NONCOMMUTATIVE RESOLUTION OF $\mathcal{SU}_C(2)$

ELIAS SINK AND JENIA TEVELEV

Abstract. We study the derived category of the moduli space $\mathcal{SU}_C(2)$ of rank 2 vector bundles on a smooth projective curve C of genus $g \geq 2$ with trivial determinant. This generalizes the recent work by Tevelev and Torres on the case with fixed odd determinant. Since $\mathcal{SU}_C(2)$ is singular, we work with its resolution of singularities, specifically with the noncommutative resolution constructed by Pădurariu and Špenko–Van den Bergh (in the more general setting of symmetric stacks). We show that this noncommutative resolution admits a semiorthogonal decomposition into derived categories of symmetric powers $\mathrm{Sym}^{2k}C$ for $2k \leq g-1$. In the case of even genus, each block appears four times. This is also true in the case of odd genus, except that the top symmetric power $\mathrm{Sym}^{g-1}C$ appears twice. In the case of even genus, the noncommutative resolution is strongly crepant in the sense of Kuznetsov and categorifies the intersection cohomology of $\mathcal{SU}_C(2)$. Since all of its components are "geometric," our semiorthogonal decomposition provides evidence for the expectation, which dates back to the work of Newstead and Tyurin, that $\mathcal{SU}_C(2)$ is a rational variety.

1. Introduction

Let C be a smooth complex projective curve of genus $g \geq 2$ and let $\mathcal{SU}_C(2)$ be the coarse moduli space of semistable rank 2 vector bundles on C with fixed determinant Λ of even degree. By tensoring with a fixed line bundle, it is easy to see that $\mathcal{SU}_C(2)$ is independent (up to isomorphism) of the choice of Λ. Common choices are $\mathcal{O}_C$ or ω_C, but it will be more convenient for us to choose an arbitrary Λ such that $\deg \Lambda = 2g$. It is well-known that $\mathcal{SU}_C(2)$ is a Gorenstein Fano variety of dimension $3g-3$ with rational singularities [DN89]. Various resolutions of singularitities of $\mathcal{SU}_C(2)$ have been studied in [Ses77, NR78, Kir86], and the relationships between these desingularizations have been worked out in [KL04, CCK05]. On the other hand, Kuznetsov [Kuz08] defines a *noncommutative resolution of singularities* of a variety X as a smooth triangulated category $\mathcal{D}$ with an adjoint pair of functors $f_* : \mathcal{D} \to D^b(X)$ and $f^* : \mathrm{Perf}(X) \to \mathcal{D}$ such that $f_* \circ f^* \cong \mathrm{Id}$. As our varieties are all projective, we also require $\mathcal{D}$ to be proper. In [KL15, Theorem 1.4], it is shown that every variety X admits such a noncommutative resolution, which is proper if X is.

The main result of this paper is the following theorem, which provides an even-degree counterpart of the main result in [TT21, Tev23] (the proof of the BGMN conjecture).

Theorem 1.1. *There exists a noncommutative resolution of singularities $\mathcal{D}$ of $\mathcal{SU}_C(2)$ with a semiorthogonal decomposition into blocks equivalent to $D^b(\mathrm{Sym}^{2k}C)$ for $2k \leq g-1$. There are four copies of each block except when g is odd, in which case the block $D^b(\mathrm{Sym}^{g-1}C)$ appears twice. The category $\mathcal{D}$ is an example of the noncommutative resolution of singularities constructed in* [Păd21, ŠVdB23] *for symmetric stacks. In particular, it is an admissible subcategory of the derived category of the Kirwan resolution of $\mathcal{SU}_C(2)$. If g is even, $\mathcal{D}$ is a strongly crepant noncommutative resolution of $\mathcal{SU}_C(2)$ in the sense of* [Kuz08].

We refer to Theorem 3.8 for a more concise and detailed statement of the main theorem. As in [TT21, Tev23], we construct $\mathcal{D}$ as an admissible subcategory in $D^b(M)$, where M is the Thaddeus moduli space of stable pairs (F, s) with F a rank 2 vector bundle on C with determinant Λ and s a non-zero global section (see [Tha94]). Furthermore, derived categories $D^b(\mathrm{Sym}^{2k}C)$ are embedded in $D^b(M)$ by means of explicit Fourier–Mukai functors with kernels given by "tensor" vector bundles twisted by various line bundles.

An easy application of the main result in [Păd21] (see Remark 3.9) shows that $\mathcal{D}$ categorifies the intersection cohomology of $\mathcal{SU}_C(2)$ in the even genus case. Indeed, our decomposition is compatible with its computation in [DB02]. By the Bondal–Orlov conjecture [BO02], we expect $\mathcal{D}$ to be an admissible subcategory of every resolution of $\mathcal{SU}_C(2)$ in even genus.

In the odd degree case ($\deg \Lambda = 2g-1$) studied in [TT21, Tev23], the moduli spaces of stable bundles and stable pairs are birational. In contrast, when $\deg \Lambda = 2g$, the morphism $M \to \mathcal{SU}_C(2)$ has generic

Date: May 14, 2024.

fibers $\mathbb{P}^1$. While M is a rational variety, the rationality of the moduli space $\mathcal{SU}_C(2)$ remains a well-known open problem, dating back to the early works of Tyurin [Tyu64, Tyu65] and Newstead [New75, New80]. Unlike the case of coprime rank and degree—where rationality has been completely settled in [KS99]—rationality for $\mathcal{SU}_C(2)$ is only known in genus 2, where the theta-morphism [Ray82] $\mathcal{SU}_C(2) \to \mathbb{P}^{2^g-1}$ happens to be an isomorphism [NR69]. In light of the "Kuznetsov rationality proposal" [Kuz16], our main result suggests that any weak factorization of a hypothetical birational map $\mathcal{SU}_C(2) \dashrightarrow \mathbb{P}^{3g-3}$ should involve blow-ups and blow-downs of even symmetric powers $\mathrm{Sym}^{2k}C$ for $2k \leq g-1$.

In the even genus case, Theorem 1.1 proves a conjecture of Belmans [Bel21] (though in odd genus it differs from that prediction). There is actually a large body of conjectures on explicit semiorthogonal decompositions of Fano manifolds, which are typically obtained by analyzing their Hodge diamonds and using the additivity of Hochschild homology in semiorthogonal decompositions [Kel99, Theorem 1.5]; see [BBF$^+$24] for another recent example related to our paper. Of course, these predictions are not easily turned into proofs. Existing proofs of such semiorthogonal decompositions follow several approaches. One general framework is to analyze a "two-ray game" given by two extremal contractions of Fano varieties $Y \dashleftarrow X \dashrightarrow Z$. In our paper, $Y = \mathbb{P}^{3g-2}$, $X = M$, and $Z = \mathcal{SU}_C(2)$. We start with a known semiorthogonal decomposition of $D^b(Y)$, extend it to a semiorthogonal decomposition of $D^b(X)$, and use the geometry of X to mutate it into a semiorthogonal decomposition of $D^b(Z)$. Various blocks appear and disappear in the process as X undergoes small birational transformations linking the contractions to Y and Z. Our approach uses weaving patterns, allowing for tight control of the Fourier–Mukai kernels for the various functors $D^b(\mathrm{Sym}^k C) \to D^b(M)$ embedding the blocks. From another perspective, mirror symmetry suggests that such mutations should exist in general, and even leads to a description of the hypothetical braid:

Conjecture 1.2 (Two-Ray Game Conjecture)**.** *Given extremal contractions $Y \dashleftarrow X \dashrightarrow Z$ of smooth Fano varieties, there exist semiorthogonal decompositions*

$$D^b(Y) = \langle \mathcal{A}_1, \dots, \mathcal{A}_s \rangle,\ D^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_s, \mathcal{P}_1, \dots, \mathcal{P}_r \rangle = \langle Q_1, \dots, \mathcal{Q}_u, \mathcal{B}_1, \dots, \mathcal{B}_t \rangle,\ D^b(Z) = \langle \mathcal{B}_1, \dots, B_t \rangle$$

compatible with pullbacks along maps $Y \dashleftarrow X \dashrightarrow Z$. The two semiorthogonal decompositions of $D^b(X)$ are related by a braid. If the contractions can be linked by a sequence of smooth small modifications $X_1, \dots, X_n$ of X in the stable base locus decomposition of the moving cone of X, the braid can be computed as the monodromy braid of eigenvalues of $c_1(X)$ acting on quantum cohomology $\mathbb{Q}H^*(X_i, \mathbb{C})$ as the base τ of small quantum cohomology varies along the path in the moving cone (with a small B-field perturbation iB added to the path τ to avoid collisions of eigenvalues). When the path crosses walls, various eigenvalues fly to or from infinity, corresponding to new blocks being added to or subtracted from the semiorthogonal decomposition.

At the moment, the only way to verify this conjecture is to compute and compare the two braids; see [Iri20, Figure 16] for a worked-out example of extremal contractions $\mathbb{P}^4 \leftarrow \mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^4 \to \mathbb{P}^2$ or [Tev24] for contractions $\mathbb{P}^3 \leftarrow \mathrm{Bl}_C\mathbb{P}^3 = \mathrm{Bl}_{\mathbb{P}^1}Q_4 \to Q_4$, where $C \hookrightarrow \mathbb{P}^3$ is a quintic curve of genus 2 and B_4 is the intersection of two quadrics in $\mathbb{P}^5$ (this is the genus 2 case of [Tev23].) It seems plausible that smoothness assumptions can be weakened by considering noncommutative resolutions of singularities as in our paper: the moduli space $\mathcal{SU}_C(2)$ is singular, but most of the blocks in the semiorthogonal decomposition of $D^b(M)$ "fly away" to leave only a noncommutative resolution of $\mathcal{SU}_C(2)$ (see Figure 3). It would be interesting to gather more evidence supporting this conjecture by connecting other Fano varieties Y and Z through a Fano variety X. One could consider toric Fano varieties, maximal flag varieties, Fano 3-folds, and various Fano moduli spaces.

The paper is organized as follows: In Section 2, we obtain several explicit semiorthogonal decompositions of the "maximal" (see Remark 2.4) Thaddeus space $M_{i_d}(d)$ for various d (see, e.g., Theorem 2.2 for a precise statement), including the case $d = 2g$ we need (Theorem 2.10). This is done by generalizing the weaving techniques developed in [Tev23] for $d = 2g-1$. This in turn requires the careful verification of a number of technical results from [Tev23] for other degrees; these checks are carried out in Section 4. The precise statement and proof of the main result are given in Section 3.

A few words regarding notation: Following [Tha94, Tev23], we often denote tensor product by juxtaposition for compactness. As in [Huy06], we usually omit R's and L's on derived functors except for emphasis (e.g., when applying derived pushfoward to a sheaf). We frequently use the same symbol to denote canonical objects on related moduli spaces when no confusion will arise, omitting explicit pullbacks (see Notation 2.1).

Acknowledgements. We are grateful to Igor Dolgachev for the suggestion to study the derived category of $\mathcal{SU}_C(2)$ in relation to its conjectural rationality, to James Hotchkiss for help with the derived categories of stacks, to Sasha Kuznetsov and Tudor Pădurariu for an insightful correspondence, and to participants of the Spring 2024 seminar on the noncommutative minimal model program at UMass Amherst for an inspirational research environment. This research was supported by NSF grants DMS-2101726 and DMS-2401387.

2. Semiorthogonal decompositions of moduli spaces of stable pairs

We begin by recalling some notation from [Tev23, TT21].

Notation 2.1. For a line bundle Λ on C of degree d and $0 \le i \le \left\lfloor \frac{d-1}{2} \right\rfloor$, we denote by $M_i(\Lambda)$ (or simply $M_i(d)$ or M_i when no confusion will arise) the moduli space of rank 2 stable pairs with determinant Λ, where i indexes the stability condition. We write $\mathcal{O}(m,n) = \mathcal{O}((m+n)H - nE)$ on any of the M_i, $i \ge 1$, where E is the exceptional divisor of the contraction $M_1 \to M_0$ and H is the pullback of the hyperplane divisor from $M_0 \cong \mathbb{P}^{d+g-2}$. (By abuse of notation, $\mathcal{O}(m,0) = \mathcal{O}(m)$ on M_0.) $\mathcal{F}$ denotes the universal vector bundle on $M_i \times C$ (for any i), $\mathcal{F}_x$ its restriction to $M_i \times \{x\} \cong M_i$ for $x \in C$, and $\boldsymbol{\Lambda}$ the line bundle $\wedge^2 \mathcal{F}_x$, which is independent of x (and not to be confused with Λ).

For any variety X and vector bundle $\mathcal{G}$ on $X \times C$, we have tensor vector bundles $\mathcal{G}^{\boxtimes k}$ and $\overline{\mathcal{G}}^{\boxtimes k}$ on $X \times \mathrm{Sym}^k C$ defined by the S_k-equivariant pushforwards $\tau_*^{S_k}(\pi_1^*\mathcal{G} \otimes \cdots \otimes \pi_k^*\mathcal{G})$ and $\tau_*^{S_k}(\pi_1^*\mathcal{G} \otimes \cdots \otimes \pi_k^*\mathcal{G} \otimes \mathrm{sgn})$, respectively, where $\tau : C^k \to \mathrm{Sym}^k C$ is the quotient by S_k, $\pi_i : C^k \to C$ are the projections, sgn is the sign character of S_k, and the S_k-action on $\pi_1^*\mathcal{G} \otimes \cdots \otimes \pi_k^*\mathcal{G}$ permutes the tensor factors (see [TT21, Section 2]). For $D \in \mathrm{Sym}^k C$, we denote by $\mathcal{G}_D^{\boxtimes k}$ and $\overline{\mathcal{G}}_D^{\boxtimes k}$ the restrictions to $X \times \{D\} \cong X$ of $\mathcal{G}^{\boxtimes k}$ and $\overline{\mathcal{G}}_D^{\boxtimes k}$, respectively. $\langle K \rangle$ denotes the essential image of a fully faithful Fourier–Mukai functor with kernel K.

The principal goal of this section is to prove the following generalization of [Tev23, Theorem 3.1]:

Theorem 2.2. *Let $d \le 2g$ with $i_d \le \left\lfloor \frac{d-1}{2} \right\rfloor$, where $i_d = \left\lceil \frac{d+g-1}{3} \right\rceil - 1$. Let $m = d + g - 1 - 3i_d \in \{1,2,3\}$, and let $m_n = 1$ if $m \le n$ or 0 otherwise. Then*

$$D^b(M_{i_d}(d)) = \left\langle \left\langle \boldsymbol{\Lambda}^{-k} \mathcal{F}^{\vee \boxtimes j} \right\rangle_{\substack{j+k \le i_d - m_2 \\ j,k \ge 0}}, \left\langle T_1 \boldsymbol{\Lambda}^{-k} \mathcal{F}^{\vee \boxtimes j} \right\rangle_{\substack{j+k \le i_d - m_1 \\ j,k \ge 0}}, \left\langle T_2 \boldsymbol{\Lambda}^{-k} \mathcal{F}^{\vee \boxtimes j} \right\rangle_{\substack{j+k \le i_d \\ j,k \ge 0}} \right\rangle \tag{2.1}$$

where the blocks within each "megablock" are ordered first by decreasing k, then by decreasing j. Here, $T_1 = \mathcal{O}(1, i_d - m_2)$, and $T_2 = \mathcal{O}(2, 2i_d - m_2 - m_1)$.

Remark 2.3. The assumptions of the theorem are equivalent to $d = 2g - \alpha$ for $\alpha \in \{0,1,2,3,5\}$, where $i_d = g - \left\lceil \frac{\alpha+2}{3} \right\rceil$. The restriction $d \le 2g$ could be removed by verifying Conjecture 4.6 below, in which case the theorem would hold in all but finitely many degrees.

Remark 2.4. It can be seen from [TT21, Proposition 3.18] that $D^b(M_i(d))$ is "largest" when $i = i_d$. Moreover, it follows from [Tha94, 5.3, 6.1] that M_{i_d} is Fano when $m = 1, 2$. When $m = 3$, $D^b(M_{i_d})$ and $D^b(M_{i_d+1})$ are equivalent, and the anticanonical bundles on M_{i_d} and M_{i_d+1} are big and nef but not ample.

2.1. Generalized weaving.

We begin with some notation. Fix $d \le 2g$ and write $M_i(d) = M_i$ for $i \le \left\lfloor \frac{d-1}{2} \right\rfloor$.

Notation 2.5. For $0 \le k \le i$, we denote by $\mathcal{D}_i^k$ the structure sheaf of the reduced subscheme

$$D_i^k = \{(D, F, s) \in \mathrm{Sym}^k C \times M_i : s|_D = 0\},$$

whose fiber over $D \in \mathrm{Sym}^k C$ is $M_{i-k}(\Lambda(-2D))$ [TT21, Remark 3.7]. For $t \in [0, i_d+1)$, let $\mathcal{D}_t^{k,s} = \mathcal{D}_{\lfloor t \rfloor}^k \otimes L_t^{k,s}$ where

$$L_t^{k,s} = \begin{cases} \mathcal{O}(s, sk) & k = \lfloor t \rfloor \\ \mathcal{O}\left(\left\lfloor \frac{s}{t-k} \right\rfloor, s + \left\lfloor \frac{s}{t-k} \right\rfloor (k-1) \right) & k < \lfloor t \rfloor. \end{cases}$$

Our first step is to prove the following:

Lemma 2.6 (cf. [Tev23, Corollary 2.10])**.** *For $t \in (0, i_d+1) \setminus \mathbb{Z}$, we have a semiorthogonal decomposition*

$$D^b(M_{\lfloor t \rfloor}) = \left\langle \mathcal{D}_t^{k,s} \right\rangle_{\substack{0 \le k \le \lfloor t \rfloor \\ 0 \le s \le d+g-3k-2}} \tag{2.2}$$

where the blocks are ordered first by increasing $x_{k,s}(t) = \frac{s}{t-k}$, then by increasing k.

We interpret t as time, with the block $\langle \mathcal{D}_t^{k,s} \rangle$ "moving" in the x-t plane with trajectory $x = x_{k,s}(t)$ (or $x = k\epsilon$ for $s = 0$, where $\epsilon \ll 1$). When the blocks cross paths, they change order and undergo mutations dictated by the line bundle $L_t^{k,s}$. When t crosses an integer level i, we embed $D^b(M_{i-1})$ into $D^b(M_i)$, introducing several new blocks as its orthogonal complement, and proceed with the process. We refer to this "weave" as the Farey Twill; see [Tev23, Section 2] for detailed illustrations in the case $d = 2g-1$. This program is facilitated by several technical lemmas, whose statements and proofs are deferred to Section 4.

To pass from M_{i-1} to M_i, we need the following "windows" embeddings, in the sense of [HL15]:

Proposition 2.7 ([TT21, Proposition 3.18]). *For $d > 0$ and $1 \le i \le i_d \le \left\lfloor \frac{d-1}{2} \right\rfloor$, there is an admissible embedding $\iota : D^b(M_{i-1}(d)) \hookrightarrow D^b(M_i(d))$ giving rise to a semiorthogonal decomposition*

$$D^b(M_i(d)) = \langle \iota(D^b(M_{i-1}(d))), \mathcal{D}_i^{i,0}, \mathcal{D}_i^{i,1}, \ldots, \mathcal{D}_i^{i,d+g-3i-2} \rangle$$

When $i > 1$, the embedding corresponds to the inclusion of objects with weights $[0, i) \subseteq [0, d+g-1-2i)$ with respect to the wall crossing $M_{i-1} \dashrightarrow M_i$.

Remark 2.8. Note that D_i^i is isomorphic to the projective bundle $\mathbb{P}W_i^+ \subset M_i$ via the second projection (see [TT21, Section 6]), and that $L_i^{i,s} = \mathcal{O}(s, si)$ restricts to $\mathcal{O}(s)$ on the fibers $M_0(d-2i) \cong \mathbb{P}^{d+g-2i-2}$ of this bundle [TT21, Remark 3.7].

Proof of Lemma 2.6. When $0 < t < 1$, (2.2) is the Beĭlinson collection $\langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(d+g-2) \rangle$ on $\mathbb{P}^{d+g-2}$. Given (2.2) for $t = i + \epsilon$ with $i \in \mathbb{Z}$, $\epsilon \ll 1$, we achieve (2.2) for all $t \in (i, i+1)$ by performing the mutations encoded in the crossings of trajectories $x_{k,s}(t)$. When blocks meet at nonintegral x, they only change order, meaning we must show that they are mutually orthogonal. Since the intersecting blocks are already ordered by $x_{s,k}(t-\epsilon)$, or equivalently by k, we need only check that $\langle \mathcal{D}_t^{k,s} \rangle \subset {}^{\perp}\langle \mathcal{D}_t^{k',s'} \rangle$ for $k < k'$, $x_{k,s}(t) = x_{k',s'}(t)$. This follows from Lemma 4.16 below.

Crossings at $x \in \mathbb{Z}$ have the form $\langle \mathcal{D}_t^{k,s}, \mathcal{D}_t^{k+1,s-x}, \ldots, \mathcal{D}_t^{\lfloor t \rfloor, s - \lfloor t \rfloor x} \rangle \to \langle \mathcal{D}_{t+\epsilon}^{\lfloor t \rfloor, s-\lfloor t \rfloor x}, \ldots, \mathcal{D}_{t+\epsilon}^{k+1,s-x}, \mathcal{D}_{t+\epsilon}^{k,s} \rangle$. Note that for $0 \le j < \lfloor t \rfloor$,

$$L_t^{k+j,s-jx} = \mathcal{O}(x, s - x(k-1))$$

is indepedent of j, while

$$L_{t+\epsilon}^{k+j,s-jx} = L_t^{k+j,s-jx}(-1, 1-j).$$

Moreover, $L_t^{\lfloor t \rfloor, s-\lfloor t \rfloor x} = L_{t+\epsilon}^{\lfloor t \rfloor, s - \lfloor t \rfloor x} = \mathcal{O}(s - \lfloor t \rfloor x, (s - \lfloor t \rfloor x) \lfloor t \rfloor)$ and $\mathcal{O}(x, s - x(k-1))$ both restrict to $\mathcal{O}(s)$ on the fibers of the projective bundle $D_{\lfloor t \rfloor}^{\lfloor t \rfloor}$, so $\langle \mathcal{D}_t^{\lfloor t \rfloor, s-\lfloor t \rfloor x} \rangle = \langle \mathcal{D}_{\lfloor t \rfloor}^{\lfloor t \rfloor}(x, s - x(k-1)) \rangle$. Hence it suffices to give a mutation

$$\langle \mathcal{D}_i^k, \ldots, \mathcal{D}_i^{i-1}, \mathcal{D}_i^i \rangle \to \langle \mathcal{D}_i^i, \mathcal{D}_i^{i-1}(-1, 2-i), \ldots, \mathcal{D}_i^k(-1, 1-k) \rangle,$$

as in Lemma 4.17.

It remains to explain how to get from $t = i - \epsilon$ to $i + \epsilon$. By Lemma 4.15 below, we have $\iota \langle \mathcal{D}_{i-\epsilon}^{k,s} \rangle = \langle \mathcal{D}_i^{k,s} \rangle$. Hence, to go from the semiorthogonal decomposition of Proposition 2.7 to (2.2) with $t = i + \epsilon$, we need only move the block $\langle \mathcal{D}_i^{i,0} \rangle$ into position, i-th from the left (we imagine this block coming horizontally from the right along $t = i$, stopping at $x = i\epsilon$). It moves past blocks with $x_{k,s}(i) \notin \mathbb{Z}$ without changing them by Lemma 4.16, while the others undergo the mutation of Lemma 4.17. ☐

It turns out that Lemma 2.6 is not quite the decomposition we need to proceed.

Lemma 2.9. *Let $m = d + g - 3i_d - 1 \in \{1, 2, 3\}$, and let $m_n = 1$ if $m \le n$ and 0 otherwise. Then*

$$D^b(M_{i_d}) = \left\langle \langle \mathbf{\Lambda}^{-j} \mathcal{D}^k \rangle_{\substack{j+k \le i_d - m_2 \\ j,k \ge 0}}, \langle T_1 \mathbf{\Lambda}^{-j} \mathcal{D}^k \rangle_{\substack{j+k \le i_d - m_1 \\ j,k \ge 0}}, \langle T_2 \mathbf{\Lambda}^{-j} \mathcal{D}^k \rangle_{\substack{j+k \le i_d \\ j,k \ge 0}} \right\rangle \tag{2.3}$$

where the blocks in each megablock are ordered first by increasing $j+k$, then by increasing j. Here, $\mathcal{D}^k = \mathcal{D}_{i_d}^k$, $T_1 = \mathcal{O}(1, i_d - m_2)$, and $T_2 = \mathcal{O}(2, 2i_d - m_2 - m_1)$.

Proof. For $m = 1, 2$, we begin with (2.2) with $t = i_d - \epsilon$. We embed with ι to obtain the following semiorthogonal decomposition:

$$D^b(M_{i_d}) = \langle \mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}, \mathcal{D}_{i_d}^{i_d,0}, \ldots, \mathcal{D}_{i_d}^{i_d,m-1} \rangle \tag{2.4}$$

where $\mathbf{I}, \mathbf{II}, \mathbf{III}, \mathbf{IV}$ are the subcategories generated by $\langle \mathcal{D}_{i_d}^{k,s} \rangle$ for $x_{k,s}(i_d) \in [0,1)$, $[1,2)$, $[2,3)$, and $[3, \infty)$, respectively.

If $m = 1$, the blocks are arranged as in [Tev23]. The blocks in $\mathbf{I}$ are the same as in the first megablock of (2.3) (with $j = s$), but they are ordered differently. We reorder them by moving $\langle \mathcal{D}^{k,s}_{i_d} \rangle$ from $x = x_{k,s}(i_d)$ to $x = \frac{s+k}{i_d}$. The moves are done in order of decreasing $s + k$, then by decreasing s. As blocks in $\mathbf{I}$ with the same $s + k$ are already ordered by increasing s, the orthogonality we need to ensure no mutations occur is $\langle \mathcal{D}^{k,s}_{i_d} \rangle \subset {}^{\perp}\langle \mathcal{D}^{k',s'}_{i_d} \rangle$ for $s' + k' < s + k$. This is checked in Lemma 4.18.

Similarly, the blocks in $\mathbf{II}$ and $\mathbf{III}$ are respectively the same as the second and third (with $j + k \leq i_d - 1$) megablocks of (2.3). (Explicitly, we have $j = s - \left\lfloor \frac{s}{i_d - k} \right\rfloor (i_d - k)$.) The same reordering procedure works, so we are left to produce the blocks in (2.3) in with $j + k = i_d$. These are exactly the blocks in $\langle \mathbf{IV}, \mathcal{D}^{i_d,0}_{i_d} \rangle = \langle \mathcal{D}^{0,3i_d}_{i_d}, \mathcal{D}^{1,3(i_d-1)}_{i_d}, \dots, \mathcal{D}^{i_d,0}_{i_d} \rangle$ after the mutation of Lemma 4.17 (note that we have $\mathcal{D}^{i_d} = \mathcal{D}^{i_d}(2, 2(i_d - 1))$ by [TT21, Remark 3.7]). This proves the lemma for $m = 1$.

For $m = 2$, the only new blocks in (2.4) compared to $m = 1$ lie in $\mathbf{IV}$. We reorder the blocks in $\mathbf{I}$ and $\mathbf{II} = \mathbf{II}_a$ just as before; this gives the first and second (with $j + k \leq i_d - 1$) megablocks in (2.3). We write $\mathbf{III} = \langle \mathbf{ii}_b, \mathbf{III}_a \rangle$, where $\mathbf{ii}_b$ contains those blocks $\langle \mathcal{D}^{k,s}_{i_d} \rangle$ with $x_{k,s}(i_d) = 2$ and $\mathbf{III}_a$ those with $x_{k,s}(i_d) \in (2, 3)$. Similar to the proof of Lemma 2.6, we move the block $\mathcal{D}^{i_d,0}_{i_d}$ past $\mathbf{IV}$, $\mathbf{III}_a$, and $\mathbf{ii}_b$. By Lemmas 4.16 and 4.17, $\mathbf{III}_a$ is unchanged, while $\mathbf{IV}$ and $\mathbf{ii}_b$ undergo some mutations. This yields

$$D^b(M_{i_d}) = \langle \mathbf{I}, \mathbf{II}_a, \mathbf{II}_b, \mathbf{III}_a, \mathbf{IV}', \mathcal{D}^{i_d,1}_{i_d} \rangle$$

where $\langle \mathbf{ii}_b, \mathcal{D}^{i_d,0}_{i_d} \rangle \to \mathbf{II}_b$ via Lemma 4.17, and $\mathbf{IV}' = \langle \mathcal{D}^{s,k}_{i_d+\epsilon} \rangle_{x_{k,s}(i_d) \geq 3}$ ordered by $x_{k,s}(i_d + \epsilon)$. Now $\mathbf{II}_b = \langle \mathcal{D}^k(1, 2i_d - k - 1) \rangle_{0 \leq k \leq i_d}$ ordered by decreasing k, so $\langle \mathbf{II}_a, \mathbf{II}_b \rangle$ forms the second megablock of (2.3).

At this point, $\mathbf{III}_a$ contains the blocks from the third megablock of (2.3) with $j + k \leq i_d - 2$; we apply the same algorithm to put them in the correct order. We write $\mathbf{IV}' = \langle \mathbf{III}_b, \mathbf{iii}_c \rangle$ with $\mathbf{III}_b$ and $\mathbf{iii}_c$ containing the blocks with $x_{k,s}(i_d) = 3$ and $x_{k,s}(i_d) > 3$. We have $\mathbf{III}_b = \langle \mathcal{D}^k(2, 3i_d - k - 2) \rangle_{0 \leq k \leq i_d - 1}$ ordered by decreasing k. The Farey Twill trajectories of blocks in $\mathbf{iii}_c$ meet with $\mathcal{D}^{i_d,1}_{i_d}$ at $(t, x) = \left(i_d + \frac{1}{3}, 3\right)$, where they undergo a final mutation $\langle \mathbf{iii}_c, \mathcal{D}^{i_d,1}_{i_d} \rangle \to \mathbf{III}_c = \langle \mathcal{D}^k(2, 3i_d - k - 1) \rangle_{0 \leq k \leq i_d}$ ordered by decreasing k. To sum up, we have

$$D^b(M_{i_d}) = \langle \mathbf{I}, \langle \mathbf{II}_a, \mathbf{II}_b \rangle, \langle \mathbf{III}_a, \mathbf{III}_b, \mathbf{III}_c \rangle \rangle\,,$$

which is exactly (2.3) with $m = 2$.

Finally, $m = 3$ is the easiest case. We begin with (2.2) with $t = i_d + 1 - \epsilon$. We have $x_{k,s}(t) \in [0, 3)$, where the blocks in $[0, 1)$, $[1, 2)$, and $[2, 3)$ correspond exactly to the respective megablocks in (2.3) with $j = s$, $j = s - (i_d + 1 - k)$, and $j = s - 2(i_d + 1 - k)$; they are in the wrong order, but this is rectified by Lemma 4.18 and the same reordering algorithm as above. □

From here, Theorem 2.2 follows exactly as in the proof of [Tev23, Theorem 3.1].

Proof of Theorem 2.2. Each megablock in (2.3) will mutate into the corresponding one in (2.1). As the megablocks differ only in size and overall line bundle twists (i.e., the shapes are the same), it will suffice to describe this mutation for the first one. We rely on the Cross Warp mutation depicted in Figure 1 and proved as Theorem 4.1(d) below. Notice that the top left portion of the mutation with top center block $\mathcal{D}^k$ is precisely the bottom right portion of the mutation with top center $\mathcal{D}^{k-1}$; similarly, the bottom left is the top right of the mutation with top center $\mathcal{D}^{k-1}$, tensored with $\mathbf{\Lambda}^{-1}$. Hence we can stack these mutations with top centers as in Figure 2. In the end, all $\mathcal{D}$'s are replaced by $\mathcal{F}$'s, resulting in (2.1). □

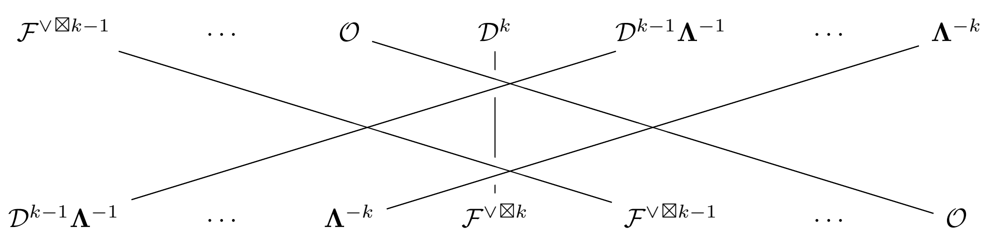


Figure 1. The basic Cross Warp mutation, cf. [Tev23, Figure 7].

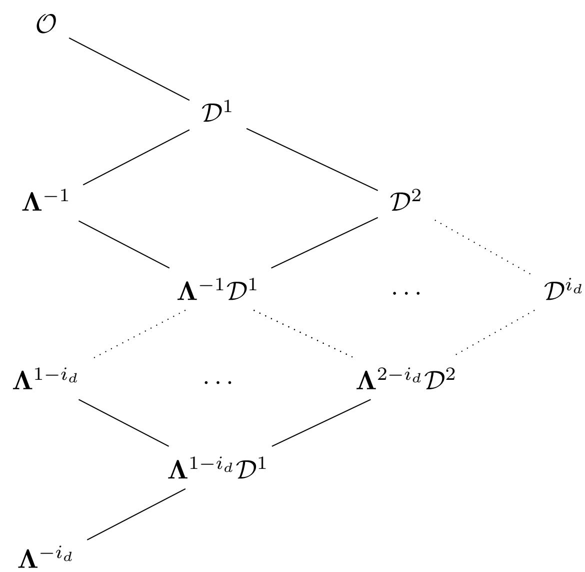


FIGURE 2. Stacking the crosswarp mutation (with $m_1 = 0$). See also [Tev23, Figure 8].

2.2. **Broken Loom for $d = 2g$.** To finish this section, we specialize to $d = 2g$. We wish to modify the semiorthogonal decomposition of Theorem 2.2 for $d = 2g$ to create as many blocks as possible of the form $\theta^j\mathbf{\Lambda}^k\mathcal{F}^{\vee\boxtimes 2k}$, where $\theta = \mathcal{O}(1, g-1)$ is the pullback of the ample generator of $\operatorname{Pic}\mathcal{SU}_C(2)$ under the forgetful morphism $M_{g-1}(2g) \to SU_C(2)$, $(F, s) \mapsto s$ (see [Tha94, 5.8]). We will see in Section 3 that such blocks form the claimed noncommutative resolution of $SU_C(2)$.

Theorem 2.10 (cf. [Tev23, Theorem 5.8])**.** *Let $M = M_{g-1}(2g)$. We have the following semiorthogonal decomposition of $D^b(M)$:*

$$\Big\langle \left\langle \theta^{-1}\mathbf{\Lambda}^{\lfloor \frac{g-2}{2}\rfloor -k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{0\le\lambda\le g-2\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor}}, \left\langle \mathbf{\Lambda}^{\lfloor\frac{g-2}{2}\rfloor-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{0\le\lambda\le 2(g-2)\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-2}},$$
$$\left\langle\theta\mathbf{\Lambda}^{\lfloor\frac{g}{2}\rfloor-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{0\le\lambda\le2(g-1)\\0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-1}}, \left\langle\theta^2\mathbf{\Lambda}^{\lfloor\frac{g}{2}\rfloor-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{g-1\le\lambda\le2(g-1)\\ \lambda-g+1\le k\le\lfloor\frac{\lambda}{2}\rfloor}}\Big\rangle.$$

Here, the blocks within each megablock are ordered first by decreasing λ, then by decreasing k.

We proceed by analogy with [Tev23, Section 5], beginning with the reordering trick. This works for any $d \le 2g$ with $i_d \le \lfloor\frac{d-1}{2}\rfloor$.

Lemma 2.11 (cf. [Tev23, Theorem 5.3])**.** *With notation as in Theorem 2.2, we have the following semiorthogonal decomposition of $D^b(M_{i_d}(d))$:*

$$\Big\langle \left\langle\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le i_d-m_2\\ \lambda-2k,k\ge0}}, \left\langle T_1\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le i_d-m_1\\ \lambda-2k,k\ge0}}, \left\langle T_2\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le i_d\\ \lambda-2k,k\ge0}}\Big\rangle. \tag{2.5}$$

The blocks within each megablock are ordered first by decreasing λ, then by decreasing k.

Proof. The blocks in each megablock are the same as those in (2.1) with $\lambda = j + 2k$. As they are already ordered by decreasing k, it suffices to show that we can move blocks with smaller λ to the right of blocks with larger λ, i.e., $\langle\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\rangle \subset {}^\perp\langle\mathbf{\Lambda}^{-k'}\mathcal{F}^{\vee\boxtimes\lambda'-2k'}\rangle$ for $\lambda < \lambda'$. This follows from Lemma 4.19 below. □

Proof of Theorem 2.10. When $d = 2g$, we have $i_d = g-1$, $m = 2$, $T_1 = \theta\mathbf{\Lambda}$, and $T_2 = \theta^2\mathbf{\Lambda}$, so (2.5) becomes

$$\Big\langle \left\langle\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-2\\ \lambda-2k,k\ge0}}, \left\langle(\theta\mathbf{\Lambda})\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-1\\ \lambda-2k,k\ge0}}, \left\langle(\theta^2\mathbf{\Lambda})\mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-1\\ \lambda-2k,k\ge0}}\Big\rangle.$$

We take the part of the third megablock with $\lambda \leq g-2$ and tensor by $\omega_{M_{g-1}(2g)} = \theta^{-3}\mathbf{\Lambda}^{-1}$, moving it to the far left. Tensoring everything by $\mathbf{\Lambda}^{\lfloor \frac{g-2}{2} \rfloor}$ proves the theorem. $\square$

3. Modified Plain Weave

3.1. **Main result.** We proceed with the Plain Weave, cf. [Tev23, Section 6]. While the spirit of the argument is the same, there are some technical complications in even degree. Notably, not every pair (F, s) with F a semistable bundle is stable, so M parameterizes only an open substack of all such pairs.

Notation 3.1. We denote by $\mathcal{N}$ the stack of rank 2 semistable bundles on C with determinant Λ (and with $\mathbb{G}_m$ as a generic inertia group) and by $\mathbb{N}$ its rigidification (with trivial generic stabilizers). Concretely, we work with the quotient stacks $\mathcal{N} = [Q/\mathrm{GL}(\mathbb{V})]$ and $\mathbb{N} = [Q/\mathrm{PGL}(\mathbb{V})]$ where $\mathbb{V}$ is a vector space of dimension $2+2m$ for some large m and Q is an appropriate locally closed subscheme of the Quot scheme parameterizing quotients of $\mathbb{V} \otimes \mathcal{O}_C(-mp)$ for some fixed point $p \in C$ (see [KT21, Section 4] for details). Here $\mathcal{N}$ and $\mathbb{N}$ are smooth algebraic stacks and Q is a smooth quasi-projective variety. The generic inertia group of $\mathcal{N}$ is identified with the center $\mathbb{G}_m \subset \mathrm{GL}(\mathbb{V})$. We have morphisms of stacks

$$M \longrightarrow \mathcal{N} \xrightarrow{\ \rho\ } \mathbb{N} \longrightarrow \mathcal{SU}_C(2),$$

where $M = M_{g-1}(\Lambda)$ is the moduli space of stable pairs, $M \to \mathcal{N}$ is the forgetful morphism, and $\mathcal{SU}_C(2)$ is the coarse moduli space (of both $\mathcal{N}$ and $\mathbb{N}$) as well as the GIT quotient of Q by $\mathrm{PGL}(\mathbb{V})$. We do not notationally distinguish between universal bundles $\mathcal{F}$ on $M \times C$ or $\mathcal{N} \times C$, nor those on other spaces that carry them appearing below; similarly for $\mathbf{\Lambda} = \det \mathcal{F}_x$ and θ, the (pullback of the) ample generator of $\operatorname{Pic} \mathcal{SU}_C(2)$ [DN89]. Unlike in the odd degree case, neither $\mathcal{F}$ nor any line bundle twist of $\mathcal{F}$ descends to $\mathbb{N}$ or any open substack of it [Ram73, Theorem 2]. However, twisted tensor vector bundles $\mathbf{\Lambda}^k \mathcal{F}^{\vee \boxtimes 2k}$ on $\mathcal{N} \times \mathrm{Sym}^{2k} C$ have weight 0 with respect to $\mathbb{G}_m$ and therefore descend to $\mathbb{N} \times \mathrm{Sym}^{2k} C$ for every $k \geq 0$.

Let $\pi_{\mathcal{N}} : \mathcal{N} \times C \to \mathcal{N}$ be the projection. We write $R\pi_{\mathcal{N}*}\mathcal{F} = [\mathcal{A} \xrightarrow{u} \mathcal{B}]$ for $\mathcal{A}$ and $\mathcal{B}$ vector bundles on $\mathcal{N}$ of ranks a and b, respectively, where $a - b = 2$ and $\mathbb{G}_m$ acts with weight 1 on the fibers of both [KT21, Lemma 4.4]. Let $\alpha : \mathcal{A} \to \mathcal{N}$, where we use the same notation for vector bundles and their total spaces. Then u gives a section of the vector bundle $\alpha^*\mathcal{B}$ over $\mathcal{A}$. Let $\mathcal{Z} \subset \mathcal{A}$ be the vanishing locus of this section, let $\mathcal{A}^\circ \subset \mathcal{A}$ be the complement of the zero section, and let $\mathcal{Z}^\circ = \mathcal{Z} \cap \mathcal{A}^\circ$, which is the stack of pairs $\{(F, s) : F \in \mathcal{N}, s \in H^0(F) \smallsetminus \{0\}\}$ (see [KT21, Lemma 4.5(i)]). We have a diagram

$$\begin{tikzcd}
 & \mathcal{Z} \arrow[r, hook] & \mathcal{A} \arrow[r, "\alpha"] & \mathcal{N} \arrow[d, "\rho"] \\
M \arrow[r, hook, "j"] & \mathcal{Z}^\circ \arrow[u] \arrow[r, hook] \arrow[rr, bend right, "\zeta"'] & \mathcal{A}^\circ \arrow[u] & \mathbb{N}.
\end{tikzcd} \tag{3.1}$$

Proposition 3.2. *In the notation of* (3.1)*:*

(a) *In $D^b(\mathcal{A}^\circ)$, $\mathcal{O}_{\mathcal{Z}^\circ}$ is isomorphic to the Koszul complex*

$$\left[\bigwedge^b \alpha^*\mathcal{B}^\vee|_{\mathcal{A}^\circ} \to \ldots \to \alpha^*\mathcal{B}^\vee|_{\mathcal{A}^\circ} \to \mathcal{O}_{\mathcal{A}^\circ} \right].$$

(b) *We have $R\zeta_*\mathcal{O}_{\mathcal{Z}^\circ} = \mathcal{O}_{\mathbb{N}}$.*

(c) *The relative dualizing sheaf for ζ is $\theta\mathbf{\Lambda}^{-1}$.*

Proof. We abuse notation and denote the morphism $\mathcal{A}^\circ \to \mathbb{N}$ by ζ. It is a (twisted) projective bundle with fiber $\mathbb{P}^{a-1}$. By [KT21, Lemma 4.5(i)], the stack $\mathcal{Z}^\circ$ is smooth of dimension $3g-2$. In particular, its codimension in $\mathcal{A}^\circ$ is equal to b. The first claim follows. Since $R\zeta_*\mathcal{O}_{\mathcal{A}^\circ} = \mathcal{O}_{\mathbb{N}}$ and $R\Gamma(\mathbb{P}^{a-1}, \mathcal{O}(-k)) = 0$ for $k = 1, \ldots, a-1$, the second claim follows from the first. Indeed, since the claim is local on $\mathbb{N}$, we can trivialize $\mathcal{B}$, and subsequently replace $\alpha^*\mathcal{B}^\vee|_{\mathcal{A}^\circ}$ with a direct sum of b copies of $\mathcal{O}_{\mathcal{A}^\circ}(-1)$. Finally, since $\mathcal{A}^\circ$ has $\mathbb{G}_m$-weight 1, the dualizing sheaf for $\mathcal{A}^\circ \to \mathbb{N}$ is isomorphic to $\alpha^* \det \mathcal{A}^\vee$. Therefore, by adjunction and ignoring pullback by α,

$$\omega_\zeta \cong \det \mathcal{A}^\vee \otimes \det \mathcal{B} \cong (\det \pi_{\mathcal{N}!}\mathcal{F})^\vee,$$

which is isomorphic to $\theta\mathbf{\Lambda}^{-1}$ by [Nar17, Proposition 2.1]. $\square$

In the diagram (3.1), j is the inclusion of the GIT-semistable locus with respect to a certain line bundle $L_\ell \otimes \chi_0^\epsilon$ on $\mathcal{Z}^\circ$ [KT21, Section 4]. This gives rise to windows embeddings $D^b(M) \cong \mathbf{G}_w \subset D^b(\mathcal{Z}^\circ)$ [HL15]. In the following proposition, we calculate the relevant weights (cf. [TT21, Lemma 3.17 and Theorem 3.21]).

Proposition 3.3. *With respect to the semistable locus $M \hookrightarrow \mathcal{Z}^\circ$:*

(a) *There is a unique Kempf–Ness stratum with associated window width $\eta = g$.*
(b) *Objects in the subcategory $\langle \theta^x \mathbf{\Lambda}^y \mathcal{F}^{\vee \boxtimes z}\rangle \subset D^b(\mathcal{Z}^\circ)$ have weights in the range $[-y, z-y]$.*

Proof. As in [TT21, Section 3], we identify $\mathcal{A}^\circ$ with the quotient stack $[X/\mathrm{PGL}(\mathbb{V})]$, where $X \subset Q \times \mathbb{P}(\mathbb{V})$ is a closed subvariety parameterizing quotients $\phi : \mathbb{V} \otimes \mathcal{O}_C(-mp) \to F$ together with a global section $s(mp) \in \mathbb{P}(\mathbb{V})$. The complement $\mathcal{U} = \mathcal{A}^\circ \smallsetminus M$ is a closed substack of pairs (F, s) such that F contains a degree g line subbundle L and $s \in H^0(C, L) \smallsetminus \{0\}$. By semistability of F, $L = \mathcal{O}_C(D)$ is unique, where $D \in \mathrm{Sym}^g C$ is the vanishing locus of s. We have a short exact sequence $0 \to \mathcal{O}_C(D) \to F \to \Lambda(-D) \to 0$. The unstable locus $S \subset X$ is the preimage of $\mathcal{U}$. For points in S, the quotient ϕ is given by a block upper-triangular matrix corresponding to a splitting $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$, where $\mathbb{V}_1 \otimes \mathcal{O}_C(-mp) = \phi^{-1}(\mathcal{O}_C(D))$ and $\dim \mathbb{V}_1 = \dim \mathbb{V}_2 = r$. Furthermore, we have $s(mp) \in \mathbb{P}(\mathbb{V}_1)$. The destabilizing one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{PGL}(\mathbb{V})$ is given by sending $\lambda(t) = \mathrm{diag}(u, \ldots, u, u^{-1}, \ldots, u^{-1})$, where $u^r = t$ (see the proof of [TT21, Lemma 3.17]). It follows that there is only one Kempf–Ness stratum (λ, Z, S), where $Z = X^\lambda$ is the locus of split quotients $\mathbb{V}_1 \otimes \mathcal{O}_C(mp) \oplus \mathbb{V}_2 \otimes \mathcal{O}_C(mp) \to \mathcal{O}_C(D) \oplus \Lambda(-D)$ with $s(mp) \in \mathbb{P}(\mathbb{V}_1)$.

Arguing as in the proof of [TT21, Lemma 3.17], the window width $\eta = \mathrm{weight}\, \mathcal{N}^*_{S/X}|_Z$ is equal to the codimension of S in X. Since $\dim \mathcal{A}^\circ = 3g-2$, to finish the proof of (a), it suffices to show that $\dim \mathcal{U} = 2g-2$. This follows from the description of $\mathcal{U}$ above. Indeed, $\mathrm{Sym}^g C$ has dimension g and the space of extensions $\mathrm{Ext}^1(\Lambda(-D), \mathcal{O}_C(D))$ has dimension $g-1$ when $2D \not\sim \Lambda$ (and g otherwise). Furthermore, points in $\mathcal{U}$ have generic stabilizers $\mathbb{G}_m$ acting trivially on $\mathbb{V}_1$ and non-trivially on $\mathbb{V}_2$. This gives $\dim \mathcal{U} = g + (g-1) + (-1) = 2g-2$ and proves (a). The proof of part (b) is entirely analogous to the proof of [TT21, Theorem 3.21] $\square$

Corollary 3.4. *The blocks in Theorem 2.10, with the kernels now regarded as objects on $D^b(\mathcal{Z}^\circ \times \mathrm{Sym}^\ell C)$, give a semiorthogonal decomposition of the windows subcategory $\mathbf{G} = \mathbf{G}_{-\lfloor g/2 \rfloor} \subset D^b(\mathcal{Z}^\circ)$.*

Proof. Using Proposition 3.3(b), one checks that objects in those blocks have weights in $[-\lfloor \frac{g}{2} \rfloor, g - \lfloor \frac{g}{2} \rfloor)$, so they are contained in $\mathbf{G}$. For example, the (λ, k) block in the first megablock has weights in the range

$$\left[k - \left\lfloor \frac{g-2}{2} \right\rfloor, \lambda - k - \left\lfloor \frac{g-2}{2} \right\rfloor\right] \subseteq \left[-\left\lfloor \frac{g-2}{2} \right\rfloor, g-2-\left\lfloor \frac{g-2}{2} \right\rfloor\right] = \left[-\left\lfloor \frac{g-2}{2} \right\rfloor, g - \left\lfloor \frac{g}{2} \right\rfloor\right).$$

The windows embedding $D^b(M) \cong \mathbf{G} \subset D^b(\mathcal{Z}^\circ)$ is right inverse to the restriction $j^* : D^b(\mathcal{Z}^\circ) \to D^b(M)$, so j^* gives an equivalence $\mathbf{G} \cong D^b(M)$ taking each block in $\mathbf{G} \subset D^b(\mathcal{Z}^\circ)$ to the corresponding block in $D^b(M)$. The result follows. $\square$

Notation 3.5. We introduce full subcategories $\mathbf{K} = \{X \in D^b(\mathcal{Z}^\circ) : \zeta_* X = 0\}$ and $\mathbf{K}^\vee = \{X^\vee : X \in \mathbf{K}\}$.

Remark 3.6. Note that $\theta \otimes \mathbf{K} = \mathbf{K}$ by the projection formula and $\mathbf{K}^\vee = \mathbf{\Lambda} \otimes \mathbf{K}$ by coherent duality and Proposition 3.2(c).

Theorem 3.7 (Plain Weave)**.** *There is a semiorthogonal decomposition*

$$\mathbf{G} = \langle \mathbf{L}, \mathbf{i}, \mathbf{ii}, \mathbf{iii}, \mathbf{iv}, \mathbf{R} \rangle \tag{3.2}$$

where $\mathbf{L} \subset \mathbf{K}$, $\mathbf{R} \subset \mathbf{K}^\vee$, and

$$\mathbf{i} = \left\langle \theta^{-1} \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \right\rangle_{0 \le m \le \lfloor \frac{g-2}{2} \rfloor}, \quad \mathbf{ii} = \left\langle \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \right\rangle_{0 \le m \le \lfloor \frac{g-2}{2} \rfloor},$$
$$\mathbf{iii} = \left\langle \theta \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \right\rangle_{0 \le m \le \lfloor \frac{g-1}{2} \rfloor}, \quad \mathbf{iv} = \left\langle \theta^2 \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \right\rangle_{0 \le m \le \lfloor \frac{g-1}{2} \rfloor},$$

with each megablock is ordered by increasing m.

We postpone the proof of Theorem 3.7 to the next subsection. We can now precisely state and prove our main result.

Theorem 3.8. *The admissible subcategory $\mathcal{D} = \langle \mathbf{i}, \mathbf{ii}, \mathbf{iii}, \mathbf{iv} \rangle \subset \mathbf{G}$ is a noncommutative resolution of singularities of $\mathcal{SU}_C(2)$. Furthermore, this resolution agrees with the resolution of* [Păd21, Theorem 1.1] *(defined there in a more general context of symmetric stacks). This resolution is strongly crepant* [Kuz08] *if g is even.*

Proof. The blocks in **i**, **ii**, **iii**, **iv** are exactly those from Theorem 2.10 which are pulled back from $\mathbb{N}$. Indeed, the bundle $\theta^i \mathbf{\Lambda}^j \mathcal{F}^{\vee \boxtimes k}$ on $\mathcal{N}$ has weight $2j-k$ with respect to the $\mathbb{G}_m$ action, so it descends to $\mathbb{N}$ exactly when $k=2j$. By Proposition 3.2(b), it follows that $\mathcal{D}$ is isomorphic (via $R\zeta_*$) to an admissible subcategory $\mathcal{D}' \subset D^b(\mathbb{N})$ given by the same Fourier–Mukai kernels as $\mathcal{D}$ (regarded as objects in $D^b(\mathbb{N} \times \mathrm{Sym}^k C)$).

By the Luna slice theorem, analytic-locally near a complex point $p \in \mathcal{SU}_C(2)$, the stack $\mathbb{N}$ is isomorphic to an analytic neighborhood of the origin in the stack $[N/G]$, where G is the stabilizer of a split bundle $F = \mathcal{O}_C(D) \oplus \Lambda(-D)$ from the S-equivalence class of p and N is its normal bundle in $\mathbb{N}$. According to [Kir86], there are two cases. In the first case, $\Lambda \cong \mathcal{O}_C(2D)$, $G = \mathrm{PGL}_2$, and $N = \mathfrak{sl}_2 \otimes \mathbb{C}^g$. In the second case, $\Lambda \ncong \mathcal{O}_C(2D)$, $G \cong \mathbb{G}_m$ is the maximal torus of PGL_2, and $N = \mathbb{C}^{g-1} \oplus \mathbb{C}^{g-1} \oplus \mathbb{C}^g$. A primitive one-parameter subgroup $\lambda : \mathbb{G}_m \to G$, given by sending $\lambda(t) = \mathrm{diag}(u, u^{-1})$ where $u^2 = t$, has weights $1, -1, 0$ on N, each with multiplicity g in the first case, and with multiplicities $g-1, g-1, g$ in the second case. It follows that $\mathbb{N}$ is a symmetric stack satisfying assumptions A, B, C of [Păd21], so all of its results apply. The window width $\eta = \mathrm{weight}_\lambda \det \mathbb{L}^{>0} = \mathrm{weight}_\lambda N^{>0} - \mathrm{weight}_\lambda \mathfrak{g}^{>0}$ is equal to $g-1$ in both cases. By [Păd21, Theorem 1.1], it follows that the full subcategory $\mathcal{D}'' = \{X \in D^b(\mathbb{N}) : -\frac{g-1}{2} \le \mathrm{weight}_\lambda X \le \frac{g-1}{2}\}$ is admissible and provides a noncommutative resolution of singularities of $\mathcal{SU}_C(2)$.

Next, we will show that $\mathcal{D}' = \mathcal{D}''$. Let $X \in \mathcal{D}'$. The computation of $\mathrm{weight}_\lambda X$ is the same as the computation of $\mathrm{weight}_\lambda \zeta^* X$ in Proposition 3.3. By Lemma 3.10 below, the weights of objects in subcategories **i**, **ii**, **iii**, **iv** are in the interval $[-\frac{g-1}{2}, \frac{g-1}{2}]$. It follows that $\mathcal{D}' \subset \mathcal{D}''$. On the other hand, let $X \in \mathcal{D}''$. Since $\mathrm{weight}_\lambda X = \mathrm{weight}_\lambda \zeta^* X$ as above, we have $\zeta^* X \in \mathbf{G}$. With respect to the semiorthogonal decomposition (3.2) of Theorem 3.7, we have $\mathrm{Hom}(\zeta^* X, \mathbf{L}) = 0$ by projection formula and $\mathrm{Hom}(\mathbf{R}, \zeta^* X) = 0$ by coherent duality. It follows that $\zeta^* X \in \mathcal{D}$, and therefore $X \in \mathcal{D}'$. We conclude that $\mathcal{D}' = \mathcal{D}''$.

It remains to show that $\mathcal{D}$ is a strongly crepant noncommutative resolution of $\mathcal{SU}_C(2)$ if g is even. By the discussion above, we can view $\mathcal{D}$ as an admissible subcategory of $D^b(M)$, $D^b(\mathcal{Z}^0)$, or $D^b(\mathbb{N})$, where in every case the pullback functor $\mathrm{Perf}(\mathcal{SU}_C(2)) \to \mathcal{D}$ is the usual pullback. Furthermore, it endows $\mathcal{D}$ with the structure of an $\mathcal{SU}_C(2)$-linear category via the usual tensor product. We will view $\mathcal{D}$ as a subcategory of $D^b(M)$. Let $f : M \to SU_C(2)$ be the forgetful morphism. For every $B \in \mathcal{D}$, the functor $\mathrm{Perf}(\mathcal{SU}_C(2)) \to \mathcal{D}$, $A \mapsto f^* A \otimes B$ has a right adjoint functor $\mathcal{D} \to D^b(\mathcal{SU}_C(2))$ given by $C \mapsto Rf_* \circ R\mathcal{H}om_M(B, C)$. According to [Kuz08], in order to show that $\mathcal{D}$ is strongly crepant, we need to show that the identity functor on $\mathcal{D}$ is a relative Serre functor for $\mathcal{D}$ over $\mathcal{SU}_C(2)$. That is, we must give a functorial isomorphism

$$R\mathcal{H}om_{\mathcal{SU}_C(2)}(Rf_* \circ R\mathcal{H}om_M(B, C), \mathcal{O}_{\mathcal{SU}_C(2)}) \cong Rf_* \circ R\mathcal{H}om_M(C, B)$$

for $B, C \in \mathcal{D}$. By coherent duality for f, it is in fact sufficient to establish a functorial isomorphism

$$Rf_* \circ R\mathcal{H}om_M(C, B \otimes \omega_f[1]) \cong Rf_* \circ R\mathcal{H}om_M(C, B),$$

where ω_f is the dualizing line bundle for f.

By Lemma 3.14, there exists a morphism $\gamma : \mathcal{O} \to \theta \otimes \mathbf{\Lambda}^{-1}[1]$ in $D^b(\mathcal{Z}^\circ)$ whose image under ζ_* is an isomorphism $\mathcal{O} \to \mathcal{O}$. We pull back γ via the open immersion $j : M \to \mathcal{Z}^\circ$ to a morphism $j^*\gamma : \mathcal{O} \to \omega_f[1]$ on M. Indeed, $\omega_f \cong \omega_M \otimes \omega_{\mathcal{SU}_C(2)}^{-1} \cong j^*(\theta \otimes \mathbf{\Lambda}^{-1})$ by [Tha94, Section 5]. We will show that $j^*\gamma$ induces a stronger isomorphism

$$R(\zeta \circ j)_* \circ R\mathcal{H}om_M(C, B) \cong R(\zeta \circ j)_* \circ R\mathcal{H}om_M(C, B \otimes \omega_f[1]). \tag{3.3}$$

By Corollary 3.4 and Lemma 3.10, the weights of objects B, C, and $B \otimes \omega_f[1]$ belong to the interval $[-\lfloor \frac{g-1}{2} \rfloor, \lfloor \frac{g-1}{2} \rfloor + 1]$, which, if g is even, has width smaller than the window width $\eta = g$ for the immersion j. The GIT construction of M is local over $\mathbb{N}$, so the quantization theorem [HL15, Theorem 3.29] gives vertical isomorphisms in the following commutative diagram:

$$\begin{tikzcd}
R(\zeta \circ j)_* \circ R\mathcal{H}om_M(C, B) \arrow[r] \arrow[d, "\cong"'] & R(\zeta \circ j)_* \circ R\mathcal{H}om_M(C, B \otimes \omega_f[1]) \arrow[d, "\cong"] \\
R\zeta_* \circ R\mathcal{H}om_{\mathcal{Z}^\circ}(C, B) \arrow[r] & R\zeta_* \circ R\mathcal{H}om_{\mathcal{Z}^\circ}(C, B \otimes \theta \otimes \mathbf{\Lambda}^{-1}[1])
\end{tikzcd}$$

The bottom horizontal morphism of this diagram is an isomorphism by the projection formula for the morphism $\mathcal{Z}^\circ \to \mathbb{N}$ and the fact that the morphism $\gamma : \mathcal{O} \to \theta \otimes \mathbf{\Lambda}^{-1}[1]$ pushes forward to an isomorphism $\zeta_*\gamma$. It follows that the top horizontal morphism is also an isomorphism, proving (3.3). □

Remark 3.9. For symmetric stacks $\mathcal{X}$ satisfying various assumptions, a noncommutative motive $\mathbf{D}^{nc}(\mathcal{X})$ is constructed in [Păd21, Section 5]. It categorifies the intersection cohomology of the coarse moduli space of $\mathcal{X}$. When g is even, $\mathbf{D}^{nc}(\mathbb{N})$ is the motive of $\mathcal{D}$ by its construction in [Păd21], since then $\frac{g-1}{2}$ is a half-integer.

Lemma 3.10. *In the setup of Corollary 3.4, objects in subcategories* **i**, **ii**, **iii**, **iv** *have weights in* $[-\frac{g-1}{2}, \frac{g-1}{2}]$.

Proof. The same calculation as the proof of Corollary 3.4 using Proposition 3.3. □

3.2. **Proof of the Plain Weave.** We require several lemmas, to be proved at the end of this section. All mutations take place in $D^b(M) \cong \mathbf{G} \subset D^b(\mathcal{Z}^\circ)$.

Lemma 3.11. *For* $0 \le \lambda \le 2(g-1)$, *there is a mutation*

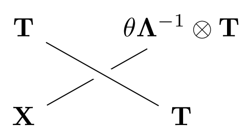
$$\left\langle \mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-1} \longrightarrow \left\langle \mathbf{\Lambda}^{-k}\overline{\mathcal{F}^{\vee\boxtimes\lambda-2k}}\right\rangle_{0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-1} \tag{3.4}$$

where the blocks are ordered by decreasing k *on the left, and by increasing* k *on the right.*

Lemma 3.12. *For* $0 \le 2k+1 \le g-1$, *we have* $\langle \mathbf{\Lambda}^k\mathcal{F}^{\vee\boxtimes 2k+1}\rangle \subset \mathbf{K}$ *and* $\langle \mathbf{\Lambda}^{k+1}\mathcal{F}^{\vee\boxtimes 2k+1}\rangle \subset \mathbf{K}^\vee$.

Lemma 3.13 (cf. [Tev23, Theorem 6.3])**.** *For* $\ell = 0, 1$ *and* $\ell \le k \le \frac{g-1}{2}$, *let* $\mathcal{D}^\ell = \mathcal{D}^\ell_{g-1} \subset D^b(M \times \mathrm{Sym}^\ell C)$ *as in Notation 2.5. Let* $\Phi : D^b(\mathcal{Z}^\circ) \to {}^\perp\langle \mathbf{\Lambda}^k\mathcal{F}^{\vee\boxtimes 2k}\rangle$ *be the semiorthogonal projector. Then* $\Phi(\langle \mathbf{\Lambda}^{\ell-k}\mathcal{D}^\ell\rangle) \subset \mathbf{K}^\vee$.

Lemma 3.14. *In* $D^b(\mathcal{Z}^\circ)$, *there exists a morphism* $\mathcal{O} \to \theta\mathbf{\Lambda}^{-1}[1]$ *whose image under* ζ_* *is an isomorphism* $\mathcal{O} \to \mathcal{O}$. *If* $\mathbf{T} \subset \zeta^* D^b(\mathbb{N})$ *is a full triangulated subcategory such that* $\theta\mathbf{\Lambda}^{-1} \otimes \mathbf{T} \subset {}^\perp\mathbf{T}$, *there is a mutation*



where $\mathbf{X} \subset \mathbf{K}$.

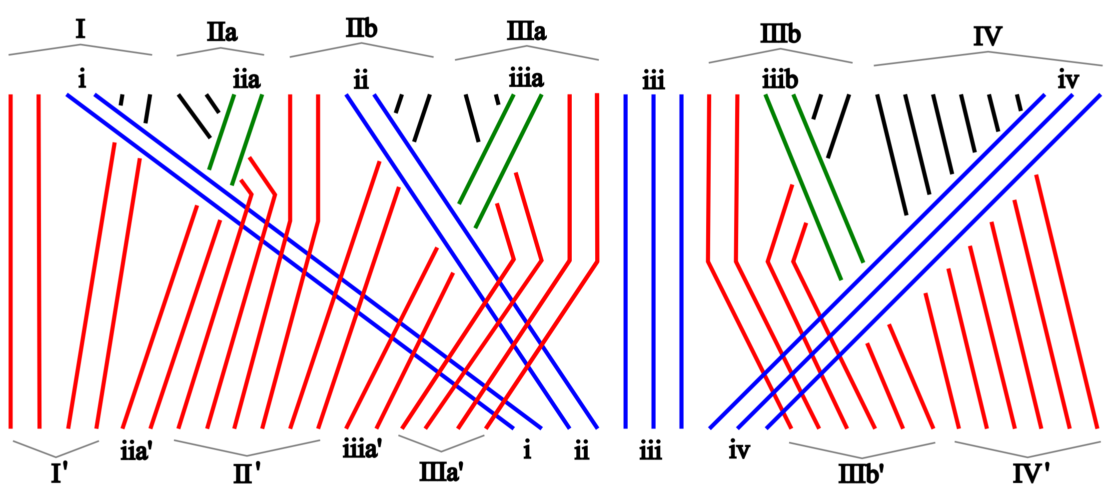


FIGURE 3. Modified Plain Weave in genus 5, cf. [Tev23, Figure 13].

Proof of Theorem 3.7. Denote by **I**, **II**, **III**, **IV** the megablocks in Theorem 2.10, regarded as subcategories of $\mathbf{G} \subset D^b(\mathcal{Z}^\circ)$ as in Corollary 3.4. We proceed as illustrated in Figure 3: we take the blocks in **i**, **ii**, **iii**, **iv**, and move them towards the center, making sure that all blocks in the way mutate into either $\mathbf{K}$ on the left or $\mathbf{K}^\vee$ on the right.

We begin with

$$\mathbf{IV} = \left\langle \theta^2\mathbf{\Lambda}^{\lfloor\frac{g}{2}\rfloor-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{g-1\le\lambda\le2(g-1)\\ \lambda-g+1\le k\le\lfloor\frac{\lambda}{2}\rfloor}}.$$

The blocks with $\lambda = 2\left\lfloor \frac{g}{2} \right\rfloor$ form megablock $\mathbf{iv}$ from the statement (with $m = \left\lfloor \frac{g}{2} \right\rfloor - k$). If g is odd, $2\left\lfloor \frac{g}{2} \right\rfloor = g-1$, so these are the rightmost blocks in $\mathbf{IV}$ as in Figure 3; if g is even, the blocks with $\lambda = g-1$ to the right of $\mathbf{iv}$ already lie in $\mathbf{K}^\vee$ by Lemma 3.12.

Claim 3.15. For $2\left\lfloor \frac{g}{2} \right\rfloor < \lambda \leq 2(g-1)$ and $\lambda - g + 1 \leq k \leq \left\lfloor \frac{\lambda}{2} \right\rfloor$, there is a mutation

$$\begin{tikzcd}
\theta^2 \mathbf{\Lambda}^{\left\lfloor \frac{g}{2} \right\rfloor - k} \mathcal{F}^{\vee \boxtimes \lambda - 2k} \arrow[dr, dash] & \mathbf{iv} \\
\mathbf{iv} \arrow[ur, dash, crossing over] & \mathbf{X}
\end{tikzcd}$$

where $\mathbf{X} \subset \mathbf{K}^\vee$.

Proof. Write $\mathbf{A} = \langle \theta^2 \mathbf{\Lambda}^{\left\lfloor \frac{g}{2} \right\rfloor - k} \mathcal{F}^{\vee \boxtimes \lambda - 2k} \rangle$. Suppose first that $\lambda = 2k$, so $\mathbf{A} = \langle \theta^2 \mathbf{\Lambda}^{\left\lfloor \frac{g}{2} \right\rfloor - k} \rangle$. Take the block $\mathbf{B} = \langle \theta^2 \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \rangle$ from $\mathbf{iv}$ with $m = k - \left\lfloor \frac{g}{2} \right\rfloor$ and write $\langle \mathbf{iv} \rangle = \langle \mathbf{B}, \mathbf{B}' \rangle$. (Note that $0 \leq m \leq \left\lfloor \frac{g-1}{2} \right\rfloor$ since $\left\lfloor \frac{g}{2} \right\rfloor < k \leq g-1$.) By Lemma 3.13, we mutate $\langle \mathbf{A}, \mathbf{B}, \mathbf{B}' \rangle \to \langle \mathbf{B}, \mathbf{A}', \mathbf{B}' \rangle$ where $\mathbf{A}' \subset \mathbf{K}^\vee$. Since $\mathbf{B}' \subset \zeta^* D^b(\mathbb{N})$, we see that $\mathbf{A}'$ and $\mathbf{B}'$ are fully orthogonal, so we move $\langle \mathbf{B}, \mathbf{A}', \mathbf{B}' \rangle \to \langle \mathbf{B}, \mathbf{B}', \mathbf{A}' \rangle = \langle \mathbf{iv}, \mathbf{A}' \rangle$ without any further mutation.

Now suppose $2k < \lambda$. Let $\mathbf{B} = \langle \theta^2 \mathbf{\Lambda}^{m-1} \mathcal{F}^{\vee \boxtimes 2m-2}, \theta^2 \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \rangle \subset \langle \mathbf{iv} \rangle$, where $m = \lambda - k - \left\lfloor \frac{g}{2} \right\rfloor$. (We have $m > \frac{\lambda}{2} - \left\lfloor \frac{g}{2} \right\rfloor \geq 0$, so $m - 1 \geq 0$, and $m \leq g - 1 - \left\lfloor \frac{g}{2} \right\rfloor \leq \left\lfloor \frac{g-1}{2} \right\rfloor$.) Denote by $\mathcal{F}^{\bullet \boxtimes \ell}_{tr} = [\mathcal{F}^{\vee \boxtimes \ell} \to \mathrm{Ker}^{1-\ell} \mathcal{F}^{\bullet \boxtimes \ell}]$ the two-step smart truncation of the complex $\mathcal{F}^{\bullet \boxtimes \ell}$ (see Notation 4.9 below). By [Tev23, Lemma 4.9, Corollary 4.10], for every $X \in \mathbf{A}$ we have exact triangles $K \to Y \to X \to$ and $H \to Y \to H' \to$ where $Y \in \langle \theta^2 \mathbf{\Lambda}^{\left\lfloor \frac{g}{2} \right\rfloor - k} \mathcal{F}^{\bullet \boxtimes \lambda - 2k}_{tr} \rangle$, $K \in \langle \theta^2 \mathbf{\Lambda}^{\left\lfloor \frac{g}{2} \right\rfloor - k - (\lambda - 2k - 1)} \rangle = \langle \theta^2 \mathbf{\Lambda}^{1-m} \rangle$, $H \in \langle \theta^2 \mathbf{\Lambda}^{-m} \rangle$, and $H' \in \langle \theta^2 \mathbf{\Lambda}^{1-m} \mathcal{D}^1 \rangle$. It follows from Lemma 3.13 that the images of K, H and H' under the semiorthogonal projector onto ${}^\perp \mathbf{B}$ lie in $\mathbf{K}^\vee$, so the same is true for X. Writing $\langle \mathbf{iv} \rangle = \langle \mathbf{B}, \mathbf{B}' \rangle$ as above, this gives mutations $\langle \mathbf{A}, \mathbf{B}, \mathbf{B}' \rangle \to \langle \mathbf{B}, \mathbf{A}', \mathbf{B}' \rangle \to \langle \mathbf{B}, \mathbf{B}', \mathbf{A}' \rangle = \langle \mathbf{iv}, \mathbf{A}' \rangle$ with $\mathbf{A}' \subset \mathbf{K}^\vee$, as claimed. $\square$

By applying this mutation to each block in $\mathbf{IV}$ with $g \leq \lambda \leq 2(g-1)$ in sequence from right to left, we obtain $\langle \mathbf{IV} \rangle = \langle \mathbf{iv}, \mathbf{IV}' \rangle$ where $\mathbf{IV}' \subset \mathbf{K}^\vee$.

The megablock

$$\mathbf{I} = \left\langle \theta^{-1} \mathbf{\Lambda}^{\left\lfloor \frac{g-2}{2} \right\rfloor - k} \mathcal{F}^{\vee \boxtimes \lambda - 2k} \right\rangle_{\substack{0 \leq \lambda \leq g-2 \\ 0 \leq k \leq \left\lfloor \frac{\lambda}{2} \right\rfloor}}$$

on the left is treated similarly, but the process is mirrored. Megablock $\mathbf{i}$ appears as the blocks with $\lambda = 2\left\lfloor \frac{g-2}{2} \right\rfloor$. If g is odd, there are additional blocks with $\lambda = g-2$ to the left of the megablock $\mathbf{i}$, but they already lie in $\mathbf{K}$ by Lemma 3.12, so we ignore them as above. We apply the mutation of Lemma 3.11 to all blocks in $\mathbf{I}$; the blocks are still ordered by decreasing λ, but now by increasing k with $\overline{\mathcal{F}^{\vee \boxtimes \lambda - 2k}}$ in place of $\mathcal{F}^{\vee \boxtimes \lambda - 2k}$.

Claim 3.16. For $0 \leq \lambda < 2\left\lfloor \frac{g-2}{2} \right\rfloor$ and $0 \leq k \leq \left\lfloor \frac{\lambda}{2} \right\rfloor$, there is a mutation

$$\begin{tikzcd}
\mathbf{i} \arrow[dr, dash] & \theta^{-1} \mathbf{\Lambda}^{\left\lfloor \frac{g-2}{2} \right\rfloor - k} \overline{\mathcal{F}^{\vee \boxtimes \lambda - 2k}} \\
\mathbf{X} \arrow[ur, dash, crossing over] & \mathbf{i}
\end{tikzcd}$$

where $\mathbf{X} \subset \mathbf{K}$.

Proof. We prove instead the dual mutation. By [Tev23, Lemma 3.4], we have

$$\langle \theta^x \mathbf{\Lambda}^y \overline{\mathcal{F}^{\vee \boxtimes z}} \rangle^\vee = \langle \theta^{-x} \mathbf{\Lambda}^{-y} (\overline{\mathcal{F}^{\vee \boxtimes z}})^\vee \rangle = \langle \theta^{-x} \mathbf{\Lambda}^{z-y} \mathcal{F}^{\vee \boxtimes z} \rangle \tag{3.5}$$

Hence it suffices to give a mutation $\langle \mathbf{A}, \mathbf{i}^\vee \rangle \to \langle \mathbf{i}^\vee, \mathbf{A}' \rangle$ where $\mathbf{A} = \langle \theta \mathbf{\Lambda}^{\lambda - k - \left\lfloor \frac{g-2}{2} \right\rfloor} \mathcal{F}^{\vee \boxtimes \lambda - 2k} \rangle$, $\mathbf{A}' \subset \mathbf{K}^\vee$, and $\mathbf{i}^\vee = \langle \theta \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \rangle_{0 \leq m \leq \left\lfloor \frac{g-2}{2} \right\rfloor}$ ordered by decreasing m.

From here, the proof is the same as for Claim 3.15. If $\lambda = 2k$, we let $\mathbf{B} = \langle \theta \mathbf{\Lambda}^m \mathcal{F}^{\vee \boxtimes 2m} \rangle$ with $m = \left\lfloor \frac{g-2}{2} \right\rfloor - k$ and proceed as above. We need only check that $0 \leq m \leq \left\lfloor \frac{g-2}{2} \right\rfloor$, which is clear. Similarly, if $2k < \lambda$, it suffices to check that $1 \leq m \leq \left\lfloor \frac{g-2}{2} \right\rfloor$ where $m = (\lambda - 2k) - \left(\lambda - k - \left\lfloor \frac{g-2}{2} \right\rfloor\right) = \left\lfloor \frac{g-2}{2} \right\rfloor - k$, which is again clear. $\square$

We thus obtain $\mathbf{I} = \langle \mathbf{I}', \mathbf{i}\rangle$ where $\mathbf{I}' \subset \mathbf{K}$. Next, let

$$\mathbf{II}_a = \left\langle \mathbf{\Lambda}^{\left\lfloor \frac{g-2}{2}\right\rfloor - k}\mathcal{F}^{\vee \boxtimes \lambda - 2k}\right\rangle_{\substack{g-1\le\lambda\le 2(g-2)\\ \lambda-g+2\le k\le\lfloor\frac{\lambda}{2}\rfloor}}, \quad \mathbf{II}_b = \left\langle \mathbf{\Lambda}^{\left\lfloor \frac{g-2}{2}\right\rfloor - k}\mathcal{F}^{\vee \boxtimes \lambda - 2k}\right\rangle_{\substack{0\le\lambda\le g-2\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor}},$$

so $\mathbf{II} = \langle \mathbf{II}_a, \mathbf{II}_b\rangle$. Then $\mathbf{II}_b = \theta \otimes \mathbf{I}$ and $\mathbf{ii} = \theta\otimes\mathbf{i}$, so $\mathbf{II}_b = \langle \mathbf{II}'_b, \mathbf{ii}\rangle$ where $\mathbf{II}'_b = \theta\otimes\mathbf{I}' \subset \mathbf{K}$. On the other hand, let

$$\mathbf{ii}_a = \left\langle \mathbf{\Lambda}^{m-1}\mathcal{F}^{\vee\boxtimes 2m}\right\rangle_{0\le m\le \left\lfloor\frac{g-3}{2}\right\rfloor}$$

ordered by increasing m, which are the blocks in $\mathbf{II}_a$ with $\lambda = 2\left\lfloor\frac{g}{2}\right\rfloor$ (where $m = \left\lfloor\frac{g}{2}\right\rfloor - k$). If g is even, there are blocks in $\mathbf{II}_a$ with $\lambda = g-1$ to the right of $\mathbf{ii}_a$, but they are contained in $\mathbf{K}$ by Lemma 3.12.

Claim 3.17. For $2\left\lfloor\frac{g}{2}\right\rfloor < \lambda \le 2(g-2)$ and $\lambda - g + 2 \le k \le \left\lfloor\frac{\lambda}{2}\right\rfloor$, there is a mutation

$$\begin{tikzcd} \mathbf{\Lambda}^{\left\lfloor\frac{g-2}{2}\right\rfloor - k}\mathcal{F}^{\vee\boxtimes\lambda-2k} \arrow[dr, dash] & \mathbf{ii}_a \arrow[dl, dash] \\ \mathbf{ii}_a & \mathbf{X} \end{tikzcd}$$

where $\mathbf{X} \subset \mathbf{K}$.

Proof. The proof is the same as Claim 3.15, but with everything tensored with $\mathbf{\Lambda}^{-1}\otimes\theta^{-2}$ and with a smaller range of λ, k, and m. We need only check that $0 \le m \le \left\lfloor\frac{g-3}{2}\right\rfloor$ with $m > 0$ if $2k < \lambda$, where $m = \lambda - k - \left\lfloor\frac{g}{2}\right\rfloor$. Indeed, we have $0 \le \frac{\lambda}{2} - \left\lfloor\frac{g}{2}\right\rfloor \le m \le g - 2 - \left\lfloor\frac{g}{2}\right\rfloor \le \left\lfloor\frac{g-3}{2}\right\rfloor$ with $\frac{\lambda}{2} - \left\lfloor\frac{g}{2}\right\rfloor < m$ if $2k < \lambda$. $\square$

Hence we can write $\mathbf{II}_a = \langle \mathbf{ii}_a, \mathbf{II}'_a\rangle$ with $\mathbf{II}'_a \subset \mathbf{K}$. Combining with $\mathbf{II}_b$ gives $\mathbf{II} = \langle \mathbf{ii}_a, \mathbf{II}', \mathbf{ii}\rangle$ where $\mathbf{II}' = \langle \mathbf{II}'_a, \mathbf{II}'_b\rangle \subset \mathbf{K}$.

Next, we have $\mathbf{III} = \langle \mathbf{III}_a, \mathbf{iii}, \mathbf{III}_b\rangle$, where

$$\mathbf{III}_a = \left\langle \theta\mathbf{\Lambda}^{\left\lfloor\frac{g}{2}\right\rfloor - k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{2\left\lfloor\frac{g}{2}\right\rfloor<\lambda\le 2(g-1)\\ \lambda-g+1\le k\le\lfloor\frac{\lambda}{2}\rfloor}}, \quad \mathbf{III}_b = \left\langle \theta\mathbf{\Lambda}^{\left\lfloor\frac{g}{2}\right\rfloor - k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{0\le\lambda<2\left\lfloor\frac{g}{2}\right\rfloor\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor}}.$$

Let $\mathbf{iii}_a = \left\langle \theta\mathbf{\Lambda}^{m+1}\mathcal{F}^{\vee\boxtimes 2m}\right\rangle_{0\le m\le\left\lfloor\frac{g-3}{2}\right\rfloor}$ be the blocks with $\lambda = 2\left\lfloor\frac{g+2}{2}\right\rfloor$. The blocks in $\mathbf{III}_a$ to the right of $\mathbf{iii}_a$ have $\lambda = 2\left\lfloor\frac{g}{2}\right\rfloor + 1$ and lie in $\mathbf{K}$ already by Lemma 3.12. The blocks to the left of $\mathbf{iii}_a$ are processed exactly as with $\mathbf{II}_a$ above (make the substitution $\lambda' = \lambda - 2$, $k' = k - 1$ and use Claim 3.17). Hence $\mathbf{III}_a = \langle \mathbf{iii}_a, \mathbf{III}'_a\rangle$ with $\mathbf{III}'_a \subset \mathbf{K}$. Similarly, let $\mathbf{iii}_b = \left\langle \theta\mathbf{\Lambda}^{m-1}\mathcal{F}^{\vee\boxtimes 2m}\right\rangle_{0\le m\le\left\lfloor\frac{g-2}{2}\right\rfloor}$ be the blocks from $\mathbf{III}_b$ with $\lambda = 2\left\lfloor\frac{g-2}{2}\right\rfloor$. The blocks in $\mathbf{III}_b$ with $\lambda = 2\left\lfloor\frac{g}{2}\right\rfloor - 1$ lie in $\mathbf{K}^\vee$; the other blocks are exactly the blocks in $\mathbf{I}$ with $\lambda \le 2\lfloor g-2\rfloor 2$, tensored by $\theta^2\mathbf{\Lambda}$. Then Claim 3.16 allows us to write $\mathbf{III}_b = \langle \mathbf{III}'_b, \mathbf{iii}_b\rangle$ with $\mathbf{III}'_b \subset \mathbf{K}^\vee$.

To summarize, we have a semiorthogonal decomposition

$$\mathbf{G} = \left\langle \mathbf{I}', \mathbf{i}, \mathbf{ii}_a, \mathbf{II}', \mathbf{ii}, \mathbf{iii}_a, \mathbf{III}'_a, \mathbf{iii}, \mathbf{III}'_b, \mathbf{iii}_b, \mathbf{iv}, \mathbf{IV}'\right\rangle$$

where $\mathbf{I}', \mathbf{II}', \mathbf{III}'_a \subset \mathbf{K}$ and $\mathbf{III}'_b, \mathbf{IV}' \subset \mathbf{K}^\vee$. It remains to mutate $\mathbf{ii}_a$, $\mathbf{iii}_a$, and $\mathbf{iii}_b$. Observe that the blocks in $\mathbf{ii}_a$ are exactly the blocks from $\mathbf{i}$ tensored by $\theta\mathbf{\Lambda}^{-1}$. One by one, we mutate each block $\theta\mathbf{\Lambda}^{-1}\otimes\mathbf{T}$ from $\mathbf{ii}_a$ past the corresponding block from $\mathbf{T}$ in $\mathbf{ii}$ using Lemma 3.14; the resulting block $\mathbf{T}' \subset \mathbf{K}$ is both left and right orthogonal to $\mathbf{T}^\perp \cap \mathbf{i}$, so we may move it to the left of $\mathbf{i}$ without further mutation. Hence we obtain $\langle \mathbf{i}, \mathbf{ii}_a\rangle = \langle \mathbf{ii}'_a, \mathbf{i}\rangle$ with $\mathbf{ii}'_a \subset \mathbf{K}$. Similarly, $\mathbf{iii}_a = \theta\mathbf{\Lambda}^{-1}\otimes\mathbf{ii}$, so $\langle \mathbf{ii}, \mathbf{iii}_a\rangle = \langle \mathbf{iii}'_a, \mathbf{ii}\rangle$ with $\mathbf{iii}'_a \subset \mathbf{K}$. On the other side, each block in $\mathbf{iii}_b$ is a block from $\mathbf{iv}$ tensored with $\theta^{-1}\mathbf{\Lambda}$. The dual of Lemma 3.14 allows us to mutate $\langle \theta^{-1}\mathbf{\Lambda}\otimes\mathbf{T}, \mathbf{T}\rangle \to \langle \mathbf{T}, \mathbf{T}'\rangle$ with $\mathbf{T}' \subset \mathbf{K}^\vee$, which gives $\langle \mathbf{iii}_b, \mathbf{iv}\rangle = \langle \mathbf{iv}, \mathbf{iii}'_b\rangle$ with $\mathbf{iii}'_b \subset \mathbf{K}^\vee$.

Put together, we have

$$\mathbf{G} = \left\langle \mathbf{I}', \mathbf{ii}'_a, \mathbf{i}, \mathbf{II}', \mathbf{iii}'_a, \mathbf{ii}, \mathbf{III}'_a, \mathbf{iii}, \mathbf{III}'_b, \mathbf{iv}, \mathbf{iii}'_b, \mathbf{IV}'\right\rangle,$$

where all primed subcategories to the left (resp. right) of $\mathbf{iii}$ lie in $\mathbf{K}$ (resp. $\mathbf{K}^\vee$). It follows that $\mathbf{ii}$ and $\mathbf{III}'_a$ are both left and right orthogonal; similarly, $\mathbf{i}$ is orthogonal to $\mathbf{II}'$, $\mathbf{iii}'_a$, and $\mathbf{III}'_a$, while $\mathbf{iv}$ is orthogonal to $\mathbf{III}'_b$. Thus we may move $\mathbf{i}$, $\mathbf{ii}$, and $\mathbf{iv}$ to the center without any mutations, giving $\mathbf{G} = \langle \mathbf{L}, \mathbf{i}, \mathbf{ii}, \mathbf{iii}, \mathbf{iv}, \mathbf{R}\rangle$ where $\mathbf{L} = \langle \mathbf{I}', \mathbf{ii}'_a, \mathbf{II}', \mathbf{iii}'_a, \mathbf{III}'_a\rangle \subset \mathbf{K}$ and $\mathbf{R} = \langle \mathbf{III}'_b, \mathbf{iii}'_b, \mathbf{IV}'\rangle \subset \mathbf{K}^\vee$. This completes the proof. $\square$

Proof of Lemma 3.11. Since all blocks on either side of (3.4) lie in $\mathbf{G}$, it suffices to perform the mutation in $D^b(M)$. We begin with the semiorthogonal decomposition of $D^b(M)$ from Lemma 2.11:

$$\left\langle \left\langle \mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-2\\ \lambda-2k,k\ge 0}}, \left\langle \theta\mathbf{\Lambda}^{1-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-1\\ \lambda-2k,k\ge 0}}, \left\langle \theta^2\mathbf{\Lambda}^{1-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-1\\ \lambda-2k,k\ge 0}}\right\rangle, \tag{3.6}$$

ordered by decreasing λ, then decreasing k. Using (3.5), we obtain a dual semiorthogonal decomposition:

$$\left\langle \left\langle \theta^{-2}\mathbf{\Lambda}^{\lambda-k-1}\overline{\mathcal{F}}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-1\\ \lambda-2k,k\ge 0}}, \left\langle \theta^{-1}\mathbf{\Lambda}^{\lambda-k-1}\overline{\mathcal{F}}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-1\\ \lambda-2k,k\ge 0}}, \left\langle \mathbf{\Lambda}^{\lambda-k}\overline{\mathcal{F}}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-2\\ \lambda-2k,k\ge 0}}\right\rangle,$$

ordered by increasing λ, then increasing k. We tensor the rightmost megablock by $\omega_M = \mathbf{\Lambda}^{-1}\theta^{-3}$, moving it to the left, then tensor everything by $\theta^3\mathbf{\Lambda}^{3-g}$ to obtain:

$$\left\langle \left\langle \mathbf{\Lambda}^{\lambda-k-g+2}\overline{\mathcal{F}}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-2\\ \lambda-2k,k\ge 0}} \left\langle \theta\mathbf{\Lambda}^{\lambda-k-g+2}\overline{\mathcal{F}}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-1\\ \lambda-2k,k\ge 0}}, \left\langle \theta^2\mathbf{\Lambda}^{\lambda-k-g+2}\overline{\mathcal{F}}^{\vee\boxtimes\lambda-2k}\right\rangle_{\substack{\lambda-k\le g-1\\ \lambda-2k,k\ge 0}}\right\rangle,$$

ordered by decreasing λ, then increasing k. Finally, we make the change of variables $\lambda' = 2(g-2)-\lambda$, $k' = k-\lambda+g-2$ in the first megablock and $\lambda' = 2(g-1)-\lambda$, $k' = k-\lambda+g-1$ in the others to obtain

$$\left\langle \left\langle \mathbf{\Lambda}^{-k'}\overline{\mathcal{F}}^{\vee\boxtimes\lambda'-2k'}\right\rangle_{\substack{\lambda'-k'\le g-2\\ \lambda'-2k',k'\ge 0}}, \left\langle \theta\mathbf{\Lambda}^{1-k'}\overline{\mathcal{F}}^{\vee\boxtimes\lambda'-2k'}\right\rangle_{\substack{\lambda'-k'\le g-1\\ \lambda'-2k',k'\ge 0}}, \left\langle \theta^2\mathbf{\Lambda}^{1-k'}\overline{\mathcal{F}}^{\vee\boxtimes\lambda'-2k'}\right\rangle_{\substack{\lambda'-k'\le g-1\\ \lambda'-2k',k'\ge 0}}\right\rangle \tag{3.7}$$

ordered by decreasing λ', then increasing k'. Denote by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ the megablocks of (3.6), and $\overline{\mathbf{A}}$, $\overline{\mathbf{B}}$, $\overline{\mathbf{C}}$ the megablocks of (3.7).

Claim 3.18. We have $\mathbf{B} = \overline{\mathbf{B}}$.

Proof. Since (3.6) and (3.7) are both full semiorthogonal decompostions of $D^b(M)$, it suffices to show that $\mathbf{B} \subset {}^\perp\overline{\mathbf{A}} \cap \overline{\mathbf{C}}^\perp$ and $\overline{\mathbf{B}} \subset {}^\perp\mathbf{A} \cap \mathbf{B}^\perp$. By [TT21, Lemma 10.2], it suffices to check that

$$R\operatorname{Hom}\left(\theta\mathbf{\Lambda}^{1-k'}\overline{\mathcal{F}}^{\vee\boxtimes\lambda'-2k'}_{D'}, \mathbf{\Lambda}^{1-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}_{D}\right) = R\operatorname{Hom}\left(\theta\mathbf{\Lambda}^{1-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}_{D}, \mathbf{\Lambda}^{1-k'}\overline{\mathcal{F}}^{\vee\boxtimes\lambda'-2k'}_{D'}\right) = 0 \tag{3.8}$$

for any $D \in \operatorname{Sym}^{\lambda-2k}C$, $D' \in \operatorname{Sym}^{\lambda'-2k'}C$, where λ, k (likewise λ', k') satisfy $k \ge 0$, $\lambda - 2k \ge 0$, and $\lambda - k \le g-1$. Using $\mathcal{F}^{\vee\boxtimes\lambda-2k}_D = \mathbf{\Lambda}^{2k-\lambda}\mathcal{F}^{\boxtimes\lambda-2k}$ and $\overline{\mathcal{F}}^{\vee\boxtimes\lambda-2k}_D = \mathbf{\Lambda}^{2k-\lambda}\overline{\mathcal{F}}^{\boxtimes\lambda-2k}$, (3.8) becomes

$$R\Gamma\left(\theta^{-1}\mathbf{\Lambda}^{\lambda'-k'-\lambda+k}(\overline{\mathcal{F}}^{\boxtimes\lambda'-2k'}_{D'})^\vee\mathcal{F}^{\boxtimes\lambda-2k}_D\right) = R\Gamma\left(\theta^{-1}\mathbf{\Lambda}^{\lambda-k-\lambda'+k'}(\mathcal{F}^{\boxtimes\lambda-2k}_D)^\vee\overline{\mathcal{F}}^{\boxtimes\lambda'-2k'}_{D'}\right) = 0.$$

Recalling that $\theta = \mathcal{O}(1, g-1)$, this follows from [TT21, Theorem 4.1, Remark 4.2] once we verify that $\lambda-2k-g < \lambda-k-(\lambda'-k') < g-\lambda+2k'$. Indeed, $-k-g < -(g-1) \le -(\lambda'-k')$ and $\lambda-k \le g-1 < g+k'$. □

Hence $\theta^{-1}\mathbf{\Lambda}^{-1}\mathbf{B} = \theta^{-1}\mathbf{\Lambda}^{-1}\overline{\mathbf{B}}$. Let for $0 \le \lambda \le 2(g-1)$, let $\mathbf{B}_\lambda = \left\langle \mathbf{\Lambda}^{-k}\mathcal{F}^{\vee\boxtimes\lambda-2k}\right\rangle_{\max(0,\lambda-g+1)\le k\le\lfloor\lambda/2\rfloor}$ (ordered by decreasing k), so $\theta^{-1}\mathbf{\Lambda}^{-1}\mathbf{B} = \langle \mathbf{B}_{2(g-1)}, \dots, \mathbf{B}_0\rangle$; likewise, $\theta^{-1}\mathbf{\Lambda}^{-1}\overline{\mathbf{B}} = \langle \overline{\mathbf{B}}_{2(g-1)}, \dots, \overline{\mathbf{B}}_0\rangle$.

Claim 3.19. For $0 \le \lambda \le 2(g-1)$, we have $\mathbf{B}_\lambda = \overline{\mathbf{B}}_\lambda$.

Proof. It suffices to show that $\mathbf{B}_\lambda \subset \overline{\mathbf{B}}^\perp_{\lambda'}$ for $\lambda < \lambda'$ and $\mathbf{B}_\lambda \subset {}^\perp\overline{\mathbf{B}}_{\lambda'}$ for $\lambda > \lambda'$. For the first, we must check

$$R\Gamma\left(\mathbf{\Lambda}^{\lambda-k-\lambda'+k'}(\mathcal{F}^{\boxtimes\lambda-2k}_D)^\vee\overline{\mathcal{F}}^{\boxtimes\lambda'-2k'}_{D'}\right) = 0$$

for any $\max(0, \lambda-g+1) \le k \le \lfloor\lambda/2\rfloor$, $\max(0, \lambda'-g+1) \le k' \le \lfloor\lambda'/2\rfloor$, $D \in \operatorname{Sym}^{\lambda-2k}C$, $D' \in \operatorname{Sym}^{\lambda'-2k'}C$. This follows from [TT21, Lemma 5.4, Remark 5.7]: we have $\lambda-2k, \lambda'-2k' \le 2g-1$ and $2(\lambda-k-\lambda'+k') < \lambda-2k-\lambda'+2k'$ immediately, and $\lambda-2k-g < \lambda-k-\lambda'+k' < g-\lambda+2k'+1$ as in the preceding claim. The second containment is proved analogously. □

It remains to show that the blocks of $\mathbf{B}_\lambda$ and $\overline{\mathbf{B}}_\lambda$ are related by a mutation. If $\lambda > g-1$, then $\mathbf{B}_\lambda = \mathbf{\Lambda}^{g-1-\lambda}\mathbf{B}_{2(g-1)-\lambda}$, so we may assume $0 \le \lambda \le g-1$. Proceed by induction on λ, with $\lambda = 0, 1$ being trivial. Assume we have the mutation $\mathbf{B}_{\lambda-2} \to \overline{\mathbf{B}}_{\lambda-2}$. Then $\mathbf{B}_\lambda = \langle \mathbf{\Lambda}^{-1}\mathbf{B}_{\lambda-2}, \mathcal{F}^{\vee\boxtimes\lambda}\rangle$ and $\overline{\mathbf{B}}_\lambda = \langle \overline{\mathcal{F}}^{\vee\boxtimes\lambda}, \mathbf{\Lambda}^{-1}\overline{\mathbf{B}}_{\lambda-2}\rangle$. Projecting $\langle\mathcal{F}^{\vee\boxtimes\lambda}\rangle$ onto $(\mathbf{\Lambda}^{-1}\mathbf{B}_{\lambda-2})^\perp = (\mathbf{\Lambda}^{-1}\overline{\mathbf{B}}_{\lambda-2})^\perp$ and mutating $\mathbf{B}_{\lambda-2} \to \overline{\mathbf{B}}_{\lambda-2}$ completes the proof. □

Proof of Lemma 3.12. Let $X \in \langle \mathbf{\Lambda}^k \mathcal{F}^{\vee \boxtimes 2k+1} \rangle$. Then X is a pullback of an object in $D^b(\mathcal{N})$ of weight -1 with respect to $\mathbb{G}_m$. By Proposition 3.2(a), it suffices to show that $R\zeta_* Y = 0$ for every object $Y \in D^b(\mathcal{A}^\circ)$ of the form $Y = \bigwedge^k \alpha^* \mathcal{B}^* \otimes^L \alpha^* Z$, where $k = 0, \dots, b$ and $Z \in D^b(\mathcal{N})$ is an object of weight -1. Since the claim is local on $\mathbb{N}$, we can replace Y with $\mathcal{O}_{\mathcal{A}^\circ}(s)$, where $s = -1, \dots, -(b+1)$. Since $R\Gamma(\mathbf{P}^{a-1}, \mathcal{O}(-s)) = 0$ for $s = 1, \dots, a-1$ and $b + 1 = a - 1$, the first statement follows. The second follows from Remark 3.6. □

Proof of Lemma 3.13. We mimic the proof of [Tev23, Theorem 6.3]. As we will need to work in both $D^b(M)$ and $D^b(\mathcal{Z}^\circ)$, we denote the windows embedding by $\iota : D^b(M) \to \mathbf{G} \subset D^b(\mathcal{Z}^\circ)$. As in [Tev23, Lemma 6.7], it suffices to show that the morphism

$$\zeta_*(\mathbf{\Lambda}^{-1} \otimes \iota(\mathbf{\Lambda}^{\ell-k} \mathcal{O}_{M(-D)})) \to \zeta_*(\mathbf{\Lambda}^{-1} \otimes \iota \circ \mathcal{P} \circ \mathcal{P}^L(\mathbf{\Lambda}^{\ell-k} \mathcal{O}_{M(-D)}))$$

is an isomorphism for any $D \in \mathrm{Sym}^\ell C$, where $\mathcal{P} : D^b(\mathrm{Sym}^k C) \to D^b(M)$ is the Fourier–Mukai functor with kernel $\mathbf{\Lambda}^k \mathcal{F}^{\vee \boxtimes 2k}$, $\mathcal{P}^L$ is its left adjoint (the Fourier–Mukai functor with kernel $(\mathbf{\Lambda}^k \mathcal{F}^{\vee \boxtimes 2k})^\vee \otimes \omega_M^\bullet$ [Huy06, Proposition 5.9]), $M(-D) \subset M$ denotes the locus of stable pairs (F, s) with $s|_D = 0$, and the morphism is induced by the unit of adjunction $\mathrm{Id} \implies \mathcal{P} \circ \mathcal{P}^L$. We compute both sides of this morphism.

Claim 3.20. We have $\mathcal{P} \circ \mathcal{P}^L(\mathbf{\Lambda}^{\ell-k} \mathcal{O}_{M(-D)}) \cong R\pi_{M*}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k})[2\ell]$, where $\pi_M : M \times \mathrm{Sym}^k C \to M$ is the projection.

Proof. Notice first that by [Huy06, Corollary 3.40] and [Tha94, 5.7, 6.1], we have

$$\mathcal{O}_{M(-D)}^\vee = \omega_{M(-D)} \otimes \omega_M^{-1}[-2\ell] = \mathbf{\Lambda}^\ell \mathcal{O}_{M(-D)}[-2\ell]$$

in $D^b(M)$. We have

$$\begin{aligned} \mathcal{P}^L(\mathbf{\Lambda}^{\ell-k} \mathcal{O}_{M(-D)}) &= R\pi_{\mathrm{Sym}^k C *}((\mathbf{\Lambda}^{2k} \mathcal{F}^{\vee \boxtimes 2k} \mathcal{O}_{M(-D)})^\vee \omega_M^\bullet)[2\ell] \\ &\cong \left(R\pi_{\mathrm{Sym}^k C *}(\mathcal{F}(-D)^{\boxtimes 2k}|_{M(-D) \times \mathrm{Sym}^k C}) \right)^\vee \otimes \mathcal{O}(-D)^{\boxtimes 2k}[2\ell] \end{aligned}$$

by coherent duality and the projection formula. Since $\mathcal{F}(-D)|_{M(-D) \times C}$ is the universal family on $M(-D) \times C$, we have $\mathcal{P}^L(\mathbf{\Lambda}^{\ell-k} \mathcal{O}_{M(-D)}) \cong \mathcal{O}(-D)^{\boxtimes 2k}[2\ell]$ by [TT21, Corollary 7.5]. Applying $\mathcal{P}$ proves the claim. □

Hence (after shifting by -2ℓ for convenience) we have a morphism

$$\mathbf{\Lambda}^{-k} \mathcal{O}_{M(-D)}^\vee \to R\pi_{M*}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k}), \tag{3.9}$$

which is unique up to scalar as in [Tev23, Remark 6.10].

It remains to show that applying the functor $\zeta_*(\mathbf{\Lambda}^{-1} \otimes \iota(-))$ to (3.9) yields an isomorphism. Since $R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k}) \in D^b(\mathcal{Z}^\circ)$ has weights in the range $[-k, k] \subseteq [-\lfloor \frac{g}{2} \rfloor, g - \lfloor \frac{g}{2} \rfloor)$ and restricts via j^* to $R\pi_{M*}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k})$, we have

$$\iota(R\pi_{M*}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k})) = R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k}).$$

On the other hand, we claim that $\iota(\mathbf{\Lambda}^{-k} \mathcal{O}_{M(-D)}^\vee) = \mathbf{\Lambda}^{-k} \mathcal{O}_{\mathcal{Z}^\circ(-D)}^\vee$ for $\ell = 0, 1$, where $\mathcal{Z}^\circ(-D) \subset \mathcal{Z}^\circ$ denotes the closed substack of pairs (F, s) with $s|_D = 0$. If $\ell = 0$, this is clear, since $\mathbf{\Lambda}^{-k} \in \mathbf{G}$. If $\ell = 1$, so $D = x \in C$, then $\mathcal{Z}^\circ(-x)$ is the codimension-2 vanishing locus of the canonical section of $\mathcal{F}_x$. We have a Koszul resolution $\mathcal{O}_{\mathcal{Z}^\circ(-D)} \cong [\mathbf{\Lambda}^{-1} \to \mathcal{F}_x^\vee \to \mathcal{O}]$, so $\mathcal{O}_{\mathcal{Z}^\circ(-D)}$ has weights in the range $[0, 1]$. Since $[k-1, k] \subseteq [-\lfloor \frac{g}{2} \rfloor, g - \lfloor \frac{g}{2} \rfloor)$ and $j^* \mathcal{O}_{\mathcal{Z}^\circ(-p)} = \mathcal{O}_{M(-p)}$, the claim holds. Thus applying ι to (3.9) gives

$$\mathbf{\Lambda}^{-k} \mathcal{O}_{\mathcal{Z}^\circ(-D)}^\vee \to R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k}), \tag{3.10}$$

again unique up to scalar. Moreover, this morphism is not zero: if it were, its cone $\Phi(\mathbf{\Lambda}^{-k} \mathcal{O}_{\mathcal{Z}^\circ(-D)}^\vee)[1]$ would have $R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k}) \in \langle \mathbf{\Lambda}^k \mathcal{F}^{\boxtimes 2k} \rangle$ as a direct summand, which is absurd.

By Proposition 3.2(c), applying the functor $(\theta \otimes R\zeta_*(\mathbf{\Lambda}^{-1} \otimes -))^\vee \cong R\zeta_*((-)^\vee)$ to (3.10) gives a morphism

$$R\zeta_* \left([R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k \mathcal{F}^\vee(-D)^{\boxtimes 2k})]^\vee \right) \to R\zeta_*(\mathbf{\Lambda}^k \mathcal{O}_{\mathcal{Z}^\circ(-D)}), \tag{3.11}$$

which we must show is an isomorphism. In fact, it suffices to show that the source and target of (3.11) are isomorphic. Indeed, write (3.10) as $X \to Y$, where $Y \cong L\zeta^* Z$ (so $R\zeta_*(Y^\vee) \cong Z^\vee$ by the projection formula). Then $\mathbb{C} \cong \mathrm{Hom}(X, Y) \cong \mathrm{Hom}(L\zeta^* Z^\vee, X^\vee) \cong \mathrm{Hom}(R\zeta_*(Y^\vee), R\zeta_*(X^\vee))$, so (3.11) is nonzero and unique up to scalar.

Recall that $\mathcal{N}$ is isomorphic to the moduli stack of rank 2 vector bundles on C with determinant $\Lambda(-2D)$, with universal family $\mathcal{F}(-D)$ on $\mathcal{N}\times C$. As in Notation 3.1, we write $R\pi_{\mathcal{N}*}(\mathcal{F}(-D)) = [\mathcal{A}' \to \mathcal{B}']$, where $\mathcal{A}'$ and $\mathcal{B}'$ are vector bundles on $\mathcal{N}$ with $\mathbb{G}_m$-weight 1 and ranks a', b', where $a'-b' = 2-2\ell$. A polystable vector bundle of the form $\mathcal{O}(D)\oplus\mathcal{O}(D)$, where D is an effective divisor of degree $g-l$, has at least a 2-dimensional space of global sections, so $a'\geq 2$. Writing $\alpha : \mathcal{Z}^\circ \to \mathcal{N}$ so that $\zeta = \rho\circ\alpha$, we have

$$\begin{aligned} R\pi_{\mathcal{Z}^\circ *}(\mathbf{\Lambda}^k\mathcal{F}^\vee(-D)^{\boxtimes 2k}) &\cong \mathbf{\Lambda}^{-k}\otimes R\pi_{\mathcal{Z}^\circ *}(\alpha\times\mathrm{id})^*(\mathcal{F}(-D)^{\boxtimes 2k}) \\ &\cong \alpha^*(\mathbf{\Lambda}^{-k}\otimes R\pi_{\mathcal{Z}^\circ *}(\mathcal{F}(-D)^{\boxtimes 2k})) \\ &\cong \alpha^*(\mathbf{\Lambda}^{-k}\mathrm{Sym}^{2k}[\mathcal{A}'\to\mathcal{B}']). \end{aligned}$$

(For the last equality, see the proof of [Tev23, Lemma 6.13]). Hence the left hand side of (3.11) is the descent of $\mathbf{\Lambda}^k\mathrm{Sym}^{2k}[\mathcal{A}'\to\mathcal{B}']^\vee$ to $\mathbb{N}$ (note that this has $\mathbb{G}_m$-weight 0). Analogous to (3.1), we have a diagram

$$\begin{tikzcd} \mathcal{Z}(-D) \arrow[r, hook] & \mathcal{A}' \arrow[r, "\alpha'"] & \mathcal{N} \arrow[d, "\rho"] \\ \mathcal{Z}^\circ(-D) \arrow[u, hook] \arrow[r, hook] \arrow[rr, bend right, "\zeta'"'] & \mathcal{A}'^\circ \arrow[u, hook] & \mathbb{N}. \end{tikzcd}$$

The right hand side of (3.11) is $R\zeta'_*(\mathbf{\Lambda}^k)$. Hence it suffices to prove:

Claim 3.21. We have $R\zeta'_*(\mathbf{\Lambda}^k)\cong\mathbf{\Lambda}^k\mathrm{Sym}^{2k}[\mathcal{A}'\to\mathcal{B}']^\vee$.

As in Proposition 3.2(a), we have a Koszul resolution in $D^b(\mathcal{A}'^\circ)$:

$$\mathcal{O}_{\mathcal{Z}^\circ(-D)} \cong \left[\bigwedge^{b'}\alpha'^*\mathcal{B}'^\vee|_{\mathcal{A}'^\circ} \to \ldots \to \alpha'^*\mathcal{B}'^\vee|_{\mathcal{A}'^\circ} \to \mathcal{O}_{\mathcal{A}'^\circ}\right].$$

As above, $\zeta' : \mathcal{A}'^\circ \to \mathbb{N}$ is a twisted projective bundle with fiber $\mathbb{P}^{a'-1}$. It follows that

$$R\zeta'_*(\mathbf{\Lambda}^k) \cong R\zeta_*\left[\bigwedge^{b'}\alpha'^*\mathcal{B}'^\vee|_{\mathcal{A}'^\circ}\otimes\mathbf{\Lambda}^k \to \ldots \to \alpha'^*\mathcal{B}'^\vee|_{\mathcal{A}'^\circ}\otimes\mathbf{\Lambda}^k \to \mathcal{O}_{\mathcal{A}'^\circ}\otimes\mathbf{\Lambda}^k\right].$$

We claim that $R^s\zeta'_*[\bigwedge^m\alpha'^*\mathcal{B}'^\vee|_{\mathcal{A}'^\circ}\otimes\mathbf{\Lambda}^k] = 0$ for all s when $2k<m\leq b'$ and for $s>0$ when $2k\geq m$. Indeed, we can work locally on $\mathbb{N}$, so $\alpha'^*\mathcal{B}'$ can be replaced by $\mathcal{O}_{\zeta'}(1)^{\oplus b'}$. Recall that $\mathbf{\Lambda}$ has $\mathbb{G}_m$-weight 2. Note that $R\Gamma(\mathbf{P}^{a'-1},\mathcal{O}(2k-m)) = 0$ for $2k<m\leq b'$, since then $-a' = 2\ell-b'-2 < 2k-m < 0$. It follows that

$$R\zeta'_*(\mathbf{\Lambda}^k) \cong \left[\zeta_*\bigwedge^{2k}\alpha'^*\mathcal{B}'^\vee|_{\mathcal{A}'^\circ}\otimes\mathbf{\Lambda}^k \to \ldots \to \zeta_*\alpha'^*\mathcal{B}'^\vee|_{\mathcal{A}'^\circ}\otimes\mathbf{\Lambda}^k \to \zeta_*\mathcal{O}_{\mathcal{A}'^\circ}\otimes\mathbf{\Lambda}^k\right]$$

(underived pushforwards, since higher cohomologies vanish). Since $\alpha'_*\mathcal{O}_{\mathcal{A}'^\circ}\cong\mathrm{Sym}^\bullet\mathcal{A}'^\vee$ (recall that $a'\geq 2$), computing the zero-weight part gives

$$R\zeta'_*(\mathbf{\Lambda}^k) \cong \left[\mathbf{\Lambda}^k\otimes\bigwedge^{2k}\mathcal{B}'^\vee \to \ldots \to \mathbf{\Lambda}^k\otimes\mathcal{B}'^\vee\otimes\mathrm{Sym}^{k-1}\mathcal{A}'^\vee \to \mathbf{\Lambda}^k\otimes\mathrm{Sym}^k\mathcal{A}'^\vee\right],$$

which is indeed isomorphic to $\mathbf{\Lambda}^k\mathrm{Sym}^{2k}[\mathcal{A}'\to\mathcal{B}']^\vee$.

□

Proof of Lemma 3.14. By Propositions 3.2(b) and 3.2(c), coherent duality gives

$$R\zeta_*R\mathcal{H}om(\mathcal{O},\theta\otimes\mathbf{\Lambda}^{-1}[1]) \cong R\mathcal{H}om(\mathcal{O},\mathcal{O}) \cong \mathcal{O}.$$

Applying $R^0\Gamma$ gives a nonzero morphism $\mathcal{O}\to\theta\otimes\mathbf{\Lambda}^{-1}[1]$ whose image under ζ_* is an isomorphism $\mathcal{O}\to\mathcal{O}$ by construction. We complete this morphism to an exact triangle $\mathcal{O}\to\theta\otimes\mathbf{\Lambda}^{-1}[1]\to K\to$, where $K\in\mathbf{K}$. Tensoring with objects of $\mathbf{T}$ gives the required mutation.

□

4. Basic weaving patterns

In this section, we prove the various technical lemmas used in Section 2. The main statements and their proofs are taken from [Tev23] (where they are proved only for $d = 2g-1$) with minor modifications and some additional details. We include them here mainly to verify that the numerical bookkeeping required to apply Theorem 4.4 below remains valid in any degree $d \leq 2g$; we omit those proofs with no dependence on d. Let $v = \lfloor \frac{d-1}{2} \rfloor$ and $M_i = M_i(d)$. We write $\mathcal{P}_K$ for the Fourier–Mukai functor with kernel K.

4.1. **Cross Warp.** Closely following [Tev23, Section 4], we prove the following:

Theorem 4.1 (Basic Cross Warp, cf. [Tev23, Theorem 3.2])**.** *For $0 \leq k \leq i \leq v$, we have:*

(a) $\mathcal{P}_{\mathcal{F}^{\vee \boxtimes k}} : D^b(\mathrm{Sym}^k C) \to D^b(M_i)$ *is fully faithful.*

(b) $\mathcal{P}_{\mathcal{D}_i^k} : D^b(\mathrm{Sym}^k C) \to D^b(M_i)$ *is fully faithful.*

(c) *If $k \leq i-1$, then $\iota\langle \mathcal{D}_{i-1}^k \rangle = \langle \mathcal{D}_i^k \rangle$ where ι is the windows embedding of Proposition 2.7. Moreover, objects in $\langle \mathcal{D}_{i-1}^k \rangle$ descend from objects with weights in the range $[0,k]$ for this wall crossing.*

(d) *There is an admissible subcategory of $D^b(M_i)$ with semiorthogonal decompositions*

$$\langle \mathcal{F}^{\vee \boxtimes k-1}, \ldots, \mathcal{O}, \mathcal{D}_i^k, \mathcal{D}_i^{k-1}\mathbf{\Lambda}^{-1}, \ldots, \mathbf{\Lambda}^{-k} \rangle$$

and

$$\langle \mathcal{D}_i^{k-1}\mathbf{\Lambda}^{-1}, \ldots, \mathbf{\Lambda}^{-k}, \mathcal{F}^{\vee \boxtimes k}, \mathcal{F}^{\vee \boxtimes k-1}, \ldots, \mathcal{O} \rangle$$

related by the mutation in Figure 1.

Remark 4.2. It follows from [TT21, Section 3] that $\iota\langle \mathcal{F}^{\vee \boxtimes k} \rangle = \langle \mathcal{F}^{\vee \boxtimes k} \rangle \subset D^b(M_i)$ for $k < i$, and that objects in this subcategory have weights in the range $[0,k]$.

Corollary 4.3. *$\langle \mathcal{D}_i^k \rangle$ is contained in the subcategory generated by $\langle \mathcal{F}^{\vee \boxtimes \ell}\mathbf{\Lambda}^{-m} \rangle$ with $0 \leq \ell \leq k$, $0 \leq m \leq k-\ell$.*

Proof. Theorem 4.1(d) and induction on k. ☐

Proof of Theorem 4.1(a). Note that since $\mathcal{F}^{\vee \boxtimes k} \cong (\Lambda^{-k} \boxtimes \mathbf{\Lambda}^{-k}) \otimes \mathcal{F}^{\boxtimes k}$ (see [Tev23, Lemma 3.4]), we may equivalently prove that $\mathcal{P}_{\mathcal{F}^{\boxtimes k}}$ is fully faithful. The case $d \leq 2g-1$ is [TT21, Theorem 9.2], and the proof for $d = 2g$ is the same. Indeed, since [TT21, Cor. 8.2, Cor. 8.4, Thm. 9.6] are proved for all $d \leq 2g+1$, we only need to verify that $k \leq 3g-2i-2$ for $k \leq i$. Since $k \leq g-1$, we have $3g-2i-2 \geq g$, so this holds. ☐

The other parts of Theorem 4.1 will be proved shortly. We require the vanishing theorems for tensor vector bundles proved in [TT21].

Theorem 4.4 ([Tev23, Theorems 3.8, 3.9, 3.10])**.** *Let d', j, a, b, and t be integers with $2 < d' \leq 2g+1$, $1 \leq j \leq \left\lfloor \frac{d'-1}{2} \right\rfloor$, and let $D \in \mathrm{Sym}^a C$, $D' \in \mathrm{Sym}^b C$.*

(a) *If $a,\ b \leq d'+g-2j-1$, $t \notin [0,a]$, and $a-j-1 < t < d'+g-2j-1-b$, then*

$$R\Gamma\left(M_j(d'), (\overline{\mathcal{F}}_D^{\boxtimes a})^\vee \otimes \overline{\mathcal{F}}_{D'}^{\boxtimes b} \otimes \mathbf{\Lambda}^t\right) = 0$$

and, if $D = \sum \alpha_i x_i$,

$$R\Gamma\left(M_j(d'), \bigotimes_i (\mathcal{F}_{x_i}^\vee)^{\otimes \alpha_i} \otimes \overline{\mathcal{F}}_{D'}^{\boxtimes b} \otimes \mathbf{\Lambda}^t\right) = 0.$$

(b) *If $a < t < d'+g-2j-1-b$, then*

$$R\Gamma\left(M_j(d'), (\overline{\mathcal{F}}_D^{\boxtimes a})^\vee \otimes \overline{\mathcal{F}}_{D'}^{\boxtimes b} \otimes \mathbf{\Lambda}^t\right) = R\Gamma\left(M_j(d'), (\mathcal{F}_D^{\boxtimes a})^\vee \otimes \mathcal{F}_{D'}^{\boxtimes b} \otimes \mathbf{\Lambda}^t\right) = 0.$$

Moreover, the same vanishing holds with $j = 0$ for any $d' > 0$.

(c) *If $a \leq j$, $b < d'+g-2j-1$, and $D \not\leq D'$ (for example, $a > b$), then*

$$R\Gamma\left(M_j(d'), (\overline{\mathcal{F}}_D^{\boxtimes a})^\vee \otimes \overline{\mathcal{F}}_{D'}^{\boxtimes b}\right) = 0.$$

Remark 4.5. Theorem 4.4(c) is not completely proved in [TT21, Tev23] for $d' = 2g, 2g+1$. It comes from [TT21, Theorem 9.6], which is proved under the inductive assumption $\mathcal{P}_{\mathcal{F}^{\boxtimes k}}$ is fully faithful for $k < a$. This is true for $d' = 2g$ by Theorem 4.1(a), and for $d' = 2g+1$ by the same argument (since then $a \le i \le g$, so $k \le g-1 = d'+g-2g-2$).

Conjecture 4.6. *Theorem 4.4 holds for any $d' > 2$ and $j \le v$ with $3j \le d'+g-1$.*

Remark 4.7. We expect this conjecture would follow from a careful analysis of the proofs given in [TT21]. The main purpose of the hypothesis $d' \le 2g+1$ is to ensure that $3j \le d'+g-1$ (needed for calculations using windows) for all $j \le \lfloor (d'-1)/2 \rfloor$; one needs to check that it is not needed otherwise (e.g., for inequalities related to ample cones).

Lemma 4.8 (cf. [Tev23, Lemma 4.1])**.** *For $1 \le k \le i$, $D^b(M_i)$ contains admissible subcategories*

$$\langle \mathcal{F}^{\vee \boxtimes k-1}, \dots, \mathcal{F}^\vee, \mathcal{O}, \mathbf{\Lambda}^{-k} \rangle \text{ and } \langle \mathbf{\Lambda}^{-k}, \mathcal{F}^{\vee \boxtimes k}, \dots, \mathcal{F}^\vee, \mathcal{O} \rangle.$$

Proof. All blocks are admissible by Theorem 4.1(a). By [TT21, Lemma 10.2], it suffices to check semi-orthogonality on skycraper sheaves of closed points $D \in \mathrm{Sym}^a C$, $D' \in \mathrm{Sym}^b C$. For $b < a \le k$, we have $b < v < d+g-2i-1$ since $v < \frac{d+g-1}{3}$. Thus

$$R\operatorname{Hom}(\mathcal{F}^{\vee}{}^{\boxtimes b}_{D'}, \mathcal{F}^{\vee}{}^{\boxtimes a}_{D}) = R\operatorname{Hom}(\overline{\mathcal{F}}^{\boxtimes a}_{D}, \overline{\mathcal{F}}^{\boxtimes b}_{D'}) = 0$$

by [Tev23, Lemma 3.4] and Theorem 4.4(c). Next, for $0 \le b \le k$, we have

$$0, l \le i < d+g-2i-1, \qquad -k \notin [0,0], \qquad -i-1 < -k < d+g-2i-1-b,$$

so

$$R\operatorname{Hom}(\mathcal{F}^{\vee}{}^{\boxtimes b}_{D'}, \mathbf{\Lambda}^{-k}) = R\Gamma(M_i, \overline{\mathcal{F}}^{\boxtimes b}_{D'} \otimes \mathbf{\Lambda}^{-k}) = 0$$

by Theorem 4.4(a). Finally, for $0 \le a < k$, we have $a < k \le v < d+g-2i-1$, so

$$R\operatorname{Hom}(\mathbf{\Lambda}^{-k}, \mathcal{F}^{\vee}{}^{\boxtimes a}_{D}) = R\Gamma(M_i, (\overline{\mathcal{F}}^{\boxtimes a}_{D})^\vee \otimes \mathbf{\Lambda}^{k}) = 0$$

by Theorem 4.4(b). □

We introduce the following complexes on $\mathrm{Sym}^k C \times M_i$, which will allow us to break the mutation of Theorem 4.1(d) into two stages (Lemmas 4.13 and 4.14):

Notation 4.9 ([Tev23, Definitions 4.2 and 4.5])**.** Let $\mathcal{F}^\bullet = [\mathcal{F}^\vee \to \mathcal{O}] \in D^b(C \times M_i)$, with $\mathcal{O}$ in degree 0 and the map given by contraction with the universal section of $\mathcal{F}$. Let

$$\mathcal{F}^{\bullet \boxtimes k} = \tau_*^{S_k}(\pi_1^* \mathcal{F}^\bullet \otimes \cdots \otimes \pi_k^* \mathcal{F}^\bullet \otimes \mathrm{sgn})$$

where $\tau : C^k \to \mathrm{Sym}^k C$ is the quotient, $\pi_\ell : C^k \to C$ are the projections, and sgn is the sign representation of S_k.

We prove parts (b), (c), and (d) Theorem 4.1 by induction on k. The following lemmas are proved for each k as part of the same induction.

Lemma 4.10. *For $0 \le \ell \le k$, $0 \le m < k$ and X, Y skyscraper sheaves at $D \in \mathrm{Sym}^\ell C$, $D' \in \mathrm{Sym}^m C$, respectively,*

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{F}^{\vee \boxtimes \ell}}(X), \mathcal{P}_{\mathcal{D}_i^m \mathbf{\Lambda}^{m-k}}(Y)) = 0 \tag{4.1}$$

and

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{D}_i^\ell \mathbf{\Lambda}^{\ell-k}}(X), \mathcal{P}_{\mathcal{F}^{\vee \boxtimes m}}(Y)) = 0. \tag{4.2}$$

Proof. We have

$$R\Gamma\left(M_{i-m}(\Lambda(-2D')), \left(\mathcal{F}^{\vee}{}^{\boxtimes \ell}_{D}\right)^\vee \mathbf{\Lambda}^{m-k}\right) = R\Gamma\left(M_{i-m}(d-2m), \overline{\mathcal{F}}^{\boxtimes \ell}_{D} \mathbf{\Lambda}^{m-k}\right) = 0$$

by Theorem 4.4(a). Indeed, we have

$$0, \ell \le v < d+g-2i-1, \qquad m-k \notin [0,0], \qquad -i+m-1 < m-k < d+g-2i-1-\ell.$$

This proves (4.1). For (4.2), we have $\omega_{M_i}|_{M_{i-\ell}(d-2\ell)} = \mathcal{O}(-3, -d-g+4+3\ell) = \omega_{M_{i-\ell}(d-2\ell)}\mathbf{\Lambda}^{-\ell}$, so

$$\begin{aligned} R\operatorname{Hom}\left(\mathcal{F}^{\vee\boxtimes m}_{D'}, \mathcal{O}_{M_{i-\ell}(d-2\ell)}\mathbf{\Lambda}^{\ell-k}\omega_{M_i}\right) &= R\Gamma\left(M_{i-\ell}(d-2\ell), \overline{\mathcal{F}}^{\boxtimes m}_{D'}\mathbf{\Lambda}^{-k}\omega_{M_{i-\ell}(d-2\ell)}\right) \\ &= R\Gamma\left(M_{i-\ell}(d-2\ell), \left(\overline{\mathcal{F}}^{\boxtimes m}_{D'}\right)^{\vee}\mathbf{\Lambda}^{k}\right)[\ldots]. \end{aligned}$$

by Serre duality (we suppress the shift in degree). This vanishes by Theorem 4.4(b), as $m < k < 3g - 2i - 1$ (note that either $j > 0$ and $d' > 2$, or $j = 0$ and $d' > 0$). (4.2) then follows from Serre duality. □

Remark 4.11. Since $\mathcal{P}_{\mathcal{F}^{\vee\boxtimes\ell}}$ and $\mathcal{P}_{\mathcal{D}_i^m\mathbf{\Lambda}^{m-k}}(Y))$ are fully faithful by Theorem 4.1(a) and the inductive hypothesis, this proves (4.1) for any $X \in D^b(\mathrm{Sym}^\ell C)$, $Y \in D^b(\mathrm{Sym}^\ell C)$ by [TT21, Lemma 10.2]. The same is true for (4.2) with $\ell < k$.

Lemma 4.12 (cf. [Tev23, Corollary 4.13])**.** *There exist $G_k, H_k \in D^b(\mathrm{Sym}^k C \times M_i)$ such that for all $X \in D^b(\mathrm{Sym}^k C)$, we have $\mathcal{P}_{G_k}(X) \in \langle \mathcal{F}^{\vee\boxtimes k-1}, \ldots, \mathcal{F}^\vee, \mathcal{O}\rangle$, $\mathcal{P}_{H_k}(X) \in \langle \mathcal{D}_i^{k-1}\mathbf{\Lambda}^{-1}, \ldots, \mathcal{D}_i^1\mathbf{\Lambda}^{1-k}, \mathbf{\Lambda}^{-k}\rangle$, and exact triangles*

$$\mathcal{P}_{G_k}(X) \to \mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}(X) \to \mathcal{P}_{\mathcal{F}^{\vee\boxtimes k}}(X)[k] \to \tag{4.3}$$

and

$$\mathcal{P}_{H_k}(X) \to \mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}(X) \to \mathcal{P}_{\mathcal{D}_i^k}(X(-B/2)) \to \tag{4.4}$$

where $B \subset \mathrm{Sym}^k C$ is the diagonal divisor.

Proof. The proofs of Lemma 4.7 through Corollary 4.13 in [Tev23] carry over to any d without change. □

Lemma 4.13 (cf. [Tev23, Lemma 4.14])**.** *$\mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}$ is fully faithful and there is a mutation*

$$\langle \mathcal{F}^{\vee\boxtimes k}, \mathcal{F}^{\vee\boxtimes k-1}, \ldots, \mathcal{O}\rangle \to \langle \mathcal{F}^{\vee\boxtimes k-1}, \ldots, \mathcal{O}, \mathcal{F}^{\bullet\boxtimes k}\rangle.$$

Proof. Applying $R\operatorname{Hom}(-, \mathcal{P}_{\mathcal{F}^{\vee\boxtimes m}}(Y))$ to (4.4) and using (4.2) gives

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}(X), \mathcal{P}_{\mathcal{F}^{\vee\boxtimes m}}(Y)) = 0 \tag{4.5}$$

for $m < k$ and X, Y skyscraper sheaves (note $R\operatorname{Hom}(\mathcal{P}_{H_k}(X), \mathcal{P}_{\mathcal{F}^{\vee\boxtimes m}}(Y)) = 0$ by Remark 4.11). Then

$$\begin{aligned} R\operatorname{Hom}(\mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}(X), \mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}(Y)) &= R\operatorname{Hom}(\mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}(X), \mathcal{P}_{\mathcal{F}^{\vee\boxtimes k}}(Y)[k]) \\ &= R\operatorname{Hom}(\mathcal{P}_{\mathcal{F}^{\vee\boxtimes k}}(X)[k], \mathcal{P}_{\mathcal{F}^{\vee\boxtimes k}}(Y)[k]) = R\operatorname{Hom}(X, Y) \end{aligned}$$

by (4.3), (4.5), and Lemma 4.8. By the Bondal–Orlov criterion, this proves full faithfulness. Lemma 4.8 and (4.5) give semiorthogonality, and (4.3) yields the claimed mutation. □

Lemma 4.14. *$\mathcal{P}_{\mathcal{D}_i^k}$ is fully faithful and there is a mutation*

$$\langle \mathcal{D}_i^{k-1}\mathbf{\Lambda}^{-1}, \ldots, \mathbf{\Lambda}^{-k}, \mathcal{F}^{\bullet\boxtimes k}\rangle \to \langle \mathcal{D}_i^k, \mathcal{D}_i^{k-1}\mathbf{\Lambda}^{-1}, \ldots, \mathbf{\Lambda}^{-k}\rangle.$$

Proof. We show

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}(X), \mathcal{P}_{\mathcal{D}_i^m\mathbf{\Lambda}^{m-k}}(Y)) = 0 \tag{4.6}$$

and

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{D}_i^m\mathbf{\Lambda}^{m-k}}(Y), \mathcal{P}_{\mathcal{D}_i^k}(X)) = 0 \tag{4.7}$$

for $m < k$ and X, Y skyscraper sheaves. (4.6) follows from (4.1) and (4.3). By the inductive hypothesis Theorem 4.1(d) and induction on m, it suffices to prove (4.7) that

$$0 = R\operatorname{Hom}\left(\mathcal{F}^{\vee\boxtimes b}_{D'}\mathbf{\Lambda}^{-t}, \mathcal{O}_{M_{i-k}(d-2k)}\right) = R\Gamma\left(M_{i-k}(d-2k), \overline{\mathcal{F}}^{\boxtimes b}_{D'}\mathbf{\Lambda}^{t}\right)$$

for $0 \le b < k$, $0 < t \le k - b$, $D \in \mathrm{Sym}^k C$, $D' \in \mathrm{Sym}^b C$ (cf. Corollary 4.3). Since $k \le i < d + g - 2i - 1$, we have $0 < t < d + g - 2i - 1 - b$, so the required vanishing follows from Theorem 4.4(b).

To prove full faithfulness, we compute

$$\begin{aligned} R\operatorname{Hom}(\mathcal{P}_{\mathcal{D}_i^k}(X), \mathcal{P}_{\mathcal{D}_i^k}(Y)) &= R\operatorname{Hom}(\mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}(X(B/2)), \mathcal{P}_{\mathcal{D}_i^k}(Y)) \\ &= R\operatorname{Hom}(\mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}(X(B/2)), \mathcal{P}_{\mathcal{F}^{\bullet\boxtimes k}}(Y(B/2))) = R\operatorname{Hom}(X, Y) \end{aligned}$$

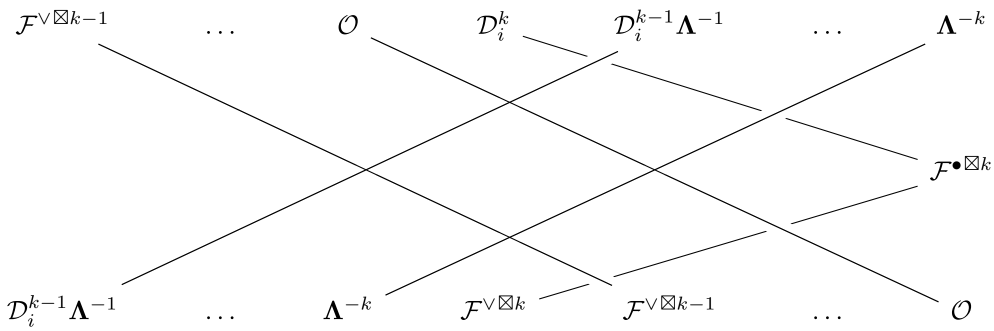


FIGURE 4. The mutation of Theorem 4.1(d) in two steps, cf. [Tev23, Figure 9].

by (4.4), (4.7), (4.6), and Lemma 4.13. Semiorthonality then follows from (4.6) and (4.7), together with the inductive hypothesis. Finally, (4.4) gives the required mutation. □

Proof of Theorem 4.1. We proved 4.1(a) above and 4.1(b) in Lemma 4.14. 4.1(d) follows from Lemmas 4.10, 4.13 and 4.14, as depicted in Figure 4. Finally, 4.1(c) follows from 4.1(d) and the inductive hypothesis. Indeed, we have $\iota\langle\mathcal{F}^{\vee\boxtimes\ell}\rangle = \langle\mathcal{F}^{\vee\boxtimes\ell}\rangle$ for $0 \le \ell \le k$ by Remark 4.2 and $\iota\langle\mathcal{D}_{i-1}^m\mathbf{\Lambda}^{m-k}\rangle = \langle\mathcal{D}_i^m\mathbf{\Lambda}^{m-k}\rangle$ by the inductive hypothesis (recall that $\mathbf{\Lambda}$ has weight -1). Performing the mutation of 4.1(d), embedding via ι, and undoing the mutation on the other side shows that $\iota\langle\mathcal{D}_{i-1}^k\rangle = \langle\mathcal{D}_i^k\rangle$. It is clear from the bottom decomposition that the entire subcategory has weights in the range $[0,k]$, so this holds for $\langle\mathcal{D}_{i-1}^k\rangle$ a fortiori. □

4.2. **Farey Twill.** We now turn to the orthogonalities and mutations required to implement the Farey Twill. Recall that for integers k,s with $0 \le k \le v$ and $t \in [k, v+1)$, we define $\mathcal{D}_t^{k,s} = \mathcal{D}_{\lfloor t\rfloor}^k \otimes L_t^{k,s}$ where

$$L_t^{k,s} = \begin{cases} \mathcal{O}(s,sk) & k = \lfloor t\rfloor \\ \mathcal{O}\left(\left\lfloor \frac{s}{t-k}\right\rfloor, s + \left\lfloor\frac{s}{t-k}\right\rfloor(k-1)\right) & k < \lfloor t\rfloor\,. \end{cases}$$

Lemma 4.15 (cf. [Tev23, Lemma 2.7])**.** *For $k \le i-1$ and $\epsilon \ll 0$, we have $\iota\langle\mathcal{D}_{i-\epsilon}^{k,s}\rangle = \langle\mathcal{D}_i^{k,s}\rangle$.*

Proof. When $i=1$, ι is simply pullback under $M_1 \to M_0$, so $\iota\langle\mathcal{D}_{1-\epsilon}^{0,s})\rangle = \langle\iota(\mathcal{O}(s))\rangle = \langle\mathcal{D}_1^{0,s}\rangle$. Suppose $i>1$. By [TT21, Remark 3.22], $\mathcal{O}(m,n)$ has weight $n+m(1-i)$ for the wall crossing $M_{i-1} \dashrightarrow M_i$. Hence $L_{i-\epsilon}^{k,s} = L_i^{k,s}$ has weight $s - \left\lfloor\frac{s}{i-k}\right\rfloor(i-k) \in [0, i-k)$. Hence the objects in $\langle(\mathcal{D}_{i-\epsilon}^{k,s})\rangle$ have weights in the range $[0,i)$. The lemma then follows from Proposition 2.7 and Theorem 4.1(c). □

Lemma 4.16 (cf. [Tev23, Lemma 2.8])**.** *Let $x = \frac{s}{t-k} \notin \mathbb{Z}$. If $k<k'$ and either $x = \frac{s'}{t-k'}$ or $s' = t-k' = 0$, then*

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{D}_t^{k,s}}(X), \mathcal{P}_{\mathcal{D}_t^{k',s'}}(Y)) = 0 \tag{4.8}$$

for any $X \in D^b(\operatorname{Sym}^k C)$, $Y \in D^b(\operatorname{Sym}^{k'} C)$.

Proof. By Corollary 4.3, it suffices to prove

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{F}^{\vee\boxtimes\ell}\mathbf{\Lambda}^{-m}L_t^{k,s}}(Z), \mathcal{P}_{\mathcal{D}_t^{k',s'}}(Y)) = 0$$

for $0 \le \ell \le k$, $0 \le m \le k-\ell$, and $Z \in D^b(\operatorname{Sym}^\ell C)$. Without loss of generality, we take Y and Z to be skyscraper sheaves, reducing (4.8) to

$$R\operatorname{Hom}\left(\mathcal{F}_D^{\vee\boxtimes\ell}\mathbf{\Lambda}^{-m}L_t^{k,s}, \mathcal{O}_{M_0(\Lambda(-2D'))}L_t^{k',s'}\right) = 0. \tag{4.9}$$

We first consider the case $k' = \lfloor t\rfloor$, so $L_t^{k',s'} = \mathcal{O}(s', s'k')$. By [TT21, Remark 3.7], $\mathcal{O}(m,n)$ restricts to $\mathcal{O}(n+m(1-k')) = \mathbf{\Lambda}^{-n-m(1-k')}$ on the fibers $M_0(d-2k') \cong \mathbb{P}^{d+g-2-2k'}$ of $D_{k'}^{k'}$ over $\operatorname{Sym}^{k'} C$. Noting that $s - s' = x(k'-k)$, we see that $(L_t^{k,s})^{-1}L_t^{k',s'}$ restricts to $\mathbf{\Lambda}^{\{x\}(k'-k)}$ where $\{x\} = x - \lfloor x\rfloor$. The case

$k' < \lfloor t \rfloor$ is similar: $\mathcal{O}(m,n)$ restricts to $\mathcal{O}(m, n-mk')$ on $M_{i-k'}(d-2k')$, so $(L_t^{k,s})^{-1}L_t^{k',s'}$ restricts to $\mathcal{O}(0,\{x\}(k-k')) = \mathbf{\Lambda}^{\{x\}(k'-k)}$. Either way, (4.9) becomes

$$R\Gamma\left(M_{\lfloor t \rfloor - k'}(d-2k'), \overline{\mathcal{F}}_D^{\boxtimes \ell}\mathbf{\Lambda}^{m+\{x\}(k'-k)}\right) = 0.$$

This follows from Theorem 4.4(b) once we verify

$$0 < m + \{x\}(k'-k) < d - 2\lfloor t \rfloor + g - 1 - \ell.$$

Since $\{x\} > 0$, the first inequality is clear. For the second, we check $m + \ell + \{x\}(k'-k) + 2\lfloor t \rfloor < d + g - 1$. Since

$$m + \ell \leq k, \quad \{x\}(k'-k) < k'-k, \quad k' \leq \lfloor t \rfloor, \text{ and } \quad 3\lfloor t \rfloor \leq 3v < d + g - 1,$$

we're done. □

Lemma 4.17 (cf. [Tev23, Lemma 2.9])**.** *For all $k \leq i$, there is a mutation*

$$\langle \mathcal{D}_i^k, \ldots, \mathcal{D}_i^{i-1}, \mathcal{D}_i^i \rangle \to \langle \mathcal{D}_i^i, \mathcal{D}_i^{i-1}(-1, 2-i), \ldots, \mathcal{D}_i^k(-1, 1-k)\rangle.$$

Proof. Semiorthogonality on the left-hand side follows from Proposition 2.7 and Theorem 4.1(c). As in [Tev23, Claim 3.9], we have for every $X \in \mathrm{Sym}^n C$ an exact triangle

$$\mathcal{P}_{\mathcal{D}_i^n}(X)(-1, 1-k) \to \mathcal{P}_{\mathcal{D}_i^n}(X) \to \mathcal{P}_{\mathcal{O}_{E_i^n}}(X) \to \tag{4.10}$$

where $\mathcal{P}_{\mathcal{O}_{E_i^n}}(X) \in \langle \mathcal{D}_i^{n+1}, \ldots, \mathcal{D}_i^i \rangle$. Here, E_i^n is the divisor $\{(D, F, s) \in D_i^n : \deg Z(s) \geq n+1\}$ where $Z(s)$ denotes the scheme of zeros. We proceed by downward induction on k, the case $i = k$ being trivial. By the inductive hypothesis, it suffices to give a mutation $\langle \mathcal{D}_i^k, \mathcal{D}_i^{k+1}, \ldots, \mathcal{D}_i^i \rangle \to \langle \mathcal{D}_i^{k+1}, \ldots, \mathcal{D}_i^i, \mathcal{D}_i^k(-1, 1-k)\rangle$. By (4.10), we are left to show that $\mathcal{D}_i^k(-1, 1-k)$ is contained in ${}^\perp\langle \mathcal{D}_i^{k+1}, \ldots, \mathcal{D}_i^i \rangle$. Again by the inductive hypothesis, this amounts to showing that

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{D}_i^k}(X)(-1, 1-k), \mathcal{P}_{\mathcal{D}_i^n}(Y)(-1, 1-n)) = 0$$

for $k < n \leq i$, $X \in D^b(\mathrm{Sym}^k C)$, $Y \in \mathrm{Sym}^n C$. (Note that $\mathcal{O}(-1, 1-i)$ has trivial restriction to $M_0(d-2i)$, so $\mathcal{D}_i^i(-1, 1-i) \cong \mathcal{D}_i^i$.) By Corollary 4.3, the required vanishing is

$$R\Gamma(M_{i-n}(d-2n), \overline{\mathcal{F}}_D^{\boxtimes \ell}\mathbf{\Lambda}^{m+n-k}) = 0$$

for $0 \leq \ell \leq k$, $0 \leq m \leq k - \ell$, $D \in \mathrm{Sym}^\ell C$. We have $0 < m + n - k < d + g - 2i - 1 - \ell$, so we're done by Theorem 4.4(b). □

Lemma 4.18. *For $k, k', j, j' \geq 0$ with $j' + k' < j + k \leq i$, we have*

$$R\operatorname{Hom}(\mathcal{P}_{\mathcal{D}_i^k \mathbf{\Lambda}^{-j}}(X), \mathcal{P}_{\mathcal{D}_i^{k'} \mathbf{\Lambda}^{-j'}}(Y)) = 0$$

for any $X \in D^b(\mathrm{Sym}^k C)$, $Y \in D^b(\mathrm{Sym}^{k'} C)$.

Proof. As above, it suffices to show that

$$R\operatorname{Hom}(\mathcal{O}_{M_{i-k}(d-2k)}\mathbf{\Lambda}^{-j}, \mathcal{F}^{\vee \boxtimes \ell'}_D \mathbf{\Lambda}^{-j'-m'}) = 0$$

for $0 \leq \ell' \leq k'$, $0 \leq m' \leq k' - \ell'$, $D \in \mathrm{Sym}^{\ell'} C$. By Serre duality applied twice (cf. the proof of Lemma 4.10), this is equivalent to

$$R\Gamma\left(M_{i-k}(d-2k), (\overline{\mathcal{F}}_D^{\boxtimes \ell'})^\vee \mathbf{\Lambda}^{j-j'-m'+k}\right) = 0.$$

This follows from Theorem 4.4(b) once $\ell' < j - j' - m' + k < d + g - 2i - 1$. For the first, we have $(j - j' + k) - (m' + \ell') > k' - k' = 0$; for the second, we have $j - j' - m' + k \leq j + k \leq i < d + g - 2i - 1$. □

4.3. **Broken Loom.** Finally, we adapt the reordering trick from [Tev23, Section 5]. We recall the following lemma:

Lemma 4.19 ([Tev23, Lemma 5.4, Remark 5.7])**.** *Suppose $2 < d' \leq 2g+1$ and $1 \leq j \leq \left\lfloor \frac{d-1}{2} \right\rfloor$. Let $D \in \mathrm{Sym}^a C$, $D' \in \mathrm{Sym}^b C$ with $a, b \leq j$ ([Tev23] has $a, b \leq \min(j, d'+g-2j-1)$, but $j \leq d'+g-2j-1$ already), and let t be an integer with $a-j-1 < t < d'+g-2j-1-b$ and $2t < a-b$. Then*

$$R\Gamma\left(M_j(d'), \left(\overline{\mathcal{F}}_D^{\boxtimes a}\right)^\vee \otimes \overline{\mathcal{F}}_{D'}^{\boxtimes b} \otimes \mathbf{\Lambda}^t\right) = 0. \tag{4.11}$$

Lemma 4.20. *Let λ, λ', k, k' be integers with $k, k', \lambda - 2k, \lambda' - 2k' \geq 0$ and $\lambda - k, \lambda' - k' \leq i_d$. If $\lambda < \lambda'$, then*

$$R\operatorname{Hom}\left(\mathbf{\Lambda}^{-k} \mathcal{F}^{\vee \boxtimes \lambda - 2k}_D, \mathbf{\Lambda}^{-k'} \mathcal{F}^{\vee \boxtimes \lambda' - 2k'}_{D'}\right) = 0$$

for any $D \in \mathrm{Sym}^{\lambda-2k} C$, $D' \in \mathrm{Sym}^{\lambda'-2k'} C$.

Proof. We need

$$R\Gamma\left(M_{i_d}(d), (\overline{\mathcal{F}}_D^{\boxtimes \lambda-2k})^\vee \overline{\mathcal{F}}_{D'}^{\boxtimes \lambda'-2k'} \mathbf{\Lambda}^{k-k'}\right) = 0,$$

which is exactly (4.11). We clearly have $\lambda - 2k, \lambda' - 2k' \leq i_d$, and $2(k-k') < \lambda' - 2k' - (\lambda - 2k')$ since $\lambda' > \lambda$. It remains to show that

$$\lambda' - 2k' - i_d - 1 < k - k' < d + g - 2i_d - 1 - \lambda + 2k.$$

We have $\lambda' - k' - i_d - 1 \leq -1 < k$ for the first inequality, and $d+g-2i_d-1-(\lambda-k) \geq d+g-3i_d-1 > 0 \geq -k$ for the second. ☐

References

[BBF+24] P. Belmans, J. Bose, S. Frei, B. Gould, J. Hotchkiss, A. Lamarche, J. Petok, C. R. Avila, and S. Shah, *On decompositions for Fano schemes of intersections of two quadrics* (2024), available at `https://arxiv.org/abs/2403.12517`.

[Bel21] P. Belmans, *Seshadri's desingularisation in the Hodge diamond cutter, and a bold proposal*, 2021. `https://pbelmans.ncag.info/blog/2021/03/21/seshadris-desingularisation-in-hodge-diamond-cutter/`.

[BO02] A. Bondal and D. Orlov, *Derived categories of coherent sheaves*, Proceedings of the ICM, Vol. II (2002), 47–56.

[CCK05] I. Choe, J. Choy, and Y.-H. Kiem, *Cohomology of the moduli space of Hecke cycles*, Topology **44** (2005), no. 3, 585–608.

[DB02] S. Del Baño, *On the motive of moduli spaces of rank two vector bundles over a curve*, Compositio Math. **131** (2002), no. 1, 1–30.

[DN89] J.-M. Drezet and M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. **97** (1989), no. 1, 53–94.

[HL15] D. Halpern-Leistner, *The derived category of a GIT quotient*, J. Amer. Math. Soc. **28** (2015), no. 3, 871–912.

[Huy06] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006.

[Iri20] H. Iritani, *Global mirrors and discrepant transformations for toric Deligne-Mumford stacks*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), Paper No. 032, 111.

[Kel99] B. Keller, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra **136** (1999), no. 1, 1–56. MR1667558

[Kir86] F. Kirwan, *On the homology of compactifications of moduli spaces of vector bundles over a Riemann surface*, Proc. London Math. Soc. (3) **53** (1986), no. 2, 237–266.

[KL04] Y.-H. Kiem and J. Li, *Desingularizations of the moduli space of rank 2 bundles over a curve*, Math. Ann. **330** (2004), no. 3, 491–518.

[KL15] A. Kuznetsov and V. A. Lunts, *Categorical resolutions of irrational singularities*, Int. Math. Res. Not. IMRN **13** (2015), 4536–4625.

[KS99] A. King and A. Schofield, *Rationality of moduli of vector bundles on curves*, Indag. Math. **10** (1999), no. 4, 519–535.

[KT21] N. Koseki and Y. Toda, *Derived categories of Thaddeus pair moduli spaces via d-critical flips*, Adv. Math. **391** (2021), 55 pp.

[Kuz08] A. Kuznetsov, *Lefschetz decompositions and categorical resolutions of singularities*, Selecta Math. **13** (2008), no. 4, 661.

[Kuz16] ———, *Derived categories view on rationality problems*, Rationality problems in algebraic geometry: Levico Terme, Italy 2015, 2016, pp. 67–104.

[Nar17] M. S. Narasimhan, *Derived categories of moduli spaces of vector bundles on curves*, J. Geom. Phys. **122** (2017), 53–58.

[New75] P. E. Newstead, *Rationality of moduli spaces of stable bundles*, Math. Ann. **215** (1975), 251–268.

[New80] P. E. Newstead, *Correction to: Rationality of moduli spaces of stable bundles*, Math. Ann. **249** (1980), 281–282.

[NR69] M. S. Narasimhan and S. Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math. (2) **89** (1969), 14–51.

[NR78] ———, *Geometry of Hecke cycles. I*, C. P. Ramanujam—a tribute, 1978, pp. 291–345.

[Păd21] T. Pădurariu, *Noncommutative resolutions and intersection cohomology for quotient singularities* (2021), available at `https://www.arxiv.org/abs/2103.06215`.

[Ram73] S. Ramanan, *The moduli spaces of vector bundles over an algebraic curve*, Math. Ann. **200** (1973), 69–84.

[Ray82] M. Raynaud, *Sections des fibrés vectoriels sur une courbe*, Bulletin de la Société Mathématique de France **110** (1982), 103–125 (fr).

[Ses77] C. S. Seshadri, *Desingularisation of the moduli varieties of vector bundles on curves*, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), 1977, pp. 155–184.

[Tev23] J. Tevelev, *Braid and phantom* (2023), available at `https://arxiv.org/abs/2304.01825`.

[Tev24] ———, *Weaving patterns in genus* 2 *and quantum cohomology*, 2024. `https://people.math.umass.edu/~tevelev/genus2braid.mp4`.

[Tha94] M. Thaddeus, *Stable pairs, linear systems and the Verlinde formula*, Invent. Math. **117** (1994), no. 2, 317–353.

[TT21] J. Tevelev and S. Torres, *The BGMN conjecture via stable pairs*, to appear in Duke Math. Journal (2021), available at `https://arxiv.org/abs/2108.11951`.

[Tyu64] A. N. Tyurin, *On the classification of two-dimensional fibre bundles over an algebraic curve of arbitrary genus*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 21–52.

[Tyu65] ———, *The classification of vector bundles over an algebraic curve of arbitrary genus*, Izv. Akad. Nauk SSSR Ser. Mat. **29** (1965), 657–688.

[ŠVdB23] Š. Špenko and M. Van den Bergh, *Comparing the Kirwan and noncommutative resolutions of quotient varieties*, J. Reine Angew. Math. **801** (2023), 1–43.