities of OSCAR [DEF$^+$24], may be adequate). While Sturmfels' approach to explicitly constructing the realization space will still work, the powerful theory of elliptic surfaces is no longer available, so a different method of proving density would be necessary.

Two-ray games such as the one used in our study of $\mathcal{S}U_C(2)$ are of great interest. It is conjectured that one can play a similar game with any Fano varieties X, Y, Z related by extremal contractions $Y \dashleftarrow X \dashrightarrow Z$, where a semiorthogonal decomposition of $D^b(Y)$ is mutated via X into a semiorthogonal decomposition of $D^b(Z)$. Indeed, mirror symmetry suggests that the braid relating these decompositions should be given by the monodromy of the eigenvalues of the first Chern class of X acting on small quantum cohomology as the base varies; this is related to the Dubrovin and Gamma conjectures [Dub98, GI19]. It would be worthwile to attempt similar investigations for other triples of Fano varieties, e.g., toric Fanos. Our result also suggests that to prove the rationality of $\mathcal{S}U_C(2)$ for $g = 3$, one should try to embed $\mathrm{Sym}^2 C$ into $\mathbb{P}^6$ and blow up; this approach has proven difficult to put into practice.

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Chapter 2

Line configurations

2.1 Introduction

As discussed in Chapter 1, a *configuration* on a finite set $\mathcal{P}$ (called the set of *points*) is a set $\mathcal{L}$ of subsets of $\mathcal{P}$ (called *lines*) such that any two lines have at most one point in common. A *realization* of $\mathcal{L}$ in $\mathbb{P}^2$ over a field k is a map $\mathcal{P} \to \mathbb{P}^2(k)$ such that for all distinct $p, q, r \in \mathcal{P}$, their images in $\mathbb{P}^2(k)$ are collinear if and only if there is a line in $\mathcal{L}$ containing p, q, r. The set of all realizations over k of a configuration $\mathcal{L}$ can be identified with the k-points of a quasiprojective variety $V \subset (\mathbb{P}^2)^n$, where $n = |\mathcal{P}|$. The condition that three points are collinear is expressed as the vanishing of a corresponding minor of the $3 \times n$ matrix of coordinates of $(\mathbb{P}^2)^n$, and V is cut out by one equation or inequation of this type for each triple. However, this variety is too large; we would like to identify realizations that are related by projective transformations of $\mathbb{P}^2$. This quotient space $V/\mathrm{PGL}(3)$ is called the *realization space* of the configuration, denoted $\mathcal{R}(\mathcal{L})$. This quotient is constructed by means of geometric invariant theory, and the choice of stability condition gives rise to various compactifications of $\mathcal{R}(\mathcal{L})$, each with its own interpretation as a moduli space of (weak) realizations of $\mathcal{L}$ (see Section 2.4).

Specializing to the case of n_3 configurations, we observe that the equation defining a line $\ell \in \mathcal{L}$ has degree 1 in the coordinates of the points on ℓ, and degree 0 for other points. Since each point lies on exactly three lines, the sum of the multidegrees of the n equations of lines in $\mathcal{L}$ is $(3, 3, \ldots, 3)$. This is exactly opposite the multidegree of the canonical class of $(\mathbb{P}^2)^n$. If the projective variety $W \supset V$ cut out by these n equations were a complete intersection of codimension n in $(\mathbb{P}^2)^n$, the adjunction formula would imply that the canonical bundle of W is trivial. The same argument applies in any open subset of $(\mathbb{P}^2)^n$; in particular, if the semistable locus of W is such a complete intersection in $(\mathbb{P}^2)^n_{ss}$, then the corresponding GIT quotient will have trivial canonical bundle as well (see Lemma 2.4.1). In other words, we expect the realization spaces of n_3 configurations to have compactifications with "Calabi–Yau type" geometry. Such varieties are of significant interest for their difficult arithmetic and their relevance to physics. For example, in the 10_3 case, $\mathcal{R}(\mathcal{L})$ has expected dimension

$$2n - |\mathcal{L}| - \dim \mathrm{PGL}(3) = 20 - 10 - 8 = 2,$$

so we would hope for a K3 surface. Similarly, 11_3 configurations would give Calabi–Yau threefolds, and so on.

As a test of this philosophy, we study the realization spaces of 10_3 configurations. It was shown by Kantor [Kan81] that there are precisely 10 such configurations (up to relabeling), and Schroeter [Sch89] found that all but one of them admit realizations over fields of characteristic 0. Following Schroeter's numbering, we refer to these configurations as $\mathcal{L}_{\mathrm{I}}, \mathcal{L}_{\mathrm{II}}, \ldots, \mathcal{L}_{\mathrm{X}}$. To illustrate our methods, we focus our discussion on

$$\mathcal{L}_{\mathrm{V}} = \{124, 138, 179, 237, 259, 350, 456, 480, 678, 690\}, \tag{2.1.1}$$

where $\mathcal{P} = \{0, \ldots, 9\}$ and $124 = \{1, 2, 4\}$, etc. This configuration was studied by Sturmfels in [Stu91], who gave a concrete description of its realization space using a geometric construction sequence. Our choice of this particular configuration was motivated by the question, left open by Sturmfels, of whether its realizations over $\mathbb{Q}$ are dense in the realizations over $\mathbb{R}$. We give a positive answer to this question for all 10_3 configurations.

Theorem 2.1.1. *For all 10_3 configurations $\mathcal{L}$, the rational realizations $\mathcal{R}(\mathcal{L})(\mathbb{Q})$ are dense in the real realizations $\mathcal{R}(\mathcal{L})(\mathbb{R})$ with respect to the classical analytic topology.*

It turns out that for many 10_3 configurations, the would-be K3 surfaces are either not of the expected dimension ($\mathcal{L}_{\mathrm{I}}$) or are reducible ($\mathcal{L}_{\mathrm{II}}$, $\mathcal{L}_{\mathrm{III}}$, $\mathcal{L}_{\mathrm{IV}}$, $\mathcal{L}_{\mathrm{VI}}$, and $\mathcal{L}_{\mathrm{VII}}$). For the other four, the Calabi–Yau dream is achieved.

Theorem 2.1.2. *Let $\mathcal{L}$ be one of the 10_3 configurations $\mathcal{L}_{\mathrm{V}}$, $\mathcal{L}_{\mathrm{VIII}}$, $\mathcal{L}_{\mathrm{IX}}$, or $\mathcal{L}_{\mathrm{X}}$.*

(i) The realization space $\mathcal{R}(\mathcal{L})$ is isomorphic to a Zariski-open subset of an elliptic K3 surface of Picard number 20 and discriminant -7, -8, -7, and -11, respectively.

(ii) This K3 surface is a fine moduli space for GIT-stable weak realizations of $\mathcal{L}$.

In Section 2.2, we observe that Sturmfels' calculation gives rise to an algebraic surface S with an elliptic fibration. We use computations in its Mordell–Weil group to prove Theorem 2.1.1 for $\mathcal{L} = \mathcal{L}_{\mathrm{V}}$. Similar techniques are used for the other configurations. In Section 2.3, we construct a K3 surface $\widetilde{S}$ as the minimal resolution of S. We compute its singular fibers, Picard number, and discriminant, which identify it as the universal elliptic curve over $\Gamma_1(7)$. The other three K3 surfaces are constructed likewise, proving Theorem 2.1.2(i). In Section 2.4, we review GIT quotients in general and for the case of $(\mathbb{P}^2)^n /\!\!/ \mathrm{PGL}(3)$. We describe the correct choice of GIT quotient for our problem and use computer algebra to prove Theorem 2.1.2(ii).

Throughout, we work with varieties over $\mathbb{C}$, though our results (except for those on analytic density) hold over any algebraically closed field of characteristic 0.

2.2 Elliptic fibrations and density of rational realizations

Sturmfels' parameterization of $\mathcal{R}(\mathcal{L}_{\mathrm{V}})$ goes as follows (see [Stu91, §2] for details): fix points 1, 2, 3, and 5 to standard coordinates $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, and $[1 : 1 : 1]$ using a projective transformation. Let $[a : b : c]$ be homogeneous coordinates for point 6, and $[0 : u : v]$ homogeneous coordinates for point 7 along the line $\overline{23}$ (both in "general position" to avoid unwanted collinearities). The positions for the other 4 points are then determined, with one condition to ensure that $6, 9, 0$ are collinear:

$$u^2a^2c - uva^2c - v^2b^3 + uvb^2c + v^2ab^2 - uvabc + uvac^2 - u^2ac^2 = 0. \tag{2.2.1}$$

It's clear that every realization of $\mathcal{L}_{\mathrm{V}}$ (up to PGL(3)) can be obtained in this manner for a unique choice of $([u : v], [a : b : c])$ satisfying (2.2.1), and that almost all choices yield such a realization. Sturmfels also gives an example of a realization of $\mathcal{L}_{\mathrm{V}}$ over $\mathbb{Q}$, so $\mathcal{R}(\mathcal{L}_{\mathrm{V}})(\mathbb{Q})$ is nonempty.

We observe that equation (2.2.1) is irreducible and homogeneous of bidegree $(2,3)$ in $[u:v]$ and $[a:b:c]$, so it defines an irreducible surface $S \subset \mathbb{P}^1 \times \mathbb{P}^2$. The above analysis shows that $\mathcal{R}(\mathcal{L}_{\mathrm{V}})$ is isomorphic (over $\mathbb{Q}$) to a dense open subset of S. Moreover, Computation 2.5.1 shows that this subset is contained in the smooth locus S_{sm} of S. Hence, to prove Theorem 2.1.1 for $\mathcal{L} = \mathcal{L}_{\mathrm{V}}$, it suffices to show that $S_{sm}(\mathbb{Q})$ is analytically dense in $S_{sm}(\mathbb{R})$.

The generic fiber E of the projection $\pi : S \to \mathbb{P}^1$ onto the first factor is a smooth cubic plane curve over the function field $K = k(\mathbb{P}^1)$ whose K-points are identified with the sections of π. That is, π gives an *elliptic fibration* of S. The point $([u:v],[1:1:1])$ lies in S for all $u:v \in \mathbb{P}^1$, so we have a section

$$o = [1:1:1] \in E(K)$$

defined over $\mathbb{Q}$. Choosing o for the identity makes E into an elliptic curve over K.

The abelian group $\mathrm{MW}(S) = E(K)$ of sections of π is called the (geometric) Mordell–Weil group of S. This group is finitely generated [SS10, Theorem 6.1] and amenable to computer calculations, which will allow us to show the existence of many rational points. As an easy demonstration of this technique, we prove the following:

Proposition 2.2.1. *$S(\mathbb{Q})$ is dense in S with respect to the Zariski topology.*

Proof. We find another section

$$s = [1:t:t] \in E(K)$$

defined over $\mathbb{Q}$, where $t = u/v \in K$. We check in Computation 2.5.2 that s is not torsion in $\mathrm{MW}(S)$, so π admits infinitely many sections defined over $\mathbb{Q}$. That is, S contains an infinite collection of irreducible curves isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}$. Their union is certainly Zariski-dense; a proper closed subset of S must have codimension at least 1 by irreducibility, and therefore can contain only finitely many irreducible curves. Since each section has a dense set of $\mathbb{Q}$-points, this proves that $S(\mathbb{Q})$ is Zariski-dense. $\square$

The proof of analytic density proceeds along similar lines.

Convention 2.2.2. For the remainder of this section only, we use the analytic topology when working with real loci. In particular, "dense", "open", and "connected" are understood with respect to this topology.

Lemma 2.2.3. *Let E be an elliptic curve over $\mathbb{R}$, and let p be an non-torsion $\mathbb{R}$-point. Then the orbit of p is dense in $E(\mathbb{R})$ if and only if either $E(\mathbb{R})$ is connected, or p does not lie in the identity component of $E(\mathbb{R})$.*

Proof. The real locus of an elliptic curve has either one or two connected components, both diffeomorphic to circles. Being a compact connected real Lie group of dimension 1, the identity component E_0 of $E(\mathbb{R})$ is a normal subgroup isomorphic (as a Lie group) to the circle group $\mathbb{R}/\mathbb{Z}$. If $E(\mathbb{R}) = E_0$, then the claim follows from the standard fact that irrational rotations have dense orbits in $\mathbb{R}/\mathbb{Z}$.

Otherwise, $E(\mathbb{R})$ has two components, E_0 and E_1, with the quotient $E(\mathbb{R})/E_0 \cong \mathbb{Z}/2$. If $p \in E_0$, then its orbit is contained in E_0 and not dense in $E(\mathbb{R})$. Otherwise, we have $E_1 = p + E_0$ and $2p \in E_0$. The orbit of $2p$ is dense in E_0 by the above, and its image under translation by p is dense in E_1. □

Lemma 2.2.4. *Suppose $f : X \to \mathbb{P}^1$ is an elliptic fibration over $\mathbb{Q}$ with identity section o. Let $F(t)$ denote the real locus of the fiber $f^{-1}(t)$ over $t \in \mathbb{P}^1(\mathbb{R})$. Suppose further that there are open sets $U_i \subseteq \mathbb{P}^1(\mathbb{R})$ and sections $s_i \in \mathrm{MW}(X)$ defined over $\mathbb{Q}$ such that*

(1) $\bigcup_i U_i$ is dense in $\mathbb{P}^1(\mathbb{R})$,

(2) the s_i are not torsion in $\mathrm{MW}(X)$, and

(3) for all $t \in U_i$ with $F(t)$ smooth, either $F(t)$ is connected or $s_i(t)$ lies in the non-identity component of $F(t)$.

Then $X_{sm}(\mathbb{Q})$ is dense in $X_{sm}(\mathbb{R})$, where X_{sm} is the smooth locus of X.

Proof. When $F(t)$ is smooth, we regard it as a real elliptic curve with identity $o(t)$. As a curve in X, any positive multiple of s_i in $\mathrm{MW}(X)$ intersects the identity section in finitely many points, so the set

$$T_{i,n} = \{t \in \mathbb{P}^1(\mathbb{Q}) \mid s_i(t) \text{ has order } n \text{ in } F(t)\}$$

is finite. By Mazur's classification of torsion subgroups of elliptic curves over $\mathbb{Q}$ [Maz77], $T_{i,n}$ can only be nonempty if $n \leq 12$, so $T_i = \bigcup_{n>0} T_{i,n}$ is finite. It follows that the set

$$D_i = (U_i \cap \mathbb{P}^1(\mathbb{Q})) \smallsetminus T_i,$$

where $s_i(t)$ is not torsion in $F(t)$, is dense in U_i. By assumption (3) and Lemma 2.2.3, the orbit of $s_i(t)$ (which consists of rational points) is dense in $F(t)$ for all $t \in D_i$.

Suppose $W \subseteq X_{sm}(\mathbb{R})$ is open and nonempty. $X_{sm}(\mathbb{R})$ is a real manifold of dimension 2, so nonempty Zariski-open sets are (analytically) dense. In particular, the differential $\mathrm{d}f$ is nonzero on such a subset, so f is a submersion on some nonempty open subset of W. Submersions are open maps, so $f(W) \subseteq \mathbb{P}^1(\mathbb{R})$ contains an open set. By (1), $f(W)$ intersects some U_i, and hence some D_i. Then W meets meets $F(t)$ for some $t \in D_i$. $F(t)$ has dense rational points, so the proof is complete. □

Corollary 2.2.5. *With notation as above, suppose there exist $s, r \in \mathrm{MW}(X)$ defined over $\mathbb{Q}$ such that s is non-torsion, r is torsion, and for all $t \in \mathbb{P}^1(\mathbb{R})$ with $F(t)$ smooth and disconnected, $r(t)$ lies in the non-identity component of $F(t)$. Then $X_{sm}(\mathbb{Q})$ is dense in $X_{sm}(\mathbb{R})$.*

Proof. Let $U = \{t \in \mathbb{P}^1(\mathbb{R}) \mid F(t) \text{ is smooth}\}$, which is a dense open subset of $\mathbb{P}^1(\mathbb{R})$. Let

$$U_0 = \left\{ t \in U \mid \begin{smallmatrix} F(t) \text{ is connected or } s(t) \text{ lies in} \\ \text{the non-identity component of } F(t) \end{smallmatrix} \right\}$$

and $U_1 = U \setminus U_0$. Both are easily seen to be open. The result follows from Lemma 2.2.4 with $s_0 = s$ and $s_1 = s + r$. □

Remark 2.2.6. By continuity, it suffices to check the condition on $r(t)$ for one t in each connected component of U. Roughly speaking, $r(t)$ cannot jump between components except when $F(t)$ degenerates.

Lemma 2.2.7. *Theorem 2.1.1 holds for $\mathcal{L} = \mathcal{L}_{\mathrm{V}}$.*

Proof. We first compute a generalized Weierstrass form for E. Computation 2.5.2 gives

$$y^2 = x^3 + a(t)x^2 + b(t)x \tag{2.2.2}$$

where

$$t = u/v \in K, \quad a(t) = \frac{8t^3 - 15t^2 + 8t}{(t-1)^2}, \quad b(t) = 16t^2.$$

(For convenience, we use affine coordinates throughout the proof.) The surface $S' \subset \mathbb{P}^1 \times \mathbb{P}^2$ given by equation (2.2.2) in (t, x, y) is birational to S over $\mathbb{Q}$. It therefore suffices to show that $S'_{sm}(\mathbb{Q})$ is dense in $S'_{sm}(\mathbb{R})$.

As with S, we have an elliptic fibration $\pi' : S' \to \mathbb{P}^1$. We study the real locus $F(t)$ of the fiber over $t \in U = \mathbb{P}^1(\mathbb{R}) \setminus \{0, 1, \infty\}$ as a real curve in $\mathbb{R}^2$ with coordinates (x, y). $F(t)$ is smooth for $t \in U$, and it has two components if and only if $x^2 + a(t)x + b(t)$ has distinct real roots, i.e.,

$$a(t)^2 - 4b(t) = \frac{t^3}{(t-1)^4}(16t^2 - 31t + 16) > 0.$$

The quadratic factor is strictly positive, so $F(t)$ has two components exactly when $t > 0$. Furthermore, $a(t)$ and $b(t)$ are positive for $t > 0$, so

$$\left(x, \pm\sqrt{x^3 + a(t)x^2 + b(t)x}\right) \tag{2.2.3}$$

is a real point for any $x \geq 0$. Since $x^3 + a(t)x^2 + b(t)x$ has a root at 0, the identity component of the fiber over $t > 0$ is exactly (2.2.3) for $x \geq 0$.

We now apply Corollary 2.2.5. We saw in the proof of Proposition 2.2.1 that $E(K)$ has a non-torsion element s defined over $\mathbb{Q}$. Computation 2.5.2 exhibits the torsion section

$$r = \left(-4t, \frac{4t^2}{t-1}\right),$$

defined over $\mathbb{Q}$. Moreover, it lies in the non-identity component of $F(t)$ when $t > 0$, so the proof is complete. □

Proof of Theorem 2.1.1. Following [Stu91], we construct the realization spaces of the other 10_3 configurations in the same fashion: choose four points $p_1, \ldots, p_4 \in \mathcal{P}$, no three on a line in $\mathcal{L}$, such that $p_1, p_2 \in \ell$ for some $\ell \in \mathcal{L}$. Fix these points to $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, and $[1:1:1]$. Let the third point on ℓ be $[u:v:0]$, and let $[a:b:c]$ be another point in general position with the first five. Nine lines suffice to determine the positions of the remaining points, and the tenth line gives a bihomogeneous equation $F_{\mathcal{L}}([u:v],[a:b:c]) = 0$ of bidegree $(2,3)$. Imposing open conditions to exclude additional collinearities presents the realization space as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^2$. This construction is carried out for each configuration in Computations 2.5.3 and 2.5.4.

Two configurations are special. One is $\mathcal{L}_{\mathrm{I}}$, the well-known Desargues configuration. Here, the nine collinearities imply the tenth (Desargues' theorem), so $F_{\mathcal{L}_{\mathrm{I}}}$ is identically 0. $\mathcal{R}(\mathcal{L}_{\mathrm{I}})$ is a Zariski-open subset of $\mathbb{P}^1 \times \mathbb{P}^2$, so it has dense rational points. The other is the unique non-realizable 10_3 configuration $\mathcal{L}_{\mathrm{IV}}$, for which the claim is vacuous.

For all other configurations, the realization space is a surface. For configurations $\mathcal{L}_{\mathrm{II}}$, $\mathcal{L}_{\mathrm{III}}$, $\mathcal{L}_{\mathrm{VI}}$, and $\mathcal{L}_{\mathrm{VII}}$, the surface given by $F_{\mathcal{L}} = 0$ is reducible. In each case, all but one component are eliminated by the open conditions, leaving a smooth rational surface as the realization space. Again, density is immediate.

For the remaining configurations $\mathcal{L}_{\mathrm{V}}$ (treated above), $\mathcal{L}_{\mathrm{VIII}}$, $\mathcal{L}_{\mathrm{IX}}$, and $\mathcal{L}_{\mathrm{X}}$, the equations $F_{\mathcal{L}} = 0$ define irreducible surfaces $S_{\mathrm{V}} = S$, S_{VIII}, S_{IX}, and S_{X} in $\mathbb{P}^1 \times \mathbb{P}^2$ with elliptic fibrations given by projection to $\mathbb{P}^1$. For these, the proof proceeds as exactly in Lemma 2.2.7: We first check that the singularities of each surface are disjoint from the corresponding realization space. We then compute a Weierstrass form for the generic fiber. We find a non-torsion section s, as well as a torsion section r which always lies in the correct component of the fiber. (Per Remark 2.2.6, this only requires checking in finitely many fibers.) With these data, which we collect and verify in Computation 2.5.4, the result follows from Corollary 2.2.5. $\square$

2.3 The K3 surfaces

In this section, we study the surface $S = S_{\mathrm{V}}$ from Section 2.2 and its elliptic fibration π in greater detail. While the general fiber of an elliptic fibration is a smooth curve of genus 1, there may be finitely many points where the fiber degenerates into something singular. Our fibration π has five such singular fibers. The fibers $\pi^{-1}([0:1])$, $\pi^{-1}([1:1])$, and $\pi^{-1}([1:0])$ are unions of lines in $\mathbb{P}^2$, and $\pi^{-1}\left(\left[\frac{31 \pm 3\sqrt{-7}}{32} : 1\right]\right)$ are a conjugate pair of nodal cubics. S itself has seven isolated singular points, all of which are contained in the first three singular fibers (see Computation 2.5.1).

In the same computation, we find that these singularities are all Du Val of type A_n for $n \leq 3$. Du Val singularities can be resolved by a finite sequence of blowups at isolated double points. The result is a minimal smooth surface $\widetilde{S}$ with a birational morphism $\varphi : \widetilde{S} \to S$. The composition $\overline{\pi} = \pi\varphi$ gives an elliptic fibration of $\widetilde{S}$ whose singular fibers are those of S with each Du Val singularity replaced by a chain of rational curves. This is depicted in Figure 2.1.

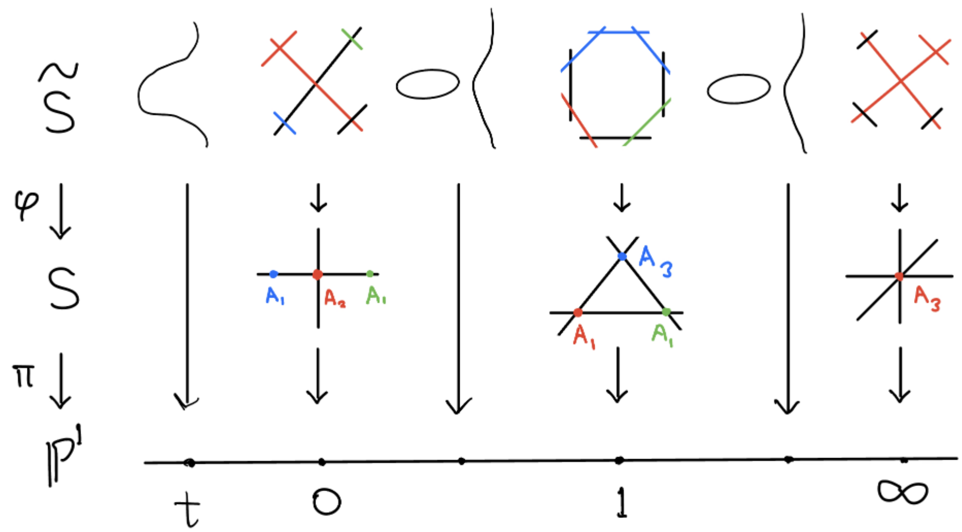


Figure 2.1: The elliptic fibrations of S and $\widetilde{S}$, drawn over $\mathbb{R}$ with coordinate $t = u/v$ on $\mathbb{P}^1$. The marked points are Du Val singularities of S and are replaced by a chain of rational curves in $\widetilde{S}$. Smooth fibers, as well as the nodal fibers (not pictured), are unchanged by φ.

A smooth surface with a minimal elliptic fibration is called an *elliptic surface* (see [SS10]). All possible singular fibers of an elliptic surface were determined by Kodaira. In his notation, the nodal cubic fibers are of type I_1. Inserting the appropriate trees of exceptional curves, we find that $\overline{\pi}^{-1}([0:1])$ and $\overline{\pi}^{-1}([1:0])$ are of type I_1^*, while $\overline{\pi}^{-1}([1:1])$ is of type I_8. (One can also compute the list of Kodaira fibers directly, which gives the same result; see Computation 2.5.2.) From this information, we obtain the topological Euler charateristic of $\widetilde{S}(\mathbb{C})$:

$$e(\widetilde{S}) = \sum_F e(F) = 1 + 1 + 7 + 7 + 8 = 24$$

where the sum is over singular fibers [SS10, Theorem 6.10]. This is the correct Euler characteristic for a *K3 surface*, i.e., a complete nonsingular surface X with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$ and irregularity $h^{1,0}(X) = h^1(X, \mathcal{O}_X) = 0$.

Lemma 2.3.1. *$\widetilde{S}$ is a K3 surface.*

Proof. We first compute the canonical bundle ω_S on S. Recall that S has bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, while the canonical bundle $\omega_{\mathbb{P}^1 \times \mathbb{P}^2}$ has bidegree $(-2,-3)$. S is a regular in codimension 1, so by the adjunction formula,

$$\omega_S = (\omega_{\mathbb{P}^1\times\mathbb{P}^2} \otimes \mathcal{O}(S))|_S = (\mathcal{O}(-2,-3) \otimes \mathcal{O}(2,3))|_S = \mathcal{O}_S.$$

We can also compute $h^1(S, \mathcal{O}_S)$ using the short exact sequence

$$0 \to \mathcal{O}(-2,-3) \to \mathcal{O} \to \mathcal{O}_S \to 0.$$

of sheaves on $\mathbb{P}^1 \times \mathbb{P}^2$. The corresponding long exact sequence in cohomology is

$$\cdots \to H^1(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}) \to H^1(S, \mathcal{O}_S) \to H^2(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(-2,-3)) \to \cdots .$$

Since $h^{1,0}(\mathbb{P}^1 \times \mathbb{P}^2) = h^{2,3}(\mathbb{P}^1 \times \mathbb{P}^2) = 0$, we have $h^1(S, \mathcal{O}_S) = 0$ as well.

Since the singularities of S are all Du Val, the resolution φ is crepant [Rei14], i.e.,

$$\omega_{\widetilde{S}} = \varphi^*\omega_S = \varphi^*\mathcal{O}_S = \mathcal{O}_{\widetilde{S}}.$$

Moreover, Du Val singularities are rational, meaning the natural map

$$\mathcal{O}_S \to R\varphi_*\mathcal{O}_{\widetilde{S}}$$

of complexes on S is a quasisomorphism. Applying $R^1\Gamma$ yields

$$h^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = h^1(S, \mathcal{O}_S) = 0,$$

so $\widetilde{S}$ is a K3 surface. □

Remark 2.3.2. The resolution φ is an isomorphism away from the singular points of S, which are disjoint from the open subset isomorphic to $\mathcal{R}(\mathcal{L}_{\mathrm{V}})$. It follows that $\widetilde{S}$ also has such a subset, so it is the compactification of $\mathcal{R}(\mathcal{L}_{\mathrm{V}})$ by an elliptic K3 surface promised in Theorem 2.1.2(i).

We now compute enough standard invariants $\widetilde{S}$ to determine it up to isomorphism. First is the *Picard number* $\rho(\widetilde{S})$, the rank of the Néron–Severi group $\mathrm{NS}(\widetilde{S})$ of divisors modulo algebraic equivalence, which we compute using the Shioda–Tate formula.

Lemma 2.3.3 ([SS10, Theorem 6.3, Corollary 6.13]). *For any elliptic surface X with identity section, we have* $\mathrm{NS}(X)/A \cong \mathrm{MW}(X)$ *where A is the subgroup generated by the classes of the identity section and fiber components. Hence,*

$$\rho(X) = 2 + \sum_F (m_F - 1) + \operatorname{rank} \mathrm{MW}(X),$$

where m_F is the number of components of the singular fiber F.

Lemma 2.3.4. *The Picard number $\rho(\widetilde{S})$ is* 20.

Proof. In the proof of Proposition 2.2.1, we exhibited a non-torsion member of $\mathrm{MW}(S) \cong \mathrm{MW}(\widetilde{S})$, so its rank is at least 1. By Lemma 2.3.3,

$$\rho(\widetilde{S}) \geq 2 + (0 + 0 + 5 + 5 + 7) + 1 = 20.$$

But 20 is the largest possible Picard number for a K3 surface [Huy16], so in fact $\operatorname{rank} \mathrm{MW}(\widetilde{S}) = 1$ and $\rho(\widetilde{S}) = 20$. □

It follows from the Torelli theorem for K3 surfaces that K3 surfaces X of Picard number 20 are determined by their *transcendental lattice* $T(X)$, the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X, \mathbb{Z})$ [Huy16, Sch10]. This is an even, positive-definite lattice of rank 2. Following [Sch10], we say that a K3 surface over $\mathbb{Q}$ has *Picard rank* 20 *over* $\mathbb{Q}$ if $\rho(X) = 20$ and $\mathrm{NS}(X)$ is generated by divisors defined over $\mathbb{Q}$. Elkies showed that there are exactly 13 such K3 surfaces, corresponding to the 13 primitive lattices of class number 1 [Elk07, Sch10]. They are determined by the discriminant d of $T(X)$, or equivalently (up to sign) the discriminant of $\mathrm{NS}(X)$.

Lemma 2.3.5. $\widetilde{S}$ *has Picard rank* 20 *over* $\mathbb{Q}$ *with discriminant* $d = -7$.

Proof. We use the Cox–Zucker formula [CZ79; Huy16, §11.3]:

$$|d| = \frac{R}{\left|\mathrm{MW}(\widetilde{S})_{\mathrm{tors}}\right|^2} \prod_F n_F.$$

Here, $\mathrm{MW}(\widetilde{S})_{\mathrm{tors}}$ is the torsion subgroup of $\mathrm{MW}(\widetilde{S})$, n_F is the number of components of the singular fiber F appearing with multipilicity 1, and R is the *regulator*, the discriminant of the Mordell–Weil lattice $\Lambda_{\mathrm{MW}}(\widetilde{S}) = \mathrm{MW}(\widetilde{S})/\mathrm{MW}(\widetilde{S})_{\mathrm{tors}}$ with respect to the height pairing $\langle -,- \rangle$ (see [Shi90]). (Note that this pairing takes values in $\mathbb{Q}$, so $\Lambda_{\mathrm{MW}}(\widetilde{S})$ is not a true lattice, despite the name.)

I_1, I_1^*, and I_8 fibers have $n_F = 1$, 4, and 8, respectively. In Computation 2.5.2, we find that the subgroup $H \subseteq \mathrm{MW}(\widetilde{S})_{\mathrm{tors}}$ consisting of sections defined over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}/4$, so $|\mathrm{MW}(\widetilde{S})_{\mathrm{tors}}| = 4k$. Since $\mathrm{MW}(\widetilde{S})$ has rank 1, R is equal to $\langle g, g \rangle$ for a generator g of $\Lambda_{\mathrm{MW}}(\widetilde{S})$. We compute $\langle s, s \rangle = 7/8$ for the section s from the proof of Proposition 2.2.1. Writing $s = ng$ for some $n \in \mathbb{Z}$, we have $R = 7/8n^2$. Putting this all together, we find that

$$|d| = \frac{7}{8n^2} \cdot \frac{1}{(4k)^2} \cdot 1 \cdot 1 \cdot 4 \cdot 4 \cdot 8 = \frac{7}{k^2 n^2}.$$

Since $T(\widetilde{S})$ is an even lattice of rank 2, d must be an integer congruent to 0 or 1 modulo 4; we deduce that $k^2 = n^2 = 1$ and $d = -7$. In particular, $\mathrm{MW}(\widetilde{S})_{\mathrm{tors}} = H$ and s generates $\Lambda_{\mathrm{MW}}(\widetilde{S})$, so the Mordell–Weil group is generated by sections defined over $\mathbb{Q}$. Since the identity section and all components of the reducible fibers are also defined over $\mathbb{Q}$, we conclude that $\widetilde{S}$ has Picard rank 20 over $\mathbb{Q}$. □

Comparing with the table of all K3 surfaces with Picard rank 20 over $\mathbb{Q}$ in [Sch10, §10], we reach the remarkable conclusion that $\widetilde{S}$ is isomorphic to the universal elliptic curve over $\Gamma_1(7)$. As a sanity check, we find our Kodaira fibers and Mordell–Weil group among the 20 elliptic fibrations of that modular surface [Lec15, Table 3, row 2], and we verify in Computation 2.5.2 that the two surfaces have isomorphic generic fibers.

Having fully analyzed $\widetilde{S} = \widetilde{S_{\mathrm{V}}}$, we turn to the other three configurations of interest.

Proof of Theorem 2.1.2(i). We follow the same steps to prove that the minimal resolutions $\widetilde{S_{\mathrm{VIII}}}$, $\widetilde{S_{\mathrm{IX}}}$, and $\widetilde{S_{\mathrm{X}}}$ are also K3 surfaces of Picard number 20. The proof that they are K3 is identical to that of Lemma 2.3.1. Since S_{VIII}, S_{IX}, and S_{X} have degree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, we need only check that they have only Du Val singularities; this is done in Computation 2.5.4. We also compute the singular fibers of each surface, and check that $\sum_F (m_F - 1) = 17$. Since each Mordell–Weil group has an explicit non-torsion element, Lemma 2.3.3 shows that the Picard number is 20.

X	Singular fibers	MW(X)	R	d
$\widetilde{S_{\mathrm{V}}} \cong \widetilde{S_{\mathrm{IX}}}$	$2\mathrm{I}_1, 2\mathrm{I}_1^*, \mathrm{I}_8$	$\mathbb{Z} \oplus \mathbb{Z}/4$	$7/8$	-7
$\widetilde{S_{\mathrm{VIII}}}$	$\mathrm{I}_0^*, 2\mathrm{I}_2^*, \mathrm{I}_2$	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	1	-8
$\widetilde{S_{\mathrm{X}}}$	$3\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_5, \mathrm{I}_6, \mathrm{I}_8$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$11/120$	-11

Table 2.1: The data required to identify each K3 surface, collected in Computation 2.5.4: the singular fibers, the Mordell-Weil group, the regulator, and the discriminant.

As in Lemma 2.3.5, we can identify these K3 surfaces $X = \widetilde{S_{\mathrm{VIII}}}$, $\widetilde{S_{\mathrm{IX}}}$, and $\widetilde{S_{\mathrm{X}}}$ up to isomorphism by computing the group $H \subseteq MW(X)_{\mathrm{tors}}$ of torsion sections defined over $\mathbb{Q}$, together with the height pairing $\langle s, s \rangle$ for some non-torsion section s defined over $\mathbb{Q}$. This is done in Computation 2.5.4, with the results collected in Table 2.1. In each case, the Cox–Zucker formula and the requirement that d be an integer congruent to 0 or 1 modulo 4 imply that $H = MW(X)_{\mathrm{tors}}$ and s generates $\Lambda_{\mathrm{MW}}(X)$. Hence they all have Picard rank 20 over $\mathbb{Q}$, and so are determined up to isomorphism by their discriminant. □

Surprisingly, we find that $\widetilde{S_{\mathrm{V}}}$ and $\widetilde{S_{\mathrm{IX}}}$ are isomorphic over $\mathbb{P}^1$, as checked explicitly in Computation 2.5.4. We do not know how to interpret this isomorphism in terms of the (nonisomorphic) configurations $\mathcal{L}_{\mathrm{V}}$ and $\mathcal{L}_{\mathrm{IX}}$. One might suspect that they are related by projective duality (exchanging points and lines), but in fact all 10_3 configurations are self-dual. If this isomorphism does arise from some combinatorial relationship, it is more subtle than this.

2.4 Moduli space interpretation

Recall that the K3 surface $\widetilde{S}$ has an open subset R (its "interior") which parameterizes the line arrangements realizing the configuration $\mathcal{L}_{\mathrm{V}}$. We would like to extend this interpretation to the complement of the interior (the "boundary"), which ought to parameterize "degenerate" realizations where additional triples become collinear. We make this precise using the machinery of geometric invariant theory (GIT), which was reviewed in Chapter 1. We will need the following fact:

Lemma 2.4.1. *Suppose the action of G on X_{ss} is free and $X_s = X_{ss}$. Then the canonical bundle ω_Y of $Y = X/\!\!/G$ is trivial if and only if $\omega_{X_{ss}}$ is trivial as a G-linearized line bundle.*

Proof. There is a G-equivariant short exact sequence

$$0 \to \rho^*\Omega_Y \to \Omega_{X_{ss}} \to \mathfrak{g}^\vee \otimes \mathcal{O}_{X_{ss}} \to 0 \tag{2.4.1}$$

where Ω_Y and $\Omega_{X_{ss}}$ denote the cotangent bundles on Y and X_{ss}, $\rho : X_{ss} \to Y$ is the quotient map, and $\mathfrak{g}$ is the Lie algebra of G with the adjoint representation (e.g., [Tor23, §2.2]). Since G acts trivially on the top exterior power of $\mathfrak{g}$, taking the top exterior power of (2.4.1) yields

$$\rho^*\omega_Y \cong \omega_{X_{ss}}$$

as G-linearized line bundles. Hence, if either bundle has a nowhere-vanishing G-invariant global section, the other does as well. Since G-invariant sections of $\rho^*\omega_Y$ are exactly sections of ω_Y, the lemma is proved. □

For our purposes, we take $X = (\mathbb{P}^2)^n$ for $n \geq 4$ and $G = \mathrm{PGL}(3)$. We refer to points of $(\mathbb{P}^2)^n$ as *arrangements* of n points in $\mathbb{P}^2$. Line bundles on $(\mathbb{P}^2)^n$ are all of the form

$$\mathcal{O}(d_1, \ldots, d_n) = \pi_1^*\mathcal{O}(d_1) \otimes \cdots \otimes \pi_n^*\mathcal{O}(d_n)$$

where π_i is the i-th projection onto $\mathbb{P}^2$. These have a canonical PGL(3)-linearization when 3 divides $\sum_i d_i$ and are ample when $d_i > 0$. It turns out that the semistable locus has a straightforward description in this case.

Lemma 2.4.2 (e.g., [Inc10, Proposition 1.1])**.** *Let $d = \sum_i d_i$ and $w_i = d_i/d$. Then $(p_1, \ldots, p_n)$ is semistable if and only if the following holds: for all $p \in \mathbb{P}^2$, $\sum_{p_i = p} w_i \leq \frac{1}{3}$, and for all lines $\ell \subset \mathbb{P}^2$, $\sum_{p_i \in \ell} w_i \leq \frac{2}{3}$. Stable points are characterized the same way, but with strict inequalities.*

We think of the w_i as weights for the n points, where an arrangement is unstable if too much weight is concentrated at a point or on a line. A choice of weights $w = (w_i)$ is called a *weighting*, and the corresponding GIT quotient is denoted $Q_w = (\mathbb{P}^2)^n /\!\!/_w \mathrm{PGL}(3)$.

Two natural weightings come to mind. For the first, we designate 4 of the n points as "heavy" and assign them weights close to 1/4; the others are given nearly zero weight. We call this the *oligarchic weighting*. Here, the sets of stable and semistable arrangements agree (there are no *strictly semistable* points). Per Lemma 2.4.2, an arrangement is stable exactly when no two of the four heavy points coincide and no three of them are collinear, with no restrictions on the other points. For this weighting, it is easy to see that the GIT quotient is isomorphic to $(\mathbb{P}^2)^{n-4}$; we just fix the four heavy points to standard positions, and the others can be anywhere.

At the other extreme, we have the *democratic weighting* δ, where all weights δ_i are equal to $1/n$. Semistability now means that there are at most $n/3$ coincident points and at most $2n/3$ points on any line. Note that this is the same as stability unless 3 divides n. The corresponding quotient Q_δ is not as easy to describe as for the oligarchic weighting. The following lemma affords us a concrete characterization of the stable part $Q_{w,s}$ of Q_w for any weighting w.

Definition 2.4.3. Four points in $\mathbb{P}^2$ are said to form a *frame* if no three are collinear. We say that an arrangement $(p_1, \ldots, p_n) \in (\mathbb{P}^2)^n$ has a frame if some choice of four p_i is a frame. The frame

$$f_1 = [1:0:0], \quad f_2 = [0:1:0], \quad f_3 = [0:0:1], \quad f_4 = [1:1:1]$$

is called the *standard frame.*

Lemma 2.4.4 ([KT06, Lemma 8.6])**.** *Every arrangement which is stable with respect to some weighting has a frame.*

Proof. It is clear that there are at least three non-collinear points in the arrangement, say p_1, p_2, p_3. If there is a fourth point not collinear with any two of p_1, p_2, p_3, then we're done, so suppose all other points lie on one of $\overline{p_1p_2}$, $\overline{p_1p_3}$, or $\overline{p_1p_3}$. Let W_1 be the combined weight of all points coincident with p_1, and similarly for W_2 and W_3. By stability, $W_i < 1/3$. Since the sum of all weights is 1 and $W_1 + W_2 + W_3 < 1$, there must be a point p_4 not coincident with p_1, p_2, or p_3. Suppose p_4 lies on $\overline{p_1p_2}$. Let $W_{12} < 2/3$ be the combined weight of all points on $\overline{p_1p_2}$. Then $W_{12} + W_3 < 1$, so there must be a point p_5 neither on the line $\overline{p_1p_2}$ nor equal to p_3; say it lies on $\overline{p_2p_3}$. Then p_1, p_3, p_4, p_5 is a frame, as required. ∎

It is worth noting that there do exist strictly semistable arrangements that do not admit a frame. For example, when 3 divides n, an arrangement with $n/3$ points at each vertex of a triangle is semistable with respect to the democratic weighting, but does not have a frame.

Corollary 2.4.5. *There exists an cover of $Q_{w,s}$ by open subsets isomorphic to*

$$U_{\mathbf{i}} = \left\{(p_i)_{i \notin \mathbf{i}} \in (\mathbb{P}^2)^{n-4} \mid (p_i) \in (\mathbb{P}^2)^n_s \text{ where } p_{i_1} = f_1, \dots, p_{i_4} = f_4\right\}$$

indexed by $\mathbf{i} = \{i_1, \dots, i_4\} \subseteq \{0, \dots, n-1\}$ *with* $i_1 < \cdots < i_4$.

Proof. Suppose $(q_i) \in (\mathbb{P}^2)^n_s$. By Lemma 2.4.4, there are $q_{i_1}, \dots, q_{i_4}$ that form a frame. Consider the PGL(3)-invariant open set

$$\left\{(p_i) \in (\mathbb{P}^2)^n \mid p_{i_1}, \dots, p_{i_4} \text{ form a frame}\right\}. \tag{2.4.2}$$

As before, the quotient of this set by PGL(3) can be identified with $(\mathbb{P}^2)^{n-4}$ by fixing $p_{i_1}, \dots, p_{i_4}$ to be the standard frame. Intersecting (2.4.2) with the stable locus $(\mathbb{P}^2)^n_s$ and passing to $Q_{w,s}$ yields an open subset of $Q_{w,s}$ containing the image of (q_i) and isomorphic to $U_{\mathbf{i}}$. Hence these subsets cover $Q_{w,s}$, as claimed. ☐

$Q_{w,s}$ carries a universal $\mathbb{P}^2$-bundle $B \to Q_{w,s}$ with n sections $P_0, \dots, P_{n-1}$ which are w-stable in each fiber $\mathbb{P}^2$. Here, "universal" means that any $\mathbb{P}^2$-bundle on a variety Y with n fiberwise w-stable sections is the pullback of B under a unique morphism $Y \to Q_{w,s}$. We say that $Q_{w,s}$ is a *fine moduli space* for w-stable arrangements. The open cover $\{U_{\mathbf{i}}\}$ gives a local trivialization of B, where the sections $P_{i_1} = f_1, \dots, P_{i_4} = f_4$ are constant and the others are given by the $n-4$ projections $U_{\mathbf{i}} \hookrightarrow (\mathbb{P}^2)^{n-4} \to \mathbb{P}^2$. This is most interesting when there are no strictly w-semistable arrangements, so $Q_{w,s} = Q_w$ is the entire GIT quotient.

Consider now a configuration $\mathcal{L}$ with n points, as in the introduction.

Definition 2.4.6. An arrangement $(p_i) \in (\mathbb{P}^2)^n$ is called a *weak realization* of $\mathcal{L}$ if for all $\ell \in \mathcal{L}$ with $i, j, k \in \mathcal{L}$, the points p_i, p_j, p_k are collinear.

The set of weak realizations forms a closed PGL(3)-invariant subvariety W of $(\mathbb{P}^2)^n$ containing the set of realizations V from the introduction as an open subset. This gives us a systematic method for furnishing compactifications of $\mathcal{R}(\mathcal{L})$, provided that we choose a weighting w such that all realizations of $\mathcal{L}$ are stable.

Definition 2.4.7. The GIT quotient $\mathcal{R}_w(\mathcal{L}) = W/\!\!/_w \mathrm{PGL}(3) \subset Q_w$ is called the *w-semistable realization space* of $\mathcal{L}$.

This is the coarse moduli space for w-semistable weak realizations of $\mathcal{L}$. When there are no strictly w-semistable arrangements, it is a fine moduli space with universal family pulled back from the one on Q_w. The closure of $\mathcal{R}(\mathcal{L})$ in $\mathcal{R}_w(\mathcal{L})$ is a compactification of $\mathcal{R}(\mathcal{L})$ whose boundary points correspond to w-semistable degenerations of realizations of $\mathcal{L}$.

With these generalities in hand, we return to our 10_3 configurations $\mathcal{L}_{\mathrm{V}}$, $\mathcal{L}_{\mathrm{VIII}}$, $\mathcal{L}_{\mathrm{IX}}$, and $\mathcal{L}_{\mathrm{X}}$. Seeing how S was constructed in Section 2.2 by fixing four points, one might hope to identify S or $\widetilde{S}$ with the semistable realization space in the corresponding oligarchic quotient $(\mathbb{P}^2)^6$. However, the rational map $S \dashrightarrow (\mathbb{P}^2)^6$ sending a point in S to its corresponding arrangement is not a morphism; its composition with $\widetilde{S} \to S$ is, but this fails to be injective. The oligarchic realization space thus turns out to lie "between" S and $\widetilde{S}$, so it does not grant the moduli interpretation we seek.

In fact, it is the democratic weighting $\delta = \left(\frac{1}{10}, \dots, \frac{1}{10}\right)$ which realizes $\widetilde{S}$ and the other K3 surfaces. One easily checks that all realizations of any 10_3 configuration are δ-stable.

Notation 2.4.8. Let $\mathcal{L}_N$ be any of $\mathcal{L}_{\mathrm{V}}$, $\mathcal{L}_{\mathrm{VIII}}$, $\mathcal{L}_{\mathrm{IX}}$, and $\mathcal{L}_{\mathrm{X}}$, and let $\widetilde{S_N}$ be the corresponding K3 surface from Theorem 2.1.2(i). For $\mathbf{i} = \{i_1, \dots, i_4\} \subset \{0, \dots, 9\}$ with $i_1 < \cdots < i_4$, let $X_{\mathbf{i}}$ be the closed subset of $(\mathbb{P}^2)^6$ corresponding to weak realizations $(p_i) \in (\mathbb{P}^2)^{10}$ of $\mathcal{L}_N$ with $p_{i_1}, \dots, p_{i_4}$ fixed to the standard frame. Let $X_{\mathbf{i},s} = X_{\mathbf{i}} \cap U_{\mathbf{i}}$ be the open subset of $X_{\mathbf{i}}$ corresponding to δ-stable arrangements.

Since 3 does not divide 10, we have $Q_\delta = Q_{\delta,s}$. By Corollary 2.4.5, the $X_{\mathbf{i},s}$ form an open cover of $\mathcal{R}_\delta(\mathcal{L}_N)$.

Lemma 2.4.9. *$\mathcal{R}(\mathcal{L}_N) \subseteq X_{\mathbf{i},s}$ as subsets of $\mathcal{R}_\delta(\mathcal{L}_N)$.*

Proof. Suppose $X_{\mathbf{i},s}$ is nonempty. Then there is an arrangement (p_i) satisfying the collinearities in $\mathcal{L}_N$ (and possibly some not in $\mathcal{L}_N$) such that $p_{i_1}, \dots, p_{i_4}$ form a frame. This means there is no line in $\mathcal{L}_N$ containing any three of those points. It follows that they form a frame in any realization of $\mathcal{L}_N$. Realizations of $\mathcal{L}_N$ are stable, so we have $\mathcal{R}(\mathcal{L}_N) \subseteq X_{\mathbf{i},s}$, as desired. $\square$

Lemma 2.4.10. *$\mathcal{R}_\delta(\mathcal{L}_N)$ is nonsingular and irreducible of dimension 2.*

Proof. In Computation 2.5.5, we find that for every $\mathbf{i}$, $X_{\mathbf{i},s}$ is either empty, or nonsingular and irreducible. Since the nonempty $X_{\mathbf{i},s}$ all intersect, it follows that $\mathcal{R}_\delta(\mathcal{L}_N)$ is nonsingular and connected, hence irreducible. Since $\mathcal{R}_\delta(\mathcal{L}_N)$ contains $\mathcal{R}(\mathcal{L}_N)$ as an open subset and $\mathcal{R}(\mathcal{L}_N)$ has dimension 2, $\mathcal{R}_\delta(\mathcal{L}_N)$ has dimension 2 as well. ☐

Proof of Theorem 2.1.2(ii). By Lemma 2.4.4, PGL(3) acts freely on $(\mathbb{P}^2)^{10}_{ss}$. From Lemma 2.4.10, we see that the preimage $W = \rho^{-1}(\mathcal{R}_\delta(\mathcal{L}_N))$ under the quotient map $\rho : (\mathbb{P}^2)^{10}_{ss} \to Q_\delta$ is smooth of dimension 10. W is cut out by the 10 equations defining the lines of $\mathcal{L}_N$, so it's a complete intersection of codimension 10. By the argument given in the introduction, W has (equivariantly) trivial canonical class; by Lemma 2.4.1, $\mathcal{R}_\delta(\mathcal{L}_N)$ also has trivial canonical class. Since $\widetilde{S_N}$ and $\mathcal{R}_\delta(\mathcal{L}_N)$ are birational (both being compactifications of $\mathcal{R}(\mathcal{L}_N)$), it follows that $\mathcal{R}_\delta(\mathcal{L}_N)$ has zero irregularity, so it is a K3 surface. Birational K3 surfaces are isomorphic, so in fact $\widetilde{S_N} \cong \mathcal{R}_\delta(\mathcal{L}_N)$. ☐

2.5 Computations

The code used in this chapter is available online in the author's Github repository [Sin23a].

Computation 2.5.1. In the magma/singularpoints file, we compute the singular locus of $S = S_{\mathrm{V}}$. The result is a list of seven schemes representing the points

$$([1:1],[1:1:1]),\quad ([1:1],[0:0:1]),\quad ([1:1],[1:0:0]),$$

$$([0:1],[1:0:1]),\quad ([0:1],[0:0:1]),\quad ([0:1],[1:0:0]),$$

$$([1:0],[0:1:0]).$$

None of these points lie in $\mathcal{R}(\mathcal{L}_{\mathrm{V}}) \subset S$, as the corresponding arrangements have unwanted collinearities; the first three have $1, 5, 7$ collinear, the middle three have $1, 3, 7$ collinear, and the last has $1, 2, 7$ collinear. We check that these singularities are all Du Val and compute their resolution graphs (all A_n for $n \leq 3$; compare Figure 2.1). We also find the singular points of fibers of $\pi : S \to \mathbb{P}^1$, which gives the list of singular fibers in Section 2.3.

Computation 2.5.2. In the magma/mordellweil file, we perform the various computations in the Mordell–Weil group $\mathrm{MW}(S) = \mathrm{MW}(\widetilde{S})$ needed for Sections 2.2 and 2.3. In particular, we compute the Weierstrass form for the generic fiber E and show that the section s from the proof of Proposition 2.2.1 is not torsion. We also directly compute the Kodaira fibers, the torsion subgroup, and the height pairing $\langle s, s\rangle$ needed to prove Lemmas 2.3.4 and 2.3.5. Finally, we check that our elliptic fibration is isomorphic to the elliptic fibration given in [Lec15, Table 3, row 2].

Computation 2.5.3. In the magma/configurations/other folder, we give construction sequences for the configurations which do not yield K3 surfaces. As described in the proof of Theorem 2.1.1, we find that the resulting equation is either trivial or reducible into rational components, and moreover that the realization space is contained in at most one such component.

Computation 2.5.4. In the magma/configurations/k3 folder, we give construction sequences for $\mathcal{L}_{\mathrm{VIII}}$, $\mathcal{L}_{\mathrm{IX}}$, and $\mathcal{L}_{\mathrm{X}}$ and repeat Computations 2.5.1 and 2.5.2 for these configurations. For $\mathcal{L}_{\mathrm{IX}}$, we check explicitly that $\widetilde{S_{\mathrm{V}}}$ and $\widetilde{S_{\mathrm{IX}}}$ have isomorphic elliptic fibrations.

Computation 2.5.5. In the macaulay2/democratic file, we show that for every $\mathbf{i}$, $X_{\mathbf{i}}$ is irreducible and $X_{\mathbf{i},s}$ is nonsingular (see Notation 2.4.8). More precisely, we show that the singular locus of $X_{\mathbf{i}}$ is contained in the unstable locus $X_{\mathbf{i}} \smallsetminus X_{\mathbf{i},s}$. This latter computation is performed in affine charts and takes several hours per configuration to complete.

Chapter 3

Vector bundles

3.1 Introduction

The main result of this chapter is the following (see Chapter 1 for definitions):

Theorem 3.1.1. *There exists a noncommutative resolution of singularities $\mathcal{D}$ of $\mathcal{SU}_C(2)$ with a semiorthogonal decomposition into blocks equivalent to $D^b(\mathrm{Sym}^{2k}C)$ for $2k \leq g-1$. There are four copies of each block except when g is odd, in which case the block $D^b(\mathrm{Sym}^{g-1}C)$ appears twice. The category $\mathcal{D}$ is an example of the noncommutative resolution of singularities constructed in* [Păd21, ŠVdB23] *for symmetric stacks. In particular, it is an admissible subcategory of the derived category of the Kirwan resolution of $\mathcal{SU}_C(2)$. If g is even, $\mathcal{D}$ is a strongly crepant noncommutative resolution of $\mathcal{SU}_C(2)$ in the sense of* [Kuz08].

A more precise statement is given as Theorem 3.3.8 below. As mentioned above, the results in this chapter will appear in the forthcoming joint article [ST24]. For brevity, some technical proofs are omitted or only sketched here; they will appear in full in [ST24].

This chapter is organized as follows: In Section 3.2, we obtain several explicit semiorthogonal decompositions of the "maximal" (see Remark 3.2.4) Thaddeus space $M_{i_d}(d)$ for various d (see, e.g., Theorem 3.2.2 for a precise statement), including the case $d = 2g$ we need (Theorem 3.2.10). The precise statement and proof of the main result are given in Section 3.3.

A few words regarding notation: Following [Tha94, Tev23], we often denote tensor product by juxtaposition for compactness. As in [Huy06], we usually omit R's and L's on derived functors except for emphasis (e.g., when applying derived pushfoward to a sheaf). We frequently use the same symbol to denote canonical objects on related moduli spaces when no confusion will arise, omitting explicit pullbacks (see Notation 3.2.1).

3.2 Semiorthogonal decompositions of spaces of stable pairs

We begin by recalling some notation from [Tev23, TT21].

Notation 3.2.1. For a line bundle Λ on C of degree d and $0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor$, we denote by $M_i(\Lambda)$ (or simply $M_i(d)$ or M_i when no confusion will arise) the moduli space of rank 2 stable pairs with determinant Λ, where i indexes the stability condition. We write $\mathcal{O}(m,n) = \mathcal{O}((m+n)H - nE)$ on any of the M_i, $i \geq 1$, where E is the exceptional divisor of the contraction $M_1 \to M_0$ and H is the pullback of the hyperplane divisor from $M_0 \cong \mathbb{P}^{d+g-2}$. (By abuse of notation, $\mathcal{O}(m,0) = \mathcal{O}(m)$ on M_0.) $\mathcal{F}$ denotes the universal vector bundle on $M_i \times C$ (for any i), $\mathcal{F}_x$ its restriction to $M_i \times \{x\} \cong M_i$ for $x \in C$, and $\mathbf{\Lambda}$ the line bundle $\wedge^2 \mathcal{F}_x$, which is independent of x (and not to be confused with Λ).

For any variety X and vector bundle $\mathcal{G}$ on $X \times C$, we have tensor vector bundles $\mathcal{G}^{\boxtimes k}$ and $\overline{\mathcal{G}}^{\boxtimes k}$ on $X \times \mathrm{Sym}^k C$ defined by the S_k-equivariant pushforwards $\tau_*^{S_k}(\pi_1^*\mathcal{G} \otimes \cdots \otimes \pi_k^*\mathcal{G})$ and $\tau_*^{S_k}(\pi_1^*\mathcal{G} \otimes \cdots \otimes \pi_k^*\mathcal{G} \otimes \mathrm{sgn})$, respectively, where $\tau : C^k \to \mathrm{Sym}^k C$ is the quotient by S_k, $\pi_i : C^k \to C$ are the projections, sgn is the sign character of S_k, and the S_k-action on $\pi_1^*\mathcal{G} \otimes \cdots \otimes \pi_k^*\mathcal{G}$ permutes the tensor factors (see [TT21, Section 2]). For $D \in \mathrm{Sym}^k C$, we denote by $\mathcal{G}_D^{\boxtimes k}$ and $\overline{\mathcal{G}}_D^{\boxtimes k}$ the restrictions to $X \times \{D\} \cong X$ of $\mathcal{G}^{\boxtimes k}$ and $\overline{\mathcal{G}}_D^{\boxtimes k}$, respectively. $\langle K \rangle$ denotes the essential image of a fully faithful Fourier–Mukai functor with kernel K.

The principal goal of this section is to prove the following generalization of [Tev23, Theorem 3.1]:

Theorem 3.2.2. *Let $d \leq 2g$ with $i_d \leq \lfloor \frac{d-1}{2} \rfloor$, where $i_d = \left\lceil \frac{d+g-1}{3} \right\rceil - 1$. Let $m = d + g - 1 - 3i_d \in \{1,2,3\}$, and let $m_n = 1$ if $m \leq n$ or 0 otherwise. Then*

$$D^b(M_{i_d}(d)) = \left\langle \left\langle \mathbf{\Lambda}^{-k} \mathcal{F}^{\vee \boxtimes j} \right\rangle_{\substack{j+k \leq i_d - m_2 \\ j,k \geq 0}}, \left\langle T_1 \mathbf{\Lambda}^{-k} \mathcal{F}^{\vee \boxtimes j} \right\rangle_{\substack{j+k \leq i_d - m_1 \\ j,k \geq 0}}, \left\langle T_2 \mathbf{\Lambda}^{-k} \mathcal{F}^{\vee \boxtimes j} \right\rangle_{\substack{j+k \leq i_d \\ j,k \geq 0}} \right\rangle \quad (3.2.1)$$

where the blocks within each "megablock" are ordered first by decreasing k, then by decreasing j. Here, $T_1 = \mathcal{O}(1, i_d - m_2)$, and $T_2 = \mathcal{O}(2, 2i_d - m_2 - m_1)$.

Remark 3.2.3. The assumptions of the theorem are equivalent to $d = 2g - \alpha$ for $\alpha \in \{0, 1, 2, 3, 5\}$, where $i_d = g - \lceil \frac{\alpha+2}{3} \rceil$.

Remark 3.2.4. It can be seen from [TT21, Proposition 3.18] that $D^b(M_i(d))$ is "largest" when $i = i_d$. Moreover, it follows from [Tha94, 5.3, 6.1] that M_{i_d} is Fano when $m = 1, 2$. When $m = 3$, $D^b(M_{i_d})$ and $D^b(M_{i_d+1})$ are equivalent, and the anticanonical bundles on M_{i_d} and M_{i_d+1} are big and nef but not ample.

3.2.1 Generalized weaving

We begin with some notation. Fix $d \leq 2g$ and write $M_i(d) = M_i$ for $i \leq \lfloor \frac{d-1}{2} \rfloor$.

Notation 3.2.5. For $0 \leq k \leq i$, we denote by $\mathcal{D}_i^k$ the structure sheaf of the reduced subscheme

$$D_i^k = \{(D, F, s) \in \mathrm{Sym}^k C \times M_i : s|_D = 0\},$$

whose fiber over $D \in \mathrm{Sym}^k C$ is $M_{i-k}(\Lambda(-2D))$ [TT21, Remark 3.7]. For $t \in [0, i_d + 1)$, let $\mathcal{D}_t^{k,s} = \mathcal{D}_{\lfloor t \rfloor}^k \otimes L_t^{k,s}$ where

$$L_t^{k,s} = \begin{cases} \mathcal{O}(s, sk) & k = \lfloor t \rfloor \\ \mathcal{O}\left(\left\lfloor \frac{s}{t-k} \right\rfloor, s + \left\lfloor \frac{s}{t-k} \right\rfloor (k-1)\right) & k < \lfloor t \rfloor \, . \end{cases}$$

Our first step is to prove the following:

Lemma 3.2.6 (cf. [Tev23, Corollary 2.10])**.** *For $t \in (0, i_d + 1) \setminus \mathbb{Z}$, we have a semiorthogonal decomposition*

$$D^b(M_{\lfloor t \rfloor}) = \left\langle \mathcal{D}_t^{k,s} \right\rangle_{\substack{0 \leq k \leq \lfloor t \rfloor \\ 0 \leq s \leq d+g-3k-2}} \tag{3.2.2}$$

where the blocks are ordered first by increasing $x_{k,s}(t) = \frac{s}{t-k}$, then by increasing k.

We interpret t as time, with the block $\langle \mathcal{D}_t^{k,s} \rangle$ "moving" in the x-t plane with trajectory $x = x_{k,s}(t)$ (or $x = k\epsilon$ for $s = 0$, where $\epsilon \ll 1$). When the blocks cross paths, they change order and undergo mutations dictated by the line bundle $L_t^{k,s}$. When t crosses an integer level i, we embed $D^b(M_{i-1})$ into $D^b(M_i)$, introducing several new blocks as its orthogonal complement, and proceed with the process. We refer to this "weave" as the Farey Twill; see [Tev23, Section 2] for detailed illustrations in the case $d = 2g - 1$. This program is facilitated by several technical lemmas, which have direct analogs in [Tev23]; we give their precise statements in Subsection 3.2.3, and refer to [ST24] for their proofs.

To pass from M_{i-1} to M_i, we need the following "windows" embeddings, in the sense of [HL15]:

Proposition 3.2.7 ([TT21, Proposition 3.18])**.** *For $d > 0$ and $1 \leq i \leq i_d \leq \lfloor \frac{d-1}{2} \rfloor$, there is an admissible embedding $\iota : D^b(M_{i-1}(d)) \hookrightarrow D^b(M_i(d))$ giving rise to a semiorthogonal decomposition*

$$D^b(M_i(d)) = \langle \iota(D^b(M_{i-1}(d))), \mathcal{D}_i^{i,0}, \mathcal{D}_i^{i,1}, \ldots, \mathcal{D}_i^{i,d+g-3i-2} \rangle$$

When $i > 1$, the embedding corresponds to the inclusion of objects with weights in $[0, i)$ with respect to the wall crossing $M_{i-1} \dashrightarrow M_i$.

Remark 3.2.8. Note that D_i^i is isomorphic to the projective bundle $\mathbb{P}W_i^+ \subset M_i$ via the second projection [TT21, Section 6], and that $L_i^{i,s} = \mathcal{O}(s, si)$ restricts to $\mathcal{O}(s)$ on the fibers $M_0(d-2i) \cong \mathbb{P}^{d+g-2i-2}$ of this bundle [TT21, Remark 3.7].

Proof of Lemma 3.2.6. When $0 < t < 1$, (3.2.2) is the Beĭlinson collection $\langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(d+g-2) \rangle$ on $\mathbb{P}^{d+g-2}$. Given (3.2.2) for $t = i + \epsilon$ with $i \in \mathbb{Z}$, $\epsilon \ll 1$, we achieve (3.2.2) for all $t \in (i, i+1)$ by performing the mutations encoded in the crossings of trajectories $x_{k,s}(t)$. When blocks meet at nonintegral x, they only change order, meaning we must show that they are mutually orthogonal. Since the intersecting blocks are already ordered by $x_{s,k}(t - \epsilon)$, or equivalently by k, we need only check that $\langle \mathcal{D}_t^{k,s} \rangle \subset {}^{\perp}\langle \mathcal{D}_t^{k',s'} \rangle$ for $k < k'$, $x_{k,s}(t) = x_{k',s'}(t)$. This follows from Lemma 3.2.14 below.

Crossings at $x \in \mathbb{Z}$ have the form

$$\langle \mathcal{D}_t^{k,s}, \mathcal{D}_t^{k+1,s-x}, \ldots, \mathcal{D}_t^{\lfloor t \rfloor, s - \lfloor t \rfloor x} \rangle \to \langle \mathcal{D}_{t+\epsilon}^{\lfloor t \rfloor, s-\lfloor t \rfloor x}, \ldots, \mathcal{D}_{t+\epsilon}^{k+1,s-x}, \mathcal{D}_{t+\epsilon}^{k,s} \rangle.$$

Note that for $0 \leq j < \lfloor t \rfloor$,

$$L_t^{k+j,s-jx} = \mathcal{O}(x, s - x(k-1))$$

is indepedent of j, while

$$L_{t+\epsilon}^{k+j,s-jx} = L_t^{k+j,s-jx}(-1, 1-j).$$

Moreover, $L_t^{\lfloor t \rfloor, s-\lfloor t \rfloor x} = L_{t+\epsilon}^{\lfloor t \rfloor, s-\lfloor t \rfloor x} = \mathcal{O}(s - \lfloor t \rfloor x, (s - \lfloor t \rfloor x) \lfloor t \rfloor)$ and $\mathcal{O}(x, s - x(k-1))$ both restrict to $\mathcal{O}(s)$ on the fibers of the projective bundle $D_{\lfloor t\rfloor}^{\lfloor t\rfloor}$, so $\langle \mathcal{D}_t^{\lfloor t\rfloor, s-\lfloor t\rfloor x}\rangle = \langle \mathcal{D}_{\lfloor t\rfloor}^{\lfloor t\rfloor}(x, s-x(k-1))\rangle$. Hence it suffices to give a mutation

$$\langle \mathcal{D}_i^k, \dots, \mathcal{D}_i^{i-1}, \mathcal{D}_i^i\rangle \to \langle \mathcal{D}_i^i, \mathcal{D}_i^{i-1}(-1, 2-i), \dots, \mathcal{D}_i^k(-1, 1-k)\rangle,$$

as in Lemma 3.2.15.

It remains to explain how to get from $t = i - \epsilon$ to $i + \epsilon$. By Lemma 3.2.13 below, we have $\iota\langle \mathcal{D}_{i-\epsilon}^{k,s}\rangle = \langle \mathcal{D}_i^{k,s}\rangle$. Hence, to go from the semiorthogonal decomposition of Proposition 3.2.7 to (3.2.2) with $t = i + \epsilon$, we need only move the block $\langle \mathcal{D}_i^{i,0}\rangle$ into position, i-th from the left (we imagine this block coming horizontally from the right along $t = i$, stopping at $x = i\epsilon$). It moves past blocks with $x_{k,s}(i) \notin \mathbb{Z}$ without changing them by Lemma 3.2.14, while the others undergo the mutation of Lemma 3.2.15. □

It turns out that Lemma 3.2.6 is not quite the decomposition we need to proceed.

Lemma 3.2.9. *Let $m = d + g - 3i_d - 1 \in \{1, 2, 3\}$, and let $m_n = 1$ if $m \le n$ and 0 otherwise. Then*

$$D^b(M_{i_d}) = \left\langle \left\langle \mathbf{\Lambda}^{-j}\mathcal{D}^k \right\rangle_{\substack{j+k\le i_d-m_2\\ j,k\ge 0}}, \left\langle T_1\mathbf{\Lambda}^{-j}\mathcal{D}^k \right\rangle_{\substack{j+k\le i_d-m_1\\ j,k\ge 0}}, \left\langle T_2\mathbf{\Lambda}^{-j}\mathcal{D}^k \right\rangle_{\substack{j+k\le i_d\\ j,k\ge 0}} \right\rangle \tag{3.2.3}$$

where the blocks in each megablock are ordered first by increasing $j+k$, then by increasing j. Here, $\mathcal{D}^k = \mathcal{D}_{i_d}^k$, $T_1 = \mathcal{O}(1, i_d - m_2)$, and $T_2 = \mathcal{O}(2, 2i_d - m_2 - m_1)$.

Proof. For $m = 1, 2$, we begin with (3.2.2) with $t = i_d - \epsilon$. We embed with ι to obtain the following semiorthogonal decomposition:

$$D^b(M_{i_d}) = \langle \mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}, \mathcal{D}_{i_d}^{i_d,0}, \dots, \mathcal{D}_{i_d}^{i_d,m-1}\rangle \tag{3.2.4}$$

where $\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}$ are the subcategories generated by $\langle \mathcal{D}_{i_d}^{k,s}\rangle$ for $x_{k,s}(i_d) \in [0,1)$, $[1,2)$, $[2,3)$, and $[3,\infty)$, respectively.

If $m = 1$, the blocks are arranged as in [Tev23]. The blocks in $\mathbf{I}$ are the same as in the first megablock of (3.2.3) (with $j = s$), but they are ordered differently. We reorder them by moving $\langle \mathcal{D}_{i_d}^{k,s}\rangle$ from $x = x_{k,s}(i_d)$ to $x = \frac{s+k}{i_d}$. The moves are done in order of decreasing $s + k$, then by decreasing s. As blocks in $\mathbf{I}$ with the same $s + k$ are already ordered by increasing s, the orthogonality we need to ensure no mutations occur is $\langle \mathcal{D}_{i_d}^{k,s} \rangle \subset {}^{\perp}\langle \mathcal{D}_{i_d}^{k',s'} \rangle$ for $s' + k' < s + k$. This is checked in Lemma 3.2.16.

Similarly, the blocks in **II** and **III** are respectively the same as the second and third (with $j + k \leq i_d - 1$) megablocks of (3.2.3). (Explicitly, we have $j = s - \left\lfloor \frac{s}{i_d - k} \right\rfloor (i_d - k)$.) The same reordering procedure works, so we are left to produce the blocks in (3.2.3) in with $j + k = i_d$. These are exactly the blocks in $\langle \mathbf{IV}, \mathcal{D}_{i_d}^{i_d,0} \rangle = \langle \mathcal{D}_{i_d}^{0,3i_d}, \mathcal{D}_{i_d}^{1,3(i_d-1)}, \ldots, \mathcal{D}_{i_d}^{i_d,0} \rangle$ after the mutation of Lemma 3.2.15 (note that we have $\mathcal{D}^{i_d} = \mathcal{D}^{i_d}(2, 2(i_d - 1))$ by [TT21, Remark 3.7]). This proves the lemma for $m = 1$.

For $m = 2$, the only new blocks in (3.2.4) compared to $m = 1$ lie in **IV**. We reorder the blocks in **I** and $\mathbf{II} = \mathbf{II}_a$ just as before; this gives the first and second (with $j + k \leq i_d - 1$) megablocks in (3.2.3). We write $\mathbf{III} = \langle \mathbf{ii}_b, \mathbf{III}_a \rangle$, where $\mathbf{ii}_b$ contains those blocks $\langle \mathcal{D}_{i_d}^{k,s} \rangle$ with $x_{k,s}(i_d) = 2$ and $\mathbf{III}_a$ those with $x_{k,s}(i_d) \in (2, 3)$. Similar to the proof of Lemma 3.2.6, we move the block $\mathcal{D}_{i_d}^{i_d,0}$ past $\mathbf{IV}$, $\mathbf{III}_a$, and $\mathbf{ii}_b$. By Lemmas 3.2.14 and 3.2.15, $\mathbf{III}_a$ is unchanged, while **IV** and $\mathbf{ii}_b$ undergo some mutations. This yields

$$D^b(M_{i_d}) = \langle \mathbf{I}, \mathbf{II}_a, \mathbf{II}_b, \mathbf{III}_a, \mathbf{IV}', \mathcal{D}_{i_d}^{i_d,1} \rangle$$

where $\langle \mathbf{ii}_b, \mathcal{D}_{i_d}^{i_d,0} \rangle \to \mathbf{II}_b$ via Lemma 3.2.15, and $\mathbf{IV}' = \langle \mathcal{D}_{i_d+\epsilon}^{s,k} \rangle_{x_{k,s}(i_d) \geq 3}$ ordered by $x_{k,s}(i_d + \epsilon)$. Now $\mathbf{II}_b = \langle \mathcal{D}^k(1, 2i_d - k - 1) \rangle_{0 \leq k \leq i_d}$ ordered by decreasing k, so $\langle \mathbf{II}_a, \mathbf{II}_b \rangle$ forms the second megablock of (3.2.3).

At this point, $\mathbf{III}_a$ contains the blocks from the third megablock of (3.2.3) with $j + k \leq i_d - 2$; we apply the same algorithm to put them in the correct order. We write $\mathbf{IV}' = \langle \mathbf{III}_b, \mathbf{iii}_c \rangle$ with $\mathbf{III}_b$ and $\mathbf{iii}_c$ containing the blocks with $x_{k,s}(i_d) = 3$ and $x_{k,s}(i_d) > 3$. We have $\mathbf{III}_b = \langle \mathcal{D}^k(2, 3i_d - k - 2) \rangle_{0 \leq k \leq i_d - 1}$ ordered by decreasing k. The Farey Twill trajectories of blocks in $\mathbf{iii}_c$ meet with $\mathcal{D}_{i_d}^{i_d,1}$ at $(t, x) = \left(i_d + \frac{1}{3}, 3\right)$, where they undergo a final mutation $\langle \mathbf{iii}_c, \mathcal{D}_{i_d}^{i_d,1} \rangle \to \mathbf{III}_c = \langle \mathcal{D}^k(2, 3i_d - k - 1) \rangle_{0 \leq k \leq i_d}$ ordered by decreasing k. To sum up, we have

$$D^b(M_{i_d}) = \langle \mathbf{I}, \langle \mathbf{II}_a, \mathbf{II}_b \rangle, \langle \mathbf{III}_a, \mathbf{III}_b, \mathbf{III}_c \rangle \rangle \,,$$

which is exactly (3.2.3) with $m = 2$.

Finally, $m = 3$ is the easiest case. We begin with (3.2.2) with $t = i_d + 1 - \epsilon$. We have

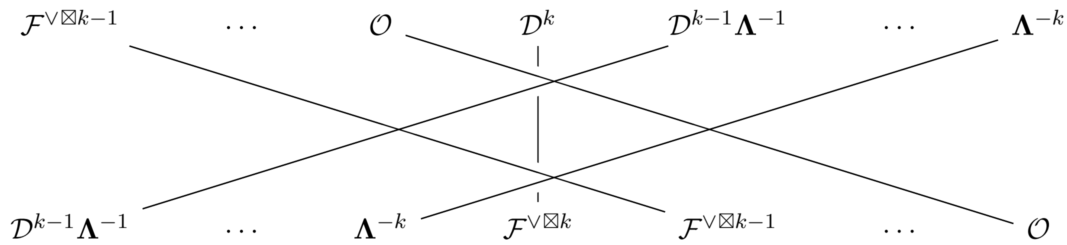


Figure 3.1: The basic Cross Warp mutation, cf. [Tev23, Figure 7].

$x_{k,s}(t) \in [0,3)$, where the blocks in $[0,1)$, $[1,2)$, and $[2,3)$ correspond exactly to the respective megablocks in (3.2.3) with $j = s$, $j = s-(i_d+1-k)$, and $j = s-2(i_d+1-k)$; they are in the wrong order, but this is rectified by Lemma 3.2.16 and the same reordering algorithm as above. □

From here, Theorem 3.2.2 follows exactly as in the proof of [Tev23, Theorem 3.1].

Proof of Theorem 3.2.2. Each megablock in (3.2.3) will mutate into the corresponding one in (3.2.1). As the megablocks differ only in size and overall line bundle twists (i.e., the shapes are the same), it will suffice to describe this mutation for the first one. We rely on the Cross Warp mutation depicted in Figure 3.1 and proved as Theorem 3.2.12(d) below. Notice that the top left portion of the mutation with top center block $\mathcal{D}^k$ is precisely the bottom right portion of the mutation with top center $\mathcal{D}^{k-1}$; similarly, the bottom left is the top right of the mutation with top center $\mathcal{D}^{k-1}$, tensored with $\mathbf{\Lambda}^{-1}$. Hence we can stack these mutations with top centers as in Figure 3.2. In the end, all $\mathcal{D}$'s are replaced by $\mathcal{F}$'s, resulting in (3.2.1). □

3.2.2 Broken Loom for $d = 2g$

To finish this section, we specialize to $d = 2g$. We wish to modify the semiorthogonal decomposition of Theorem 3.2.2 for $d = 2g$ to create as many blocks as possible of the form $\theta^j\mathbf{\Lambda}^k\mathcal{F}^{\vee\boxtimes 2k}$, where $\theta = \mathcal{O}(1, g-1)$ is the pullback of the ample generator of $\operatorname{Pic}\mathcal{SU}_C(2)$ under the forgetful morphism $M_{g-1}(2g) \to SU_C(2)$, $(F,s) \mapsto s$ (see [Tha94, 5.8]). We will see in Section 3.3 that such blocks form the claimed noncommutative resolution of $SU_C(2)$.

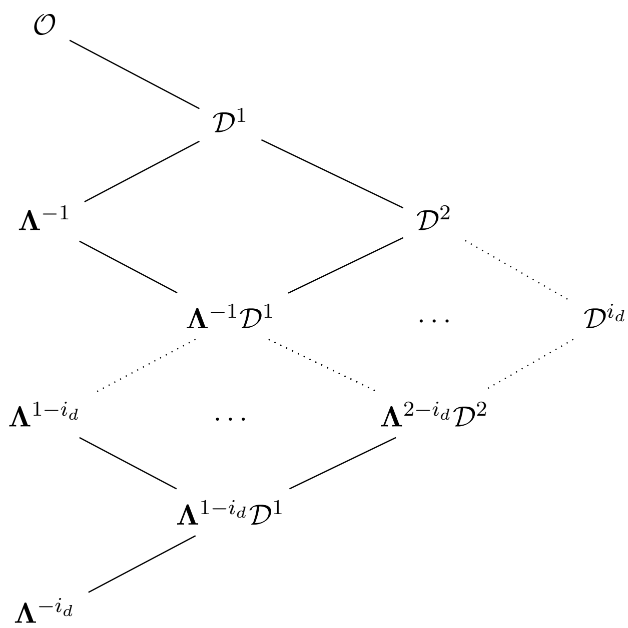


Figure 3.2: Stacking the crosswarp mutation (with $m_1 = 0$). See also [Tev23, Figure 8].

Theorem 3.2.10 (cf. [Tev23, Theorem 5.8])**.** *Let $M = M_{g-1}(2g)$. We have the following semiorthogonal decomposition of $D^b(M)$:*

$$\left\langle \left\langle \theta^{-1}\mathbf{\Lambda}^{\lfloor \frac{g-2}{2}\rfloor -k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{0\le\lambda\le g-2\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor}}, \left\langle \mathbf{\Lambda}^{\lfloor\frac{g-2}{2}\rfloor-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{0\le\lambda\le 2(g-2)\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-2}}, \right.$$
$$\left. \left\langle \theta\mathbf{\Lambda}^{\lfloor\frac{g}{2}\rfloor-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{0\le\lambda\le 2(g-1)\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-1}}, \left\langle \theta^{2}\mathbf{\Lambda}^{\lfloor\frac{g}{2}\rfloor-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{g-1\le\lambda\le 2(g-1)\\ \lambda-g+1\le k\le\lfloor\frac{\lambda}{2}\rfloor}} \right\rangle.$$

Here, the blocks within each megablock are ordered first by decreasing λ, then by decreasing k.

We proceed by analogy with [Tev23, Section 5], beginning with the reordering trick. This works for any $d \le 2g$ with $i_d \le \left\lfloor \frac{d-1}{2}\right\rfloor$.

Lemma 3.2.11 (cf. [Tev23, Theorem 5.3])**.** *With notation as in Theorem 3.2.2, we have the following semi-orthogonal decomposition of $D^b(M_{i_d}(d))$:*

$$\left\langle \left\langle \mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le i_d-m_2\\ \lambda-2k,k\ge 0}}, \left\langle T_1\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le i_d-m_1\\ \lambda-2k,k\ge 0}}, \left\langle T_2\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le i_d\\ \lambda-2k,k\ge 0}} \right\rangle. \quad (3.2.5)$$

The blocks within each megablock are ordered first by decreasing λ, then by decreasing k.

Proof. The blocks in each megablock are the same as those in (3.2.1) with $\lambda = j + 2k$. As they are already ordered by decreasing k, it suffices to show that we can move blocks with smaller λ to the right of blocks with larger λ, i.e., $\langle \mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\rangle \subset {}^{\perp}\langle \mathbf{\Lambda}^{-k'}\mathcal{F}^{\vee\boxtimes\lambda'-2k'}\rangle$ for $\lambda < \lambda'$. This follows from Lemma 3.2.17 below. $\square$

Proof of Theorem 3.2.10. When $d = 2g$, we have $i_d = g-1$, $m = 2$, $T_1 = \theta\mathbf{\Lambda}$, and $T_2 = \theta^2\mathbf{\Lambda}$, so (3.2.5) becomes

$$\left\langle \left\langle \mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\leq g-2\\ \lambda-2k,k\geq 0}}, \left\langle (\theta\mathbf{\Lambda})\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\leq g-1\\ \lambda-2k,k\geq 0}}, \left\langle (\theta^2\mathbf{\Lambda})\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\leq g-1\\ \lambda-2k,k\geq 0}} \right\rangle.$$

We take the part of the third megablock with $\lambda \leq g-2$ and tensor by $\omega_{M_{g-1}(2g)} = \theta^{-3}\mathbf{\Lambda}^{-1}$, moving it to the far left. Tensoring everything by $\mathbf{\Lambda}^{\lfloor \frac{g-2}{2} \rfloor}$ proves the theorem. $\square$

3.2.3 Technical lemmas

In this subsection, we collect the various technical results used above. Their proofs (which are minor modifications of their analogs in [Tev23]) will appear in [ST24]. Recall that $\mathcal{P}_K$ denotes the Fourier–Mukai functor with kernel K.

Theorem 3.2.12 (Basic Cross Warp, cf. [Tev23, Theorem 3.2])**.** *For $0 \leq k \leq i \leq v$, we have:*

(a) $\mathcal{P}_{\mathcal{F}^{\vee\boxtimes k}} : D^b(\mathrm{Sym}^k C) \to D^b(M_i)$ *is fully faithful.*

(b) $\mathcal{P}_{\mathcal{D}_i^k} : D^b(\mathrm{Sym}^k C) \to D^b(M_i)$ *is fully faithful.*

(c) *If $k \leq i-1$, then $\iota\langle \mathcal{D}_{i-1}^k\rangle = \langle \mathcal{D}_i^k\rangle$ where ι is the windows embedding of Proposition 3.2.7. Moreover, objects in $\langle \mathcal{D}_{i-1}^k\rangle$ descend from objects with weights in the range $[0,k]$ for this wall crossing.*

(d) *There is an admissible subcategory of $D^b(M_i)$ with semiorthogonal decompositions*

$$\langle \mathcal{F}^{\vee\boxtimes k-1}, \ldots, \mathcal{O}, \mathcal{D}_i^k, \mathcal{D}_i^{k-1}\mathbf{\Lambda}^{-1}, \ldots, \mathbf{\Lambda}^{-k}\rangle$$

and

$$\langle \mathcal{D}_i^{k-1}\mathbf{\Lambda}^{-1}, \ldots, \mathbf{\Lambda}^{-k}, \mathcal{F}^{\vee\boxtimes k}, \mathcal{F}^{\vee\boxtimes k-1}, \ldots, \mathcal{O}\rangle$$

related by the mutation in Figure 3.1.

Lemma 3.2.13 (cf. [Tev23, Lemma 2.7])**.** *For $k \le i-1$ and $\epsilon \ll 0$, we have $\iota\langle \mathcal{D}^{k,s}_{i-\epsilon}\rangle = \langle \mathcal{D}^{k,s}_i\rangle$.*

Lemma 3.2.14 (cf. [Tev23, Lemma 2.8])**.** *Let $x = \frac{s}{t-k} \notin \mathbb{Z}$. If $k < k'$ and either $x = \frac{s'}{t-k'}$ or $s' = t-k' = 0$, then*

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{D}^{k,s}_t}(X), \mathcal{P}_{\mathcal{D}^{k',s'}_t}(Y)) = 0 \tag{3.2.6}$$

for any $X \in D^b(\mathrm{Sym}^k C)$, $Y \in D^b(\mathrm{Sym}^{k'} C)$.

Lemma 3.2.15 (cf. [Tev23, Lemma 2.9])**.** *For all $k \le i$, there is a mutation*

$$\langle \mathcal{D}^k_i, \dots, \mathcal{D}^{i-1}_i, \mathcal{D}^i_i\rangle \to \langle \mathcal{D}^i_i, \mathcal{D}^{i-1}_i(-1, 2-i), \dots, \mathcal{D}^k_i(-1, 1-k)\rangle.$$

Lemma 3.2.16. *For $k, k', j, j' \ge 0$ with $j' + k' < j + k \le i$, we have*

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{D}^k_i \mathbf{\Lambda}^{-j}}(X), \mathcal{P}_{\mathcal{D}^{k'}_i \mathbf{\Lambda}^{-j'}}(Y)) = 0$$

for any $X \in D^b(\mathrm{Sym}^k C)$, $Y \in D^b(\mathrm{Sym}^{k'} C)$.

Lemma 3.2.17 ([Tev23, Lemma 5.4, Remark 5.7])**.** *Suppose $2 < d' \le 2g+1$ and $1 \le j \le \left\lfloor \frac{d-1}{2} \right\rfloor$. Let $D \in \mathrm{Sym}^a C$, $D' \in \mathrm{Sym}^b C$ with $a, b \le j$ (*[Tev23] *has $a, b \le \min(j, d'+g-2j-1)$, but $j \le d'+g-2j-1$ already), and let t be an integer with $a-j-1 < t < d'+g-2j-1-b$ and $2t < a-b$. Then*

$$R\Gamma\left(M_j(d'), \left(\overline{\mathcal{F}}^{\boxtimes a}_D\right)^\vee \otimes \overline{\mathcal{F}}^{\boxtimes b}_{D'} \otimes \mathbf{\Lambda}^t\right) = 0. \tag{3.2.7}$$

Lemma 3.2.18. *Let λ, λ', k, k' be integers with $k, k', \lambda - 2k, \lambda' - 2k' \ge 0$ and $\lambda - k, \lambda' - k' \le i_d$. If $\lambda < \lambda'$, then*

$$R\operatorname{Hom}\left(\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}_D, \mathbf{\Lambda}^{-k'}\mathcal{F}^{\vee\boxtimes\lambda'-2k'}_{D'}\right) = 0$$

for any $D \in \mathrm{Sym}^{\lambda-2k} C$, $D' \in \mathrm{Sym}^{\lambda'-2k'} C$.

3.3 Modified Plain Weave

3.3.1 Main result

We proceed with the Plain Weave, cf. [Tev23, Section 6]. While the spirit of the argument is the same, there are some technical complications in even degree. Notably, not every pair (F, s) with F a semistable bundle is stable, so M parameterizes only an open substack of all such pairs.

Notation 3.3.1. We denote by $\mathcal{N}$ the stack of rank 2 semistable bundles on C with determinant Λ (and with $\mathbb{G}_m$ as a generic inertia group) and by $\mathbb{N}$ its rigidification (with trivial generic stabilizers). Concretely, we work with the quotient stacks $\mathcal{N} = [Q/\mathrm{GL}(\mathbb{V})]$ and $\mathbb{N} = [Q/\mathrm{PGL}(\mathbb{V})]$ where $\mathbb{V}$ is a vector space of dimension $2 + 2m$ for some large m and Q is an appropriate locally closed subscheme of the Quot scheme parameterizing quotients of $\mathbb{V} \otimes \mathcal{O}_C(-mp)$ for some fixed point $p \in C$ (see [KT21, Section 4] for details). Here $\mathcal{N}$ and $\mathbb{N}$ are smooth algebraic stacks and Q is a smooth quasi-projective variety. The generic inertia group of $\mathcal{N}$ is identified with the center $\mathbb{G}_m \subset \mathrm{GL}(\mathbb{V})$. We have morphisms of stacks

$$M \longrightarrow \mathcal{N} \xrightarrow{\rho} \mathbb{N} \longrightarrow \mathcal{S}U_C(2),$$

where $M = M_{g-1}(\Lambda)$ is the moduli space of stable pairs, $M \to \mathcal{N}$ is the forgetful morphism, and $\mathcal{S}U_C(2)$ is the coarse moduli space (of both $\mathcal{N}$ and $\mathbb{N}$) as well as the GIT quotient of Q by $\mathrm{PGL}(\mathbb{V})$. We do not notationally distinguish between universal bundles $\mathcal{F}$ on $M \times C$ or $\mathcal{N} \times C$, nor those on other spaces that carry them appearing below; similarly for $\mathbf{\Lambda} = \det \mathcal{F}_x$ and θ, the (pullback of the) ample generator of $\operatorname{Pic} \mathcal{S}U_C(2)$ [DN89]. Unlike in the odd degree case, neither $\mathcal{F}$ nor any line bundle twist of $\mathcal{F}$ descends to $\mathbb{N}$ or any open substack of it [Ram73, Theorem 2]. However, twisted tensor vector bundles $\mathbf{\Lambda}^k \mathcal{F}^{\vee \boxtimes 2k}$ on $\mathcal{N} \times \mathrm{Sym}^{2k} C$ have weight 0 with respect to $\mathbb{G}_m$ and therefore descend to $\mathbb{N} \times \mathrm{Sym}^{2k} C$ for every $k \geq 0$.

Let $\pi_{\mathcal{N}} : \mathcal{N} \times C \to \mathcal{N}$ be the projection. We write $R\pi_{\mathcal{N}*}\mathcal{F} = [\mathcal{A} \xrightarrow{u} \mathcal{B}]$ for $\mathcal{A}$ and $\mathcal{B}$ vector bundles on $\mathcal{N}$ of ranks a and b, respectively, where $a - b = 2$ and $\mathbb{G}_m$ acts with weight 1 on the fibers of both [KT21, Lemma 4.4]. Let $\alpha : \mathcal{A} \to \mathcal{N}$, where we use the same notation for vector bundles and their total spaces. Then u gives a section of the vector bundle $\alpha^*\mathcal{B}$ over $\mathcal{A}$. Let $\mathcal{Z} \subset \mathcal{A}$ be the vanishing locus of this section, let $\mathcal{A}^\circ \subset \mathcal{A}$ be the complement of the zero section, and let $\mathcal{Z}^\circ = \mathcal{Z} \cap \mathcal{A}^\circ$, which is the stack of pairs $\{(F,s) : F \in \mathcal{N}, s \in H^0(F) \smallsetminus \{0\}\}$ (see [KT21, Lemma 4.5(i)]). We have a diagram

$$\begin{array}{ccccccc} & & \mathcal{Z} & \hookrightarrow & \mathcal{A} & \xrightarrow{\alpha} & \mathcal{N} \\ & & \uparrow & & \uparrow & & \downarrow{\scriptstyle \rho} \\ M & \xhookrightarrow{j} & \mathcal{Z}^\circ & \hookrightarrow & \mathcal{A}^\circ & & \mathbb{N}. \end{array} \qquad \zeta : \mathcal{Z}^\circ \to \mathbb{N} \tag{3.3.1}$$

Proposition 3.3.2. *In the notation of* (3.3.1)*:*

(a) *In $D^b(\mathcal{A}^\circ)$, $\mathcal{O}_{\mathcal{Z}^\circ}$ is isomorphic to the Koszul complex*

$$\left[\bigwedge^b \alpha^* \mathcal{B}^\vee|_{\mathcal{A}^\circ} \to \ldots \to \alpha^* \mathcal{B}^\vee|_{\mathcal{A}^\circ} \to \mathcal{O}_{\mathcal{A}^\circ} \right].$$

(b) *We have $R\zeta_* \mathcal{O}_{\mathcal{Z}^\circ} = \mathcal{O}_{\mathbb{N}}$.*

(c) *The relative dualizing sheaf for ζ is $\theta \mathbf{\Lambda}^{-1}$.*

Sketch of proof. Part (a) follows from [KT21, Lemma 4.5(i)]. For part (b), use the fact that $\mathcal{A}^\circ$ is a (twisted) projective bundle over $\mathbb{N}$ of rank $a-1 = b+1$, and B can be replaced $\mathcal{O}(1)^{\oplus b}$ locally over $\mathbb{N}$. For part (c), use adjunction and [Nar17, Proposition 2.1]. □

In the diagram (3.3.1), j is the inclusion of the GIT-semistable locus with respect to a certain line bundle $L_\ell \otimes \chi_0^\epsilon$ on $\mathcal{Z}^\circ$ [KT21, Section 4]. This gives rise to windows embeddings $D^b(M) \cong \mathbf{G}_w \subset D^b(\mathcal{Z}^\circ)$ [HL15]. The relevant weights are given by the following proposition (cf. [TT21, Lemma 3.17 and Theorem 3.21]).

Proposition 3.3.3. *With respect to the semistable locus $M \hookrightarrow \mathcal{Z}^\circ$:*

(a) *There is a unique Kempf–Ness stratum with associated window width $\eta = g$.*

(b) *Objects in the subcategory $\langle \theta^x \mathbf{\Lambda}^y \mathcal{F}^{\vee \boxtimes z} \rangle \subset D^b(\mathcal{Z}^\circ)$ have weights in the range $[-y, z-y]$.*

Proof. See [ST24]. □

Corollary 3.3.4. *The blocks in Theorem 3.2.10, with the kernels now regarded as objects on $D^b(\mathcal{Z}^\circ \times \mathrm{Sym}^\ell C)$, give a semiorthogonal decomposition of the windows subcategory $\mathbf{G} = \mathbf{G}_{-\lfloor g/2 \rfloor} \subset D^b(\mathcal{Z}^\circ)$.*

Proof. Using Proposition 3.3.3(b), one checks that objects in those blocks have weights in the interval $[-\lfloor \frac{g}{2} \rfloor, g - \lfloor \frac{g}{2} \rfloor)$, so they are contained in $\mathbf{G}$. For example, the (λ, k) block in the first megablock has weights in the range

$$\left[k - \left\lfloor \frac{g-2}{2} \right\rfloor, \lambda - k - \left\lfloor \frac{g-2}{2} \right\rfloor\right] \subseteq \left[-\left\lfloor \frac{g-2}{2} \right\rfloor, g - 2 - \left\lfloor \frac{g-2}{2} \right\rfloor\right] = \left[-\left\lfloor \frac{g-2}{2} \right\rfloor, g - \left\lfloor \frac{g}{2} \right\rfloor\right).$$

The windows embedding $D^b(M) \cong \mathbf{G} \subset D^b(\mathcal{Z}^\circ)$ is right inverse to the restriction $j^* : D^b(\mathcal{Z}^\circ) \to D^b(M)$, so j^* gives an equivalence $\mathbf{G} \cong D^b(M)$ taking each block in $\mathbf{G} \subset D^b(\mathcal{Z}^\circ)$ to the corresponding block in $D^b(M)$. The result follows. □

Notation 3.3.5. We introduce full subcategories $\mathbf{K} = \{X \in D^b(\mathcal{Z}^\circ) : \zeta_* X = 0\}$ and $\mathbf{K}^\vee = \{X^\vee : X \in \mathbf{K}\}$.

Remark 3.3.6. Note that $\theta \otimes \mathbf{K} = \mathbf{K}$ by the projection formula and $\mathbf{K}^\vee = \mathbf{\Lambda} \otimes \mathbf{K}$ by coherent duality and Proposition 3.3.2(c).

Theorem 3.3.7 (Plain Weave)**.** *There is a semiorthogonal decomposition*

$$\mathbf{G} = \langle \mathbf{L}, \mathbf{i}, \mathbf{ii}, \mathbf{iii}, \mathbf{iv}, \mathbf{R} \rangle \tag{3.3.2}$$

where $\mathbf{L} \subset \mathbf{K}$, $\mathbf{R} \subset \mathbf{K}^\vee$, *and*

$$\mathbf{i} = \left\langle \theta^{-1} \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \right\rangle_{0 \le m \le \lfloor \frac{g-2}{2} \rfloor}, \quad \mathbf{ii} = \left\langle \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \right\rangle_{0 \le m \le \lfloor \frac{g-2}{2} \rfloor},$$
$$\mathbf{iii} = \left\langle \theta \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \right\rangle_{0 \le m \le \lfloor \frac{g-1}{2} \rfloor}, \quad \mathbf{iv} = \left\langle \theta^2 \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \right\rangle_{0 \le m \le \lfloor \frac{g-1}{2} \rfloor},$$

with each megablock is ordered by increasing m.

We postpone the proof of Theorem 3.3.7 to the next subsection. We can now precisely state and prove our main result.

Theorem 3.3.8. *The admissible subcategory* $\mathcal{D} = \langle \mathbf{i}, \mathbf{ii}, \mathbf{iii}, \mathbf{iv} \rangle \subset \mathbf{G}$ *is a noncommutative resolution of sin-gularities of* $\mathcal{SU}_C(2)$. *Furthermore, this resolution agrees with the resolution of* [Păd21, Theorem 1.1] *(defined there in a more general context of symmetric stacks). This resolution is strongly crepant* [Kuz08] *if* g *is even.*

Proof sketch. The blocks in **i**, **ii**, **iii**, **iv** are exactly those from Theorem 3.2.10 which are pulled back from $\mathbb{N}$. Indeed, the bundle $\theta^i \mathbf{\Lambda}^j \mathcal{F}^{\vee \boxtimes k}$ on $\mathcal{N}$ has weight $2j - k$ with respect to the $\mathbb{G}_m$ action, so it descends to $\mathbb{N}$ exactly when $k = 2j$. By Proposition 3.3.2(b), it follows that $\mathcal{D}$ is isomorphic (via $R\zeta_*$) to an admissible subcategory $\mathcal{D}' \subset D^b(\mathbb{N})$ given by the same Fourier–Mukai kernels as $\mathcal{D}$ (regarded as objects in $D^b(\mathbb{N} \times \mathrm{Sym}^k C)$).

It can be shown [ST24] that $\mathbb{N}$ is a symmetric stack satisfying assumptions A–C in [Păd21], and that the window width is $\eta = g - 1$. By [Păd21, Theorem 1.1], it follows that the full subcategory $\mathcal{D}'' = \{X \in D^b(\mathbb{N}) : -\frac{g-1}{2} \leq \mathrm{weight}_\lambda X \leq \frac{g-1}{2}\}$ is admissible and provides a noncommutative resolution of singularities of $\mathcal{SU}_C(2)$.

Next, we will show that $\mathcal{D}' = \mathcal{D}''$. Let $X \in \mathcal{D}'$. The computation of $\mathrm{weight}_\lambda X$ is the same as the computation of $\mathrm{weight}_\lambda \zeta^* X$ in Proposition 3.3.3. By Lemma 3.3.9 below, the weights of objects in subcategories **i**, **ii**, **iii**, **iv** are in the interval $[-\frac{g-1}{2}, \frac{g-1}{2}]$. It follows that $\mathcal{D}' \subset \mathcal{D}''$. On the other hand, let $X \in \mathcal{D}''$. Since $\mathrm{weight}_\lambda X = \mathrm{weight}_\lambda \zeta^* X$ as above, we have $\zeta^* X \in \mathbf{G}$. With respect to the semiorthogonal decomposition (3.3.2) of Theorem 3.3.7, we have $\mathrm{Hom}(\zeta^* X, \mathbf{L}) = 0$ by projection formula and $\mathrm{Hom}(\mathbf{R}, \zeta^* X) = 0$ by coherent duality. It follows that $\zeta^* X \in \mathcal{D}$, and therefore $X \in \mathcal{D}'$. We conclude that $\mathcal{D}' = \mathcal{D}''$.

It remains to show that $\mathcal{D}$ is a strongly crepant noncommutative resolution of $\mathcal{SU}_C(2)$ if g is even. By the discussion above, we can view $\mathcal{D}$ as an admissible subcategory of $D^b(M)$, $D^b(\mathcal{Z}^0)$, or $D^b(\mathbb{N})$, where in every case the pullback functor $\mathrm{Perf}(\mathcal{SU}_C(2)) \to \mathcal{D}$ is the usual pullback. Furthermore, it endows $\mathcal{D}$ with the structure of an $\mathcal{SU}_C(2)$-linear category via the usual tensor product. We will view $\mathcal{D}$ as a subcategory of $D^b(M)$. Let $f : M \to SU_C(2)$ be the forgetful morphism. For every $B \in \mathcal{D}$, the functor $\mathrm{Perf}(\mathcal{SU}_C(2)) \to \mathcal{D}$, $A \mapsto f^* A \otimes B$ has a right adjoint functor $\mathcal{D} \to D^b(\mathcal{SU}_C(2))$ given by $C \mapsto Rf_* \circ R\mathcal{H}om_M(B, C)$. According to [Kuz08], in order to show that $\mathcal{D}$ is strongly crepant, we need to show that the identity functor on $\mathcal{D}$ is a relative Serre functor for $\mathcal{D}$ over $\mathcal{SU}_C(2)$. That is, we must give a functorial isomorphism

$$R\mathcal{H}om_{\mathcal{SU}_C(2)}(Rf_* \circ R\mathcal{H}om_M(B, C), \mathcal{O}_{\mathcal{SU}_C(2)}) \cong Rf_* \circ R\mathcal{H}om_M(C, B)$$

for $B, C \in \mathcal{D}$. By coherent duality for f, it is in fact sufficient to establish a functorial isomorphism

$$Rf_* \circ R\mathcal{H}om_M(C, B \otimes \omega_f[1]) \cong Rf_* \circ R\mathcal{H}om_M(C, B),$$

where ω_f is the dualizing line bundle for f.

By Lemma 3.3.13, there exists a morphism $\gamma : \mathcal{O} \to \theta \otimes \mathbf{\Lambda}^{-1}[1]$ in $D^b(\mathcal{Z}^\circ)$ whose image under ζ_* is an isomorphism $\mathcal{O} \to \mathcal{O}$. We pull back γ via the open immersion $j : M \to \mathcal{Z}^\circ$ to a morphism $j^*\gamma : \mathcal{O} \to \omega_f[1]$ on M. Indeed, $\omega_f \cong \omega_M \otimes \omega^{-1}_{\mathcal{S}U_C(2)} \cong j^*(\theta \otimes \mathbf{\Lambda}^{-1})$ by [Tha94, Section 5]. We will show that $j^*\gamma$ induces a stronger isomorphism

$$R(\zeta \circ j)_* \circ R\mathcal{H}om_M(C, B) \cong R(\zeta \circ j)_* \circ R\mathcal{H}om_M(C, B \otimes \omega_f[1]). \tag{3.3.3}$$

By Corollary 3.3.4 and Lemma 3.3.9, the weights of objects B, C, and $B \otimes \omega_f[1]$ belong to the interval $[-\lfloor \frac{g-1}{2} \rfloor, \lfloor \frac{g-1}{2} \rfloor + 1]$, which, if g is even, has width smaller than the window width $\eta = g$ for the immersion j. The GIT construction of M is local over $\mathbb{N}$, so the quantization theorem [HL15, Theorem 3.29] gives vertical isomorphisms in the following commutative diagram:

$$\begin{array}{ccc} R(\zeta \circ j)_* \circ R\mathcal{H}om_M(C, B) & \longrightarrow & R(\zeta \circ j)_* \circ R\mathcal{H}om_M(C, B \otimes \omega_f[1]) \\ \cong\downarrow & & \downarrow\cong \\ R\zeta_* \circ R\mathcal{H}om_{\mathcal{Z}^\circ}(C, B) & \longrightarrow & R\zeta_* \circ R\mathcal{H}om_{\mathcal{Z}^\circ}(C, B \otimes \theta \otimes \mathbf{\Lambda}^{-1}[1]) \end{array}$$

The bottom horizontal morphism of this diagram is an isomorphism by the projection formula for the morphism $\mathcal{Z}^\circ \to \mathbb{N}$ and the fact that the morphism $\gamma : \mathcal{O} \to \theta \otimes \mathbf{\Lambda}^{-1}[1]$ pushes forward to an isomorphism $\zeta_*\gamma$. It follows that the top horizontal morphism is also an isomorphism, proving (3.3.3). □

Lemma 3.3.9. *In the setup of Corollary 3.3.4, objects in subcategories* **i**, **ii**, **iii**, **iv** *have weights in* $[-\frac{g-1}{2}, \frac{g-1}{2}]$.

Proof. The same calculation as the proof of Corollary 3.3.4 using Proposition 3.3.3. □

3.3.2 Proof of the Plain Weave

We require several lemmas. All mutations take place in $D^b(M) \cong \mathbf{G} \subset D^b(\mathcal{Z}^\circ)$.

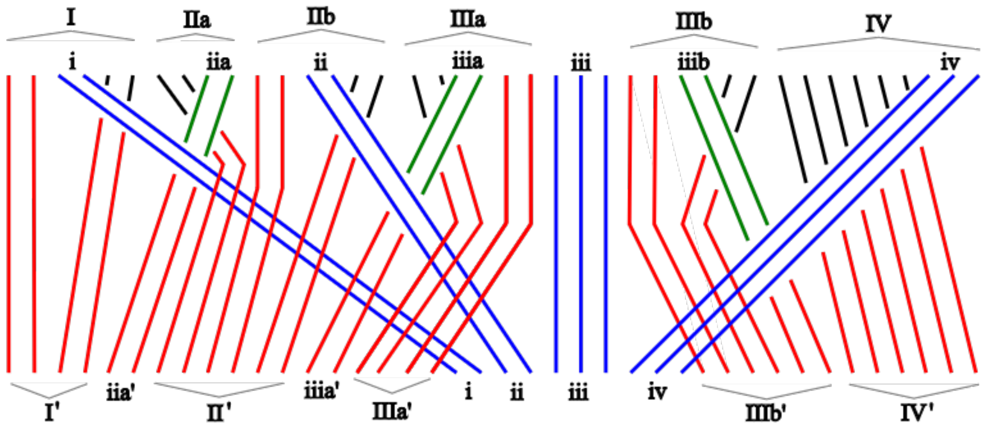


Figure 3.3: Modified Plain Weave in genus 5, cf. [Tev23, Figure 13].

Lemma 3.3.10. *For $0 \le \lambda \le 2(g-1)$, there is a mutation*

$$\left\langle \mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-1} \longrightarrow \left\langle \mathbf{\Lambda}^{-k}\overline{\mathcal{F}^{\vee\boxtimes\lambda-2k}}\right\rangle_{0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-1} \tag{3.3.4}$$

where the blocks are ordered by decreasing k on the left, and by increasing k on the right.

Lemma 3.3.11. *For $0 \le 2k+1 \le g-1$, we have $\langle \mathbf{\Lambda}^k\mathcal{F}^{\vee\boxtimes 2k+1}\rangle \subset \mathbf{K}$ and $\langle \mathbf{\Lambda}^{k+1}\mathcal{F}^{\vee\boxtimes 2k+1}\rangle \subset \mathbf{K}^\vee$.*

Lemma 3.3.12 (cf. [Tev23, Theorem 6.3])**.** *For $\ell = 0,1$ and $\ell \le k \le \frac{g-1}{2}$, let $\mathcal{D}^\ell = \mathcal{D}^\ell_{g-1} \subset D^b(M \times \mathrm{Sym}^\ell C)$ as in Notation 3.2.5. Let $\Phi : D^b(\mathcal{Z}^\circ) \to {}^\perp\langle \mathbf{\Lambda}^k\mathcal{F}^{\vee\boxtimes 2k}\rangle$ be the semiorthogonal projector. Then $\Phi(\langle \mathbf{\Lambda}^{\ell-k}\mathcal{D}^\ell\rangle) \subset \mathbf{K}^\vee$.*

Lemma 3.3.13. *In $D^b(\mathcal{Z}^\circ)$, there exists a morphism $\mathcal{O} \to \theta\mathbf{\Lambda}^{-1}[1]$ whose image under ζ_* is an isomorphism $\mathcal{O} \to \mathcal{O}$. If $\mathbf{T} \subset \zeta^* D^b(\mathbb{N})$ is a full triangulated subcategory such that $\theta\mathbf{\Lambda}^{-1} \otimes \mathbf{T} \subset {}^\perp\mathbf{T}$, there is a mutation $\langle \mathbf{T}, \theta\mathbf{\Lambda}^{-1} \otimes \mathbf{T}\rangle \to \langle \mathbf{X}, \mathbf{T}\rangle$ where $\mathbf{X} \subset \mathbf{K}$.*

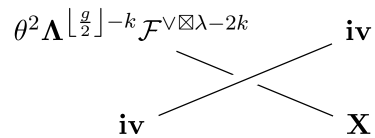
Proof of Theorem 3.3.7. Denote by $\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}$ the megablocks in Theorem 3.2.10, regarded as subcategories of $\mathbf{G} \subset D^b(\mathcal{Z}^\circ)$ as in Corollary 3.3.4. We proceed as illustrated in Figure 3.3: we take the blocks in $\mathbf{i}, \mathbf{ii}, \mathbf{iii}, \mathbf{iv}$, and move them towards the center, making sure that all blocks in the way mutate into either $\mathbf{K}$ on the left or $\mathbf{K}^\vee$ on the right.

We begin with

$$\mathbf{IV} = \left\langle \theta^2\mathbf{\Lambda}^{\lfloor\frac{g}{2}\rfloor-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{g-1\le\lambda\le 2(g-1)\\ \lambda-g+1\le k\le\lfloor\frac{\lambda}{2}\rfloor}}.$$

The blocks with $\lambda = 2\left\lfloor \frac{g}{2} \right\rfloor$ form megablock **iv** from the statement (with $m = \left\lfloor \frac{g}{2} \right\rfloor - k$). If g is odd, $2\left\lfloor \frac{g}{2} \right\rfloor = g - 1$, so these are the rightmost blocks in **IV** as in Figure 3.3; if g is even, the blocks with $\lambda = g - 1$ to the right of **iv** already lie in $\mathbf{K}^\vee$ by Lemma 3.3.11.

Claim 3.3.14. For $2\left\lfloor \frac{g}{2} \right\rfloor < \lambda \leq 2(g-1)$ and $\lambda - g + 1 \leq k \leq \left\lfloor \frac{\lambda}{2} \right\rfloor$, there is a mutation



where $\mathbf{X} \subset \mathbf{K}^\vee$.

Proof. Write $\mathbf{A} = \langle \theta^2 \mathbf{\Lambda}^{\lfloor \frac{g}{2} \rfloor - k} \mathcal{F}^{\vee \boxtimes \lambda - 2k} \rangle$. Suppose first that $\lambda = 2k$, so $\mathbf{A} = \langle \theta^2 \mathbf{\Lambda}^{\lfloor \frac{g}{2} \rfloor - k} \rangle$. Take the block $\mathbf{B} = \langle \theta^2 \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \rangle$ from **iv** with $m = k - \left\lfloor \frac{g}{2} \right\rfloor$ and write $\langle \mathbf{iv} \rangle = \langle \mathbf{B}, \mathbf{B}' \rangle$. (Note that $0 \leq m \leq \left\lfloor \frac{g-1}{2} \right\rfloor$ since $\left\lfloor \frac{g}{2} \right\rfloor < k \leq g - 1$.) By Lemma 3.3.12, we mutate $\langle \mathbf{A}, \mathbf{B}, \mathbf{B}' \rangle \to \langle \mathbf{B}, \mathbf{A}', \mathbf{B}' \rangle$ where $\mathbf{A}' \subset \mathbf{K}^\vee$. Since $\mathbf{B}' \subset \zeta^* D^b(\mathbb{N})$, we see that $\mathbf{A}'$ and $\mathbf{B}'$ are fully orthogonal, so we move $\langle \mathbf{B}, \mathbf{A}', \mathbf{B}' \rangle \to \langle \mathbf{B}, \mathbf{B}', \mathbf{A}' \rangle = \langle \mathbf{iv}, \mathbf{A}' \rangle$ without any further mutation.

Now suppose $2k < \lambda$. Let $\mathbf{B} = \langle \theta^2 \mathbf{\Lambda}^{m-1} \mathcal{F}^{\vee \boxtimes 2m-2}, \theta^2 \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \rangle \subset \langle \mathbf{iv} \rangle$, where $m = \lambda - k - \left\lfloor \frac{g}{2} \right\rfloor$. (We have $m > \frac{\lambda}{2} - \left\lfloor \frac{g}{2} \right\rfloor \geq 0$, so $m - 1 \geq 0$, and $m \leq g - 1 - \left\lfloor \frac{g}{2} \right\rfloor \leq \left\lfloor \frac{g-1}{2} \right\rfloor$.) Denote by $\mathcal{F}^{\bullet \boxtimes \ell}_{tr} = [\mathcal{F}^{\vee \boxtimes \ell} \to \mathrm{Ker}^{1-\ell} \mathcal{F}^{\bullet \boxtimes \ell}]$ the two-step smart truncation of the complex $\mathcal{F}^{\bullet \boxtimes \ell}$ (see [Tev23, Definitions 4.2 and 4.5]). By [Tev23, Lemma 4.9, Corollary 4.10], for every $X \in \mathbf{A}$ we have exact triangles $K \to Y \to X \to$ and $H \to Y \to H' \to$ where $Y \in \langle \theta^2 \mathbf{\Lambda}^{\lfloor \frac{g}{2} \rfloor - k} \mathcal{F}^{\bullet \boxtimes \lambda - 2k}_{tr} \rangle$, $K \in \langle \theta^2 \mathbf{\Lambda}^{\lfloor \frac{g}{2} \rfloor - k - (\lambda - 2k - 1)} \rangle = \langle \theta^2 \mathbf{\Lambda}^{1-m} \rangle$, $H \in \langle \theta^2 \mathbf{\Lambda}^{-m} \rangle$, and $H' \in \langle \theta^2 \mathbf{\Lambda}^{1-m} \mathcal{D}^1 \rangle$. It follows from Lemma 3.3.12 that the images of K, H and H' under the semiorthogonal projector onto ${}^\perp\mathbf{B}$ lie in $\mathbf{K}^\vee$, so the same is true for X. Writing $\langle \mathbf{iv} \rangle = \langle \mathbf{B}, \mathbf{B}' \rangle$ as above, this gives mutations $\langle \mathbf{A}, \mathbf{B}, \mathbf{B}' \rangle \to \langle \mathbf{B}, \mathbf{A}', \mathbf{B}' \rangle \to \langle \mathbf{B}, \mathbf{B}', \mathbf{A}' \rangle = \langle \mathbf{iv}, \mathbf{A}' \rangle$ with $\mathbf{A}' \subset \mathbf{K}^\vee$, as claimed. $\square$

By applying this mutation to each block in **IV** with $g \leq \lambda \leq 2(g-1)$ in sequence from right to left, we obtain $\langle \mathbf{IV} \rangle = \langle \mathbf{iv}, \mathbf{IV}' \rangle$ where $\mathbf{IV}' \subset \mathbf{K}^\vee$.

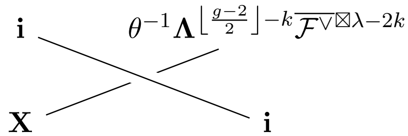
The megablock

$$\mathbf{I} = \left\langle \theta^{-1} \mathbf{\Lambda}^{\lfloor \frac{g-2}{2} \rfloor - k} \mathcal{F}^{\vee \boxtimes \lambda - 2k} \right\rangle_{\substack{0 \leq \lambda \leq g-2 \\ 0 \leq k \leq \lfloor \frac{\lambda}{2} \rfloor}}$$

on the left is treated similarly, but the process is mirrored. Megablock **i** appears as the blocks with

$\lambda = 2\left\lfloor \frac{g-2}{2} \right\rfloor$. If g is odd, there are additional blocks with $\lambda = g-2$ to the left of the megablock $\mathbf{i}$, but they already lie in $\mathbf{K}$ by Lemma 3.3.11, so we ignore them as above. We apply the mutation of Lemma 3.3.10 to all blocks in $\mathbf{I}$; the blocks are still ordered by decreasing λ, but now by increasing k with $\overline{\mathcal{F}^{\vee\boxtimes\lambda-2k}}$ in place of $\mathcal{F}^{\vee\boxtimes\lambda-2k}$.

Claim 3.3.15. For $0 \le \lambda < 2\left\lfloor \frac{g-2}{2} \right\rfloor$ and $0 \le k \le \left\lfloor \frac{\lambda}{2} \right\rfloor$, there is a mutation



where $\mathbf{X} \subset \mathbf{K}$.

Proof. We prove instead the dual mutation. By [Tev23, Lemma 3.4], we have

$$\langle \theta^x \mathbf{\Lambda}^y \overline{\mathcal{F}^{\vee\boxtimes z}} \rangle^\vee = \langle \theta^{-x}\mathbf{\Lambda}^{-y}(\overline{\mathcal{F}^{\vee\boxtimes z}})^\vee \rangle = \langle \theta^{-x}\mathbf{\Lambda}^{z-y}\mathcal{F}^{\vee\boxtimes z} \rangle \tag{3.3.5}$$

Hence it suffices to give a mutation $\langle \mathbf{A}, \mathbf{i}^\vee \rangle \to \langle \mathbf{i}^\vee, \mathbf{A}' \rangle$ where $\mathbf{A} = \langle \theta \mathbf{\Lambda}^{\lambda - k - \left\lfloor \frac{g-2}{2} \right\rfloor} \mathcal{F}^{\vee\boxtimes\lambda-2k} \rangle$, $\mathbf{A}' \subset \mathbf{K}^\vee$, and $\mathbf{i}^\vee = \langle \theta \mathbf{\Lambda}^m \mathcal{F}^{\vee\boxtimes 2m} \rangle_{0 \le m \le \left\lfloor \frac{g-2}{2} \right\rfloor}$ ordered by decreasing m.

From here, the proof is the same as for Claim 3.3.14. If $\lambda = 2k$, we let $\mathbf{B} = \langle \theta \mathbf{\Lambda}^m \mathcal{F}^{\vee\boxtimes 2m} \rangle$ with $m = \left\lfloor \frac{g-2}{2} \right\rfloor - k$ and proceed as above. We need only check that $0 \le m \le \left\lfloor \frac{g-2}{2} \right\rfloor$, which is clear. Similarly, if $2k < \lambda$, it suffices to check that $1 \le m \le \left\lfloor \frac{g-2}{2} \right\rfloor$ where $m = (\lambda - 2k) - \left(\lambda - k - \left\lfloor \frac{g-2}{2} \right\rfloor\right) = \left\lfloor \frac{g-2}{2} \right\rfloor - k$, which is again clear. ◻

We thus obtain $\mathbf{I} = \langle \mathbf{I}', \mathbf{i} \rangle$ where $\mathbf{I}' \subset \mathbf{K}$. Next, let

$$\mathbf{II}_a = \left\langle \mathbf{\Lambda}^{\left\lfloor \frac{g-2}{2} \right\rfloor - k} \mathcal{F}^{\vee\boxtimes\lambda-2k} \right\rangle_{\substack{g-1 \le \lambda \le 2(g-2), \\ \lambda - g + 2 \le k \le \left\lfloor \frac{\lambda}{2} \right\rfloor}}, \quad \mathbf{II}_b = \left\langle \mathbf{\Lambda}^{\left\lfloor \frac{g-2}{2} \right\rfloor - k} \mathcal{F}^{\vee\boxtimes\lambda-2k} \right\rangle_{\substack{0 \le \lambda \le g-2, \\ 0 \le k \le \left\lfloor \frac{\lambda}{2} \right\rfloor}},$$

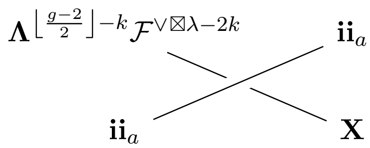
so $\mathbf{II} = \langle \mathbf{II}_a, \mathbf{II}_b \rangle$. Then $\mathbf{II}_b = \theta \otimes \mathbf{I}$ and $\mathbf{ii} = \theta \otimes \mathbf{i}$, so $\mathbf{II}_b = \langle \mathbf{II}_b', \mathbf{ii} \rangle$ where $\mathbf{II}_b' = \theta \otimes \mathbf{I}' \subset \mathbf{K}$. On the other hand, let

$$\mathbf{ii}_a = \left\langle \mathbf{\Lambda}^{m-1} \mathcal{F}^{\vee\boxtimes 2m} \right\rangle_{0 \le m \le \left\lfloor \frac{g-3}{2} \right\rfloor}$$

ordered by increasing m, which are the blocks in $\mathbf{II}_a$ with $\lambda = 2\left\lfloor \frac{g}{2} \right\rfloor$ (where $m = \left\lfloor \frac{g}{2} \right\rfloor - k$). If g is even, there are blocks in $\mathbf{II}_a$ with $\lambda = g-1$ to the right of $\mathbf{ii}_a$, but they are contained in $\mathbf{K}$ by

Lemma 3.3.11.

Claim 3.3.16. For $2\left\lfloor \frac{g}{2}\right\rfloor < \lambda \le 2(g-2)$ and $\lambda - g + 2 \le k \le \left\lfloor \frac{\lambda}{2}\right\rfloor$, there is a mutation



where $\mathbf{X} \subset \mathbf{K}$.

Proof. The proof is the same as Claim 3.3.14, but with everything tensored with $\mathbf{\Lambda}^{-1} \otimes \theta^{-2}$ and with a smaller range of λ, k, and m. We need only check that $0 \le m \le \left\lfloor \frac{g-3}{2}\right\rfloor$ with $m > 0$ if $2k < \lambda$, where $m = \lambda - k - \left\lfloor \frac{g}{2}\right\rfloor$. Indeed, we have $0 \le \frac{\lambda}{2} - \left\lfloor \frac{g}{2}\right\rfloor \le m \le g - 2 - \left\lfloor \frac{g}{2}\right\rfloor \le \left\lfloor \frac{g-3}{2}\right\rfloor$ with $\frac{\lambda}{2} - \left\lfloor \frac{g}{2}\right\rfloor < m$ if $2k < \lambda$. ☐

Hence we can write $\mathbf{II}_a = \langle \mathbf{ii}_a, \mathbf{II}'_a\rangle$ with $\mathbf{II}'_a \subset \mathbf{K}$. Combining with $\mathbf{II}_b$ gives $\mathbf{II} = \langle \mathbf{ii}_a, \mathbf{II}', \mathbf{ii}\rangle$ where $\mathbf{II}' = \langle \mathbf{II}'_a, \mathbf{II}'_b\rangle \subset \mathbf{K}$.

Next, we have $\mathbf{III} = \langle \mathbf{III}_a, \mathbf{iii}, \mathbf{III}_b\rangle$, where

$$\mathbf{III}_a = \left\langle \theta\mathbf{\Lambda}^{\left\lfloor \frac{g}{2}\right\rfloor - k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{2\left\lfloor \frac{g}{2}\right\rfloor < \lambda \le 2(g-1) \\ \lambda - g + 1 \le k \le \left\lfloor \frac{\lambda}{2}\right\rfloor}}, \quad \mathbf{III}_b = \left\langle \theta\mathbf{\Lambda}^{\left\lfloor \frac{g}{2}\right\rfloor - k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{0 \le \lambda < 2\left\lfloor \frac{g}{2}\right\rfloor \\ 0 \le k \le \left\lfloor \frac{\lambda}{2}\right\rfloor}}.$$

Let $\mathbf{iii}_a = \left\langle \theta\mathbf{\Lambda}^{m+1}\mathcal{F}^{\vee\boxtimes 2m}\right\rangle_{0 \le m \le \left\lfloor \frac{g-3}{2}\right\rfloor}$ be the blocks with $\lambda = 2\left\lfloor \frac{g+2}{2}\right\rfloor$. The blocks in $\mathbf{III}_a$ to the right of $\mathbf{iii}_a$ have $\lambda = 2\left\lfloor \frac{g}{2}\right\rfloor + 1$ and lie in $\mathbf{K}$ already by Lemma 3.3.11. The blocks to the left of $\mathbf{iii}_a$ are processed exactly as with $\mathbf{II}_a$ above (make the substitution $\lambda' = \lambda - 2$, $k' = k - 1$ and use Claim 3.3.16). Hence $\mathbf{III}_a = \langle \mathbf{iii}_a, \mathbf{III}'_a\rangle$ with $\mathbf{III}'_a \subset \mathbf{K}$. Similarly, let $\mathbf{iii}_b = \left\langle \theta\mathbf{\Lambda}^{m-1}\mathcal{F}^{\vee\boxtimes 2m}\right\rangle_{0 \le m \le \left\lfloor \frac{g-2}{2}\right\rfloor}$ be the blocks from $\mathbf{III}_b$ with $\lambda = 2\left\lfloor \frac{g-2}{2}\right\rfloor$. The blocks in $\mathbf{III}_b$ with $\lambda = 2\left\lfloor \frac{g}{2}\right\rfloor - 1$ lie in $\mathbf{K}^\vee$; the other blocks are exactly the blocks in $\mathbf{I}$ with $\lambda \le 2\lfloor g-2\rfloor 2$, tensored by $\theta^2\mathbf{\Lambda}$. Then Claim 3.3.15 allows us to write $\mathbf{III}_b = \langle \mathbf{III}'_b, \mathbf{iii}_b\rangle$ with $\mathbf{III}'_b \subset \mathbf{K}^\vee$.

To summarize, we have a semiorthogonal decomposition

$$\mathbf{G} = \left\langle \mathbf{I}', \mathbf{i}, \mathbf{ii}_a, \mathbf{II}', \mathbf{ii}, \mathbf{iii}_a, \mathbf{III}'_a, \mathbf{iii}, \mathbf{III}'_b, \mathbf{iii}_b, \mathbf{iv}, \mathbf{IV}'\right\rangle$$

where $\mathbf{I}', \mathbf{II}', \mathbf{III}'_a \subset \mathbf{K}$ and $\mathbf{III}'_b, \mathbf{IV}' \subset \mathbf{K}^\vee$. It remains to mutate $\mathbf{ii}_a$, $\mathbf{iii}_a$, and $\mathbf{iii}_b$. Observe that the blocks in $\mathbf{ii}_a$ are exactly the blocks from $\mathbf{i}$ tensored by $\theta\mathbf{\Lambda}^{-1}$. One by one, we mutate each block $\theta\mathbf{\Lambda}^{-1}\otimes\mathbf{T}$ from $\mathbf{ii}_a$ past the corresponding block from $\mathbf{T}$ in $\mathbf{ii}$ using Lemma 3.3.13; the resulting block $\mathbf{T'}\subset\mathbf{K}$ is both left and right orthogonal to $\mathbf{T}^\perp\cap\mathbf{i}$, so we may move it to the left of $\mathbf{i}$ without further mutation. Hence we obtain $\langle\mathbf{i},\mathbf{ii}_a\rangle=\langle\mathbf{ii}'_a,\mathbf{i}\rangle$ with $\mathbf{ii}'_a\subset\mathbf{K}$. Similarly, $\mathbf{iii}_a=\theta\mathbf{\Lambda}^{-1}\otimes\mathbf{ii}$, so $\langle\mathbf{ii},\mathbf{iii}_a\rangle=\langle\mathbf{iii}'_a,\mathbf{ii}\rangle$ with $\mathbf{iii}'_a\subset\mathbf{K}$. On the other side, each block in $\mathbf{iii}_b$ is a block from $\mathbf{iv}$ tensored with $\theta^{-1}\mathbf{\Lambda}$. The dual of Lemma 3.3.13 allows us to mutate $\langle\theta^{-1}\mathbf{\Lambda}\otimes\mathbf{T},\mathbf{T}\rangle\to\langle\mathbf{T},\mathbf{T'}\rangle$ with $\mathbf{T'}\subset\mathbf{K}^\vee$, which gives $\langle\mathbf{iii}_b,\mathbf{iv}\rangle=\langle\mathbf{iv},\mathbf{iii}'_b\rangle$ with $\mathbf{iii}'_b\subset\mathbf{K}^\vee$.

Put together, we have

$$\mathbf{G}=\left\langle\mathbf{I'},\mathbf{ii}'_a,\mathbf{i},\mathbf{II'},\mathbf{iii}'_a,\mathbf{ii},\mathbf{III}'_a,\mathbf{iii},\mathbf{III}'_b,\mathbf{iv},\mathbf{iii}'_b,\mathbf{IV'}\right\rangle,$$

where all primed subcategories to the left (resp. right) of $\mathbf{iii}$ lie in $\mathbf{K}$ (resp. $\mathbf{K}^\vee$). It follows that $\mathbf{ii}$ and $\mathbf{III}'_a$ are both left and right orthogonal; similarly, $\mathbf{i}$ is orthogonal to $\mathbf{II'}$, $\mathbf{iii}'_a$, and $\mathbf{III}'_a$, while $\mathbf{iv}$ is orthogonal to $\mathbf{III}'_b$. Thus we may move $\mathbf{i}$, $\mathbf{ii}$, and $\mathbf{iv}$ to the center without any mutations, giving $\mathbf{G}=\langle\mathbf{L},\mathbf{i},\mathbf{ii},\mathbf{iii},\mathbf{iv},\mathbf{R}\rangle$ where $\mathbf{L}=\langle\mathbf{I'},\mathbf{ii}'_a,\mathbf{II'},\mathbf{iii}'_a,\mathbf{III}'_a\rangle\subset\mathbf{K}$ and $\mathbf{R}=\langle\mathbf{III}'_b,\mathbf{iii}'_b,\mathbf{IV'}\rangle\subset\mathbf{K}^\vee$. This completes the proof. □

Proof sketch of Lemma 3.3.10. Compare the semiorthogonal decomposition of Lemma 3.2.11 with its dual using vanishing theorems from [TT21]. See [ST24] for details. □

Proof of Lemma 3.3.11. Let $X\in\langle\mathbf{\Lambda}^k\mathcal{F}^{\vee\boxtimes 2k+1}\rangle$. Then X is a pullback of an object in $D^b(\mathcal{N})$ of weight -1 with respect to $\mathbb{G}_m$. By Proposition 3.3.2(a), it suffices to show that $R\zeta_*Y=0$ for every object $Y\in D^b(\mathcal{A}^\circ)$ of the form $Y=\bigwedge^k\alpha^*\mathcal{B}^*\otimes^L\alpha^*Z$, where $k=0,\dots,b$ and $Z\in D^b(\mathcal{N})$ is an object of weight -1. Since the claim is local on $\mathbb{N}$, we can replace Y with $\mathcal{O}_{\mathcal{A}^\circ}(s)$, where $s=-1,\dots,-(b+1)$. Since $R\Gamma(\mathbf{P}^{a-1},\mathcal{O}(-s))=0$ for $s=1,\dots,a-1$ and $b+1=a-1$, the first statement follows. The second follows from Remark 3.3.6. □

Proof of Lemma 3.3.12. We mimic the proof of [Tev23, Theorem 6.3]. As we will need to work in both $D^b(M)$ and $D^b(\mathcal{Z}^\circ)$, we denote the windows embedding by $\iota:D^b(M)\to\mathbf{G}\subset D^b(\mathcal{Z}^\circ)$. As in [Tev23, Lemma 6.7], it suffices to show that the morphism

$$\zeta_*(\mathbf{\Lambda}^{-1}\otimes\iota(\mathbf{\Lambda}^{\ell-k}\mathcal{O}_{M(-D)}))\to\zeta_*(\mathbf{\Lambda}^{-1}\otimes\iota\circ\mathcal{P}\circ\mathcal{P}^L(\mathbf{\Lambda}^{\ell-k}\mathcal{O}_{M(-D)}))$$

is an isomorphism for any $D \in \mathrm{Sym}^\ell C$, where $\mathcal{P} : D^b(\mathrm{Sym}^k C) \to D^b(M)$ is the Fourier–Mukai functor with kernel given by $\mathbf{\Lambda}^k \mathcal{F}^{\vee \boxtimes 2k}$, $\mathcal{P}^L$ is its left adjoint (the Fourier–Mukai functor with kernel $(\mathbf{\Lambda}^k \mathcal{F}^{\vee\boxtimes 2k})^\vee \otimes \omega_M^\bullet$ [Huy06, Proposition 5.9]), $M(-D) \subset M$ denotes the locus of stable pairs (F, s) with $s|_D = 0$, and the morphism is induced by the unit of adjunction $\mathrm{Id} \implies \mathcal{P} \circ \mathcal{P}^L$. We compute both sides of this morphism.

Claim 3.3.17. We have $\mathcal{P} \circ \mathcal{P}^L(\mathbf{\Lambda}^{\ell-k}\mathcal{O}_{M(-D)}) \cong R\pi_{M*}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k})[2\ell]$, where π_M denotes the projection $M \times \mathrm{Sym}^k C \to M$.

Proof. Notice first that by [Huy06, Corollary 3.40] and [Tha94, 5.7, 6.1], we have

$$\mathcal{O}_{M(-D)}^\vee = \omega_{M(-D)} \otimes \omega_M^{-1}[-2\ell] = \mathbf{\Lambda}^\ell \mathcal{O}_{M(-D)}[-2\ell]$$

in $D^b(M)$. We have

$$\begin{aligned} \mathcal{P}^L(\mathbf{\Lambda}^{\ell-k}\mathcal{O}_{M(-D)}) &= R\pi_{\mathrm{Sym}^k C*}((\mathbf{\Lambda}^{2k}\mathcal{F}^{\vee\boxtimes 2k}\mathcal{O}_{M(-D)})^\vee \omega_M^\bullet)[2\ell] \\ &\cong \left(R\pi_{\mathrm{Sym}^k C*}(\mathcal{F}(-D)^{\boxtimes 2k}|_{M(-D)\times \mathrm{Sym}^k C})\right)^\vee \otimes \mathcal{O}(-D)^{\boxtimes 2k}[2\ell] \end{aligned}$$

by coherent duality and the projection formula. Since $\mathcal{F}(-D)|_{M(-D)\times C}$ is the universal family on $M(-D) \times C$, we have $\mathcal{P}^L(\mathbf{\Lambda}^{\ell-k}\mathcal{O}_{M(-D)}) \cong \mathcal{O}(-D)^{\boxtimes 2k}[2\ell]$ by [TT21, Corollary 7.5]. Applying $\mathcal{P}$ proves the claim. □

Hence (after shifting by -2ℓ for convenience) we have a morphism

$$\mathbf{\Lambda}^{-k}\mathcal{O}_{M(-D)}^\vee \to R\pi_{M*}(\mathbf{\Lambda}^k\mathcal{F}^\vee(-D)^{\boxtimes 2k}), \tag{3.3.6}$$

which is unique up to scalar as in [Tev23, Remark 6.10].

It remains to show that applying the functor $\zeta_*(\mathbf{\Lambda}^{-1} \otimes \iota(-))$ to (3.3.6) yields an isomorphism. Since $R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k\mathcal{F}^\vee(-D)^{\boxtimes 2k}) \in D^b(\mathcal{Z}^\circ)$ has weights in the range $[-k, k] \subseteq [-\lfloor \frac{g}{2} \rfloor, g - \lfloor \frac{g}{2} \rfloor)$ and restricts via j^* to $R\pi_{M*}(\mathbf{\Lambda}^k\mathcal{F}^\vee(-D)^{\boxtimes 2k})$, we have

$$\iota(R\pi_{M*}(\mathbf{\Lambda}^k\mathcal{F}^\vee(-D)^{\boxtimes 2k})) = R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k\mathcal{F}^\vee(-D)^{\boxtimes 2k}).$$

On the other hand, we claim that $\iota(\mathbf{\Lambda}^{-k}\mathcal{O}^\vee_{M(-D)}) = \mathbf{\Lambda}^{-k}\mathcal{O}^\vee_{\mathcal{Z}^\circ(-D)}$ for $\ell = 0, 1$, where $\mathcal{Z}^\circ(-D) \subset \mathcal{Z}^\circ$ denotes the closed substack of pairs (F, s) with $s|_D = 0$. If $\ell = 0$, this is clear, since $\mathbf{\Lambda}^{-k} \in \mathbf{G}$. If $\ell = 1$, so $D = x \in C$, then $\mathcal{Z}^\circ(-x)$ is the codimension-2 vanishing locus of the canonical section of $\mathcal{F}_x$. We have a Koszul resolution $\mathcal{O}_{\mathcal{Z}^\circ(-D)} \cong [\mathbf{\Lambda}^{-1} \to \mathcal{F}_x^\vee \to \mathcal{O}]$, so $\mathcal{O}_{\mathcal{Z}^\circ(-D)}$ has weights in the range $[0, 1]$. Since $[k-1, k] \subseteq [-\lfloor \frac{g}{2} \rfloor, g - \lfloor \frac{g}{2} \rfloor)$ and $j^*\mathcal{O}_{\mathcal{Z}^\circ(-p)} = \mathcal{O}_{M(-p)}$, the claim holds.

Thus applying ι to (3.3.6) gives

$$\mathbf{\Lambda}^{-k}\mathcal{O}^\vee_{\mathcal{Z}^\circ(-D)} \to R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k}), \tag{3.3.7}$$

again unique up to scalar. Moreover, this morphism is nonzero: if not, its cone $\Phi(\mathbf{\Lambda}^{-k}\mathcal{O}^\vee_{\mathcal{Z}^\circ(-D)})[1]$ would have $R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k}) \in \langle \mathbf{\Lambda}^k \mathcal{F}^{\boxtimes 2k} \rangle$ as a direct summand, which is absurd.

By Proposition 3.3.2(c), applying the functor $(\theta \otimes R\zeta_*(\mathbf{\Lambda}^{-1} \otimes -))^\vee \cong R\zeta_*((-)^\vee)$ to (3.3.7) gives a morphism

$$R\zeta_*\left([R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k})]^\vee\right) \to R\zeta_*(\mathbf{\Lambda}^k \mathcal{O}_{\mathcal{Z}^\circ(-D)}), \tag{3.3.8}$$

which we must show is an isomorphism. In fact, it suffices to show that the source and target of (3.3.8) are isomorphic. Indeed, write (3.3.7) as $X \to Y$, where $Y \cong L\zeta^* Z$ (so $R\zeta_*(Y^\vee) \cong Z^\vee$ by the projection formula). Then $\mathbb{C} \cong \mathrm{Hom}(X, Y) \cong \mathrm{Hom}(L\zeta^* Z^\vee, X^\vee) \cong \mathrm{Hom}(R\zeta_*(Y^\vee), R\zeta_*(X^\vee))$, so (3.3.8) is nonzero and unique up to scalar.

Recall that $\mathcal{N}$ is isomorphic to the moduli stack of rank 2 vector bundles on C with determinant $\Lambda(-2D)$, with universal family $\mathcal{F}(-D)$ on $\mathcal{N} \times C$. As in Notation 3.3.1, we write $R\pi_{\mathcal{N}*}(\mathcal{F}(-D)) = [\mathcal{A}' \to \mathcal{B}']$, where $\mathcal{A}'$ and $\mathcal{B}'$ are vector bundles on $\mathcal{N}$ with $\mathbb{G}_m$-weight 1 and ranks a', b', where $a' - b' = 2 - 2\ell$. A polystable vector bundle of the form $\mathcal{O}(D) \oplus \mathcal{O}(D)$, where D is an effective divisor of degree $g - l$, has at least a 2-dimensional space of global sections, so $a' \geq 2$. Writing $\alpha : \mathcal{Z}^\circ \to \mathcal{N}$ so that $\zeta = \rho \circ \alpha$, we have

$$\begin{aligned} R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k}) &\cong \mathbf{\Lambda}^{-k} \otimes R\pi_{\mathcal{Z}^\circ *}(\alpha \times \mathrm{id})^*(\mathcal{F}(-D)^{\boxtimes 2k}) \\ &\cong \alpha^*(\mathbf{\Lambda}^{-k} \otimes R\pi_{\mathcal{Z}^\circ *}(\mathcal{F}(-D)^{\boxtimes 2k})) \\ &\cong \alpha^*(\mathbf{\Lambda}^{-k} \mathrm{Sym}^{2k}[\mathcal{A}' \to \mathcal{B}']). \end{aligned}$$

(For the last equality, see the proof of [Tev23, Lemma 6.13]). Hence the left hand side of (3.3.8) is the descent of $\mathbf{\Lambda}^k \mathrm{Sym}^{2k}[\mathcal{A}' \to \mathcal{B}']^\vee$ to $\mathbb{N}$ (note that this has $\mathbb{G}_m$-weight 0). Analogous to (3.3.1), we have a diagram

$$\begin{array}{ccccc} \mathcal{Z}(-D) & \hookrightarrow & \mathcal{A}' & \xrightarrow{\alpha'} & \mathcal{N} \\ \uparrow & & \uparrow & & \downarrow \rho \\ \mathcal{Z}^\circ(-D) & \hookrightarrow & \mathcal{A}'^\circ & & \mathbb{N}. \end{array} \qquad \zeta' : \mathcal{Z}^\circ(-D) \to \mathbb{N}$$

The right hand side of (3.3.8) is $R\zeta'_*(\mathbf{\Lambda}^k)$. Hence it suffices to prove:

Claim 3.3.18. We have $R\zeta'_*(\mathbf{\Lambda}^k) \cong \mathbf{\Lambda}^k \mathrm{Sym}^{2k}[\mathcal{A}' \to \mathcal{B}']^\vee$.

As in Proposition 3.3.2(a), we have a Koszul resolution in $D^b(\mathcal{A}'^\circ)$:

$$\mathcal{O}_{\mathcal{Z}^\circ(-D)} \cong \left[\bigwedge^{b'} \alpha'^* \mathcal{B}'^\vee|_{\mathcal{A}'^\circ} \to \ldots \to \alpha'^* \mathcal{B}'^\vee|_{\mathcal{A}'^\circ} \to \mathcal{O}_{\mathcal{A}'^\circ} \right].$$

As above, $\zeta' : \mathcal{A}'^\circ \to \mathbb{N}$ is a twisted projective bundle with fiber $\mathbb{P}^{a'-1}$. It follows that

$$R\zeta'_*(\mathbf{\Lambda}^k) \cong R\zeta_* \left[\bigwedge^{b'} \alpha'^* \mathcal{B}'^\vee|_{\mathcal{A}'^\circ} \otimes \mathbf{\Lambda}^k \to \ldots \to \alpha'^* \mathcal{B}'^\vee|_{\mathcal{A}'^\circ} \otimes \mathbf{\Lambda}^k \to \mathcal{O}_{\mathcal{A}'^\circ} \otimes \mathbf{\Lambda}^k \right].$$

We claim that $R^s \zeta'_*[\bigwedge^m \alpha'^* \mathcal{B}'^\vee|_{\mathcal{A}'^\circ} \otimes \mathbf{\Lambda}^k] = 0$ for all s when $2k < m \leq b'$ and for $s > 0$ when $2k \geq m$. Indeed, we can work locally on $\mathbb{N}$, so $\alpha'^* \mathcal{B}'$ can be replaced by $\mathcal{O}_{\zeta'}(1)^{\oplus b'}$. Recall that $\mathbf{\Lambda}$ has $\mathbb{G}_m$-weight 2. Note that $R\Gamma(\mathbf{P}^{a'-1}, \mathcal{O}(2k - m)) = 0$ for $2k < m \leq b'$, since then $-a' = 2\ell - b' - 2 < 2k - m < 0$.

It follows that

$$R\zeta'_*(\mathbf{\Lambda}^k) \cong \left[\zeta_* \bigwedge^{2k} \alpha'^* \mathcal{B}'^\vee|_{\mathcal{A}'^\circ} \otimes \mathbf{\Lambda}^k \to \ldots \to \zeta_* \alpha'^* \mathcal{B}'^\vee|_{\mathcal{A}'^\circ} \otimes \mathbf{\Lambda}^k \to \zeta_* \mathcal{O}_{\mathcal{A}'^\circ} \otimes \mathbf{\Lambda}^k \right]$$

(underived pushforwards, since higher cohomologies vanish). Since $\alpha'_* \mathcal{O}_{\mathcal{A}'^\circ} \cong \mathrm{Sym}^\bullet \mathcal{A}'^\vee$ (recall that $a' \geq 2$), computing the zero-weight part gives

$$R\zeta'_*(\mathbf{\Lambda}^k) \cong \left[\mathbf{\Lambda}^k \otimes \bigwedge^{2k} \mathcal{B}'^\vee \to \ldots \to \mathbf{\Lambda}^k \otimes \mathcal{B}'^\vee \otimes \mathrm{Sym}^{k-1} \mathcal{A}'^\vee \to \mathbf{\Lambda}^k \otimes \mathrm{Sym}^k \mathcal{A}'^\vee \right],$$

which is indeed isomorphic to $\mathbf{\Lambda}^k \mathrm{Sym}^{2k}[\mathcal{A}' \to \mathcal{B}']^\vee$.

□

Proof of Lemma 3.3.13. By Propositions 3.3.2(b) and 3.3.2(c), coherent duality gives

$$R\zeta_* R\mathcal{H}om(\mathcal{O}, \theta \otimes \mathbf{\Lambda}^{-1}[1]) \cong R\mathcal{H}om(\mathcal{O}, \mathcal{O}) \cong \mathcal{O}.$$

Applying $R^0\Gamma$ gives a nonzero morphism $\mathcal{O} \to \theta \otimes \mathbf{\Lambda}^{-1}[1]$ whose image under ζ_* is an isomorphism $\mathcal{O} \to \mathcal{O}$ by construction. We complete this morphism to an exact triangle $\mathcal{O} \to \theta \otimes \mathbf{\Lambda}^{-1}[1] \to K \to$, where $K \in \mathbf{K}$. Tensoring with objects of $\mathbf{T}$ gives the required mutation. □

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