MODULI SPACES AND INVARIANT THEORY

JENIA TEVELEV

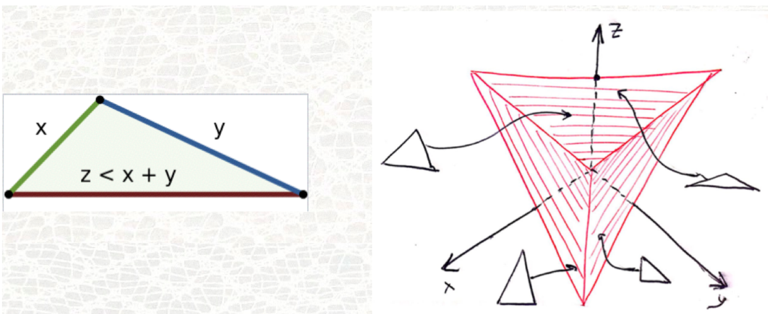
Contents

§1. **Syllabus**

§1.1. **Prerequisites.** An introductory course in algebraic geometry including algebraic curves, projective varieties and divisors (e.g. based on Fulton's "Algebraic Curves" or Shafarevich's "Basic Algebraic Geometry").

§1.2. **Course description.** A *moduli space* is a space that parametrizes all geometric objects of some sort. For example, a plane triangle is uniquely determined by its sides x, y, z, which have to satisfy triangle inequalities. So the moduli space of all triangles is a subset of $\mathbb{R}^3$ given by inequalities

$$0 < x < y + z, \quad 0 < y < x + z, \quad 0 < z < x + y.$$



A guiding principle is that a nice moduli space should have geometry similar to geometry of parametrized objects. For example, the moduli space of convex triangles is itself a convex cone, which is connected and simply-connected. This implies that every triangle can be continuously deformed into another and any two such deformations are homotopy equivalent. So knowing the moduli space is useful! One can also compactify this moduli space by making all inequalities non-strict – points of the boundary will correspond to “degenerate” triangles.

We will focus on moduli spaces of algebraic varieties such as elliptic curves. They are classified by the j-invariant, so the moduli space of elliptic curves is the line (with coordinate j). More generally, there exists a moduli space $\mathcal{M}_g$, which parametrizes all projective algebraic curves of genus g (equivalently, all compact Riemann surfaces of genus g). Connectedness of $\mathcal{M}_g$ is a deep theorem of Deligne and Mumford, who also introduced its compactification $\overline{M}_g$, the moduli space of stable curves.

An example of a different kind is the Jacobian, which is a moduli space that classifies complex line bundles of degree 0 on a fixed Riemann surface. One can consider moduli spaces of vector bundles, coherent sheaves, etc.

The study of moduli spaces is an old branch of algebraic geometry with an abundance of technical tools: classical algebraic theory, geometric invariant theory, period domains and variation of Hodge structures, stacks, etc. But I believe that a lot can be learned by studying examples using minimal machinery, as a motivation to learn more sophisticated tools. We will start with the Grassmannian $G(2,n)$, the moduli space of projective lines in the fixed projective space $\mathbb{P}^{n-1}$, and use it as a template to introduce various constructions of moduli spaces. We will discuss how to use global geometry of the moduli space to extract information about families of geometric objects. For example, we will see that $G(2,4)$ is a quadric hypersurface in $\mathbb{P}^5$ and therefore there exist exactly two lines in $\mathbb{P}^3$ that intersect four given general lines - can you prove this without moduli spaces?

§1.3. **Course grading and expectations.** The course grade will be based on two components, the homework (5-6 biweekly sets) and the project presented at the end of the semester.

§1.4. **Tentative topics.**

(1) Grassmannian.
(2) Representable functors and fine moduli spaces.
(3) Coarse moduli spaces. J-invariant. Elliptic fibrations.
(4) Moduli space of stable rational curves.
(5) Quotients by finite groups. Quotient singularities.
(6) Linear algebraic groups.
(7) Reductive groups. Hilbert's finite generation theorem.
(8) Geometric and categorical quotient.
(9) Weighted projective space.
(10) GIT quotients and stability.
(11) Hilbert–Mumford criterion. Stability of smooth hypersurfaces.
(12) Hilbert scheme.
(13) Riemann-Roch analysis for families of algebraic curves.
(14) Moduli spaces of algebraic curves.
(15) Jacobians. Moduli spaces of vector bundles.

§1.5. **Textbooks.** There is no required textbook and I will provide lecture notes. We will draw heavily from the following sources, which are recommended for further study. Many of them are freely available on-line.

REFERENCES

[D] I. Dolgachev, *Lectures on Invariant Theory*
[GH] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*
[HM] J. Harris, I. Morrison, *Moduli of Curves*
[Ha] R. Hartshorne, *Algebraic Geometry*

[Mu] S. Mukai, *An introduction to Invariants and Moduli*
[M1] D. Mumford, *Curves and their Jacobians*
[M2] D. Mumford, *Geometric Invariant Theory*
[MS] D. Mumford, K. Suominen, *Introduction to the theory of moduli*
[PV] V. Popov, E. Vinberg, *Invariant Theory*
[R1] M. Reid, *Surface cyclic quotient singularities and Hirzebruch–Jung resolutions*
[R2] M. Reid, *Graded rings and varieties in weighted projective space*
[St] B. Sturmfels, *Algorithms in Invariant Theory*

§2. **Geometry of lines**

Let's start with a familiar example. Recall that the *Grassmannian* $G(r,n)$ parametrizes r-dimensional linear subspaces of $\mathbb{C}^n$. For example,

$$G(1,n)=\mathbb{P}^{n-1}$$

is the *projective space*. Its point, a 1-dimensional subspace $L\subset\mathbb{C}^n$, can be represented by any non-zero vector $(x_1,\ldots,x_n)\in L$, which is defined uniquely up to rescaling (multiplying by a non-zero constant). Thus every point of $\mathbb{P}^{n-1}$ has homogeneous coordinates $[x_1:\ldots:x_n]$ which are not all equal to zero and defined uniquely up to rescaling:

$$[x_1:\ldots:x_n]=[\lambda x_1:\ldots:\lambda x_n].$$

It is also easy to understand $G(n-1,n)$, the set of *hyperplanes* in $\mathbb{C}^n$. Indeed, every hyperplane is given by a linear equation

$$a_1x_1+\ldots+a_nx_n=0.$$

The corresponding *covector* $(a_1,\ldots,a_n)$ is defined uniquely up to rescaling. Thus $G(n-1,n)$ is also a projective space, the *dual projective space* $(\mathbb{P}^{n-1})^*$.

The first non-trivial example is $G(2,4)$. What is it? Almost by definition, projectivization gives a bijection between r-dimensional linear subspaces of $\mathbb{C}^n$ and $(r-1)$-dimensional projective subspaces of $\mathbb{P}^{n-1}$. Thus

2.0.1. PROPOSITION. *$G(r,n)$ is the set of all $(r-1)$-dimensional projective subspaces of the projective space $\mathbb{P}^{n-1}$.*

For example, $G(1,n)$ is the set of points in $\mathbb{P}^{n-1}$ and $G(2,n)$ is the set of lines in $\mathbb{P}^{n-1}$. In particular, $G(2,4)$ is the set of lines in $\mathbb{P}^3$.

§2.1. **Grassmannian as a complex manifold.** Thinking about $G(k,n)$ as a set is not very useful, we need to introduce geometric structures on it. A distinguished feature of algebraic geometry (and difficulty for beginners) is abundance of these structures. Paraphrasing Claire Voisin, algebraic geometry involves studying the variety of perspectives from which one can see the same object, and using the *"constant moving back and forth several geometries and several types of tools to prove results in one field or another."* We will embrace and try to understand this diversity of approaches such as complex analytic geometry, geometry of algebraic varieties, etc.

A basic object of complex analytic geometry is a *complex manifold*:

2.1.1. DEFINITION. A complex manifold is a topological space[1] X with a covering by open sets X_i called *charts* homeomorphic to open subsets of $\mathbb{C}^n$:

$$\phi_i : X_i \hookrightarrow \mathbb{C}^n.$$

Coordinate functions on $\mathbb{C}^n$ restricted to $\phi_i(X_i)$ are called *local coordinates* in the chart. On the overlaps $X_i \cap X_j$ we thus have two competing systems of coordinates, and the main requirement is that the *transition functions*

$$\phi_j \circ \phi_i^{-1} : \phi_i(X_i \cap X_j) \to \phi_j(X_i \cap X_j)$$

between these coordinate systems are *holomorphic functions*.

2.1.2. EXAMPLE. The projective space $\mathbb{P}^{n-1}$ is covered by n charts

$$X_i = \{[x_1 : \ldots : x_n], \quad x_i \neq 0\} \subset \mathbb{P}^{n-1}.$$

The coordinate maps $\phi_i : X_i \hookrightarrow \mathbb{C}^{n-1}$ are given by

$$\phi_i([x_1 : \ldots : x_n]) = (x_1/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i)$$

(here and thereafter the ˆ sign indicates omission of something). In fact ϕ_i is a bijection, $X_i \simeq \mathbb{C}^{n-1}$, with the inverse map ϕ_i^{-1} given by

$$(z_1, \ldots, z_{n-1}) \mapsto [z_1, \ldots, z_{i-1}, 1, z_i, \ldots, z_n].$$

The transition functions $\phi_j \circ \phi_i^{-1} : \phi_i(X_i \cap X_j) \to \phi_j(X_i \cap X_j)$ for $i < j$ are as follows

$$(z_1, \ldots, z_{n-1}) \to [z_1, \ldots, z_{i-1}, 1, z_i, \ldots, z_n] \to$$
$$\to (z_1/z_j, \ldots z_{i-1}/z_j, 1/z_j, z_i/z_j, \ldots, \widehat{z_j/z_j}, \ldots, z_n/z_j)$$

Components of this functions are holomorphic functions, which means by definition that transition functions themselves are holomorphic.

In this example (and in general) the structure of a topological space on X can be introduced simultaneously with constructing charts: a subset $U \subset X$ is declared open if its intersection with every chart $U \cap X_i$ is open. We will leave it as an exercise to check that $\mathbb{P}^{n-1}$ is indeed a second countable Hausdorff topological space.

Let's generalize homogeneous coordinates $[x_1 : \ldots : x_n]$ and charts X_i for the Grassmannian $G(r, n)$. Every r-dimensional subspace $U \subset \mathbb{C}^n$ is a row space of a $r \times n$ matrix A of rank r. This matrix is not unique but we know from linear algebra that matrices A and A' of rank r have the same row space if and only if there exists an invertible $r \times r$ matrix Λ such that

$$A' = \Lambda A.$$

Let $[n]$ denote the set $\{1, \ldots, n\}$. For every subset $I \subset [n]$ of cardinality r, let A_I denote the $r \times r$ submatrix of A with columns indexed by I and let

$$p_I = \det A_I$$

be the corresponding minor. These numbers p_I are called *Plücker coordinates* of U. When $r = 1$ we recover homogenous coordinates on $\mathbb{P}^{n-1}$.

Since $\operatorname{rank} A = r$, we can find some subset I such that $p_I \neq 0$. Then $A' = A_I^{-1} A$ has a special form: A'_I is the identity matrix E.

[1]Technically speaking, it has to be *second countable*, i.e. admit a countable basis of open sets, and *Hausdorff*, i.e. any two distinct points have disjoint neighborhoods.

2.1.3. LEMMA–DEFINITION. *$G(r,n)$ is a complex manifold covered by $\binom{n}{r}$ charts X_I indexed by subsets $I \subset [n]$ of cardinality r:*

$$X_I = \{U \in G(r,n) \quad | \quad U = \text{row space}(A), \quad A_I = E.\}$$

Each subspace in X_I is a row space of a unique matrix with $A_I = E$, in particular we have bijections $\phi_I : X_I \to \mathbb{C}^{r(n-r)}$. Local coordinates on X_I are just entries of the matrix A such that $A_I = E$.

Proof. We have to check that transition functions between charts X_I and $X_{I'}$ are holomorphic. Let $U \in X_I \cap X_{I'}$. In the chart X_I, U is represented by a matrix A with $A_I = E$. The matrix representing U in $X_{I'}$ will be $(A_{I'})^{-1}A$. Its matrix entries depend holomorphically (in fact rationally) on the matrix entries of A. □

§2.2. **Moduli space or a parameter space?** One can argue that $G(2,n)$ is not truly a moduli space because all lines in $\mathbb{P}^{n-1}$ are isomorphic. So maybe it's better to call it a *parameter* space: it doesn't classify geometric objects up to isomorphism but rather it classifies sub-objects (lines) in a fixed ambient space (projective space). This distinction is of course purely philosophical. A natural generalization of the Grassmannian is given by a *Hilbert scheme*, which parametrizes all algebraic subvarieties (more precisely subschemes) of the projective space. As a rule, parameter spaces are easier to construct than moduli spaces. But then to construct a moduli space $\mathcal{M}$, one can

- embed our objects in some ambient space;
- construct a "parameter space" $\mathcal{H}$ for embedded objects;
- Then $\mathcal{M}$ will be the set of equivalence classes for the following equivalence relation on $\mathcal{H}$: two embedded objects are equivalent if they are abstractly isomorphic.

In many cases there will be a group G acting on $\mathcal{H}$ and equivalence classes will be just orbits for the group action. So we will have to learn how to construct an *orbit space* $\mathcal{M} = \mathcal{H}/G$ and a *quotient map* $\mathcal{H} \to \mathcal{M}$ that sends each point to its orbit. These techniques are provided by the *invariant theory* – the second component from the title of this course.

For example, suppose we want to construct $\mathcal{M}_3$, the moduli space of curves of genus 3. These curves come in two flavors, they are either hyperelliptic or not. The following theorem is well-known:

2.2.1. THEOREM.

- *A hyperelliptic curve of genus 3 is a double cover of $\mathbb{P}^1$ ramified at 8 points. These points are determined uniquely upto the action of $\mathrm{PGL}_2 \curvearrowright \mathbb{P}^1$.*
- *A non-hyperelliptic curve of genus 3 is isomorphic to a smooth quartic curve in $\mathbb{P}^2$. Two smooth quartic curves are isomorphic as algebraic varieties if and only if they belong to the same PGL_3-orbit.*

Let $\mathcal{H}_3 \subset \mathcal{M}_3$ be the set of isomorphism classes of hyperelliptic curves and let $\mathcal{M}_3 \setminus \mathcal{H}_3$ be the complement. We can construct both of them as quotient spaces:

$$\mathcal{H}_3 = \left[\mathbb{P}(\mathrm{Sym}^8 \mathbb{C}^2) \setminus \mathcal{D}\right] / \mathrm{PGL}_2$$

and

$$\mathcal{M}_3 \setminus \mathcal{H}_3 = \left[\mathbb{P}(\mathrm{Sym}^4 \mathbb{C}^3) \setminus \mathcal{D}\right] / \mathrm{PGL}_3 .$$

Here $\mathcal{D}$ is the discriminant locus. In the first case it parametrizes degree 8 polynomials in 2 variables without a multiple root (and hence such that its zero locus is 8 distinct points in $\mathbb{P}^1$) and in the second case degree 4 polynomials in 3 variables such that its zero locus is a smooth curve in $\mathbb{P}^2$.

An easy dimension count shows that

$$\dim \mathcal{H}_3 = \dim \mathbb{P}(\mathrm{Sym}^8 \mathbb{C}^2) - \dim \mathrm{PGL}_2 = 7 - 3 = 5$$

and

$$\dim \mathcal{M}_3 \setminus \mathcal{H}_3 = \dim \mathbb{P}(\mathrm{Sym}^4 \mathbb{C}^3) - \dim \mathrm{PGL}_3 = 14 - 8 = 6.$$

It is quite remarkable that there exists an irreducible moduli space $\mathcal{M}_3$ which contains $\mathcal{H}_3$ as a hypersurface. We will return to this example later.

§2.3. **Stiefel coordinates.** It's interesting that one can construct the Grassmannian as a quotient by the group action. The motivating idea is that $G(1, n)$ is a quotient:

$$\mathbb{P}^{n-1} = [\mathbb{C}^n \setminus \{0\}]/\mathbb{C}^*.$$

Coordinates in $\mathbb{C}^n$ are the *homogeneous coordinates* on $\mathbb{P}^{n-1}$ defined uniquely up to a common scalar factor $\lambda \in \mathbb{C}^*$. As we have already seen above, there are two ways to generalize them to $r > 1$, *Stiefel coordinates* and *Plücker coordinates*. Let

$$\mathrm{Mat}^0_{r,n} \subset \mathrm{Mat}_{r,n}$$

be an open subset (in Zariski or complex topology) of matrices of rank r. We have a map

$$\Psi : \mathrm{Mat}^0_{r,n} \to G(r, n) \tag{2.3.1}$$

which sends a matrix to its row space. Ψ can be interpreted as the quotient map for the action of GL_r on $\mathrm{Mat}^0_{r,n}$ by left multiplication. Indeed, two rank r matrices A and A' have the same row space if and only if $A' = \Lambda A$ for some matrix $\Lambda \in \mathrm{GL}_r$. Matrix coordinates on $\mathrm{Mat}^0_{r,n}$ are sometimes known as *Stiefel coordinates* on the Grassmannian. The are determined up to a left multiplication with an invertible $r \times r$ matrix Λ.

§2.4. **Complete system of (semi-)invariants.** Let me use (2.3.1) as an example to explain how to use invariants to construct quotient maps.

2.4.1. DEFINITION. We start with a very general situation: let G be a group acting on a set X. A function $f : X \to \mathbb{C}$ is called an *invariant function* if it is constant along G-orbits, i.e. if

$$f(gx) = f(x) \quad \text{for every } x \in X,\ g \in G.$$

We say that invariant functions $f_1, \ldots, f_r$ form a *complete system of invariants* if they *separate orbits*. This means that for any two orbits O_1 and O_2, there exists at least one function f_i for $i = 1, \ldots, r$ such that $f_i|_{O_1} \neq f_i|_{O_2}$.

2.4.2. LEMMA. *If $f_1, \ldots, f_r$ is a complete system of invariants then the map*

$$F : X \to \mathbb{C}^r, \quad F(x) = (f_1(x), \ldots, f_r(x))$$

is a quotient map *(onto its image) in the sense that its fibers are exactly the orbits.*

Proof. Indeed, if $x_1, x_2 \in X$ belong to different orbits then $f_i(x_1) \neq f_i(x_2)$ for some $i = 1, \ldots, r$ and so x_1 and x_2 belong to different fibers of F. $\square$

Often we want to have a quotient map with target $\mathbb{P}^r$ rather than $\mathbb{C}^r$. Thus we need the following generalization:

2.4.3. DEFINITION. Fix a homomorphism (called *character* or *weight*)

$$\chi : G \to \mathbb{C}^*.$$

A function $f : X \to \mathbb{C}$ is called a *semi-invariant of weight* χ if

$$f(gx) = \chi(g)f(x) \quad \text{for every } x \in X,\ g \in G.$$

(An invariant function is a semi-invariant function of weight $\chi = 1$). Suppose $f_0, \ldots, f_r$ are semi-invariants of <u>the same weight χ</u>. We will call them a *complete system of semi-invariants of weight* χ if

- for any $x \in X$, there exists a function f_i such that $f_i(x) \neq 0$;
- for any two points $x, x' \in X$ not in the same orbit, we have

$$[f_0(x) : \ldots : f_r(x)] \neq [f_0(x') : \ldots : f_r(x')].$$

The first condition means that we have a map

$$F : X \to \mathbb{P}^r, \quad F(x) = [f_0(x) : \ldots : f_r(x)],$$

which is clearly constant along G-orbits:

$$[f_0(gx) : \ldots : f_r(gx)] = [\chi(g)f_0(x) : \ldots : \chi(g)f_r(x)] = [f_0(x) : \ldots : f_r(x)].$$

The second condition means that F is a quotient map onto its image in the sense that its fibers are precisely the orbits.

§2.5. **Plücker coordinates.** Let $G = \mathrm{GL}_r$ act by left multiplication on $\mathrm{Mat}^0_{r,n}$. Consider the $r \times r$ minors p_I as functions on $\mathrm{Mat}^0_{r,n}$.

2.5.1. PROPOSITION. *The minors p_I form a complete system of semi-invariants on $\mathrm{Mat}^0_{r,n}$ of weight*

$$\det : \mathrm{GL}_r \to \mathbb{C}^*.$$

Proof. For any $A \in \mathrm{Mat}^0_{r,n}$, at least one of the minors p_I's does not vanish. So we have a map

$$F : \mathrm{Mat}^0_{r,n} \to \mathbb{P}^{\binom{n}{r}-1}$$

given by $\binom{n}{r}$ minors p_I. We have

$$p_I(gA) = \det(g)p_I(A) \quad \text{for any} \quad g \in \mathrm{GL}_r,\ A \in \mathrm{Mat}^0_{r,n}.$$

It follows that p_I's are semi-invariants of weight $\det$ and therefore the map F is GL_r-equivariant, i.e. constant on orbits.

It remains to show that fibers are precisely the orbits. Take $A, A' \in \mathrm{Mat}^0_{r,n}$ such that $F(A) = F(A')$. We have to show that A and A' have the same row space, i.e. are in the same GL_r-orbit. Fix a subset I such that $p_I(A) \neq 0$, then of course $p_I(A') \neq 0$. By acting on A and A' by A_I^{-1} (resp. $(A'_I)^{-1}$), we can assume that both A_I and A'_I are identity matrices. To show that $A = A'$, it suffices to prove the following claim:

2.5.2. CLAIM. *Suppose A_I is the identity matrix. Then every matrix entry a_{ij} of A is equal to the minor A_J for some subset J of cardinality r. In particular, if $F(A) = F(A')$ and $A_I = A'_I$ is the identity matrix then $A = A'$.*

Without loss of generality, we can take $I = \{1, \dots, r\}$. In particular,

$$p_I(A) = p_I(A') = 1.$$

Since $F(A) = F(A')$, we now have

$$p_J(A) = p_J(A') \quad \text{for every subset } J.$$

It remains to notice that if

$$J = \{1, \dots, r\} \setminus \{i\} \cup \{j\}$$

then $a_{ij} = \pm A_J$. □

2.5.3. DEFINITION. The inclusion

$$i : G(r, n) = \mathrm{Mat}^0_{r,n} / GL_r \hookrightarrow \mathbb{P}^{\binom{n}{r}-1}$$

is called the *Plücker embedding*.

2.5.4. PROPOSITION. *This embedding is an immersion of complex manifolds.*

Proof. We have already proved that i is an inclusion of sets. Let x_I be homogeneous coordinates on $\mathbb{P}^{\binom{n}{r}-1}$ for all subsets $I \subset [n]$ of cardinality r. Each chart $p_I \neq 0$ of the Grassmannian is mapped to the corresponding affine chart $x_I \neq 0$ of the projective space. The map i in this chart is given by the remaining $r \times r$ minors of a matrix A such that A_I is the identity matrix. These minors are of course holomorphic functions. By definition, this means that i is a *holomorphic map* of complex manifolds.

To show that i is an *immersion* of complex manifolds, we have to verify more, namely that the *Jacobian matrix* of this map has maximal rank (equal to the dimension of $G(r, n)$) at every point. Without loss of generality we can work in the chart $p_{1,\dots,r} \neq 0$ of the Grassmannian. Local coordinates are given by $r \times (n - r)$ matrix entries a_{ij} for $j > r$ of a matrix A such that $A_{1,\dots,r}$ is the identity matrix. Local coordinates in the chart $x_{1,\dots,r} \neq 0$ of the projective space are $\binom{n}{r} - 1$ coordinates x_J for $J \neq \{1, \dots, r\}$.

By the Claim 2.5.2, every matrix entry a_{ij} is equal to a minor $p_J(A)$ for some J. Therefore all local coordinates of the chart $p_{1,\dots,r} \neq 0$ are some of the components of the map i and so its Jacobian matrix contains the identity submatrix of rank $r(n - r)$. So the Jacobian matrix has maximal rank $r(n - r)$. □

§2.6. **First Fundamental Theorem.** Why the minors p_I are a natural choice for a complete system of semi-invariants? Let's consider all possible semi-invariants on $\mathrm{Mat}^0_{r,n}$ which are polynomials in matrix entries. By continuity, this is the same as polynomial semi-invariants on $\mathrm{Mat}_{r,n}$. Let

$$\mathcal{O}(\mathrm{Mat}_{r,n}) = \mathbb{C}[a_{ij}]_{1 \leq i \leq r, \quad 1 \leq j \leq n}$$

be the algebra of polynomial functions on $\mathrm{Mat}_{r,n}$. It is well-known that the only holomorphic homomorphisms $\mathrm{GL}_r(\mathbb{C}) \to \mathbb{C}^*$ are powers of the determinant.

Let

$$R_i = \mathcal{O}(\mathrm{Mat}_{r,n})^{\mathrm{GL}_r}_{\det^i}$$

be a subset of polynomial semi-invariants of weight $\det^i$. Notice that the scalar matrix $t\,\mathrm{Id}_r$ acts on R_i by multiplying it by $\det^i(t\,\mathrm{Id}_r) = t^{ri}$. On the other hand, if $f \in \mathcal{O}(\mathrm{Mat}_{r,n})$ is a polynomial of degree d then

$$f(t\,\mathrm{Id}_r \cdot A) = f(tA) = t^d f(A).$$

It follows that all polynomials in R_i have degree $d = ri$, in particular

$$R_i = 0 \quad \text{for } i < 0, \quad R_0 = \mathbb{C}.$$

We assemble all semi-invariants in one package (*algebra of semi-invariants*):

$$R = \bigoplus_{i \geq 0} R_i \subset \mathcal{O}(\mathrm{Mat}_{r,n}).$$

Since the product of semi-invariants of weights χ and χ' is a semi-invariant of weight $\chi \cdot \chi'$, R is a *graded subalgebra* of $\mathcal{O}(\mathrm{Mat}_{r,n})$.

2.6.1. THEOREM (First Fundamental Theorem of invariant theory). *The algebra R is generated by the minors p_I.*

Thus considering only minors p_I's makes sense: all semi-invariants can't separate orbits any more effectively than the generators. We will skip the proof of this theorem. But it raises some general questions:

- is the algebra of polynomial invariants (or semi-invariants) always finitely generated?
- do these basic invariants separate orbits?
- how to compute these basic invariants?

We will see that the answer to the first question is positive under very general assumptions (*Hilbert-Mumford finite generation theorem*). The answer to the second question is typically negative but a detailed analysis of which orbits are separated is available (Hilbert–Mumford's *stability* and the *numerical criterion* for it). As far as the last question is concerned, the situation is worse: the generators can be computed effectively and explicitly only in a handful of cases.

§2.7. **Equations of the Grassmannian.** How to describe the image of the Grassmannian in the Plücker embedding? We will show that it is a projective algebraic variety and describe its defining equations.

We are going to focus on $G(2,n)$, the Grassmannian of lines in $\mathbb{P}^{n-1}$.

2.7.1. DEFINITION. Let $f_1, \ldots, f_r \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous polynomials (possibly of various degrees). The vanishing set

$$X = V(f_1, \ldots, f_r) = \{x \in \mathbb{P}^n \mid f_1(x) = \ldots = f_r(x) = 0\}$$

is called a *projective algebraic variety*.

2.7.2. REMARK. We will frequently do calculations in charts: for each chart

$$U_i = \{x_i \neq 0\} \subset \mathbb{P}^n, \quad U_i \simeq \mathbb{A}^n,$$

$X \cap U_i \subset \mathbb{A}^n$ is an *affine algebraic variety* given by vanishing of polynomials

$$\tilde{f}_j = f_j(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n), \quad j = 1, \ldots, r.$$

These polynomials are called *dehomogenizations* of $f_1, \ldots, f_r$.

2.7.3. THEOREM. *$i(G(2,n))$ is a projective algebraic variety in $\mathbb{P}^{\binom{n}{2}-1}$ given by vanishing of the following polynomials called Plücker quadrics:*

$$x_{ij}x_{kl}-x_{ik}x_{jl}+x_{il}x_{jk},\quad i<j<k<l.$$

Here we use homogeneous coordinates x_{ij} for $1\le i<j\le n$.

Proof. Let U be a row space of a matrix

$$A=\begin{bmatrix}a_{11} & \ldots & a_{1n}\\ a_{21} & \ldots & a_{2n}\end{bmatrix}.$$

In order to show that the Plücker quadrics vanish along the image of the Grassmannian, it suffices to show that minors of the matrix A satisfy the relation

$$p_{ij}(A)p_{kl}(A)-p_{ik}(A)p_{jl}(A)+p_{il}(A)p_{jk}(A)=0,$$

which can be verified directly. A more conceptual approach is to consider a bivector

$$b=(a_{11}e_1+\ldots+a_{1n}e_n)\wedge(a_{21}e_1+\ldots+a_{2n}e_n)=\sum_{i<j}p_{ij}(A)e_i\wedge e_j\in\Lambda^2\mathbb{C}^n.$$

If we identify $\mathbb{P}^{\binom{n}{2}-1}$ with the projectivization of $\Lambda^2\mathbb{C}^n$, the Plücker embedding sends a subspace $U\in G(2,n)$ (spanned by vectors $u,v\in\mathbb{C}^n$) to the line $\Lambda^2U\subset\Lambda^2\mathbb{C}^n$ (spanned by $u\wedge v$). Bivectors of this form are called *decomposable*. We see that the image of the Plücker embedding is the projectivization of the subset of *decomposable* bivectors. Notice that we have

$$b\wedge b=(u\wedge v)\wedge(u\wedge v)=-u\wedge u\wedge v\wedge v=0.$$

On the other hand,

$$b\wedge b=2\sum_{i<j<k<l}(p_{ij}p_{kl}-p_{ik}p_{jl}+p_{il}p_{jk})e_i\wedge e_j\wedge e_k\wedge e_l.$$

Thus Plücker quadrics indeed vanish along $i(G(2,n))$.

Next we have to show that the vanishing set

$$X=V(x_{ij}x_{kl}-x_{ik}x_{jl}+x_{il}x_{jk})\subset\mathbb{P}^{\binom{n}{2}-1}$$

does not contain any other points. We can do this in affine charts of $\mathbb{P}^{\binom{n}{2}-1}$. Without loss of generality it suffices to consider the chart $U_{12}=\{x_{12}=1\}$.

What are the equations of $X\cap U_{12}$? Taking Plücker quadrics with $ij=12$ and dehomogenizing gives equations

$$x_{kl}=x_{1k}x_{2l}-x_{1l}x_{2k},\quad 2<k<l\le n. \tag{2.7.4}$$

It follows that homogeneous coordinates of any point $x\in X\cap U_{12}$ are minors of the matrix

$$A=\begin{bmatrix}1 & 0 & -x_{23} & -x_{24} & \ldots & -x_{2n}\\ 0 & 1 & x_{13} & x_{14} & \ldots & x_{1n}\end{bmatrix}.$$

It follows that x corresponds to the row space of A. $\square$

2.7.5. COROLLARY. *$i(G(2,n))$ is a compact complex submanifold of $\mathbb{P}^{\binom{n}{2}-1}$.*

Proof. Every projective algebraic variety is closed and hence compact in the Euclidean topology. Since we already know that the Plücker embedding i is an immersion, $i(G(2,n))$ is a compact complex submanifold of $\mathbb{P}^{\binom{n}{2}-1}$. □

2.7.6. REMARK. In general, not every complex algebraic variety X is a complex manifold simply because it can have *singularities*. But if X is a non-singular complex algebraic variety then it admits a structure of a complex manifold of the same dimension denoted by X^{an}. A remarkable fact is a Chow's theorem: every compact complex submanifold of the projective space is a projective algebraic varierty.

§2.8. **Homogeneous ideal.** By Th. 2.7.3, the Grassmannian $G(2,n)$ is the vanishing set of Plücker quadrics. We would like to describe *all* homogenous polynomials in $\binom{n}{2}$ variables x_{ij} that vanish along $G(2,n)$, in other words the *homogeneous ideal* $I \subset \mathbb{C}[x_{ij}]$ of $G(2,n)$. The following theorem was classically known as the *second fundamental theorem* of invariant theory.

2.8.1. THEOREM. *The ideal I of the polynomial ring is generated by Plücker quadrics*

$$x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}, \qquad i < j < k < l.$$

Proof. The proof consists of several steps.

Step 1. One can think about elements of I as relations between 2×2 minors of a general $2 \times n$ matrix A. Indeed, by (2.3.1), $G(2,n)$ is the image of the map

$$\Psi : \mathrm{Mat}^0_{2,n} \to G(2,n).$$

Thus I is the kernel of the homomorphism of polynomial algebras

$$\psi : \mathbb{C}[x_{ij}]_{1\le i<j\le n} \to \mathbb{C}[a_{1i}, a_{2i}]_{1\le i\le n}, \quad x_{ij} \mapsto p_{ij}(A).$$

We proved in Prop. 2.7.3 that Plücker quadrics are in I. Let $I' \subset I$ be the ideal generated by the Plücker relations. The goal is to show that $I = I'$.

Step 2 is called the *straightening law*. It was introduced by Alfred Young (who has Young diagrams named after him). We encode each monomial $x_{i_1j_1}\dots x_{i_kj_k} \in \mathbb{C}[x_{ij}]$ as a *Young tableaux*

$$\begin{bmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{bmatrix} \tag{2.8.2}$$

Note that entries in the columns of this matrix are increasing, $i_r < j_r$ for every r. The Young tableaux is called *standard* if the entries in each row are non-decreasing:

$$i_1 \le i_2 \le \dots \le i_k \quad \text{and} \quad j_1 \le j_2 \le \dots \le j_k.$$

In this case the corresponding monomial $x_{i_1j_1}\dots x_{i_kj_k}$ is called a *standard monomial*. We claim that every monomial is equivalent modulo I' (i.e. modulo Plücker quadrics) to a linear combination of standard monomials.

Suppose $\mathbf{x} = x_{i_1j_1}\dots x_{i_kj_k}$ is a non-standard monomial. Then it contains a pair of variables x_{ab}, x_{cd} such that $a < c$ but $b > d$. Then we have

$$a < c < d < b.$$

We have

$$x_{ab}x_{cd} = x_{ac}x_{db} - x_{ad}x_{cb} \mod I'. \tag{2.8.3}$$

We can plug-in the right-hand-side into $\mathbf{x}$ instead of $x_{ab}x_{cd}$and rewrite it as a sum of two monomials. We can continue this procedure every time one of the remaining monomials in the sum is still not standard. However, we need show that this *straightening algorithm* terminates.

To every monomial $\mathbf{x} = x_{i_1j_1}\ldots x_{i_kj_k}$, we associate a sequence of positive integers

$$j_1 - i_1, \ldots, j_k - i_k,$$

reordered in non-increasing order. We order all monomials by ordering these sequences of integers *lexicographically* (and if monomials have the same sequences, for example monomials x_{12} and x_{34}, order them randomly). We argue by induction on lexicographical order that every non-standard monomial is equivalent modulo Plücker quadrics to a linear combination of standard monomials. Suppose this is verified for all monomials smaller than $\mathbf{x} = x_{i_1j_1}\ldots x_{i_kj_k}$. Do the substitution (2.8.3) and notice that $c-a$, $b-d$, $d-a$, and $b-c$ are all strictly less than $b-a$. Therefore both of these monomials are smaller than $\mathbf{x}$ in lexicographic order and so can be re-written as linear combinations of standard monomials modulo I' by the inductive assumption.

Step 3. Finally, we claim that standard monomials are linearly independent modulo I, and in particular $I = I'$. Concretely, we claim that

$$\{\psi(\mathbf{x})\,|\,\mathbf{x} \text{ is a standard monomial}\}$$

is a linearly independent subset of $\mathbb{C}[a_{1i}, a_{2i}]_{1\le i\le n}$. A cool idea is to order the variables as follows:

$$a_{11} < a_{12} < \ldots < a_{1n} < a_{21} < a_{22} < \ldots < a_{2n}$$

and to consider the corresponding lexicographic ordering of monomials in $\mathbb{C}[a_{1i}, a_{2i}]_{1\le i\le n}$. For any polynomial f, let $\mathrm{in}(f)$ be the *initial monomial* of f (i.e. the smallest monomial for lexicographic ordering). Notice that $\mathrm{in}(f)$ is multiplicative:

$$\mathrm{in}(fg) = \mathrm{in}(f)\,\mathrm{in}(g) \tag{2.8.4}$$

for any two (non-zero) polynomials. We have

$$\mathrm{in}\, p_{ij} = a_{1i}a_{2j},$$

and therefore

$$\mathrm{in}(\psi(\mathbf{x})) = \mathrm{in}(p_{i_1j_1}\ldots p_{i_kj_k}) = a_{1i_1}a_{1i_2}\ldots a_{1i_k}a_{2j_1}a_{2j_2}\ldots a_{2j_k}.$$

Notice that a standard monomial $\mathbf{x}$ is completely determined by $\mathrm{in}(\psi(\mathbf{x}))$. Now we argue by contradiction that the set of polynomials $\{\psi(\mathbf{x})\}$, for all standard monomials $\mathbf{x}$, is linearly independent. Indeed, consider a trivial linear combination

$$\lambda_1\psi(\mathbf{x}_1) + \ldots + \lambda_r\psi(\mathbf{x}_r) = 0$$

where all $\lambda_i \neq 0$. Then the minimum of initial terms $\mathrm{in}(\psi(\mathbf{x}_i))$, $i = 1, \ldots, r$ should be attained at least twice, a contradiction with the fact they are all different. $\square$

Many calculations in commutative algebra can be reduced to similar manipulations with polynomials by considering different types of orderings of monomials. Computer algebra packages like Macaulay 2 run on Groebner bases algorithms based on these ideas.

§2.9. **Hilbert polynomial.** Equations of a projective variety are usually not known so explicitly as in Theorem 2.8.1 for the Grassmannian. But some numerical information about them is often available and sufficient.

We start with a general situation: let $X \subset \mathbb{P}^n$ be a projective variety and let $I \subset \mathbb{C}[x_0, \ldots, x_n]$ be a *homogeneous ideal* of polynomials that vanish along X. The algebra $R = \mathbb{C}[x_0, \ldots, x_n]/I$ is known as a *homogeneous coordinate algebra* of X. Note that R is graded by degree of polynomials

$$R = \bigoplus_{j \geq 0} R_j, \quad R_0 = \mathbb{C},$$

and R is generated by R_1 as an algebra. The function

$$h(k) = \dim R_k$$

is called the *the Hilbert function* of X. Notice that knowing $h(k)$ is equivalent to knowing $\dim I_k$ for every k:

$$h(k) + \dim I_k = \binom{n+k}{k}.$$

Recall the following fundamental theorem:

2.9.1. THEOREM. *There exists a unique polynomial $H(t)$ such that*

$$h(k) = H(k) \quad \text{for} \quad k \gg 0.$$

It is called the Hilbert polynomial. It has the following form:

$$\frac{d}{r!}t^r + (\text{lower terms}),$$

where r is the dimension and d is the degree of X, i.e. the number of points in the intersection of X with a general projective subspace of codimension r.

The word "general" here is used in the standard sense of algebraic geometry: there exists a dense Zariski open subset $W \subset G(n+1-r, n+1)$ (the Grassmannian of all projective subspaces of codimension r) such that $X \cap L$ is a finite set of d points for every projective subspace $L \in W$.

2.9.2. EXAMPLE. Let $\mathbb{P}^l \subset \mathbb{P}^n$ be a projective subspace, for example given by $x_{l+1} = \ldots = x_n = 0$. Restricting polynomials in $n+1$ variable to this subspace gives an exact sequence

$$0 \to I \to \mathbb{C}[x_0, \ldots, x_n] \to \mathbb{C}[x_0, \ldots, x_l] \to 0,$$

where $I = \langle x_{l+1}, \ldots x_n \rangle$ is the homogeneous ideal of $\mathbb{P}^l$ and $R = \mathbb{C}[x_0, \ldots, x_l]$ is its homogeneous coordinate algebra. The Hilbert function is

$$h(k) = \dim \mathbb{C}[x_0, \ldots, x_l]_k = \binom{l+k}{k} =$$

$$= \frac{k(k-1)\ldots(k-l+1)}{l!} = \frac{1}{l!}k^l + \ldots$$

This agrees with the fact $\mathbb{P}^l$ has degree 1 and dimension l.

It is not difficult to show that projective subspaces are the only projective varieties of degree 1. In particular, $G(l+1, n+1)$ parametrizes all projective subvarieties of $\mathbb{P}^n$ with Hilbert polynomial $H(k) = \binom{l+k}{k}$. More generally, fix a polynomial $H(t)$. There exists a projective scheme

$$\operatorname{Hilb}_{H(t)} \mathbb{P}^n$$

called the *Hilbert scheme* that parametrizes all projective subschemes of $\mathbb{P}^n$ with Hilbert polynomial $H(t)$. Of course one has to define "schemes", explain the meaning of "parametrize" and prove existence and projectivity – all deep accomplishments of Grothendieck.

2.9.3. PROPOSITION. *Let $n \geq 3$. The Hilbert function of $G(2, n)$ in the Plücker embedding is*

$$h(k) = \binom{n+k-1}{k}^2 - \binom{n+k}{k+1}\binom{n+k-2}{k-1}. \tag{2.9.4}$$

The degree of $G(2, n)$ in the Plücker embedding is the Catalan number

$$\frac{1}{n-1}\binom{2n-4}{n-2} = 1, 2, 5, 14, 42, 132, \ldots$$

Proof. While proving Theorem 2.8.1 we have already established that $h(k)$ is equal to the number of standard monomials of degree k, i.e. to the number of standard Young tableaux with k columns. Let N_l be the number of non-decreasing sequences $1 \leq i_1 \leq \ldots \leq i_l \leq n$. Then

$$N_l = \binom{n+l-1}{l}.$$

Indeed, this is just the number of ways to choose l objects from $\{1, \ldots, n\}$ with repetitions (so it is for example equal to the dimension of the space of polynomials in n variables of degree l). The number of all tableaux

$$\begin{bmatrix} i_1 & i_2 & \ldots & i_k \\ j_1 & j_2 & \ldots & j_k \end{bmatrix}, \quad 1 \leq i_1 \leq \ldots \leq i_k \leq n,\ 1 \leq j_1 \leq \ldots \leq j_k \leq n$$

is then

$$N_k^2 = \binom{n+k-1}{k}^2.$$

To prove (2.9.4), it suffices to prove the following

2.9.5. CLAIM. *The number of non-standard tableaux is*

$$\binom{n+k}{k+1}\binom{n+k-2}{k-1}.$$

More precisely, there is a bijection between the set of nonstandard tableaux and the set of pairs (A, B), where A is a non-decreasing sequence of length $k+1$ and B is a non-decreasing sequence of length $k-1$.

Proof of the Claim. Suppose that l is the number of the first column where $i_l \geq j_l$. Then we can produce two sequences:

$$j_1 \leq \ldots \leq j_l \leq i_l \leq \ldots \leq i_k$$

of length $k+1$ and

$$i_1 \leq \ldots \leq i_{l-1} \leq j_{l+1} \leq \ldots \leq j_k$$

of length $k-1$. In the opposite direction, suppose we are given sequences

$$i_1 \leq \ldots \leq i_{k+1} \quad \text{and} \quad j_1 \leq \ldots \leq j_{k-1}.$$

Let l be the minimal index such that $i_l \leq j_l$ and take a tableaux

$$\begin{bmatrix} j_1 & \ldots & j_{l-1} & i_{l+1} & i_{l+2} & \ldots & i_k \\ i_1 & \ldots & i_{l-1} & i_l & j_l & \ldots & j_{k-1} \end{bmatrix}$$

If $i_l > j_l$ for any $l \leq k-1$, then take the tableaux

$$\begin{bmatrix} j_1 & \ldots & j_{k-1} & i_k \\ i_1 & \ldots & i_{k-1} & i_{k+1} \end{bmatrix}.$$

This proves the Claim. □

After some manipulations, (2.9.4) can be rewritten as

$$\binom{n+k-1}{n-1}^2 - \binom{n+k}{n-1}\binom{n+k-2}{n-1} =$$

$$\frac{(n+k-1)^2 \ldots (k+1)^2}{(n-1)!^2} - \frac{(n+k)\ldots(k+2)}{(n-1)!}\frac{(n+k-2)\ldots k}{(n-1)!} =$$

$$\frac{(k+n-1)(k+n-2)^2 \ldots (k+2)^2(k+1)}{(n-1)!^2}\left[(n+k-1)(k+1)-(n+k)k\right].$$

$$\frac{1}{(n-1)!(n-2)!}(k+n-1)(k+n-2)^2 \ldots (k+2)^2(k+1).$$

This is a polynomial in k of degree $2n-4$ with leading coefficient $\frac{1}{(n-1)!(n-2)!}$. Thus the degree of $G(2,n)$ is equal to $(2n-4)!$ multiplied by the leading coefficient of $h(k)$, which is indeed the Catalan number. □

§2.10. **Enumerative geometry.** Why study moduli spaces? To extract interesting information. For example, let's relate the degree of the Grassmannian (i.e. the Catalan number) to geometry of lines.

2.10.1. THEOREM. *The number of lines in $\mathbb{P}^{n-1}$ that intersect $2n-4$ general*[2] *codimension 2 subspaces is equal to the Catalan number*

$$\frac{1}{n-1}\binom{2n-4}{n-2},$$

the degree of the Grassmannian in the Plücker embedding.

[2]The precise meaning of the word "general", as before, is the following. There exists a dense Zariski open subset

$$U \subset G(n-2,n) \times \ldots \times G(n-2,n) \quad (2n-4 \text{ copies})$$

such that for every $(2n-4)$-tuple of codimension two subspaces

$$(W_1, \ldots, W_{2n-4}) \in U,$$

the number of lines that intersect $W_1, \ldots, W_{2n-4}$ is the Catalan number.

For example, there is only one line in $\mathbb{P}^2$ passing through 2 general points, 2 lines in $\mathbb{P}^3$ intersecting 4 general lines, 5 lines in $\mathbb{P}^4$ intersecting 6 general planes, and so on. This is a typical problem from *enumerative geometry*, which was described by H. Schubert around 1870s as a branch of algebraic geometry concerned with questions like: How many geometric figures of some type satisfy certain given conditions? The enumerative geometry of lines, or more generally projective subspaces, can be reduced to questions about the Grassmannian known as *Schubert calculus*, which is nowdays more or less understood. Enumerative geometry of curves of higher degree is encapsulated in theory of *Gromov–Witten invariants* which had many recent advances in moduli theory and physics.

In order to show that the degree of the Grassmannian solves an enumerative problem, we need some preparation.

2.10.2. DEFINITION. Let $X \subset \mathbb{P}^{n-1}$ be a subvariety of codimension l. A *Chow form* (or an *associated hypersurface*)

$$D_X \subset G(l,n)$$

is the locus of all projective subspaces $\mathbb{P}^{l-1} \subset \mathbb{P}^{n-1}$ that intersect X. For example, if X is a hypersurface ($l=1$) then $D_X = X$. An amazing fact is that D_X is always a hypersurface in the Grassmannian and X is uniquely determined by D_X. An equation of D_X in Plücker coordinates of the Grassmannian is a convenient way of presenting X.

The simplest example is the following: let $W \subset \mathbb{P}^{n-1}$ be a projective subspace of codimension 2, then

$$D_W \subset G(2,n)$$

is the locus of all lines that intersect W. Incidentally, D_W is also the most basic example of a *Schubert variety*. We claim that D_W can be described as the intersection with a hyperplane:

$$D_W = G(2,n) \cap H_W, \qquad H_W \subset \mathbb{P}^{\binom{n}{2}-1}.$$

Indeed, without loss of generality we can take

$$W = \{x_1 = x_2 = 0\} = \mathbb{P}\langle e_3, \ldots e_n\rangle.$$

Lines intersecting W are projectivizations of 2-dimensional subspaces U that intersect a linear subspace $\langle e_3, \ldots e_n\rangle$ nontrivially. Let $\pi : \mathbb{C}^n \to \langle e_1, e_2\rangle$ be a linear projection with kernel $\langle e_3, \ldots e_n\rangle$. Then $U \in D_W$ iff $\operatorname{Ker} \pi|_U \neq 0$ iff $\dim \operatorname{Im} \pi|_U < 2$ iff the Plücker minor $p_{12}(U) = 0$. So D_W is precisely the intersection of $G(2,n)$ with the hyperplane

$$H_W = \{x_{12} = 0\}.$$

We will show that if

$$W_1, \ldots, W_{2n-4} \subset \mathbb{P}^{n-1}$$

are sufficiently general codimension 2 subspaces then the intersection

$$D_{W_1} \cap \ldots \cap D_{W_{2n-4}} = G(2,n) \cap \left(H_{W_1} \cap \ldots \cap H_{W_{2n-4}}\right)$$

is a finite union of the Catalan number of points. We are going to see that $H_{W_1} \cap \ldots \cap H_{W_{2n-4}}$ is a codimension $2n-4$ projective subspace and would like to invoke the definition of the degree. The problem, however, is that this subspace is not going to be "general". So we need a refined notion of the degree that can be verified for a specific subspace.

§2.11. **Transversality.** Recall that any algebraic variety X carries a *structure sheaf* $\mathcal{O}_X$: for every Zariski open subset $U \subset X$, $\mathcal{O}_X(U)$ is the ring of all regular functions on U. If $x \in X$, then we also have

- The *local ring* $\mathcal{O}_{X,x}$ of functions regular at x.
- $\mathfrak{m}_{X,x} \subset \mathcal{O}_{X,x}$ is the *maximal ideal* of functions vanishing at x.
- $T^*_{X,x} = \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$ is the *Zariski cotangent space* of x. For every function $f \in \mathfrak{m}_{X,x}$, its coset $f + \mathfrak{m}^2_{X,x} \in T^*_{X,x}$ is called the differential df. This is a usual differential if $X = \mathbb{A}^n$.
- $T_{X,x} = (\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x})^*$ is the *Zariski tangent space* of x.

We say that X is *smooth* (or *non-singular*) at x if

$$\dim_{\mathbb{C}} T_{X,x} = \dim X$$

(and otherwise $\dim_{\mathbb{C}} T_{X,x} > \dim X$). The locus of all smooth points is called the *smooth locus* of X. It is Zariski open and dense. Its complement is called the *singular locus*.

Let $Y \subset X$ be an algebraic subvariety defined by a sheaf of radical ideals

$$\mathcal{I}_Y \subset \mathcal{O}_X.$$

Namely, $\mathcal{I}_Y(U) \subset \mathcal{O}_X(U)$ is the ideal of functions that vanish along $Y \cap U$. For every $y \in Y$, we have the following.

- $\mathcal{O}_{Y,y} = \mathcal{O}_{X,y}/\mathcal{I}_{Y,y}$.
- $T_{Y,y} \subset T_{X,y}$ is a vector subspace defined by vanishing of differentials $df \in T^*_{X,y}$ of all functions $f \in \mathcal{I}_{Y,y}$.
- If X is smooth at y and $\operatorname{codim}_X Y = r$ then Y is smooth at y if and only if one can find r functions $f_1, \ldots, f_r \in \mathcal{I}_{Y,y}$ with linearly independent differentials $df_1, \ldots, df_r \in T^*_{X,y}$ (the *Jacobian criterion*).

For example, let's consider $D_W = G(2, n) \cap H_W$, where $H_W = \{x_{12} = 0\}$. Let's describe this intersection in charts of the Grassmannian. D_W is exactly the complement of the chart $U_{12} \subset G(2, n)$. In charts U_{1i} and U_{2i} for $i \geq 3$, the minor p_{12} reduces to a linear equation, so intersection of D_W with any of these charts is non-singular. In the remaining charts, D_W is given by a quadric $a_{11}a_{22} - a_{12}a_{21} = 0$. Its singular locus is

$$\{a_{11} = a_{22} = a_{12} = a_{21} = 0\},$$

a codimension 4 linear subspace. We see that D_W is irreducible and its singular locus has codimension 4.

2.11.1. DEFINITION. Let X be an algebraic variety with subvarieties Y and Z. We say that Y and Z *intersect properly* if $Y \cap Z$ is either empty or

$$\operatorname{codim}_X(Y \cap Z) = \operatorname{codim}_X Y + \operatorname{codim}_X Z.$$

These subvarieties intersect *transversally* at $x \in Y \cap Z$ if X and Y and Z are smooth at x and

$$T_{X,x} = T_{Y,x} + T_{Z,x}.$$

One can prove that in this case $Y \cap Z$ is smooth at x and

$$T_{Y\cap Z,x} = T_{Y,x} \cap T_{Z,x}.$$

2.11.2. LEMMA. *The Grassmannian $G(2,n)$ intersects the hyperplane H_W properly. The intersection is transversal away from the singular locus of D_W.*

Proof. This is essentially the same argument as above. The intersection is not transversal at some point x if and the only if the tangent space to $G(2,n)$ at x is contained in the tangent space to H_W at x. One can work in the affine charts $x_{ij} \neq 0$ of $\mathbb{P}^{\binom{n}{2}-1}$ and the corresponding charts $p_{ij} \neq 0$ of the Grassmannian. The tangent space to H_W is a hyperplane $H = \{x_{12} = 0\}$. The tangent space to $G(2,n)$ is contained in this hyperplane if and only if the differential of the restriction of x_{12} to the Grassmannian vanishes. But this restriction is the Plücker minor p_{12} and its differential vanishes precisely along the singular locus of D_W. $\square$

We are going to invoke a powerful

2.11.3. THEOREM (Kleiman–Bertini). *Let G be a complex algebraic group acting regularly and transitively on an algebraic variety X[3]. Let $Y, Z \subset X$ be irreducible subvarieties. Then, for a sufficiently general[4] $g \in G$, subvarieties Y and gZ intersect properly. Moreover, if $x \in Y \cap gZ$ is a smooth point of both Y and gZ then these subvarieties will intersect transversally at x.*

2.11.4. COROLLARY (Bertini's Theorem). *An irreducible subvariety $Y \subset \mathbb{P}^n$ intersects a general hyperplane $H \subset \mathbb{P}^n$ (or a general projective subspace of fixed dimension) properly and transversally at all smooth points of Y.*

Proof. Apply Kleiman–Bertini to $X = \mathbb{P}^n$, $G = \mathrm{GL}_{n+1}$, and $Z = H$. $\square$

2.11.5. REMARK. In the set-up of Bertini Theorem one can prove more: $Y \cap H$ is non-empty (unless Y is a point) and irreducible (unless Y is a curve).

Applying Bertini's theorem repeatedly, we see that if $X \subset \mathbb{P}^n$ be a projective variety of dimension r then a general projective subspace of codimension r intersects X transversally in all intersection points. The following refined definition of degree is a special case of a general principle called "conservation of number".

2.11.6. THEOREM. *Let $X \subset \mathbb{P}^n$ be a projective variety of dimension r. If L is a projective subspace of codimension r that intersects X transversally in all intersection points then $X \cap L$ is a finite union d points, where $d = \deg X$.*

Proof of Theorem 2.10.1. We need to show that for sufficiently general codimension 2 subspaces

$$W_1, \ldots, W_{2n-4} \subset \mathbb{P}^{n-1},$$

the subspace

$$L = H_{W_1} \cap \ldots \cap H_{W_{2n-4}} \subset \mathbb{P}^{\binom{n}{2}-1}$$

has codimension $2n-4$ and the intersection $G(2,n) \cap L$ is transversal at all intersection points. Arguing by induction, it suffices to show

[3]We are going to discuss algebraic groups and their actions later. The only relevant case for now is the transitive action of $G = \mathrm{GL}_n(\mathbb{C})$ on $X = G(k,n)$, for example on $\mathbb{P}^{n-1}$.

[4]There exists a dense Zariski open subset $U \subset G$ such that this is true for every $g \in U$.

2.11.7. CLAIM. *Let*

$$X_i = G(2,n) \cap H_{W_1} \cap \ldots \cap H_{W_i},$$

then X_i and $H_{W_{i+1}}$ intersect properly and transversally in $\mathbb{P}^{\binom{n}{2}-1}$ at all points of an open Zariski dense subset of X_{i+1}.

Equivalently, we need to show that X_i and $D_{W_{i+1}}$ intersect properly and transversally in $G(2,n)$ at all points of an open Zariski dense subset of X_{i+1}. We note that GL_n acts transitively on $G(2,n)$ and

$$gD_{W_{i+1}} = D_{gW_{i+1}}$$

for every $g \in \mathrm{GL}_n$. By the Kleiman–Bertini theorem, not only $X_i \cap D_{W_{i+1}}$ but also the intersections $X_i \cap \operatorname{Sing} D_{W_{i+1}}$ and $\operatorname{Sing} X_i \cap D_{W_{i+1}}$ will be proper for a sufficiently general subset W_{i+1}. In particular, they will have higher codimension than $X_i \cap D_{W_{i+1}}$, and therefore X_i and $D_{W_{i+1}}$ will in fact be smooth (and hence intersect transversally) at all points of an open Zariski dense subset of $X_i \cap D_{W_{i+1}}$. □

§2.12. **Homework 1.**

Problem 1. (2 points) Show that $G(k,n)$ (as defined in the lecture notes) is a second countable Hausdorff topological space.

Problem 2. (2 points) Let $L_1, L_2, L_3 \subset \mathbb{P}^3$ be general lines. (a) Show that there exists a unique smooth quadric surface $S \subset \mathbb{P}^3$ which contains L_1, L_2, L_3. (b) Use the previous part to give an alternative proof of the fact that 4 general lines in $\mathbb{P}^3$ intersect exactly two lines.

Problem 3. (1 point) Identify $\mathbb{C}^{\binom{n}{2}}$ with the space of skew-symmetric $n \times n$ matrices in such a way that $G(2,n)$ becomes the projectivization of the set of skew-symmetric matrices of rank 2 and Plücker quadrics become 4×4 sub-Pfaffians of an $n \times n$ skew-symmetric matrix.

Problem 4. (2 point) For any line $L \subset \mathbb{P}^3$, let $[L] \in \mathbb{C}^6$ be its Plücker vector. The Grassmannian $G(2,4) \subset \mathbb{P}^5$ is a quadric, and therefore can be described as the vanishing set of a quadratic form Q, which has an associated inner product such that $Q(v) = v \cdot v$. Describe this inner product in coordinates and show that $[L_1] \cdot [L_2] = 0$ if and only if L_1 and L_2 intersect.

Problem 5. (3 points) In the notation of the previous problem, show that five given lines $L_1, \ldots, L_5$ all intersect some line if and only if

$$\det \begin{bmatrix} 0 & [L_1]\cdot[L_2] & [L_1]\cdot[L_3] & [L_1]\cdot[L_4] & [L_1]\cdot[L_5] \\ [L_2]\cdot[L_1] & 0 & [L_2]\cdot[L_3] & [L_2]\cdot[L_4] & [L_2]\cdot[L_5] \\ [L_3]\cdot[L_1] & [L_3]\cdot[L_2] & 0 & [L_3]\cdot[L_4] & [L_3]\cdot[L_5] \\ [L_4]\cdot[L_1] & [L_4]\cdot[L_2] & [L_4]\cdot[L_3] & 0 & [L_4]\cdot[L_5] \\ [L_5]\cdot[L_1] & [L_5]\cdot[L_2] & [L_5]\cdot[L_3] & [L_5]\cdot[L_4] & 0 \end{bmatrix} = 0$$

Problem 6. (2 points) Find a point of $G(3,6)$ with exactly 14 non-vanishing Plücker minors.

Problem 7. (2 points) Prove multiplicativity of initial terms

$$\mathrm{in}(fg) = \mathrm{in}(f)\,\mathrm{in}(g)$$

of polynomials in lexicographic ordering.

Problem 8. (1 point) Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d. Compute its Hilbert function and Hilbert polynomial.

Problem 9. (1 point) Let $p_1, p_2, p_3 \in \mathbb{P}^2$ be different points which (a) don't lie on a line or (b) lie on line. Compute the Hilbert function and the Hilbert polynomial of the union $X = \{p_1, p_2, p_3\}$.

Problem 10. (2 points) Let $L_1, L_2 \subset \mathbb{P}^3$ be skew lines. Compute the Hilbert function and the Hilbert polynomial of the union $X = L_1 \cup L_2$.

Problem 11. (3 points) Let $X \subset \mathbb{P}^n$ be a hypersurface and let $F_X \subset G(2, n)$ be the subset of all lines contained in X. Show that F_X is a projective algebraic variety.

Problem 12. (3 points) For any point $p \in \mathbb{P}^3$ (resp. any plane $H \subset \mathbb{P}^3$) let $L_p \subset G(2, 4)$ (resp. $L_H \subset G(2, 4)$) be a subset of lines containing p (resp. contained in H). (a) Show that every L_p and L_H is isomorphic to $\mathbb{P}^2$ in the Plücker embedding of $G(2, 4)$. (b) Show that any $\mathbb{P}^2$ contained in $G(2, 4) \subset \mathbb{P}^5$ has a form L_p or L_H for some p or H.

Problem 13. (4 points) Consider the Segre map

$$\psi : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n^2-1} = \mathbb{P}(\mathrm{Mat}_{n,n}),$$
$$\psi([x_1 : \ldots : x_n], [y_1 : \ldots : y_n]) = [x_1 y_1 : \ldots : x_i y_j : \ldots : x_n y_n]$$

Its image is called the Segre variety. (a) Show that this map is an embedding of complex manifolds. (b) Show that the homogeneous ideal of the Segre variety in $\mathbb{P}(\mathrm{Mat}_{n,n})$ is generated by 2×2 minors $a_{ij}a_{kl} - a_{il}a_{kj}$. (c) Compute the Hilbert polynomial and the degree of the Segre variety.

Problem 14. (1 point) Let the symmetric group S_n act on $\mathbb{C}^n$ by permuting coordinates. Show that the elementary symmetric functions

$$\sigma_1 = x_1 + \ldots + x_n, \quad \ldots, \quad \sigma_n = x_1 \ldots x_n$$

form a complete set of invariants for this action.

Problem 15. (2 points) An alternative way of thinking about a matrix

$$X = \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ x_{21} & \ldots & x_{2n} \end{bmatrix}$$

is that it gives n points $p_1, \ldots, p_n$ in $\mathbb{P}^1$ (with homogeneous coordinates $[x_{11} : x_{21}]$, ..., $[x_{1n} : x_{2n}]$) as long as X has no zero columns. Suppose $n = 4$ and consider the rational normal curve $f : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.

(a) Show that points $f(p_1), \ldots, f(p_4)$ lie on a plane if and only

$$F(X) = \det \begin{bmatrix} x_{11}^3 & x_{11}^2 x_{21} & x_{11} x_{21}^2 & x_{21}^3 \\ x_{12}^3 & x_{12}^2 x_{22} & x_{12} x_{22}^2 & x_{22}^3 \\ x_{13}^3 & x_{13}^2 x_{23} & x_{13} x_{23}^2 & x_{23}^3 \\ x_{14}^3 & x_{14}^2 x_{24} & x_{14} x_{24}^2 & x_{24}^3 \end{bmatrix} = 0.$$

(b) Express $F(X)$ as a polynomial in 2×2 minors of the matrix X.

Problem 16. (3 points) Let $V_i = \langle e_1, \ldots, e_i \rangle \subset \mathbb{C}^n$ for $i = 0, \ldots, n$,

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n.$$

Fix integers $n - 2 \geq a \geq b \geq 0$ and define the Schubert variety

$$W_{a,b} \subset G(2, n) \quad \text{consists of all subspaces } U \text{ such that}$$

$$\dim(U \cap V_k) = \begin{cases} 0 & \text{if} \quad k < n-1-a \\ 1 & \text{if} \quad n-1-a \le k < n-b \\ 2 & \text{if} \quad n-b \le k. \end{cases}$$

(a) Show that $\overline{W}_{1,0}$ is $D_{\langle e_1,\ldots,e_{n-2}\rangle}$ from the lecture notes.
(b) Show that $W_{a,b}$ is isomorphic to $\mathbb{C}^{2(n-2)-a-b}$.
(c) Use part (b) to compute the topological Euler characteristic of $G(2,n)$.

§3. **Fine moduli spaces**

In this section we introduce a functorial (or categorical) language for dealing with moduli problems. First we remind basic definitions.

§3.1. **Categories.** Most mathematical theories deal with situations when there are some maps between objects. The set of objects is usually somewhat static (and so boring), and considering maps makes the theory more dynamic (and so more fun). Usually there are some natural restrictions on what kind of maps should be considered: for example, it is rarely interesting to consider any map from one group to another: usually we require this map to be a homomorphism. The notion of a *category* was introduced by Samuel Eilenberg and Saunders MacLane to capture tsituations when we have both *objects* and *morphisms* between objects. This notion is a bit abstract (hence the moniker *abstract nonsense*), but extremely useful in moduli theory. Before we give a rigorous definition, here are some examples of categories. For each category we describe objects and morphisms:

3.1.1. EXAMPLE.

- **Sets**: objects are sets, morphisms are functions between sets.
- **Groups**: groups, homomorphisms of groups.
- **Ab**: abelian groups, homomorphisms of abelian groups
- **Rings**: rings, homomorphisms of rings.
- $\mathbf{Vect}_k$: k-vector spaces, linear maps.
- $\mathbf{Mod}_R$: R-modules, homomorphisms of R-modules.
- **Top**: topological spaces, continuous functions.
- **Mflds**: smooth manifolds, differentiable maps
- **CpxMflds**: complex manifolds, holomorphic maps
- $\mathbf{Var}_k$: algebraic varieties over a field k, regular maps

In all these examples composition of morphisms is well-defined and associative (because in these examples morphisms are functions and composition of functions is associative). Associativity of composition is a sacred cow of mathematics, and essentially the only axiom of a category:

3.1.2. DEFINITION. A category C consists of the following data:

- The set of objects $\mathrm{Ob}(C)$. Instead of writing "X is an object in C", we can write $X \in \mathrm{Ob}(C)$, or even $X \in C$.
- The set of morphisms $\mathrm{Mor}(C)$. Every morphism f is a morphism *from* an object X *to* an object Y. It is common to denote a morphism by an arrow $X \xrightarrow{f} Y$. Formally, $\mathrm{Mor}(C)$ is a disjoint union of subsets $\mathrm{Mor}_C(X,Y)$ over all $X, Y \in C$.

- Composition law for morphisms, i.e. a function
$$\mathrm{Mor}_C(X,Y)\times\mathrm{Mor}_C(Y,Z)\to\mathrm{Mor}_C(X,Z),\qquad (f,g)\mapsto g\circ f$$
which takes $X\xrightarrow{f}Y$ and $Y\xrightarrow{g}Z$ to the morphism denoted $X\xrightarrow{g\circ f}Z$ even though it doesn't have to be composition of functions.
- For each object $X\in C$, we have an *identity morphism* $X\xrightarrow{\mathrm{Id}_X}X$.

These data should satisfy the following axioms:

- Composition is associative.
- Composition of any morphism $X\xrightarrow{f}Y$ with $X\xrightarrow{\mathrm{Id}_X}X$ (resp. with $Y\xrightarrow{\mathrm{Id}_Y}Y$) is equal to f.

Beginners typically focus on objects, but morphisms are more important. In fact, one can define very interesting categories with just one object:

3.1.3. EXAMPLE. Let G be a group. Then we can define a category C with just one object (let's denote it by $\cdot$) and with
$$\mathrm{Mor}(C)=\mathrm{Mor}(\cdot,\cdot)=G.$$
The composition law is just the composition law in the group and the identity element Id_O is just the identity element of G.

3.1.4. DEFINITION. A morphism $X\xrightarrow{f}Y$ is called an *isomorphism* if there exists a morphism $Y\xrightarrow{g}X$ (called an inverse of f) such that
$$f\circ g=\mathrm{Id}_Y\qquad\text{and}\qquad g\circ f=\mathrm{Id}_X.$$

In the example above, every morphism is an isomorphism. Namely, an inverse of any element of $\mathrm{Mor}(C)=G$ is its inverse in G. A category where every morphism is an isomorphism is called a *groupoid*, because every groupoid with one object corresponds to some group G. Indeed, axioms of the group (associativity, existence of a unit, existence of an inverse) translate into axioms of the groupoid (associativity of the composition, existence of an identity morphism, existence of an inverse morphism).

Of course not every category with one object is a groupoid and not every groupoid has only one object.

3.1.5. EXAMPLE. Fix a field k and a positive integer n. We can define a category C with just one object (let's denote it by $\cdot$) and with
$$\mathrm{Mor}(C)=\mathrm{Mat}_{n,n}\,.$$
The composition law is given by the multiplication of matrices. The identity element Id_O is just the identity matrix. In this category, a morphism is an isomorphism if and only if the corresponding matrix is invertible.

Here is an interesting example of a category with a different flavor:

3.1.6. EXAMPLE. Recall that a *partially ordered set*, or a *poset*, is a set I with an order relation $\preceq$ which is

- reflexive: $i\preceq i$ for any $i\in I$,
- transitive: $i\preceq j$ and $j\preceq k$ implies $i\preceq k$, and
- anti-symmetric: $i\preceq j$ and $j\preceq i$ implies $i=j$.

For example, we can take the usual order relation $\leq$ on real numbers, or divisibility relation $a|b$ on natural numbers ($a|b$ if a divides b). Note that in this last example not any pair of elements can be compared.

Interestingly, we can view any poset as a category C. Namely, $\mathrm{Ob}(C) = I$ and for any $i, j \in I$, $\mathrm{Mor}(i, j)$ is an empty set if $i \not\preceq j$ and $\mathrm{Mor}(i, j)$ is a set with one element if $i \preceq j$. The composition of morphisms is defined using transitivity of $\preceq$: if $\mathrm{Mor}(i, j)$ and $\mathrm{Mor}(j, k)$ is non-empty then $i \preceq j$ and $j \preceq k$, in which case $i \preceq k$ by transitivity, and therefore $\mathrm{Mor}(i, k)$ is non-empty. In this case $\mathrm{Mor}(i, j)$, $\mathrm{Mor}(j, k)$, and $\mathrm{Mor}(i, k)$ consist of one element each, and the composition law $\mathrm{Mor}(i, j) \times \mathrm{Mor}(j, k) \to \mathrm{Mor}(i, k)$ is defined in a unique way. Notice also that, by reflexivity, $i \preceq i$ for any i, hence $\mathrm{Mor}(i, i)$ contains a unique morphism: this will be our identity morphism.

3.1.7. EXAMPLE. Let X be a topological space. Let I be the set of open subsets of X ordered by inclusion of open sets $U \subset V$. This is a poset. The corresponding category is denoted by $\mathbf{Top}(X)$.

§3.2. **Functors.**

3.2.1. DEFINITION. A *covariant functor* F from a category C to a category D is a rule that, for each object $X \in C$, associates an object $F(X) \in D$, and for each morphism $X \xrightarrow{f} Y$, associates a morphism $F(X) \xrightarrow{F(f)} F(Y)$. Two axioms have to be satisfied:

- $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for any $X \in C$.
- F preserves composition: for any $X \xrightarrow{g} Y$ and $Y \xrightarrow{f} Z$, we have $F(f \circ g) = F(f) \circ F(g)$.

3.2.2. DEFINITION. A *contravariant*) *functor* F from a category C to a category D is a rule that, for each object $X \in C$, associates an object $F(X) \in D$, and for each morphism $X \xrightarrow{f} Y$, associates a morphism $F(Y) \xrightarrow{F(f)} F(X)$. Two axioms have to be satisfied:

- $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for any $X \in C$.
- F preserves composition: for any $X \xrightarrow{g} Y$ and $Y \xrightarrow{f} Z$, we have $F(f \circ g) = F(g) \circ F(f)$.

3.2.3. REMARK. One can avoid discussion of contravariant functors by defining an *opposite category* D^{op} which has the same objects as D and all arrows are reversed: $\mathrm{Mor}_{D^{op}}(X, Y) := \mathrm{Mor}_D(Y, X)$. A contravariant functor from C to D is the same thing as a covariant functor from C to D^{op}. However, the most important functor we are going to be interested in (the functor of points) is contravariant so we prefer to use this terminology.

3.2.4. EXAMPLE. Let's give some examples of functors.

- Inclusion of a subcategory, for example a functor
$$\mathbf{Ab} \to \mathbf{Groups}$$
sends every Abelian group G to itself (viewed simply as a group) and every homomorphism $G \xrightarrow{f} H$ of Abelian groups to itself (viewed as a homomorphism of groups).

- More generally, we have various *forgetful functors* $C \to D$. This is not a rigorous notion, it simply means that objects (and morphisms) of C are objects (and morphisms) of D with some extra data and some restrictions on this data. The forgetful functor simply 'forgets' extra data. For example, a forgetful functor $\mathbf{Vect}_k \to \mathrm{Sets}$ sends every vector space to the set of its vectors and every linear map to itself (viewed as a function from vectors to vectors). Here we 'forget' that we can add vectors and multiply them by scalars, and that linear maps are linear!
- The duality $\mathbf{Vect}_k \to \mathbf{Vect}_k$ is a contravariant functor that sends every vector space V to the vector space V^* of linear functions on V. A linear map $L : V \to U$ goes to a linear map $L^* : U^* \to V^*$, which sends a linear function $f \in U^*$ to a linear function $v \mapsto f(L(v))$.
- A similar contravariant functor is a functor $\mathbf{Var}_k \to \mathbf{Rings}$ that sends an algebraic variety X to its coordinate ring $k[X]$ and a regular map $X \xrightarrow{f} Y$ to the *pull-back homomorphism* $f^* : k[Y] \to k[X]$ (just compose a function on Y with f to get a function on X).
- Here is an important variation: fix an algebraic variety X and consider a functor $\mathbf{Top}(X) \to \mathbf{Rings}$ that sends every Zariski open subset $U \subset X$ to the ring $\mathcal{O}_X(U)$ of functions regular on U. For every inclusion $U \subset V$ of open sets, the pull-back homomorphism $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is just the restriction of regular functions.
- A contravariant functor $\mathbf{Top}(X) \to \mathbf{Sets}$ (or $\to \mathbf{Ab}$, $\to \mathbf{Rings}$, $\ldots$) is called a presheaf of sets (of abelian groups, of rings, $\ldots$) on the topological space X. For example, $\mathcal{O}_X$ is a presheaf. In fact it satisfies additional gluing axioms of a *sheaf*.

§3.3. **Equivalence of Categories.** It is tempting to consider a category of all categories with functors as morphisms. Indeed, we can define composition of functors $C \xrightarrow{F} D$ and $D \xrightarrow{G} E$ in an obvious way, and we have identity functors $C \xrightarrow{\mathrm{Id}_C} C$ that do not change objects or morphisms. There are set-theoretic issues with this super category, but we are going to ignore them. An interesting question though is when to consider two categories C and D equivalent? Categories C and D are *isomorphic* if there exist functors $C \xrightarrow{F} D$ and $D \xrightarrow{G} C$ that are inverses of each other. However, this definition is too restrictive to be useful. Here is a typical example why:

3.3.1. EXAMPLE. Let D be a category of finite-dimensional k-vector spaces and let C be its subcategory that has only one object in each dimension n, namely the standard vector space k^n of column vectors. Notice that

$$\mathrm{Mor}_C(k^n, k^m) = \mathrm{Mat}_{m,n}$$

in the usual way of linear algebra. The categories C and D are not isomorphic, because D contains all sorts of vector spaces in every dimension and C contains only k^n. However, the main point of linear algebra is that C is sufficient to do any calculation, because any n-dimensional vector space V is isomorphic to k^n "after we choose a basis in V". It turns out that this reflects the fact that C and D are *equivalent categories*.

3.3.2. DEFINITION. A covariant functor $C \xrightarrow{F} D$ is called an *equivalence of categories* if

- F is *essentially surjective*, i.e. every object in D is isomorphic to an object of the form $F(X)$ for some $X \in C$.
- F is *fully faithful*, i.e.
$$\mathrm{Mor}_C(X, Y) = \mathrm{Mor}_D(F(X), F(Y))$$
for any objects $X, Y \in C$.

3.3.3. EXAMPLE. Consider "linear-algebra" categories above. We claim that an obvious inclusion functor $F : C \to D$ is an equivalence of categories. To show that F is essentially surjective, take $V \in D$, an n-dimensional vector space. Then V is isomorphic to k^n, indeed any choice of a basis $e_1, \ldots, e_n \in V$ gives an isomorphism $V \to k^n$ which sends $v \in V$ to the column vector of its coordinates in the basis $\{e_i\}$. Notice that F is also fully faithful: linear maps from k^n to k^m are the same in categories C and D. So F is an equivalence of categories.

Our definition is not very intuitive because it is not clear that equivalence of categories is an equivalence relation! We postpone the general statement until exercises and just look at our example: is there an equivalence of categories $G : D \to C$? For every n-dimensional vector space V, there is only one obvious candidate for $G(V)$, namely k^n. Are we done? No, because we also have to define $G(L)$ for every linear map $L : V \to U$. This has to be a matrix, but to write a matrix of L we need to make a choice of a basis in *every vector space* V. In other words, let's choose a linear isomorphism $I_V : V \to k^n$ for every n-dimensional vector space V. Then we can define $G(L) : k^n \to k^m$ as the composition
$$k^n \xrightarrow{I_V^{-1}} V \xrightarrow{G} U \xrightarrow{I_U} k^m.$$
In more down-to-earth terms, $G(L)$ is a matrix of L in coordinates associated to our choice of bases in V and in U. It is immediate that G is essentially surjective (in fact just surjective) and fully faithful: linear maps from V to U are identified with linear maps from k^n to k^m. This shows that there is no "canonical" choice for G: unlike F, G is not unique!

3.3.4. EXAMPLE. A similar example from algebraic geometry is equivalence of the category of irreducible affine algebraic sets $X \subset \mathbb{A}^n$ and the category of finitely generated k-algebras without zero-divisors and homomorphisms between them. Recall that a morphism from $X \subset \mathbb{A}^n$ to $Y \subset \mathbb{A}^m$ is simply the restriction of a polynomial mapping $F : \mathbb{A}^n \to \mathbb{A}^m$ such that $F(X) \subset Y$. In one direction, the functor associates to $X \subset \mathbb{A}^n$ the coordinate ring $k[X] = k[x_1, \ldots, x_n]/I(X)$ and to morphism $f : X \to Y$ the pullback homomorphism $f^* : k[Y] \to k[X]$. It is straightforward to see that this functor is essentially surjective and fully faithful, and therefore an equivalence of categories. To construct an equivalence in the opposite direction, one needs to make a choice of generators in every finitely generated k-algebra R without zero-divisors. This choice allows to present R as the quotient of the polynomial algebra $k[x_1, \ldots, x_n]$ by some ideal I, the algebraic set X will be $V(I)$, the vanishing set of that ideal.

§3.4. **Representable Functors.** Now we got to the core of applications of categories to moduli problems and to algebraic geometry in general: a brilliant idea of Grothendieck to study any object $X \in C$ by poking it with other objects of C. As a motivation, if X is an algebraic variety then we can easily recover the set of points of X as

$$\mathrm{Mor}_{\mathbf{Var}}(pt, X).$$

But of course we will lose all geometric information about X. As it turns out, one can recover all of it by considering morphisms from all algebraic varieties Y to X, not just the point. A morphism from Y to X is called a Grothendieck's *Y-point of X*. This is packaged into the *functor of points h_X*:

3.4.1. DEFINITION. A *contravariant functor of points h_X* is a functor

$$h_X : C \to \mathbf{Sets}$$

that sends every $Y \in C$ to the set of morphisms $\mathrm{Mor}_C(Y, X)$ and every morphism $Y_1 \xrightarrow{f} Y_2$ to the function

$$\mathrm{Mor}(Y_2, X) \to \mathrm{Mor}(Y_1, X), \quad \alpha \mapsto \alpha \circ f.$$

The main game in town is to start with a contravariant functor and try to guess, is it a functor of points of some object X?

3.4.2. EXAMPLE. Take a functor $\mathbf{Var}_k \to \mathbf{Sets}$ that sends every algebraic variety X to its coordinate ring $k[X]$ and every regular map $X \xrightarrow{f} Y$ to the pull-back homomorphism. Is it a functor of points of some algebraic variety? A regular function f on X is the same data as a regular map $f : X \to \mathbb{A}^1$. It follows that the functor is functor of points of $\mathbb{A}^1$!

§3.5. **Natural Transformations.** As Maclane famously said: "I did not invent category theory to talk about functors. I invented it to talk about natural transformations." So what is a natural transformation? It is a map form one functor to another! Let's start with an example that explains why we might need such a thing.

3.5.1. EXAMPLE. Recall that for every vector space V, we have a "natural" linear map

$$\alpha_V : V \to V^{**}$$

(in fact an isomorphism if $\dim V < \infty$) that sends a vector $v \in V$ to the linear functional $f \mapsto f(v)$ on V^*. What's so "natural" about this map? One explanation is that α_V does not depend on any choices. But this is a "linguistic" explanation, can we *define* naturality mathematically?

Let's study the effect of α_V on morphisms. Let $U \xrightarrow{L} V$ be a linear map. We also have our "natural" linear maps $\alpha_U : U \to U^{**}$ and $\alpha_V : V \to V^{**}$. By taking a dual linear map twice, we have a linear map $U^{**} \xrightarrow{L^{**}} V^{**}$. To summarize, we have a square of linear maps:

$$\begin{tikzcd} U \arrow[r, "\alpha_U"] \arrow[d, "L"'] & U^{**} \arrow[d, "L^{**}"] \\ V \arrow[r, "\alpha_V"'] & V^{**} \end{tikzcd} \tag{3.5.2}$$

A key observation is that this diagram is commutative. Indeed, pick $u \in U$. Then we claim that

$$\alpha_V(L(u)) = L^{**}(\alpha_U(u)).$$

Both sides of this equation are elements of V^{**}, i.e. linear functionals on V^*. The functional on the LHS takes $f \in V^*$ to $f(L(u))$. The functional on the RHS takes $f \in V^*$ to

$$\alpha_U(u)(L^*(f)) = L^*(f)(u) = f(L(u)).$$

If this calculation looks confusing, just redo it yourself!

Now let's give a general definition.

3.5.3. DEFINITION. Let $F, G : C \to D$ be two covariant functors. A *natural transformation* $\alpha : F \to G$ between them is a rule that, for each object $X \in C$, assigns a morphism $F(X) \xrightarrow{\alpha_X} G(X)$ in D such that for any morphism $X_1 \xrightarrow{f} X_2$ in C, the following diagram is commutative:

$$\begin{array}{ccc} F(X_1) & \xrightarrow{\alpha_{X_1}} & G(X_1) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(X_2) & \xrightarrow[\alpha_{X_2}]{} & G(X_2) \end{array} \tag{3.5.4}$$

If α_X is an isomorphism for any X then α is called a *natural isomorphism*. Natural transformation of contravariant functors is defined similarly.

3.5.5. EXAMPLE. Let $\mathbf{Vect}_k$ be the category of vector spaces over k. Consider two functors: the identity functor $\mathrm{Id} : \mathbf{Vect}_k \to \mathbf{Vect}_k$ and the "double dual" functor $D : \mathbf{Vect}_k \to \mathbf{Vect}_k$ that sends every vector space V to V^{**} and every linear map $L : U \to V$ to a double dual linear map $L^{**} : U^{**} \to V^{**}$. We claim that there is a natural transformation from Id to D (and in fact a natural isomorphism if we restrict to a subcategory of finite-dimensinal vector spaces). All we need is a rule α_V for every vector space V: it should be a morphism, i.e. a linear map, from $\mathrm{Id}(V) = V$ to $D(V) = V^{**}$ such that (3.5.4) is satisfied for any morphism $U \to V$. This is exactly the linear map constructed above.

3.5.6. DEFINITION. Let $F : C \to \mathbf{Sets}$ be a contravariant functor. If F is naturally isomorphic to a functor of points h_X of some object $X \in C$ then we say that F is *representable by X*.

3.5.7. DEFINITION. A *moduli functor* is a contravariant functor

$$\mathcal{M} : \mathbf{Var}_k \to \mathbf{Sets}.$$

If this functor is representable by an algebraic variety M, we say that M is a *fine moduli space* of the functor $\mathcal{M}$.

§3.6. **Yoneda's Lemma.** A powerful general result is that h_X determines X up to isomorphism. The following is a weak version:

3.6.1. LEMMA. *Let X, Y be two objects in a category C. Suppose we have a natural isomorphism of representable functors $\alpha : h_X \to h_Y$. Then X and Y are isomorphic, in fact α gives a canonical choice of such an isomorphism.*

Proof. Indeed, α gives, for any object Z in C, a bijection

$$\alpha_Z:\ \mathrm{Mor}(Z,X)\to\mathrm{Mor}(Z,Y)$$

such that for each morphism $Z_1\to Z_2$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Mor}(Z_1,X) \arrow[r, "\alpha_{Z_1}"] & \mathrm{Mor}(Z_1,Y) \\ \mathrm{Mor}(Z_2,X) \arrow[u] \arrow[r, "\alpha_{Z_2}"'] & \mathrm{Mor}(Z_2,Y) \arrow[u] \end{tikzcd}$$

where the vertical arrows are obtained by composing with $Z_1\to Z_2$. In particular, we have bijections

$$\mathrm{Mor}(X,X)\xrightarrow{\alpha_X}\mathrm{Mor}(X,Y)\quad\text{and}\quad\mathrm{Mor}(Y,X)\xrightarrow{\alpha_Y}\mathrm{Mor}(Y,Y).$$

We define morphisms

$$f=\alpha_X(\mathrm{Id}_X)\in\mathrm{Mor}(X,Y)\quad\text{and}\quad g=\alpha_Y^{-1}(\mathrm{Id}_Y)\in\mathrm{Mor}(Y,X).$$

We claim that f and g are inverses of each other, and in particular X and Y are isomorphic via f and g. Indeed, consider the commutative square above when $Z_1=Y$, $Z_2=X$, and the morphism from Y to X is g. It gives

$$\begin{tikzcd} \mathrm{Mor}(Y,X) \arrow[r, "\alpha_Y"] & \mathrm{Mor}(Y,Y) \\ \mathrm{Mor}(X,X) \arrow[u, "\circ g"] \arrow[r, "\alpha_X"'] & \mathrm{Mor}(X,Y) \arrow[u, "\circ g"'] \end{tikzcd}$$

Let's take $\mathrm{Id}_X\in\mathrm{Mor}(X,X)$ and compute its image in $\mathrm{Mor}(Y,Y)$ in two different ways. If we go horizontally, we first get $\alpha_X(\mathrm{Id}_X)=f\in\mathrm{Mor}(X,Y)$. Then we take its composition with $Y\xrightarrow{g}X$ to get $f\circ g\in\mathrm{Mor}(Y,Y)$. If we go vertically first, we get $g\in\mathrm{Mor}(Y,X)$. Then we get $\alpha_Y(g)=\mathrm{Id}_Y$, because $g=\alpha_Y^{-1}(\mathrm{Id}_Y)$. So we see that $f\circ g=\mathrm{Id}_Y$. Similarly, one can show that $f\circ g=\mathrm{Id}_X$, i.e. f and g are inverses of each other. □

3.6.2. COROLLARY. *A moduli functor can have only one fine moduli space up to (canonical) isomorphism.*

Even better, one has the following full version of Yoneda's lemma:

3.6.3. LEMMA. *Let C be a category. For every object X of C, consider its functor of points $h_X\,:\,C\to\mathrm{Sets}$. For every morphism $X_1\xrightarrow{f}X_2$, consider a natural transformation $h_{X_1}\to h_{X_2}$ defined as follows: for any object Y of C, the function*

$$\alpha_Y:\ h_{X_1}(Y)=\mathrm{Mor}(Y,X_1)\xrightarrow{f\circ}\mathrm{Mor}(Y,X_2)=h_{X_2}(Y)$$

is just a composition of $g\in\mathrm{Mor}(Y,X_1)$ with $X_1\xrightarrow{f}X_2$. This gives a functor from C to the category of contravariant functors $C\to\mathrm{Sets}$ (with natural transformations as morphisms). This functor is fully faithful, i.e. the set of morphisms $\mathrm{Mor}_C(X_1,X_2)$ is identified with the set of natural transformations $h_{X_1}\to h_{X_2}$.

Proof. Suppose we are given a natural transformation $\alpha : h_{X_1} \to h_{X_2}$. Applying α_{X_1} to $\mathrm{Id}_{X_1} \in \mathrm{Mor}(X_1, X_1)$ gives some morphism $f \in \mathrm{Mor}(X_1, X_2)$. We claim that this establishes a required bijection between $\mathrm{Mor}(X_1, X_2)$ and natural transformations $h_{X_1} \to h_{X_2}$.

Start with $f \in \mathrm{Mor}(X_1, X_2)$. Then $\alpha_{X_1} : \mathrm{Mor}(X_1, X_1) \to \mathrm{Mor}(X_1, X_2)$ is obtained by composing with f. In particular, $\alpha_{X_1}(\mathrm{Id}_{X_1}) = f$.

Now let us start with a natural transformation $\alpha : h_{X_1} \to h_{X_2}$. Then

$$f := \alpha_{X_1}(\mathrm{Id}_{X_1}) \in \mathrm{Mor}(X_1, X_2).$$

This morphism in turn defines a natural transformation $\beta : h_{X_1} \to h_{X_2}$. We have to show that $\alpha = \beta$, i.e. that for any $Y \in C$, the map $\alpha_Y : \mathrm{Mor}(Y, X_1) \to \mathrm{Mor}(Y, X_2)$ is just a composition with f. The argument is the same as in the previous Lemma. Start with any $g \in \mathrm{Mor}(Y, X_1)$ and consider a commutative square

$$\begin{tikzcd}
\mathrm{Mor}(Y, X_1) \arrow[r, "\alpha_Y"] & \mathrm{Mor}(Y, X_2) \\
\mathrm{Mor}(X_1, X_1) \arrow[u] \arrow[r, "\alpha_{X_1}"'] & \mathrm{Mor}(X_1, X_2) \arrow[u]
\end{tikzcd}$$

where the vertical maps are compositions with $Y \xrightarrow{g} X_1$. Take Id_{X_1} and chase it along the diagram. We get

$$\begin{tikzcd}
g \arrow[r, "\alpha_Y"] & \alpha_Y(g) = f \circ g \\
\mathrm{Id}_{X_1} \arrow[u] \arrow[r, "\alpha_{X_1}"'] & f \arrow[u]
\end{tikzcd}$$

So $\alpha_Y(g)$ is exactly what we want: composition of f with g. $\square$

Constructing morphisms of algebraic varieties $X \to Y$ can be tricky. Yoneda's Lemma shows that if we have a good grasp of h_X and h_Y, we can instead construct a natural transformation $\alpha : h_X \to h_Y$. This is especially useful for constructing morphisms between fine moduli spaces.

§3.7. **Grassmannian as a fine moduli space.** Let's re-examine the Grassmannian as a fine example of a fine moduli space. What is the corresponding moduli functor? A point of $G(k, n)$, i.e. a morphism

$$\mathrm{pt} \to G(k, n)$$

is given by a k-dimensional subspace of $\mathbb{C}^n$. Likewise, we would like to think about a Grothendieck's point of $G(k, n)$, i.e. a morphism

$$X \to G(k, n)$$

as a a "family" of k-dimensional subspaces of $\mathbb{C}^n$ parametrized by X. To make this rigorous, let's recall the definition of a vector bundle.

3.7.1. DEFINITION. A *trivial vector bundle* over an algebraic variety X with the fiber $\mathbb{C}^r$ is the product $X \times \mathbb{C}^r$ along with the projection

$$\pi : X \times \mathbb{C}^r \to X.$$

More generally, an r-dimensional *vector bundle* over an algebraic variety X is an algebraic variety E and a surjective morphism

$$\pi : E \to X$$

with the following additional data and properties:

- There exists a covering $X = \bigcup U_\alpha$ called an *atlas* and
- isomorphisms (called *trivializations*)

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r, \quad p_2 \circ \psi_\alpha = \pi$$

- Such that *transition functions*, i.e. functions

$$(U_\alpha \cap U_\beta) \times \mathbb{C}^r \overset{\psi_\beta \circ \psi_\alpha^{-1}}{\longrightarrow} (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

  over overlaps $U_\alpha \cap U_\beta$ are $\mathcal{O}_X$-linear, i.e. take

$$(x, v) \mapsto (x, \phi_{\alpha\beta}(x)v),$$

  where $\phi_{\alpha\beta}(x)$ is an invertible $r \times r$ matrix with entries in $\mathcal{O}(U_\alpha \cap U_\beta)$.

An atlas is of course not unique, for example any covering finer than an atlas is also an atlas. The number r is called *the rank* of a vector bundle. A vector bundle of rank 1 is called a *line bundle*.

3.7.2. REMARK. Given an atlas $X = \bigcup U_\alpha$ and regular functions

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r,$$

a vector bundle with transition functions $\phi_{\alpha\beta}$ exist if and only if they satisfy the *cocycle condition*

$$\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}$$

on all triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$.

3.7.3. DEFINITION. Vector bundles on an algebraic variety X form a category with morphisms defined as follows: a *map of vector bundles*

$$(E, p_E) \to (F, p_F)$$

is a regular map of underlying algebraic varieties $L : E \to F$ which commutes with projections

$$\pi_F \circ L = \pi_E$$

and such that L is given by linear transformations in one (and therefore any) atlas $\{U_\alpha\}$ which trivializes both E and F.

Concretely, if $\psi_\alpha : \pi_E^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ and $\phi_\alpha : \pi_F^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^s$ are trivializations of E and F then the map

$$\phi_\alpha \circ L \circ \psi_\alpha^{-1} : U_\alpha \times \mathbb{C}^r \to U_\alpha \times \mathbb{C}^s$$

takes

$$(x, v) \mapsto (x, L_\alpha(x)v),$$

where $L_\alpha(x)$ is an $s \times r$ matrix with entries in $\mathcal{O}(U_\alpha)$. If $E \subset F$ and the inclusion is the map of vector bundles then E is called a *sub-bundle* of F.

3.7.4. EXAMPLE. The *universal (or tautological) bundle* of rank k on the Grassmannian $G(k,n)$ is defined as follows:

$$\mathcal{U} = \{([U], v) \mid v \in U\} \subset G(k,n) \times \mathbb{C}^n.$$

Its fiber at a point that corresponds to a subspace $U \subset \mathbb{C}^n$ is U itself. We claim that $\mathcal{U}$ is a vector bundle, in fact a subbundle of the trivial bundle with fiber $\mathbb{C}^n$. Indeed, it is trivialized in standard affine charts U_I of the Grassmannian where the Plücker coordinate $p_I \neq 0$. For example, let's take $k = 2$ and consider the chart $U = U_{12}$ defined by $p_{12} \neq 0$. At any point of this chart, rows (v_1, v_2) of the matrix

$$A = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_{13} & a_{14} & \dots & a_{1n} \\ 0 & 1 & a_{23} & a_{24} & \dots & a_{2n} \end{bmatrix}$$

give a basis of the corresponding subspace in $\mathbb{C}^n$. The trivialization of $\mathcal{U}$ is defined as follows:

$$\phi^{-1} : U \times \mathbb{C}^2 \to \pi^{-1}(U),$$

$$(u, z_1, z_2) \mapsto (u, z_1 v_1 + z_2 v_2) \in U \times \mathbb{C}^n.$$

In the matrix form,

$$(z_1, z_2) \mapsto (z_1, z_2)A = (z_1, z_2)A_I^{-1}A.$$

The last formula has an advantage that we don't have to assume that the matrix A has a standard form with $A_I = \mathrm{Id}$. What are the transition functions? Trivialization in the chart $p_J \neq 0$ has form

$$(z_1', z_2') \mapsto (z_1', z_2')A_J^{-1}A.$$

Therefore, the transition functions are

$$(z_1, z_2) \mapsto (z_1', z_2') = (z_1, z_2)A_I^{-1}A_J, \quad A_I^{-1}A_J \in \mathrm{GL}_2.$$

To the define the Grassman functor, we also need the notion of pull-back for vector bundles:

3.7.5. DEFINITION. Let $\pi : E \to Y$ be a vector bundle and $f : X \to Y$ a morphism of algebraic varieties. We define the pull-back vector bundle on X as follows:

$$f^*E = \{(x, v) \mid f(x) = \pi(v)\} \subset X \times E.$$

The projection map $f^*\pi : f^*E \to X$ is induced by the first projection $\mathrm{pr}_1 : X \times E \to X$. To see that f^*E is indeed a vector bundle, we can first assume that E is trivial (by trivializing it), in which case f^*E is clearly also a trivial vector bundle of the same rank. A trivializing atlas for f^*E can be obtained by taking preimages of open sets in a trivializing atlas of E and transition functions for f^*E are pullbacks of transition functions for E. If $E \subset F$ is a subbundle then f^*E is also naturally a subbundle of f^*E.

3.7.6. DEFINITION. A contravariant *Grassmann functor*

$$\mathcal{G}(k,n) : Algebraic\,Varieties \to Sets$$

sends every algebraic variety X to the set $\mathcal{G}(k,n)(X)$ of all rank k subbundles E of the trivial vector bundle $X \times \mathbb{C}^n$. A morphism $f : X \to Y$ gives a function $\mathcal{G}(k,n)(Y) \to \mathcal{G}(k,n)(X)$, namely the pull-back $E \mapsto f^*E$.

Notice that as sets

$$G(k,n) = \mathcal{G}(k,n)(point)$$

because both sides parametrize k-dimensional subspaces of $\mathbb{C}^n$.

3.7.7. PROPOSITION. *$\mathcal{G}(k,n)$ is represented by $G(k,n)$. Thus the Grassmannian is a fine moduli space of the Grassmann functor.*

Proof. Let E be a rank k subbundle E of the trivial vector bundle $X \times \mathbb{C}^n$. Then we have a map of sets

$$f_E : X \to G(k,n), \quad x \mapsto \pi^{-1}(x) \subset \mathbb{C}^n.$$

What are the properties of this map? Let $U \subset X$ be a trivializing chart with trivialization $\psi : \pi_E^{-1}(U) \to U \times \mathbb{C}^k$. Composing ψ^{-1} with the embedding of E into $X \times \mathbb{C}^n$ gives a map of trivial vector bundles

$$U \times \mathbb{C}^k \to U \times \mathbb{C}^n,$$

which is given by an $k \times n$ matrix A with coefficients in $\mathcal{O}_X(U)$.

In other words, restriction of the map $f_E : X \to G(k,n)$ to $U \subset X$ factors as the composition of the map $U \to \mathrm{Mat}^0(k,n)$ given by the matrix A and the map $\mathrm{Mat}^0(k,n) \to G(k,n)$ which sends a matrix to its row space. In particular, $f_E|_U$ is a morphism of algebraic varieties, and therefore the same is true for f_E.

It is clear from the definitions that f_E completely determines E, specifically E is the pull-back of the universal bundle of the Grassmannian (inside the trivial bundle):

$$E = f_E^* \mathcal{U} \subset X \times \mathbb{C}^n.$$

To summarize, given a k-dimensional subbundle (E, π) of a trivial bundle $X \times \mathbb{C}^n$, there exists a unique map $f_E : X \to G(2,n)$ such that $f_E^* \mathcal{U} = E$ as a subbundle of the trivial bundle $X \times \mathbb{C}^n = f_E^* [G(2,n) \times \mathbb{C}^n]$.

By definition of an isomorphism of functors, we have to associate to every algebraic variety X a bijection

$$\eta_X : \mathrm{Mor}(X, G(k,n)) \to \mathcal{G}(k,n)(X).$$

This is exactly what we have done above: $\eta_X(f) := f^* \mathcal{U}$. This bijection should be such that, for every morphism $g : X \to Y$, we have

$$\eta_Y \circ h_{G(2,n)}(g) = \mathcal{G}(2,n)(g) \circ \eta_X,$$

which translates into $(f \circ g)^* \mathcal{U} = g^* f^* \mathcal{U}$. □

3.7.8. EXAMPLE. Let's re-examine the projective space $\mathbb{P}^n = G(1, n+1)$ from this point of view. The universal line bundle is denoted by

$$\mathcal{O}_{\mathbb{P}^n}(-1) = \{(L, v) \,|\, v \in L\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}.$$

So $\mathbb{P}^n$ represents a functor

$$Algebraic\,Varieties \to Sets$$

which sends every algebraic variety X to the set of all line sub-bundles L of the trivial vector bundle $X \times \mathbb{C}^{n+1}$. Given this subbundle, we have an obvious map $X \to \mathbb{P}^n$ which sends $x \in X$ to the fiber of L over x (viewed as a line in $\mathbb{C}^{n+1}$). And L is then a pull-back of $\mathcal{O}_{\mathbb{P}^n}(-1)$ inside the trivial line bundle $X \times \mathbb{C}^{n+1}$.

In algebraic geometry it is more common to use an isomorphic functor. To define it, we need the following standard definitions.

3.7.9. DEFINITION. Let $\pi : E \to X$ be a vector bundle on an algebraic variety. A morphism $s : X \to E$ is called a *global section* if $\pi \circ s = \mathrm{Id}_X$. All global sections form a $\mathbb{C}$-vector space denoted by $H^0(X, E)$. Any linear map of vector bundles $L : E \to F$ on X induces a linear map

$$H^0(X, E) \to H^0(X, F), \quad s \mapsto L \circ s.$$

3.7.10. DEFINITION. A line bundle $\pi : L \to X$ is called *globally generated* if for every $x \in X$ there exists a global section $s \in H^0(X, E)$ such that

$$s(x) \neq 0.$$

3.7.11. THEOREM. *$\mathbb{P}^n$ represents a functor*

$$Algebraic\,Varieties \to Sets$$

which sends every algebraic variety X to the set of isomorphism classes of data

$$\{L^*; s_0, \ldots, s_n\},$$

where L^* is a globally generated line bundle on X and

$$s_0, \ldots, s_n \in H^0(X, L^*)$$

have the property that for every $x \in X$, there exists at least one s_i such that $s_i(x) \neq 0$. The universal family on $\mathbb{P}^n$ is given by $\{\mathcal{O}_{\mathbb{P}^n}(1); z_0, \ldots, z_n\}$

Given a datum $\{L^*; s_0, \ldots, s_n\}$ on X, the corresponding morphism to $\mathbb{P}^n$ sends $x \in X$ to a point with homogeneous coordinates $[s_0(x) : \ldots : s_n(x)]$, where we identify the fiber of L^* over x with $\mathbb{C}$ linearly.

Proof. We just have to construct an isomorphism of the new functor with our old functor of all possible inclusions $i : L \hookrightarrow X \times \mathbb{C}^{n+1}$. First we dualize the inclusion to obtain the surjection of vector bundles

$$\alpha : X \times (\mathbb{C}^{n+1})^* \twoheadrightarrow L^*$$

(surjection on each fiber!). Given α, we can define $n + 1$ global sections $s_0, \ldots, s_n$ of L^* by taking images of constant global sections of $X \times (\mathbb{C}^{n+1})^*$ which send every point $x \in X$ to $z_0, \ldots, z_n \in (\mathbb{C}^{n+1})^*$, standard coordinate functions on $\mathbb{C}^{n+1}$. And vice versa, suppose we have global sections

$$s_0, \ldots, s_n \in H^0(X, L^*)$$

such that for every $x \in X$ we have $s_i(x) \neq 0$ for some i. Then we can define α by sending (x, z_i) to $(x, s_i(x))$ and extending linearly. $\square$

3.7.12. EXAMPLE. Let's re-examine the Plücker embedding from the functorial point of view. According to Yoneda's lemma, constructing a morphism

$$G(k, n) \to \mathbb{P}^{\binom{n}{k}-1}$$

is equivalent to describing a natural transformation from the Grassmann functor to the projective space functor. Thus, for every algebraic variety X, we have to construct a function η_X that sends every rank k subbundle E of the trivial bundle $X \times \mathbb{C}^n$ to a line subbundle of the trivial bundle $X \times \mathbb{C}^{\binom{n}{k}}$. This is given by the top exterior power of the vector bundle

$$\Lambda^k E \subset \Lambda^k \mathbb{C}^n.$$

Indeed, applying this to the universal bundle of the Grassmannian takes a point $U \in G(k,n)$ given by a $k \times n$ matrix A to the top exterior power of its row space (the fiber of the universal bundle). Concretely, A goes to

$$(a_{11}e_1 + \ldots + a_{1n}e_n) \wedge \ldots \wedge (a_{k1}e_1 + \ldots + a_{kn}e_n) = \sum_I p_I(A) e_{i_1} \wedge \ldots \wedge e_{i_k},$$

which is exactly the Plücker embedding.

So far we defined moduli functors as arbitrary contravariant functors

$$\mathcal{M} : \, Algebraic\, Varieties \to Sets,$$

which is the only possible rigorous mathematical definition. In practice, $\mathcal{M}(X)$ is usually the set of isomorphism classes of "families of geometric objects" parametrized by X, which often (but not always) mean morphisms

$$\pi : \, E \to X$$

such that "fibers" of π are geometric objects of interest. Typically there are various additional assumptions on π and E. In the case of $\mathcal{G}(k,n)$, E has to be a subbundle of the trivial bundle $\mathbb{C}^n$. For a morphism $f : X \to Y$, the corresponding function

$$\mathcal{M}(f) : \, \mathcal{M}(Y) \to \mathcal{M}(X)$$

is often called the *pull-back* and denoted by f^*. Specifically, in many cases,

$$f^*E = X \times_Y E = \{(x,e) \,|\, f(x) = \pi(e)\} \subset X \times E$$

is the *fiber product*, which we will discuss in detail later. We can recover the set of isomorphism classes of our objects as $\mathcal{M}(point)$. If this set-up, the fine moduli space M (if it exists) should have a "universal" family $\mathcal{U} \in \mathcal{M}(M)$ such that every family E on X is a pull-back of $\mathcal{U}$ with respect to a unique morphism $f : \, X \to M$.

3.7.13. EXAMPLE. For a simple example of a moduli functor without a fine moduli space, consider the functor

$$\mathcal{P} : \, Algebraic\, Varieties \to Sets$$

which sends every algebraic variety X to its Picard group $\mathrm{Pic}(X)$ of all line bundles on X modulo isomorphism. To a morphism $f : \, X \to Y$ we associate a pull-back of line bundles $f^* : \, \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$, which turns $\mathcal{P}$ into a functor. We claim that it is not representable. Indeed, suppose it is representable by an algebraic variety P. The set of points of P will be identified with $\mathrm{Pic}(point)$, which is just a one-element set, the trivial line bundle $\mathbb{C}$ on the point. So $P = point$. But then every line bundle on every X will be a pull-back of the trivial line bundle on the point, i.e. will be trivial, which of course is not the case.

3.7.14. REMARK. The same argument will work for every moduli functor of families with isomorphic fibers, for example for the functor of isomorphism classes of vector bundles, $\mathbb{P}^1$-bundles, etc.

3.7.15. REMARK. I am not assuming familiarity with schemes, in fact I will use moduli problems as an excuse to introduce schemes. But if you know what they are, notice the argument above will be a bit harder if one works with algebraic schemes instead of algebraic varieties. Indeed, there is only one algebraic variety with one point, but there are plenty of schemes with one point, for example $\operatorname{Spec}\mathbb{C}[t]/(t^k)$ for every $k > 0$.

After the projective line $\mathbb{P}^1$, the easiest algebraic curve to understand is an elliptic curve (a Riemann surface of genus 1). Let

$$M_1 = \{\text{isom. classes of elliptic curves}\}.$$

We are going to assign to each elliptic curve a number, called its *j-invariant* and prove that as a set

$$M_1 = \mathbb{A}^1_j.$$

We will define the moduli functor $\mathcal{M}_1$ of families of elliptic curves and show that it has no fine moduli space! We will discuss various ways to fix this. For example, we will show that $\mathbb{A}^1$ is a *coarse* moduli space of $\mathcal{M}_1$.

More generally, we introduce

$$M_g = \{\text{isom. classes of smooth projective curves of genus } g\}$$

and

$$M_{g,n} = \left\{ \begin{matrix} \text{isom. classes of smooth projective curves } C \text{ of genus } g \\ \text{with n distinct points } p_1, \ldots, p_n \in C \end{matrix} \right\}.$$

In order to understand these spaces, we need to study GIT. The only exception is $M_{0,n}$, which is very easy to describe. Indeed, it is well-known that every 3 distinct points $p_{n-2}, p_{n-1}, p_n \in \mathbb{P}^1$ can be send to $0, 1, \infty$ by a unique automorphism of $\mathbb{P}^1$ (fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$). Thus $M_{0,n}$ can be identified with an open subset of $(\mathbb{P}^1)^{n-3}$ of points $(p_1, \ldots, p_{n-3})$ such that $p_i \neq 0, 1, \infty$ for every i and $p_i \neq p_j$ for every $i \neq j$.

§4. Algebraic curves and Riemann surfaces

We start with a review of basic facts about projective algebraic curves. Over complex numbers, the theory is equivalent to the study of compact Riemann surfaces and we frequently use both approaches.

§4.1. Elliptic and Abelian integrals.

§4.1. Elliptic and Abelian integrals. The theory of algebraic curves has its roots in analysis. In 1655 Wallis began to study the arc length of an ellipse

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1.$$

The equation can be solved for Y

$$Y = (B/A)\sqrt{(A^2 - X^2)},$$

differentiated

$$Y' = \frac{-BX}{A\sqrt{A^2 - X^2}},$$

squared and put into the integral $L=\int\sqrt{1+(Y')^2}\,dX$ for the arc length. Now the substitution $x=X/A$ results in

$$L/A=\int_0^{X/A}\sqrt{\frac{1-e^2x^2}{1-x^2}}\,dx,$$

where

$$e=\sqrt{1-(b/a)^2}$$

is the eccentricity. We can rewrite this integral as an *elliptic integral*

$$\int\frac{1-e^2x^2}{\sqrt{(1-e^2x^2)(1-x^2)}}\,dx=\int u(x,y)\,dx,$$

where $u(x,y)$ is a rational function and y is a solution of the equation

$$y^2=(1-e^2x^2)(1-x^2),$$

which defines an *elliptic curve* in $\mathbb{A}^2$. More generally,

4.1.1. DEFINITION. An *algebraic function* $y=y(x)$ is a solution of an equation

$$y^n+a_1(x)y^{n-1}+\ldots+a_n(x)=0, \tag{4.1.2}$$

where $a_i(x)\in\mathbb{C}(x)$ are rational functions. Without loss of generality, we can assume that this polynomial is irreducible over $\mathbb{C}(x)$.[5]

An *Abelian integral* is an integral of the form

$$\int u(x,y)\,dx$$

where $y=y(x)$ is an algebraic function and $u(x,y)$ is a rational function.

§4.2. **Finitely generated fields of transcendence degree** 1. All functions of the form $u(x,y)$, where u is a rational function in 2 variables and $y=y(x)$ is a solution of (4.1.2), form a field K.

4.2.1. LEMMA. *K is finitely generated and has transcendence degree 1 over $\mathbb{C}$. Every finitely generated field K with $\mathrm{tr.deg.}_{\mathbb{C}}K=1$ can be obtained in this way.*

Proof. Since x and y are algebraically dependent and x is transcendental over $\mathbb{C}$, we have $\mathrm{tr.deg.}_{\mathbb{C}}\mathbb{C}(x,y)=1$.

Now let K be any finitely generated field of transcendence degree 1 over $\mathbb{C}$. Choose any $x\in K$ transcendental over $\mathbb{C}$. Then $K/\mathbb{C}(x)$ is a *finitely generated*, *algebraic* (hence *finite*), and *separable* (because we are in characteristic 0) field extension. By a *theorem on the primitive element*, we indeed have $K=\mathbb{C}(x,y)$, where y is a root of an irreducible polynomial of the form (4.1.2). $\square$

Notice that of course there are many choices for x and y in K, thus the equation (4.1.2) is not determined by the field extension $K/\mathbb{C}$. But it turns out that this choice is not important from the perspective of computing integrals because we can always do substitutions: any integral of the form

[5]Let us point out for clarity that after Abel and Galois we know that for $n\geq 5$ not every algebraic function is a nested radical function like $y(x)=\sqrt[3]{x^3-7x\sqrt{x}}$.

$\int f\,dg$ for $f, g \in K$ is an Abelian integral. On a purely algebraic level we should study the moduli problem

$$\mathcal{M} = \{\text{isom. classes of f.g. field extensions } K/\mathbb{C} \text{ with tr.deg.}_{\mathbb{C}} K = 1.\}$$

Clearing denominators in (4.1.2) gives an irreducible affine plane curve

$$C = \{f(x, y) = 0\} \subset \mathbb{A}^2$$

and its *projective completion*, an irreducible plane curve in $\mathbb{P}^2$. Recall that the *field of rational functions* $\mathbb{C}(X)$ on an irreducible affine variety X is the quotient field of its ring of regular functions $\mathbb{C}[X]$. The field of rational functions on an arbitrary algebraic variety X is the field of rational functions of any of its affine charts. In our case

$$\mathbb{C}[C] = \mathbb{C}[x, y]/(f) \quad \text{and so} \quad \mathbb{C}(C) = K.$$

Recall by the way that the word *curve* means "of dimension 1", and *dimension* of an irreducible affine or projective variety is by definition the transcendence degree of its field of rational functions. So we can restate our moduli problem as

$$\mathcal{M} = \{\text{birational equivalence classes of irreducible plane curves.}\}$$

Here we use the following definition

4.2.2. DEFINITION. Irreducible algebraic varieties X and Y are called *birational* if their fields of rational functions $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ are isomorphic. Equivalently, there exist dense Zariski open subsets $U \subset X$ and $V \subset Y$ such that U and V are isomorphic.

More generally, we can consider arbitrary irreducible affine or projective curves because every curve C is birational to a curve in $\mathbb{A}^2$ by Lemma 4.2.1.[6] Thus our moduli problem is equivalent to the study of

$$\mathcal{M} = \{\text{birational equivalence classes of irreducible algebraic curves.}\}$$

4.2.3. THEOREM. *For any algebraic curve C, there exists a smooth projective curve C' birational to C.*

Sketch. One can assume that C is projective by taking the projective closure. There are two ways to proceed. A constructive argument involves finding a projective plane curve $\tilde{C}$ birational to C and then *resolving singularities* of $\tilde{C}$ by consecutively *blowing-up* $\mathbb{P}^2$ in remaining singular points of the proper transform of $\tilde{C}$. The difficulty is to show that the process terminates: the proper transform is eventually non-singular. Nevertheless, this is the only approach known to work in higher dimensions. Hironaka's celebrated *resolution of singularities theorem* states that, for every projective algebraic variety $X \subset \mathbb{P}^n$ in characteristic 0, there exists a sequence of blow-ups of $\mathbb{P}^n$ (but not only in points of course, one has to blow-up smooth subvarieties of higher dimension) such that eventually the proper transform of X is non-singular (and birational to X). The proof is quite involved and its existence in characteristic p is still an open question.

[6]A constructive approach to find a birational model of C in $\mathbb{P}^2$ is to take the image of C after a general *linear projection* $\mathbb{P}^n \dashrightarrow \mathbb{P}^2$. How is this related to the standard proof of the primitive element theorem?

Another approach is to construct the *normalization* of C. Recall that if X is an irreducible affine variety then its normalization $\tilde{X}$ has coordinate algebra given the *integral closure* of $\mathbb{C}[X]$ in its field of fractions $\mathbb{C}(X)$ (the fact the integral closure is finitely generated is non-trivial). If X is arbitrary then one can cover X by affine charts, normalize the charts, and then glue them back together using the fact that the integral closure commutes with localization. One can show that if X is a projective variety then $\tilde{X}$ is also projective (despite apriori being defined abstractly, by gluing charts). It turns out that *the singular locus of a normal variety has codimension at least* 2. In particular, an *algebraic curve C is normal iff it is non-singular*. □

Thus our moduli problem can be restated as the study of

$$\mathcal{M} = \{\text{birational equivalence classes of smooth projective algebraic curves.}\}$$

So far everything we said could have been done in any dimension. But the last step is specific for curves. It is known that

4.2.4. THEOREM ([**?**, 2.3.3]). *If X is a smooth algebraic variety and $f : X \to \mathbb{P}^n$ is a rational map then the indeterminancy locus of f has codimension* 2.

Thus if C is a smooth curve and $f : C \to \mathbb{P}^n$ is a rational map then f is regular. Suppose now that C and C' are birational projective curves. Then we have a birational map $C \dashrightarrow C'$, which has to be regular by the above theorem. So birational curves are in fact isomorphic,

To summarize, we have the following

4.2.5. THEOREM. *There is a bijection between*

$$\mathcal{M} = \{\text{isom. classes of smooth projective algebraic curves}\}$$

and

$$\left\{\begin{matrix}\text{isomorphism classes of finitely generated} \\ \text{field extensions } K/\mathbb{C} \text{ with } \text{tr.deg.}_{\mathbb{C}} K = 1\end{matrix}\right\},$$

which sends a curve C to its field of rational functions $\mathbb{C}(C)$.

§4.3. **Analytic approach.** Instead of complex algebraic curves, we can consider *compact Riemann surfaces* (one–dimensional compact complex manifolds) instead of algebraic curves. It turns out that this gives the same moduli problem:

$$\mathcal{M}\left\{\text{biholomorphic isom. classes of compact Riemann surfaces}\right\}.$$

Even better we have the following theorem:

4.3.1. THEOREM. *Categories of smooth projective curves and of compact Riemann surfaces are equivalent.*

Indeed, if X is a smooth projective algebraic curve then X^{an} is a compact Riemann surface and any morphism $X \to Y$ of smooth algebraic curves gives a holomorphic map $X^{an} \to Y^{an}$. Thus we have a functor, the *analytification*, from one category to another.

4.3.2. LEMMA. *Analytification is fully faithful, that is every holomorphic map $f : X \to Y$ between two smooth projective algebraic curves is a regular morphism.*

Sketch. Let $G \subset X \times Y$ be the graph of f. We embed $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$, and

$$G \subset X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{nm+n+m}$$

by the Segre embedding. Thus G is a complex submanifold of $\mathbb{P}^{nm+n+m}$. But very general *Chow embedding theorem* [GH] asserts that every complex submanifold of $\mathbb{P}^N$ is algebraic[7]. Thus $G \subset X \times Y$ is algebraic and therefore the map $f : X \to G \to Y$ is regular. □

4.3.3. EXAMPLE. Every *meromorphic function* f on a smooth projective algebraic curve C can be viewed as a holomorphic map from C to $\mathbb{P}^1$. Thus f is in fact a rational function.

A difficult part is to show that analytification is *essentially surjective*: any compact Riemann surface is biholomorphic to a smooth projective curve. It is hard to construct a single meromorphic function, but once this is done the rest is straightforward. It is enough to find a harmonic function (*why?*). Klein (following Riemann) *"covers the surface with tin foil... Suppose the poles of a galvanic battery are placed at the points A_1 and A_2. A current arises whose potential u is single-valued, continuous, and satisfies the equation $\Delta u = 0$ across the entire surface, except for the points A_1 and A_2, which are discontinuity points of the function."* A modern treatment can be found in [GH].

§4.4. **Genus and meromorphic forms.** In the language of Riemann surfaces, an Abelian integral is the integral of a meromorphic form. Indeed, it turns out that all meromorphic forms are rational:

4.4.1. LEMMA. *Every meromorphic form ω on a smooth projective algebraic curve (viewed as a complex Riemann surface) is rational, i.e. can be written as $\omega = f\,dg$, where f and g are rational functions.*

Proof. This follows from Example 4.3.3. Indeed, let g be a non-constant rational function on C. Then dg is a non-zero meromorphic form. So we can write $\omega = f\,dg$, where f is a meromorphic function. But all meromorphic functions are rational. □

In particular, the vector space of *holomorphic differential forms*, i.e. meromorphic differential forms without poles is the same as the vector space of *regular differential forms*, i.e. *rational differential forms* without poles. The fundamental result is that this space is finite-dimensional and its dimension is equal to the *genus* g, the number of handles on a Riemann surface.

Another way to compute the genus is to use the *genus formula*:

$$2g - 2 = (\text{number of zeros of } \omega) - (\text{number of poles of } \omega) \tag{4.4.2}$$

of any meromorphic (=rational) differential form $\omega = f\,dg$.

4.4.3. EXAMPLE. Consider a form $\omega = dx$ on $\mathbb{P}^1$. It has no zeros or poles in the x-chart of 0. In the y-chart at infinity we have

$$dx = d(1/y) = -(1/y^2)dy.$$

So it has a pole of order 2 at infinity, which shows that $g(\mathbb{P}^1) = 0$ by (4.4.2).

[7] In fact the theorem says that every analytic (i.e. locally given as a vanishing set of holomorphic functions) subset of $\mathbb{P}^N$ is algebraic. It doesn't have to be a submanifold.

4.4.4. EXAMPLE. A smooth plane curve $C \subset \mathbb{P}^2$ of degree d has genus

$$g = \frac{(d-1)(d-2)}{2}. \tag{4.4.5}$$

In algebraic geometry, it is more natural to write this as

$$2g - 2 = d(d-3) \tag{4.4.6}$$

because this can be generalized to an *adjunction formula* which computes the genus of a curve on any algebraic surface S, not just $\mathbb{P}^2$.

There is a nice choice of a holomorphic form on C: suppose $C \cap \mathbb{A}^2_{x,y}$ is given by the equation $f(x,y) = 0$. Differentiating this equation shows that

$$\omega := \frac{dx}{f_y} = -\frac{dy}{f_x}$$

along C, where the first (resp. second) expression is valid where $f_y \neq 0$ (resp. $f_x \neq 0$), i.e. where x (resp. y) is a holomorphic coordinate. Thus ω has no zeros or poles in $C \cap \mathbb{A}^2$. We will show that ω has zeros at each of the d intersection points of C with the line at infinity and each zero has multiplicity $d-3$. Combined with (4.4.2), this will give (4.4.6). Indeed, switching from the chart $(x,y) = [x:y:1]$ to the chart $(x,z) = [x:1:z]$ gives

$$\omega = -\frac{dy}{f_x(x,y)} = -\frac{d\frac{1}{z}}{f_x(\frac{x}{z},\frac{1}{z})} = \frac{dz}{z^2\frac{1}{z^{d-1}}g(x,z)} = z^{d-3}\frac{dz}{g(x,z)},$$

where we can arrange homogeneous coordinates from the start so that at every point at infinity g doesn't vanish and z is a holomorphic coordinate.

§4.5. **Divisors and linear equivalence.** A (Weil) *divisor* D on a smooth projective curve C is an integral linear combination $\sum a_i P_i$ of points $P_i \in C$. Divisors form an Abelian group, denoted by $\operatorname{Div} C$, freely generated by classes of points. There is a homomorphism $\deg : \operatorname{Div} C \to \mathbb{Z}$, called the *degree*, namely

$$\deg D = \sum a_i.$$

4.5.1. DEFINITION. A *principal divisor* of a rational function f on C is

$$(f) = \sum_{P \in C} \operatorname{ord}_P(f) P,$$

where $\operatorname{ord}_P(f)$ is the *order of zeros or poles* of f at P.

Analytically, if z is a holomorphic coordinate centered at P then in some neighborhood of P

$$f(z) = z^n g(z),$$

where $g(z)$ is holomorphic and does not vanish at p. Then $\operatorname{ord}_P(f) = n$.

Algebraically, instead of choosing a holomorphic coordinate we choose a *local parameter*, i.e. a rational function z regular at P, vanishes at P, and such that any rational function f on C can be written (uniquely) as

$$f = z^n g,$$

where g is regular at P and does not vanish there (see [**?**, 1.1.5]). This is an instance of a general strategy in Algebraic Geometry: if there is some useful analytic concept (e.g. a holomorphic coordinate) that does not exist algebraically, one should look for its desirable properties (e.g. a factorization $f = z^n g$ as above). Often it is possible to find a purely algebraic object (e.g. a local parameter) satisfying the same properties.

A local parameter at x can also be described as follows:

- a *uniformizer* of the DVR $\mathcal{O}_{C,P}$ (with valuation ord);
- any element in $\mathfrak{m}_{C,P} \setminus \mathfrak{m}_{C,P}^2$;
- a coordinate of an affine chart $\mathbb{A}^n$ such that the tangent space $T_P C$ projects onto the corresponding coordinate axis;
- a rational function z such that $\mathrm{ord}_P(z) = 1$.

4.5.2. DEFINITION. A *canonical divisor* of a meromorphic (=rational) form ω is defined as follows:

$$(\omega) = \sum_{P \in C} \mathrm{ord}_P(\omega) P,$$

where if z is a holomorphic coordinate (or a local parameter) at P then we can write $\omega = f\,dz$ and $\mathrm{ord}_P(\omega) := \mathrm{ord}_P(f)$.

We can rewrite (4.4.2) as

$$\deg(\omega) = 2g - 2.$$

4.5.3. DEFINITION. Two divisors D and D' are called *linearly equivalent* if $D - D'$ is a principal divisor. Notation: $D \sim D'$.

For example, any two canonical divisors are linearly equivalent. Indeed, if ω and ω' are two rational forms then $\omega = g\omega'$ for some rational function g and it is easy to see that $(\omega) = (g) + (\omega')$. A linear equivalence class of a canonical divisor is denoted by K_C.

The quotient of the divisor group $\mathrm{Div}\, C$ by a subgroup of principal divisor is called the *Picard group* $\mathrm{Pic}\, C$.

§4.6. **Branched covers and Riemann–Hurwitz formula.** Theorem 4.2.5 can be upgraded to a (contragredient) equivalence of the category of smooth projective algebraic curves with *non-constant* morphisms and the category of finitely generated field extensions $K/\mathbb{C}$ of $\mathrm{tr.deg.}_{\mathbb{C}} K = 1$. Morphisms in this category are inclusions of fields over $\mathbb{C}$. A non-constant map $f : C \to C'$ of smooth projective algebraic curves is called a *branched cover*. The corresponding field extension $\mathbb{C}(C') \hookrightarrow \mathbb{C}(C)$ is just the pull-back f^*.

4.6.1. LEMMA–DEFINITION. *The degree of the branched cover can be computed*

- *topologically: number of points in the preimage of a general point.*
- *algebraically: degree of the field extension* $\mathbb{C}(C)/\mathbb{C}(C')$.
- *with multiplicities: if* $f^{-1}(P) = \{Q_1, \ldots, Q_r\}$ *then*

$$\deg f = \sum_{i=1}^{r} e_{Q_i},$$

where

$$e_{Q_i} = \mathrm{ord}_{Q_i} f^*(z),$$

where z is a local parameter at P. The multiplicity e_{Q_i} is called the ramification index. If $e_{Q_i} > 1$ then P is called a branch point *and Q_i is called a* ramification point. *The divisor*

$$R = \sum_{Q \in C} (e_Q - 1)[Q]$$

is called the ramification divisor.

In particular, viewing a rational function f on C as a map $f : C \to \mathbb{P}^1$, its degree is equal to both the number of zeros and the number of poles of f (counted with multiplicities), and so

$$\deg(f) = (\deg f) - (\deg f) = 0 \quad \text{for every } f \in k(C).$$

In particular, linearly equivalent divisors have the same degree and therefore we have a degree homomorphism

$$\deg : \operatorname{Pic} C \to \mathbb{Z}.$$

For example, $\deg$ gives an isomorphism $\operatorname{Pic}\mathbb{P}^1 \simeq \mathbb{Z}$ because any two points $P, Q \in \mathbb{P}^1$ are linearly equivalent using a Möbius function $\frac{z-P}{z-Q} \in \mathbb{C}(\mathbb{P}^1)$.

We extend the map $[P] \mapsto \sum\limits_{i=1}^{r} e_{Q_i}[Q_i]$ by linearity to a homomorphism

$$f^* : \operatorname{Div} C' \to \operatorname{Div} C.$$

If $\alpha \in \mathbb{C}(C')$ then we have a suggestive formula

$$f^*(\alpha) = (f^*\alpha)$$

because every zero (or pole) P of α contributes to a zero (or pole) Q of $f^*\alpha$ with multiplicity e_Q for every $Q \in f^{-1}(P)$. In particular, f^* sends principal divisors to principal divisors and thus induces a homomorphism

$$f^* : \operatorname{Pic} C' \to \operatorname{Pic} C.$$

Moreover, $\deg f^*D = (\deg f) \deg D$ for every divisor D on C'.

4.6.2. THEOREM (Riemann–Hurwitz). *For every branched cover $f : C \to C'$,*

$$K_C \sim f^*K_{C'} + R.$$

and comparing the degrees and using (4.4.2),

$$2g(C) - 2 = (\deg f)(2g(C') - 2) + \sum_{Q \in C} (e_Q - 1).$$

Proof. Choose a meromorphic form ω on C' without zeros or poles at branch points. Then $K_{C'} = (\omega)$ and $K_C = (f^*\omega)$. Every zero (resp. pole) of ω contributes to $\deg f$ zeros (resp. poles) of $f^*\omega$. In addition, if t is a local parameter at a ramification point Q and z is a local parameter at $P = f(Q)$ then $f^*(z) = t^{e_Q} g$, where g is regular at Q. Then

$$f^*(dz) = d(t^{e_Q} g) = e_Q t^{e_Q - 1} g\, dt + t^{e_Q}\, dg,$$

which shows that each ramification point is a zero of $f^*\omega$ of order $e_Q - 1$. So

$$(f^*\omega) = f^*(\omega) + \sum_{Q \in C} (e_Q - 1)[Q],$$

which proves the Riemann–Hurwitz formula. $\square$

§4.7. **Riemann–Roch formula.** The divisor $D = \sum a_i P_i$ is called *effective* (notation $D \geq 0$) if $a_i \geq 0$ for every i.

4.7.1. THEOREM (Riemann–Roch). *For every divisor D on C, we have*

$$l(D) - i(D) = 1 - g + \deg D,$$

where

$$l(D) = \dim \mathcal{L}(D), \quad \text{where} \quad \mathcal{L}(D) = \{f \in \mathbb{C}(C) \,|\, (f) + D \geq 0\}$$

and

$$i(D) = \dim K(D), \quad \text{where} \quad K(D) = \{\text{meromorphic forms } \omega \,|\, (\omega) \geq D\}.$$

4.7.2. EXAMPLE. If $D = 0$ then $l(D) = 1$ and therefore $i(D) = g$. Notice that $K(0)$ is the space of holomorphic differentials.

Here $l(D) = 1$ because the only rational functions which are regular everywhere are constants. Analytically, this is *Liouville's Theorem* for Riemann surfaces (see also the *maximum principle* for harmonic functions). Algebraically, this is

4.7.3. THEOREM. *If X is an irreducible projective variety then the only functions regular on X are constants.*

Proof. A regular function is a regular morphism $X \to \mathbb{A}^1$. Composing it with the inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ gives a regular morphism $f : X \to \mathbb{P}^1$ such that $f(X) \subset \mathbb{A}^1$. But the image of a projective variety under any morphism is closed, thus $f(X)$ must be closed in $\mathbb{P}^1$ and so $f(X)$ must be a point. $\square$

4.7.4. EXAMPLE. If $D = K$ then $L(D)$ is the space of holomorphic forms and $K(D)$ is the space of holomorphic functions. So in this case Riemann–Roch gives the genus formula (4.4.2).

4.7.5. EXAMPLE. Suppose $g(C) = 0$. Let $D = P$ be a point. Then RR gives

$$l(P) = i(P) + 2 \geq 2.$$

It follows that $\mathcal{L}(D)$ contains a non-constant rational function f, which then has a unique simple pole at P. This function gives an branch cover $C \to \mathbb{P}^1$ of degree 1, therefore an isomorphism. So

$$\mathcal{M}_0 = \{pt\}.$$

§4.8. **Linear systems.**

4.8.1. DEFINITION. A *linear system of divisors*

$$|D| = \{(f) + D \,|\, f \in \mathcal{L}(D)\} = \{D' \,|\, D' \sim D,\ D' \geq 0\},$$

consists of all effective divisors linearly equivalent to D. Note that D' defines f uniquely up to a constant, therefore

$$|D| \simeq \mathbb{P}\mathcal{L}(D).$$

More generally, if $L \subset \mathcal{L}(D)$ is a vector subspace then an *incomplete linear system* $|L| \subset |D|$ consists of all divisors of the form $\{(f) + D \,|\, f \in L\}$.

Choosing a basis $f_0, \dots, f_r$ of $\mathcal{L}(D)$ gives a map

$$\phi_D : C \to \mathbb{P}^r, \qquad \phi_D(x) = [f_0, \dots, f_r].$$

Since C is a smooth curve, this map is regular. More generally, we define a similar map $\phi_{|L|}$ for every incomplete linear system L.

4.8.2. DEFINITION. Let D be an effective divisor. Its *base locus* is the intersection of all divisors in the linear system $|D|$. Its *fixed part* is a maximal effective divisor E such that $D' - E \geq 0$ for every $D' \in |D|$. Notice that E is a sum of points in the base locus with positive multiplicities.[8]

4.8.3. LEMMA. *We have*

$$|D| = |D - E| + E,$$
$$\mathcal{L}(D) = \mathcal{L}(D - E)$$

and

$$\phi_D = \phi_{D-E}.$$

4.8.4. LEMMA. *D has no base points iff $l(D - P) = l(D) - 1$ for every $P \in C$.*

Proof. If P is in the base locus then $l(D - P) = l(D)$. If the base locus is empty then $|D - P| + P$ is strictly contained in $|D|$ and so $l(D - P) < l(D)$. Thus we can assume that D is effective and doesn't contain P. In this case, we can identify $\mathcal{L}(D-P)$ with a hyperplane in $\mathcal{L}(D)$ of all rational functions that in addition vanish at P. $\square$

If we are interested in maps ϕ_D then we can always assume that D is effective and base-point-free, i.e. its base locus is empty. In addition, D is called *very ample* if ϕ_D is an embedding $C \subset \mathbb{P}^r$. One has the following very useful criterion generalizing the previous lemma.

4.8.5. THEOREM. *A divisor D is very ample if and only if*

- ϕ_D separates any points $P, Q \in C$, *i.e.* $l(D - P - Q) = l(D) - 2$;
- ϕ_D separates tangents, *i.e.* $l(D - 2P) = l(D) - 2$ *for any point* $P \in C$.

4.8.6. PROPOSITION. *Every morphism $\phi : C \to \mathbb{P}^r$ is given by a base-point-free linear system (possibly incomplete) as long as $\phi(C)$ is not contained in a projective subspace of $\mathbb{P}^r$ (in which case we can just switch from $\mathbb{P}^r$ to $\mathbb{P}^s$ for $s < r$).*

Proof. Indeed, ϕ is obtained by choosing rational functions

$$f_0, \dots, f_r \in k(C).$$

Consider their divisors $(f_0), \dots, (f_r)$ and let D be the smallest effective divisor such that $(f_i) + D$ is effective for every i. Then of course every $f_i \in \mathcal{L}(D)$ and D is base-point-free (otherwise it's not the smallest). $\square$

Divisors $(f_0) + D, \dots, (f_r) + D$ have very simple meaning: they are just "pull-backs" of coordinate hyperplanes in $\mathbb{P}^r$. More precisely, suppose h is a local parameter at a point $P \in C$, which contributes nP to D. Then ϕ (in the neighborhood of P) can be written as

$$[f_0 h^n : \dots f_r h^n],$$

[8]This is a special feature of algebraic curves as in higher dimension the base locus is not necessarily a divisor.

where at least one of the functions does not vanish. So pull-backs of coordinate hyperplanes near P are given by divisors $(f_0)+nP,\dots,(f_r)+nP$.

§5. **Moduli of elliptic curves**

§5.1. **Curves of genus 1.** Let us recall the following basic result.

5.1.1. THEOREM. *Let C be a smooth projective curve. TFAE:*

(1) *C has a plane model in $\mathbb{A}^2$ given by the Weierstrass equation*
$$y^2=4x^3-g_2x-g_3,\quad \Delta=g_2^3-27g_3^2\neq 0.$$
(2) *C is isomorphic to a cubic curve in $\mathbb{P}^2$.*
(3) *C admits a $2:1$ cover of $\mathbb{P}^1$ ramified at 4 points.*
(4) *C has genus 1.*

In addition, every cubic curve in $\mathbb{P}^2$ has a flex point and admits a unique (up to scalar) regular form ω. Moreover, this form has no zeros.

Proof. **Easy steps.**

(1) $\Rightarrow$ (2). It is easy to check that the projective closure is smooth at all points including $[0:1:0]$, which is the only point at infinity. The line at infinity $z=0$ is a flex line (or inflection line): it intersects the curve at $[0:1:0]$ with multiplicity 3. The curve has genus 1 by the formula for the genus of a plane curve.

(2) $\Rightarrow$ (3) A double cover can be obtained as a linear projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ from any point $p\in C$ (projecting from points away from C gives a triple cover). Use Riemann–Hurwitz to find the number of branch points.

(3) $\Rightarrow$ (4) Use Riemann–Hurwitz to compute the genus. Then $\mathcal{L}(K)$ is one-dimensional. Let ω be a generator. Since $\deg K=0$, ω has no zeros.

Riemann–Roch analysis. Assume (4). Let D be a divisor of positive degree. Since $\deg D>\deg K$, we have $i(D)=0$ and
$$l(D)=\deg D.$$
It follows that D is base-point-free if $\deg D\geq 2$ and very ample if $\deg D\geq 3$.

Fix a point $P\in C$. Notice that $\mathcal{L}(kP)\subset\mathcal{L}(lP)$ for $k\leq l$ and that $\mathcal{L}(kP)\cdot\mathcal{L}(lP)\subset\mathcal{L}((k+l)P)$. Thus we have a graded algebra
$$R(C,P)=\bigoplus_{k\geq 0}\mathcal{L}(kP)\subset\mathbb{C}(C).$$

$\mathcal{L}(0)=\mathcal{L}(P)$ is spanned by 1. $\mathcal{L}(2P)$ is spanned by 1 and by some non-constant function, which we will call x. Since $2P$ has no base-points, we have a $2:1$ map
$$\psi_{2P}:\,C\to\mathbb{P}^1$$
given by $[x:1]$. This shows (4) $\Rightarrow$ (3). P is one of the ramification points.

$\mathcal{L}(3P)$ is spanned by $1,x$, and a new function, which we will call y. Since $3P$ is very ample, we have an embedding
$$\psi_{3P}:\,C\to\mathbb{P}^2,$$
given by $[x:y:1]$. The image is a curve of degree 3. Moreover, P is a flex point. This shows that (4) $\Rightarrow$ (2).

Notice that $\mathcal{L}(6P)$ has dimension 6 but contains seven functions

$$1, x, y, x^2, xy, x^3, y^2.$$

Thus, they are linearly dependent. Moreover, x^3 and y^2 are the only functions on the list that have a pole of order 6 at P. Therefore, they must both contribute to the linear combination. After rescaling them by constants, we can assume that the equation has form

$$y^2 + \alpha xy = 4x^3 - g_1x^2 - g_2x - g_3.$$

After making the changes of variables $y \mapsto y - \frac{\alpha}{2}x$ and $x \mapsto x + \frac{g_1}{12}$, we get the Weierstrass form. This shows that (4) $\Rightarrow$ (1).

Logically unnecessary but fun implications:

(2) $\Rightarrow$ (1). Prove existence of a flex point directly, by intersecting with the *Hessian cubic*. Then move a flex point to $[0 : 1 : 0]$ by a change of variable, then make the line at infinity $z = 0$ the flex line, etc. (this is analogous in spirit to the Riemann-Roch analysis above but more tedious).

(1) $\Rightarrow$ (3). Project $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_x$. Three ramification points are at the roots of $4x^3 - g_2x - g_3 = 0$, the last ramification point is at ∞. $\square$

From the complex-analytic perspective, we have the following

5.1.2. THEOREM. *Let C be a compact Riemann surface. TFAE:*

(1) *C is a smooth projective cubic curve in $\mathbb{P}^2$.*
(2) *C is a compact Riemann surface of genus 1.*
(3) *C is biholomorphic to a complex torus*

$$\mathbb{C}/\Lambda, \quad \text{where } \Lambda \simeq \mathbb{Z} \oplus \mathbb{Z}\tau,\ \mathrm{Im}\,\tau > 0.$$

Proof. (1) $\Rightarrow$ (2). Apply analytification and genus formula.

(3) $\Rightarrow$ (2). $\mathbb{C}/\Lambda$ is topologically a torus and has a structure of a Riemann surface induced from a translation-invariant complex structure on $\mathbb{C}$. Also notice that dz descends to a non-vanishing holomorphic form.

(3) $\Rightarrow$ (1). One can invoke a general theorem about the equivalence of categories here, but it's more instructive to show directly that every complex torus is biholomorphic to a projective cubic curve. Let $P \in \mathbb{C}/\Lambda$ be the image of $0 \in \mathbb{C}$. From the Riemann–Roch analysis, we should expect to find a meromorphic function in with pole of order 2 at P and holomorphic elsewhere. Its pull-back to $\mathbb{C}$ will be a doubly-periodic (i.e. Λ-invariant) meromorphic function on $\mathbb{C}$ with poles only of order 2 and only at lattice points. Luckily, this function was constructed explicitly by Weierstrass:

$$\wp(z) = \frac{1}{z^2} + \sum_{\gamma \in \Lambda,\ \gamma \neq 0} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right).$$

Notice that $\wp'(z)$ has poles of order 3 at lattice points, and therefore

$$\{1, \wp(z), \wp'(z)\}$$

should be a basis of $\mathcal{L}(3P)$. Indeed, one can check directly that the map

$$\mathbb{C} \to \mathbb{C}^2, \quad z \mapsto [\wp(z) : \wp'(z) : 1]$$

gives an embedding $C\subset\mathbb{P}^2$ as a cubic curve. In fact it is easy to see that $\wp'$ and $\wp$ satisfy the Weierstrass equation

$$(\wp')^2=4\wp^3-g_2\wp-g_3,$$

which explains a traditional factor of 4.

Periods. (2) $\Rightarrow$ (3). Let's assume that C is compact Riemann surface which has a nowhere vanishing holomorphic form ω and topological genus 1. We fix a point $P\in C$ and consider a multi-valued holomorphic map

$$\pi:\,C\to\mathbb{C},\quad z\mapsto\int_P^z\omega.$$

It is multi-valued because it depends on the choice of a path of integration. Notice however that near every point $z\in C$ we can choose a branch of π by specifying paths of integration (say connecting P to z and then z to a nearby point by a segment in a holomorphic chart) and this branch of π is conformal (because ω has no zeros and $\int dz=z$.)[9]

Topologically, we can obtain C by gluing opposite sides of the rectangle, in other words we have a homeomorphism $\mathbb{C}/\mathbb{Z}^2\to C$ which sends segments $\alpha,\beta\subset\mathbb{C}$ connecting the origin to $(1,0)$ and $(0,1)$ to generators of the first *homology group* $H_1(C)=\mathbb{Z}\alpha+\mathbb{Z}\beta$. We can then define *periods*

$$A=\int_\alpha\omega\quad\text{and}\quad B=\int_\beta\omega.$$

They generate a subgroup $\Lambda\subset\mathbb{C}$. Integrals along paths in C are uniquely defined modulo Λ.

5.1.3. LEMMA. *A and B are linearly independent over $\mathbb{R}$.*

Proof. If not then we can assume that $\Lambda\subset\mathbb{R}$ (by multiplying ω by a constant). Then $\operatorname{Im}\pi$ is a single-valued *harmonic function*, which must be constant by the *maximum principle* because C is a compact Riemann surface. This is a contradiction: a branch of π is a local isomorphism near P. $\square$

So Λ is a lattice and π induces a holomorphic map

$$f:\,C\to\mathbb{C}/\Lambda.$$

Notice that its composition with a homeomorphism $\mathbb{C}/\mathbb{Z}^2\to C$ is a homeomorphism $\mathbb{C}/\mathbb{Z}^2\to\mathbb{C}/\Lambda$. Indeed, it is induced by the integration map $\mathbb{C}\to\mathbb{C}$ which sends $\alpha\mapsto A$ and $\beta\mapsto B$. Thus f is bijective. $\square$

5.1.4. REMARK. An important generalization is a beautiful *Klein–Poincare Uniformization Theorem*: a universal cover of a compact Riemann surface is

- $\mathbb{P}^1$ if $g=0$;
- $\mathbb{C}$ if $g=1$;
- $\mathbb{H}$ (the upper half-plane) if $g\geq 2$.

[9]If C is a cubic curve in the Weierstrass normal form then $\omega=\frac{dx}{y}$ (see Example 4.4.4) and so these integrals are elliptic integrals

$$\int\frac{dx}{\sqrt{4x^3-g_2x-g_3}}.$$

In other words, every compact Riemann surface of genus ≥ 2 is isomorphic to a quotient of $\mathbb{H}$ by a discrete subgroup
$$\Gamma \subset \mathrm{Aut}(\mathbb{H}) = \mathrm{PGL}_2(\mathbb{R}),$$
which acts freely on $\mathbb{H}$.

§5.2. **J-invariant.** Now we would like to classify elliptic curves up to isomorphism, i.e. to describe M_1 as a set. As we will see many times in this course, automorphisms of geometric objects can cause problems for constructing moduli spaces and so it's good to know what they are. A curve of genus 1 has a lot of automorphisms: a complex torus $\mathbb{C}/\Lambda$ admits translations by vectors in $\mathbb{C}$. These translations are biholomorphic, and therefore regular, automorphisms. In fact $\mathbb{C}/\Lambda$ is an algebraic group:

5.2.1. DEFINITION. An algebraic variety X with a group structure is called an *algebraic group* if the multiplication $X \times X \to X$ and the inverse $X \to X$ maps are morphisms of algebraic varieties.

In a cubic plane curve realization, this group structure is a famous "three points on a line" group. We can eliminate translations by fixing a point.

5.2.2. DEFINITION. An *elliptic curve* is a pair (C, P), where C is a smooth projective curve of genus 1 and $P \in C$. It is convenient to choose P to be the unity of the group structure on C if one cares about it.

Of course as a set we have
$$M_1 = M_{1,1}.$$

Every pointed curve (C, P) still has at least one automorphism, namely the involution given by permuting the two branches of the double cover
$$\phi_{2P} : C \to \mathbb{P}^1.$$
In the complex torus model this involution is given by the formula $z \mapsto -z$, which reflects the fact that the Weierstrass $\wp$-function is even.

Let's work out when two elliptic curves are isomorphic and when the automorphism group $\mathrm{Aut}(C, P)$ is larger than $\mathbb{Z}/2\mathbb{Z}$.

5.2.3. THEOREM.
(1) *Curves with Weierstrass equations $y^2 = 4x^3 - g_2x - g_3$ and $y^2 = 4x^3 - g_2'x - g_3'$ are isomorphic if and only if there exists $t \in \mathbb{C}^*$ such that $g_2' = t^2g_2$ and $g_3' = t^3g_3$.*
(2) *Two smooth cubic curves C and C' are isomorphic if and only if they are projectively equivalent: $C = A(C')$ for some $A \in \mathrm{PGL}_3(\mathbb{C})$.*
(3) *Let C (resp. C') be a double cover of $\mathbb{P}^1$ with a branch locus $p_1, \ldots, p_4$ (resp. $p_1', \ldots, p_4'$). Then $C \simeq C'$ if and only if there exists $g \in \mathrm{PGL}_2(\mathbb{C})$ such that*
$$\{p_1', p_2', p_3', p_4'\} = g\{p_1, p_2, p_3, p_4\}.$$
In particular, we can always assume that branch points are $0, 1, \lambda, \infty$.
(4) *$\mathbb{C}/\Lambda \simeq \mathbb{C}/\Lambda'$ if and only if $\Lambda = \alpha\Lambda'$ for some $\alpha \in \mathbb{C}^*$. If $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ and $\Lambda' = \mathbb{Z} \oplus \mathbb{Z}\tau'$ with $\mathrm{Im}\,\tau, \mathrm{Im}\,\tau' > 0$ then this is equivalent to*
$$\tau' = \frac{a\tau + b}{c\tau + d} \quad \text{for some} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \tag{5.2.4}$$

There are only two curves with special automorphisms:

$$\mathrm{Aut}(y^2 = x^3 + 1) = \mathbb{Z}/6\mathbb{Z} \quad \text{and} \quad \mathrm{Aut}(y^2 = x^3 + x) = \mathbb{Z}/4\mathbb{Z}.$$

Their lattices in $\mathbb{C}$ are the hexagonal and the square lattices. These curves are double covers of $\mathbb{P}^1$ branched at $0, 1, \lambda, \infty$, with $\lambda = e^{\frac{\pi i}{3}}$ and -1, respectively.

Proof. Let C be and C' be two plane smooth cubic curves, which are abstractly isomorphic. Let $P \in C$ and $P' \in C'$ be flex points. Then embeddings $C \hookrightarrow \mathbb{P}^2$ and $C' \hookrightarrow \mathbb{P}^2$ are given by linear systems $\mathcal{L}(3P)$ and $\mathcal{L}(3P')$, respectively. After translation by an element C', we can assume that an isomorphism $\phi : C \to C'$ takes P to P'. Then $\mathcal{L}(3P) = \phi^*\mathcal{L}(3P')$. Applying projective transformations to C and C' is equivalent to choosing bases in the linear systems. If we choose a basis in $\mathcal{L}(3P')$ and pull it back to the basis of $\mathcal{L}(3P)$, we will have

$$\phi_{3P} = \phi_{3P'} \circ \phi,$$

i.e. C and C' are equal cubic curves. This proves (2). In the Weierstrass form, the only possible linear transformations are $x \mapsto tx$ and $y \mapsto \pm t^{3/2}y$, which proves (1).

A similar argument proves (3). Notice that in this case $\mathrm{Aut}(C, P)$ modulo the hyperelliptic involution acts on $\mathbb{P}^1$ by permuting branch points. In fact, λ is simply the cross-ratio:

$$\lambda = \frac{(p_4 - p_1)}{(p_2 - p_1)} \frac{(p_2 - p_3)}{(p_4 - p_3)},$$

but branch points are not ordered, so we have an action of S_4 on possible cross-ratios. However, it is easy to see that the Klein's four-group V does not change the cross-ratio. The quotient $S_4/V \simeq S_3$ acts non-trivially:

$$\lambda \mapsto \{\lambda, 1 - \lambda, 1/\lambda, (\lambda - 1)/\lambda, \lambda/(\lambda - 1), 1/(1 - \lambda)\} \tag{5.2.5}$$

Special values of λ correspond to cases when some of the numbers in this list are equal. For example, $\lambda = 1/\lambda$ implies $\lambda = -1$ and the list of possible cross-ratios boils down to $-1, 2, 1/2$ and $\lambda = 1/(1 - \lambda)$ implies $\lambda = e^{\frac{\pi i}{3}}$, in which case the only possible cross-ratios are $e^{\frac{\pi i}{3}}$ and $e^{\frac{-\pi i}{3}}$. To work out the actual automorphism group, we look at the Weierstrass models. For example, if $\lambda = -1$ then the branch points are $0, 1, -1, \infty$ and the equation is $y^2 = x^3 - x$. One can make a change of variables $x \mapsto ix$, $y \mapsto e^{3\pi i/4}$, then the equation becomes

$$y^2 = x^3 + x$$

as required and the branch points now are $0, i, -i, \infty$. An automorphism of $\mathbb{P}^1$ permuting these branch points is $x \mapsto -x$. To keep the curve in the Weierstrass form, we also have to adjust $y \mapsto iy$. This gives an automorphism of C and its square is an automorphism $x \mapsto x$, $y \mapsto -y$, i.e. an involution permuting branches of the double cover.

(4) Consider an isomorphism $f : \mathbb{C}/\Lambda' \to \mathbb{C}/\Lambda$. Composing it with translations on the source and on the target, we can assume that $f(0+\Lambda') = 0+\Lambda$. Then f induces a holomorphic map $\mathbb{C} \to \mathbb{C}/\Lambda$ with kernel Λ', and its lift to the universal cover gives an isomorphism $F : \mathbb{C} \to \mathbb{C}$ such that $F(\Lambda') = \Lambda$.

But it is proved in complex analysis that all automorphisms of $\mathbb{C}$ preserving the origin are maps $z \mapsto \alpha z$ for $\alpha \in \mathbb{C}^*$. So we have

$$\mathbb{Z} + \mathbb{Z}\tau = \alpha(\mathbb{Z} + \mathbb{Z}\tau'),$$

which gives

$$\alpha\tau' = a + b\tau, \quad \alpha = c + d\tau,$$

which gives (5.2.4). □

5.2.6. THEOREM. *We can define the j-invariant by any of the two formulas:*

$$j = 1728g_2^3/\Delta = 256\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}. \tag{5.2.7}$$

The j-invariant uniquely determines an isomorphism class of an elliptic curve. The special values of the j-invariant are $j = 0$ (for $\mathbb{Z}/6\mathbb{Z}$) and $j = 1728$ (for $\mathbb{Z}/4\mathbb{Z}$).

Proof. It is easy to see that the expression $256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$ does not change under the transformations (5.2.5). Thus, for fixed j, the polynomial

$$(\lambda^2 - \lambda + 1)^3 - \frac{1}{256}j\lambda^2(\lambda - 1)^2 \tag{1}$$

has 6 roots related by transformations (5.2.5). So the j-invariant uniquely determines an isomorphism class of an elliptic curve. The rest is left to the homework exercises. □

§5.3. **Monstrous Moonshine.** Trying to compute the j-invariant in terms of the lattice parameter τ produces some amazing mathematics. Notice that $j(\tau)$ is invariant under the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{H}$. This group is called the *modular group*. It is generated by two transformations,

$$S : z \mapsto -1/z \quad \text{and} \quad T : z \mapsto z + 1$$

It has a fundamental domain (see the figure). The j-invariant maps the fundamental domain to the plane $\mathbb{A}^1$.

Since the j-invariant is invariant under $z \mapsto z + 1$, it can be expanded in a variable $q = e^{2\pi i\tau}$:

$$j = q^{-1} + 744 + 196884q + 21493760q^2 + \dots$$

According to the classification of finite simple groups, there are a few infinite families (like alternating groups A_n) and several sporadic groups. The largest sporadic group F_1 is called the Monster. It has about 10^{54} elements. Its existence was predicted by Robert Griess and Bernd Fischer in 1973 and it was eventually constructed by Griess in 1980 as the automorphism group of a certain (commutative, non-associative) algebra of dimension 196884. In other words, the Monster has a natural 196884-dimensional representation, just like S_n has a natural n-dimensional representation. This dimension appears as one of the coefficients of $j(q)$ and In fact all coefficients in this q-expansion are related to representations of the Monster group. This is a *Monstrous Moonshine Conjecture* of McKay, Conway, and Norton proved in 1992 by Borcherds (who won a Fields medal for this work).

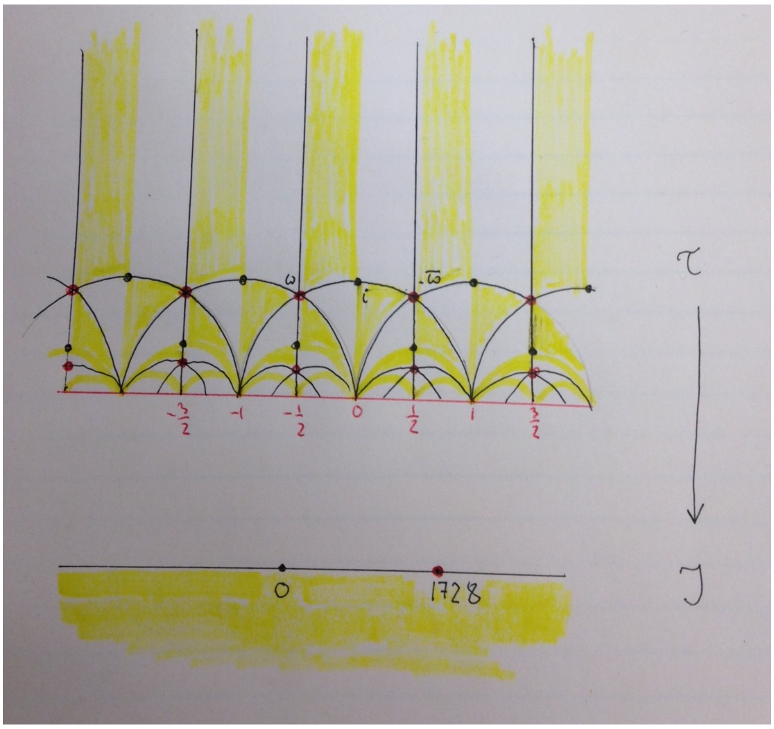


FIGURE 1. Modular group.

§5.4. **Families of elliptic curves.** We would like to upgrade M_g and $M_{g,n}$ to moduli functors. What is a family of smooth projective curves? It should be a morphism $f : X \to B$ such that every fiber of f is a smooth projective curve. We have to impose a technical condition: the morphism f must be *smooth* and *proper*. We postpone definitions until later and focus on the following good properties, which are sufficient for most applications and will allow us to define the moduli functor.

5.4.1. THEOREM.

- *If $X \to B$ is a smooth morphism of algebraic varieties then all fibers are non-singular and have the same dimension* $\dim X - \dim B$*. If B is non-singular then X is also non-singular.*
- *Let $f : X \to B$ be a morphism of non-singular complex algebraic varieties. Then f is smooth if and only if the induced map of analytifications $X^{an} \to B^{an}$ is a submersion of complex manifolds, i.e. its differential is surjective at every point.*
- *Let $f : X \to B$ be a smooth morphism and let $g : B' \to B$ be any morphism of algebraic varieties. Consider the fiber product*
$$X \times_B B' = \{(x, b') \mid f(x) = g(b')\} \subset X \times B'.$$
Then $X \times_B B' \to B'$ is a smooth morphism of algebraic varieties.

5.4.2. THEOREM. *Let $f : X \to B$ be a morphism of non-singular algebraic varieties. Then f is proper iff the corresponding map of analytifications $X^{an} \to B^{an}$ is a proper holomorphic map, i.e. the preimage of every compact set is compact.*

5.4.3. DEFINITION. A family of smooth projective curves of genus g is a smooth proper morphism of algebraic varieties $f : X \to B$ such that all fibers are smooth projective curves of genus g. The moduli functor

$$\mathcal{M}_g : AlgebraicVarieties \to Sets$$

sends every algebraic variety B to the set of isomorphism classes of families $f : X \to B$ of smooth projective curves of genus g and every morphism $B' \to B$ a pull-back function $\mathcal{M}_{g,n}(B) \to \mathcal{M}_{g,n}(B')$, $X \mapsto X \times B'$.

Likewise, a family of smooth projective curves of genus g with n marked points is a family $f : X \to B$ of smooth projective curves of genus g with n disjoint sections, i.e. morphisms $s_1, \ldots, s_n : B \to X$ such that $f \circ s_i = \mathrm{Id}_B$ for every i and such that $s_i(b) \neq s_j(b)$ for every $b \in B$ and $i \neq j$.

The moduli functor

$$\mathcal{M}_{g,n} : AlgebraicVarieties \to Sets$$

sends every algebraic variety B to the set of isomorphism classes of families $f : X \to B$ of smooth projective curves of genus g with n marked points and every morphism $B' \to B$ a pull-back function $\mathcal{M}_{g,n}(B) \to \mathcal{M}_{g,n}(B')$, $X \mapsto X \times B'$. (Try to define the pullback of sections $s_1, \ldots, s_n$ yourself.)

5.4.4. DEFINITION. A family of smooth projective curves of genus 1 with a marked point is also called an *elliptic fibration*.

It would be nice to have a better structure theory of elliptic fibrations. We will later show the following:

5.4.5. THEOREM. *A morphism $\pi : X \to B$ with a section $\sigma : B \to X$ is an elliptic fibration if and only if every point $b \in B$ has an affine neighborhood $U = \operatorname{Spec} R$ such that $\pi^{-1}(U)$ is isomorphic to a subvariety of $U \times \mathbb{P}^2_{[x:y:z]}$ given by the Weierstrass equation*

$$y^2 z = 4x^3 - g_2(u)xz^2 - g_3(u)z^3,$$

where $g_2, g_3 \in R = \mathcal{O}(U)$ are regular functions such that $\Delta = g_2^3 - 27g_3^2 \in R^*$ is invertible. Moreover, g_2 and g_3 are defined uniquely up to transformations

$$g_2 \mapsto t^4 g_2, \quad g_3 \mapsto t^6 g_3 \tag{5.4.6}$$

for some invertible function $t \in R^*$.

5.4.7. REMARK. Dependence on t comes from the following basic observation: multiplying y by t^3 and x by t^2 will induce transformation (5.4.6). Notice that if $t \in \mathcal{O}^*(pt) = \mathbb{C}^*$ then we can take a square root $\sqrt{t}$ and multiply by it instead. This gives Theorem 5.2.3 (1). But if t is a non-constant regular function in $\mathcal{O}(U)$ then the square root may not exist. For example, take t to be a coordinate in $\mathbb{A}^1_t \setminus \{0\}$.

§5.5. **The j-line is a coarse moduli space.** We are going to see that the functor of elliptic fibrations $\mathcal{M}_{1,1}$ doesn't have a fine moduli space. Indeed, if M is a fine moduli space then its points should bijectively correspond to isomorphism classes of elliptic curves, i.e. to different values of the j-invariant. In other words, M should be bijective to $\mathbb{A}^1$ with coordinate j. Moreover, the identity map $M \to M$ should come from the universal family $\pi : U \to M$ with a section $\sigma : M \to U$ such that every elliptic fibration is a pull-back of the universal family. In particular, every elliptic curve should appear as exactly one fiber of the universal elliptic fibrations. As a first approximation, let's show

5.5.1. LEMMA. *There is no elliptic fibration over* $\mathbb{A}^1_j$ *such that the* j*-invariant of the fiber over* j *is* j*.*

Proof. Suppose this fibration exists. Then Theorem 5.4.5 would be applicable to it and so locally at every point $j_0 \in \mathbb{A}^1$ we would have

$$j = 1728\frac{g_2^3(j)}{g_2^3(j) - 27g_3^2(j)}$$

for some rational functions $g_2, g_3 \in \mathbb{C}(j)$ regular at j_0. Taking $j_0 = 0$ gives a contradiction because j has zero of multiplicity 1 at 0, whereas the RHS has zero of multiplicity divisible by 3. Taking $j_0 = 1728$ gives another contradiction:

$$j - 1728 = 1728\frac{27g_3^2(j)}{g_2^3(j) - 27g_3^2(j)}.$$

The LHS has simple zero at 1728 (multiplicity 1) but the RHS has a zero of even multiplicity. □

To turn lemons into lemonade, let's show that $\mathbb{A}^1$ is a coarse moduli space of the functor of elliptic fibrations.

5.5.2. DEFINITION. We say that an algebraic variety M is a *coarse moduli space* of the moduli functor $\mathcal{M} : Algebraic\,Varieties \to Sets$ if

(1) We have a natural transformation of functors $\mathcal{M} \to h_M$ (but not necessarily an equivalence), which induces a bijection of sets

$$\mathcal{M}(point) = \mathrm{Mor}(point, M) = M.$$

More concretely, points of M correspond to isomorphism classes of objects and every X-point $E \in \mathcal{M}(X)$ induces a morphism $X \to M$.

(2) Suppose M' is another algebraic variety satisfying (1). There is an obvious map $M \to M'$ because points of both varieties correspond to the same isomorphism classes. We require that this map is a morphism of algebraic varieties. This condition guarantees that the coarse moduli space is unique up to an isomorphism (if it exists).

If $\mathcal{M}$ admits a fine moduli space M then M is also a coarse moduli space. Indeed, by Yoneda's lemma any natural transformation $\mathcal{M} \simeq h_M \to h_{M'}$ comes from a morphism $M \to M'$, which gives (2).

5.5.3. THEOREM. *The* j*-line* $\mathbb{A}^1$ *is a coarse moduli space for* $\mathcal{M}_{1,1}$*.*

Proof. Suppose we have an elliptic fibration $\pi : X \to B$. Then we have a function $j_B : B \to \mathbb{A}^1$ which sends every $b \in B$ to the j-invariant of $\pi^{-1}(b)$. We have to show that this function is a morphism of algebraic varieties. This can be checked in affine charts on B, and thus by Theorem 5.4.5 we can assume that the fibration is in the Weierstrass normal form. But then j_B can be computed by the usual formula (5.2.7). Since g_2 and g_3 are regular functions in the chart, j_B is a regular function as well.

A tricky part is to check the second condition in the definition of a coarse moduli space. Suppose we have another variety Z and a natural transformation $\mathcal{M}_{1,1} \to h_Z$ such that points of Z are in 1-1 correspondence with isomorphism classes of elliptic curves. Every elliptic fibration $\pi : X \to B$ then gives a regular morphism $j_Z : B \to Z$. We want to show that it factors through a morphism $\mathbb{A}^1_j \to Z$. Let $I \subset Z \times \mathbb{A}^1_j$ be the locus of pairs corresponding to curves with the same j-invariant. It is a bijective correspondence between Z and $\mathbb{A}^1$. Suppose we know that I is closed. Then both projections $I \to Z$ and $I \to \mathbb{A}^1_j$ are bijective morphisms of closed algebraic sets. Since $\mathbb{A}^1_j$ is irreducible, I and Z are irreducible as well. It follows that both projections $I \to Z$ and $I \to \mathbb{A}^1_j$ are bijective morphisms of algebraic curves. But $\mathbb{A}^1_j$ is a smooth curve, and therefore $I \to \mathbb{A}^1_j$ is an isomorphism. Thus I is a graph of a morphism $\mathbb{A}^1 \to Z$.

It remains to show that I is closed. For this we need to remind

5.5.4. DEFINITION. Recall that a morphism of algebraic varieties $\pi : X \to Y$ is called *finite* if for some (and therefore any) affine covering $Y = \cup_i U_i$, every pre-image $V_i := \pi^{-1}(U_i)$ is an affine variety and $k[V_i]$ is a finitely generated $k[U_i]$ module. Equivalently, $k[V_i]$ is generated (as a k-algebra) by finitely many elements which are roots of monic polynomials with coefficients in $k[U_i]$. The basic properties of finite morphisms include:

(1) a finite morphism has finite fibers.
(2) a finite morphism is surjective.

It is easy to construct an elliptic fibration $X \to B$ over a smooth algebraic curve such that every elliptic curve appears as one of the fibers. Just take $B = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and define $X \subset B \times \mathbb{P}^2_{[x:y:z]}$ by the Weierstrass equation

$$y^2 z = x(x - z)(x - \lambda z).$$

The j-invariant in this case is the map

$$\mathbb{P}^1 \setminus \{0, 1, \infty\} = \operatorname{Spec} \mathbb{C}\left[\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}\right] \to \mathbb{A}^1_j$$

given by (5.2.7). Notice that this morphism is finite. Indeed,

$$k[\mathbb{P}^1 \setminus \{0, 1, \infty\}] = k\left[\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}\right]$$

and λ is a root of a monic polynomial (1) with coefficients in $\mathbb{C}[j]$, with other roots given by (5.2.5), which include $\frac{1}{\lambda}$ and $\frac{1}{\lambda - 1}$. By one of the homework problems, $j_Z \times j : B \to Z \times \mathbb{A}^1_j$ is also finite and therefore its image I is closed. $\square$

§5.6. **The j-line is not a fine moduli space.**

5.6.1. PROPOSITION. *$\mathcal{M}_{1,1}$ does not admit a fine moduli space.*

Proof. Indeed, if $\mathcal{M}_{1,1}$ admits a fine moduli space then it should be $\mathbb{A}^1_j$, because a fine moduli space is also automatically a coarse one. But the universal family over $\mathbb{A}^1_j$ can't exist by Lemma 5.5.1.

There are other ways to reach the same conclusion. Let $B = \mathbb{A}^1_s$ or more generally let $B = \operatorname{MaxSpec} R$, where R contans an element s without a square root (for example take $R = \mathbb{C}[s]$). Consider any elliptic fibration $f : E \to B$ with Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3$$

and let $E' \to B$ be a "twisted" fibration with Weierstrass equation

$$y^2 = 4x^3 - s^2g_2x - s^3g_3.$$

These fibrations have isomorphic fibers (over every $b \in B$) hence give the same maps into the coarse moduli space $B \to \mathbb{A}^1_j$. However, they are not isomorphic. If they were, we would have

$$s^2 = t^4, \quad s^3 = t^6$$

for some $t \in R$ by Theorem 5.4.5. Thus $s = t^2$, a contradiction. □

§5.7. **Homework 2.**

Problem 1. (2 points) Using a birational isomorphism between $\mathbb{P}^1$ and the circle $\{x^2 + y^2 = 1\} \subset \mathbb{A}^2$ given by stereographic projection from $(0, 1)$, describe an algorithm for computing integrals of the form

$$\int P(x, \sqrt{1 - x^2})\, dx$$

where $P(x, y)$ is an arbitrary rational function.

Problem 2. (3 points) The formula $j = 256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$ gives a $6 : 1$ cover $\mathbb{P}^1_\lambda \to \mathbb{P}^1_j$. Thinking about $\mathbb{P}^1$ as a Riemann sphere, color $\mathbb{P}^1_j$ in two colors: color the upper half-plane $\mathcal{H}$ white and the lower half-plane $-\mathcal{H}$ black. Draw the pull-back of this coloring to $\mathbb{P}^1_\lambda$.

Problem 3. (1 points) Let $F : Sets \to Sets$ be a contravariant functor that sends a set S to the set of subsets of S and any function $f : S \to S'$ to a function that sends $U \subset S'$ to $f^{-1}(U) \subset S$. Show that F is representable.

Problem 4. (2 points) Let F be a contravariant) functor from the category of topological spaces to $Sets$ which sends a topological space X to the set of open sets of X and any continuous function $f : X \to X'$ to a function that sends $U \subset X'$ to $f^{-1}(U) \subset X$. Is it representable?

Problem 5. (2 points) Let $S \subset R$ be a multiplicative system in the commutative ring R. Consider the covariant functor from commutative rings to sets that sends a ring A to the set of all homomorphisms $f : R \to A$ such that $f(s)$ is a unit for any $s \in S$ (describe its action of homomorphisms $A \to A'$ yourself). Show that this functor is representable.

Problem 6. (2 points) Let $F : AlgebraicVarieties \to Sets$ be a functor which assigns to each X the subset $S \subset \mathcal{O}(X)$ of all regular functions f which an be written as a square $f = g^2$ of a regular function. Describe the action of F on morphisms $X \to Y$ of algebraic varieties. Is F representable?

Problem 7. (2 points) Let X and Y be irreducible quasi-projective varieties with fields of rational functions $\mathbb{C}(X)$ and $\mathbb{C}(Y)$. Show that these fields are isomorphic if and only if there exist non-empty affine open subsets $U \subset X$ and $V \subset Y$ such that U is isomorphic to V.

Problem 8. (3 points) Let C be an algebraic curve. Then one can describe morphisms $C \to \mathbb{P}^r$ either using Theorem 3.7.11 (line bundles) or using

Proposition 4.8.6 (linear systems of divisors). Explain how these methods are related.

Problem 9. (2 points) Compute j-invariants of elliptic curves

$$(a)\ y^2 + y = x^3 + x; \quad (b)\ y^2 = x^4 + bx^3 + cx.$$

Problem 10. (1 point) Show that every elliptic curve is isomorphic to a curve of the form $y^2 = (1-x^2)(1-e^2x^2)$.

Problem 11. (2 points) Show that the two formulas in (5.2.7) agree.

Problem 12. (3 points) Let (C, P) be an elliptic curve. (a) By considering a linear system $\phi_{|4P|}$, show that C embeds in $\mathbb{P}^3$ as a curve of degree 4. (b) Show that quadrics in $\mathbb{P}^3$ containing C form a pencil $\mathbb{P}^1$ with 4 singular fibers. (c) These four singular fibers define 4 points in $\mathbb{P}^1$. Relate their cross-ratio to the j-invariant of C.

Problem 13. (2 points). (a) Compute the j-invariant of an elliptic curve

$$y^2 + xy = x^3 - \frac{36}{q-1728}x - \frac{1}{q-1728},$$

where q is some parameter. (b) Show that $\mathbb{A}^1_j \setminus \{0, 1728\}$ carries a family of elliptic curves with j-invariant j.

Problem 14. (2 points) Solve a cross-word puzzle.

§6. **Families of algebraic varieties**

Our next goal is to sketch the proof of Theorem 5.4.5 following [MS]. Recall that our goal is to describe explicitly any smooth proper morphism $\pi : X \to B$ with a section σ such that all of the fibers are elliptic curves. In order to do that, we have to introduce a more advanced viewpoint on families of algebraic varieties. For starters, we can shrink B to an affine open neighborhood of a point $b \in B$ and assume that B is an affine variety with ring of functions $R = k[B]$. We can think about X as "an elliptic curve over the ring R" generalizing "an elliptic curve E over the field $\mathbb{C}$". Eventually we will write a Weierstrass equation for it with coefficients in R.

We adopt the same strategy as in the proof of Theorem 5.1.1, namely the Riemann–Roch analysis of linear systems $\mathcal{L}(kP)$ on E. But we will have to upgrade our technology so that we can work with a family over the base B and with a section $A = \sigma(B) \subset X$ instead of a single point $P \in E$.

We will go back and forth between introducing general techniques and filling the gaps in the proof of Theorem 5.4.5.

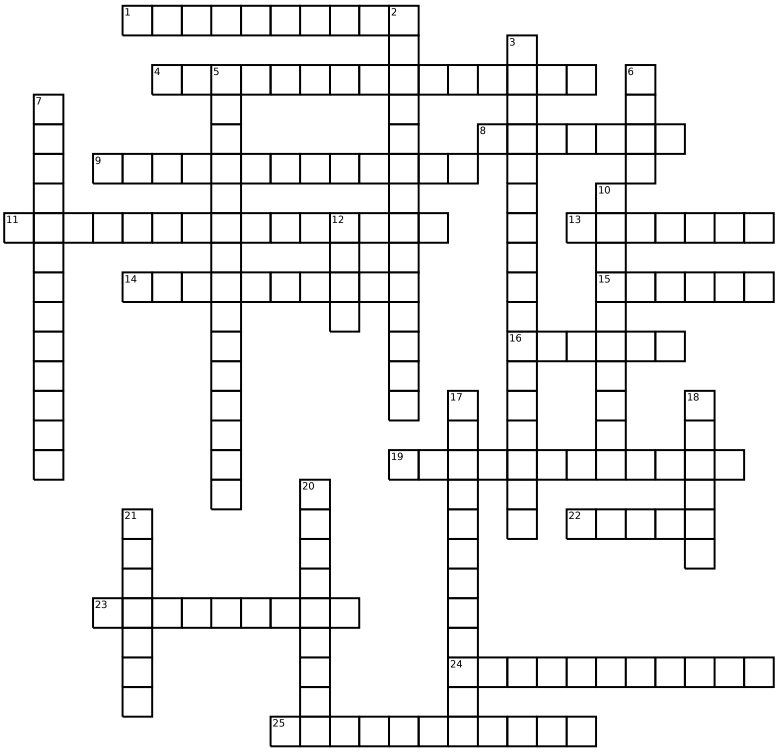
§6.1. **Short exact sequence associated with a subvariety.**

Let X be an algebraic variety and let $Z \subset X$ be a subvariety. Algebraically, it is given by a *sheaf of (radical) ideals* $\mathcal{I}_Z \subset \mathcal{O}_X$. Namely,

$$\mathcal{I}_Z(U) = \{f \in \mathcal{O}_X(U) \mid f|_{Z\cap U} = 0\} \subset \mathcal{O}_X(U)$$

for every open subset $U \subset X$. These sheaves sit in the following very useful short exact sequence of sheaves

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0, \tag{2}$$



Across

1 A linear system that corresponds to a proper subset of a basis in L(D) is called ...
4 Theorem of zeros
8 A friendly giant hiding inside the j-function
9 All effective divisors linearly equivalent to a given one
11 An algebraic curve over C is also known as a ...
13 He coined a term "coarse moduli space"
14 A cross-ratio is also known as an ... ratio
15 Word used by Riemann for a number of parameters
16 An attribute of a divisor contributing to the Riemann-Roch formula
19 Example of a fine moduli space
22 Dimension of the space of holomorphic differentials on the algebraic curve
23 Something you can do with families
24 An algebraic variety is called ... if it is not a union of two proper closed subsets
25 A normal form of a cubic curve is named after him

Down

2 Type of an algebraic curve most revered by number theorists
3 Divisor of a meromorphic form
5 An algebraic geometer's analogue of a holomorphic coordinate
6 He made important contributions to the study of integrals of algebraic functions
7 A fancy word used by algebraic geometers instead of "compact"
10 Holomorphic map with surjective differential
12 The best kind of a moduli space
17 Image of a ramification point
18 The second best kind of a moduli space
20 A divisor with non-negative multiplicities is called ...
21 An everywhere defined rational map is called ...

where $i: Z \hookrightarrow X$ is the inclusion map and i_* is the *push-forward* of sheaves[10]. If $U \subset X$ is an affine open subset with ring of functions $R = k[U]$ then $I = \mathcal{I}_Z(U)$ is an ideal of R and R/I is a coordinate ring of $Z \cap U$. Taking sections of sheaves in (2) over U gives a short exact sequence

$$0 \to I \to R \to R/I \to 0.$$

[10] Recall that if $X \to Y$ is a continuous map of topological spaces and $\mathcal{F}$ is a sheaf on X then the pushforward $f_*\mathcal{F}$ has sections $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for every open set $U \subset Y$.

For example, if $P \in C$ is a point of a smooth curve then (2) becomes

$$0 \to \mathcal{I}_P \to \mathcal{O}_C \to \mathbb{C}_P \to 0, \tag{3}$$

where $\mathbb{C}_P$ is the skyscraper sheaf of the point P.

§6.2. **Cartier divisors and invertible sheaves.**

6.2.1. DEFINITION. A *Cartier divisor* D on an algebraic variety X is an (equivalence class of) data

$$(U_\alpha, f_\alpha), \quad \alpha \in I,$$

where $X = \cup U_\alpha$ is an open covering and $f_\alpha \in \mathbb{C}(X)$ are non-zero rational functions such that f_α/f_β is an invertible function on each overlap $U_\alpha \cap U_\beta$. A Cartier divisor is called *effective* if $f_\alpha \in \mathcal{O}_X(U_\alpha)$ for every α.

We can think about a Cartier divisor D as a divisor given by equation $f_\alpha = 0$ in each chart U_α. The fact that f_α/f_β is invertible on every overlap makes this consistent. More precisely, we have the following definition:

6.2.2. DEFINITION. Suppose the singular locus of X has codimension at least 2[11]. Then an associated divisor is

$$\sum \operatorname{ord}_H(f_\alpha)[H],$$

the summation over all prime divisors (=irreducible hypersurfaces) $H \subset X$ and $\operatorname{ord}_H(f_\alpha)$ is the order of zeros–poles along H. The order is defined using any f_α such that $H \cap U_\alpha \neq \emptyset$. The fact that f_α/f_β is invertible on $U_\alpha \cap U_\beta$ makes this definition independent of α. The sum of course turns out to be finite.

In general, not every prime divisor is Cartier: locally the former correspond to prime ideals of height 1 and the latter to locally principal ideals. If X is non-singular then every divisor is Cartier[12].

6.2.3. EXAMPLE. Suppose $n_1P_1+\ldots+n_rP_r$ is a divisor on a curve C. Choose a covering $C = U_0 \cup U_1 \cup \ldots \cup U_r$, where $U_0 = C \setminus \{P_1, \ldots, P_r\}$ and U_i is defined as follows: choose a local parameter g_i for P_i and define U_i to be C with removed points P_j, $j \neq i$ and removed zeros and poles of g_i except for its simple zero at P_i[13]. Finally, define $f_i = g_i^{n_i}$.

A Cartier divisor D has *an associated line bundle $X = L_D$*, which has trivializing atlas $\cup U_\alpha$ and transition functions f_α/f_β. In algebraic geometry we often bypass this line bundle and work with its sheaf of sections, denoted by $\mathcal{O}_X(D)$. One can define it directly: for every open subset $U \subset X$,

$$\mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(X) \mid ff_\alpha \in \mathcal{O}_X(U \cap U_\alpha).$$

The sheaf $\mathcal{O}_X(D)$ is *invertible*, i.e. every point of X has a neighborhood U such that $\mathcal{O}_X(D)|_U \simeq \mathcal{O}_U$. Namely, for every α,

$$\mathcal{O}_X(D)|_{U_\alpha} = \frac{1}{f_\alpha}\mathcal{O}_X|_{U_\alpha}.$$

[11]Recall that this is always the case if X is a normal variety.

[12]A difficult step of the proof is that non-singular varieties are *locally factorial*, i.e. every local ring is a UFD. It is easy to show that on locally factorial varieties all divisors are Cartier.

[13]In the analytic category one can take U_i to be a small neighborhood of P_i and g_i a corresponding local coordinate.

6.2.4. EXAMPLE. One very useful application is a special case of the exact sequence (2) when $Z = D$ is a prime divisor, which happens to be Cartier:

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0.$$

For example, the exact sequence (3) can be rewritten as

$$0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to \mathbb{C}_P \to 0.$$

6.2.5. EXAMPLE. The linear system $\mathcal{L}(D)$ in this language is the space of global sections:

$$\mathcal{L}(D) = \Gamma(X, \mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(D)).$$

§6.3. **Morphisms with a section.** A section is always a subvariety:

6.3.1. LEMMA. *Suppose a morphism of algebraic varieties $\pi : X \to B$ has a section $\sigma : B \to X$. Then $A = \sigma(B)$ is a closed subvariety isomorphic to B.*

Proof. It suffices to show that A is closed because then $\pi|_A$ and σ give a required isomorphism. Arguing by contradiction, take $x \in \bar{A} \setminus A$. Let $b = \pi(x)$. Then $y = \sigma(b) \neq x$. Since X is quasiprojective, there exists an open subset U of X, which contains both x and y, and a function $f \in \mathcal{O}(U)$ such that $f(x) = 1$ and $f(y) = 0$. But this gives a contradiction: the function

$$f - f \circ \sigma \circ \pi$$

vanishes identically along A, and therefore has to vanish at $x \in \bar{A}$ but its value at x is equal to $f(x) - f(y) = 1$.[14]. □

§6.4. **Morphisms with reduced fibers.** Let $\pi : X \to B$ be a morphism of algebraic varieties, $b \in B$. What are the equations of the fiber $X_b = \pi^{-1}(b)$? Notice that $x \in X_b$ if and only if $\pi^* f(x) = 0$ for every function f on B regular and vanishing at b. Therefore the ideal $J \subset \mathcal{O}_{X,x}$ of the fiber is equal to the radical of the ideal in $\mathcal{O}_{X,x}$ generated by $\pi^*(\mathfrak{m}_{B,b})$.

6.4.1. DEFINITION. The *fiber X_b is reduced at $x \in X_b$* if $J = \mathcal{O}_{X,x}\pi^*(\mathfrak{m}_{B,b})$ is a radical ideal. We say that X_b is *a reduced fiber* if it is reduced at every point.

6.4.2. EXAMPLE. Let $f : C \to D$ be a non-constant morphism of algebraic curves. For every point of a curve, its maximal ideal in the local ring is generated by a local parameter at that point. Thus the fiber $f^{-1}(y)$ is reduced at x if and only if f is unramified at x.

6.4.3. LEMMA. *Under the assumptions of Lemma 6.3.1, let $x \in A$, $b = \pi(x)$. Suppose that the fiber X_b is reduced. Under the restriction homomorphism*

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X_b,x},$$

the maximal ideal $\mathfrak{m}_{X_b,x} \subset \mathcal{O}_{X_b,x}$ is the image of the ideal $I \subset \mathcal{O}_{X,x}$ of the section.

[14] The proof reflects the fact that every quasi-projective algebraic variety is *separated*. Moreover, an algebraic variety is separated whenever any two points $x, y \in X$ are contained in an open subset U which admits a regular function $f \in \mathcal{O}(U)$ separating x and y, i.e. such that $f(x) \neq f(y)$. It is easy to construct an unseparated variety by gluing affine open varieties. A simple example is an affine line with two origins X obtained by gluing two copies of $\mathbb{A}^1$ along an open subset $\{x \neq 0\}$ using the identity map. The two origins cannot be separated by a function. Notice by the way that a projection map $X \to \mathbb{A}^1$ has a section with the non-closed image! In the category of manifolds, the "separation of points by a continuous function" property is equivalent to Hausdorffness.

Proof. It is clear that I restricts to the ideal $\bar{I} \subset \mathcal{O}_{X_b,x}$ contained in $\mathfrak{m}_{X_b,x}$. So it suffices to prove that $\mathcal{O}_{X_b,x}/\bar{I} = \mathbb{C}$. Since the fiber is reduced, we have $\mathcal{O}_{X_b,x} = \mathcal{O}_{X,x}/J$, where $J = \mathcal{O}_{X,x}\pi^*(\mathfrak{m}_{B,b})$. Thus it suffices to show that

$$\mathcal{O}_{X,x}/(J+I) = \mathbb{C}.$$

And indeed, doing factorization in a different order gives

$$\mathcal{O}_{X,x}/(I+J) = \mathcal{O}_{A,x}/\pi^*(\mathfrak{m}_{B,b}) \simeq \mathcal{O}_{B,b}/\mathfrak{m}_{B,b} = \mathbb{C}.$$

This proves the Lemma. □

If all fibers of π are curves then

$$\dim A = \dim B = \dim X - 1,$$

i.e. the section A is a divisor. We claim that it is a Cartier divisor if π has reduced fibers and all of them are smooth curves.

6.4.4. LEMMA. *Let $\pi : X \to B$ be a morphism with a section σ, reduced fibers, and such that all fibers are smooth curves. Then $A = \sigma(B)$ is a Cartier divisor.*

Proof. We have to show that every point $x = \sigma(b) \in A$ has an affine neighborhood U such that $\mathcal{I}_A(U)$ is a principal ideal. It suffices to prove that its localization, $I := \mathcal{I}_{A,x} \subset \mathcal{O}_{X,x}$ is a principal ideal. Let $J := \mathcal{I}_{X_b,x} \subset \mathcal{O}_{X,x}$ be the vanishing ideal of the fiber.

By Lemma 6.4.3, the maximal ideal $\mathfrak{m}_{X_b,x}$ of x in the fiber is equal to the restriction of I. Since the fiber is a non-singular curve, $\mathfrak{m}_{X_b,x}$ is generated by a local parameter z (algebraically, $\mathcal{O}_{X_b,x}$ is a discrete valuation ring and z is a uniformizer). Choose $f \in I$ that restricts to $z \in \mathfrak{m}_{X_b,x}$.

We claim that $I = (f)$, or equivalently $I/(f) = 0$. Since $I/(f)$ is a finitely generated module of the local ring $\mathcal{O}_{X,x}$, Nakayama's lemma applies, and thus $I/(f) = 0$ if and only if

$$I/(f) = \mathfrak{m}_{X,x}I/(f),$$

i.e.

$$I = (f) + \mathfrak{m}_{X,x}I.$$

Choose $\alpha \in I$ and choose $gf \in (f)$ which has the same restriction to the local ring $\mathcal{O}_{X_b,x}$ of the fiber as α. Then $\alpha - gf$ restricts trivially, i.e. belongs to J. Thus it suffices to prove that $I \cap J \subset \mathfrak{m}_{X,x}I$. In fact we claim that

$$I \cap J = IJ.$$

The pull-back by π^* and by σ^* give an isomorphism of $\mathcal{O}_{B,b}$ modules

$$\mathcal{O}_{X,x} = \pi^*\mathcal{O}_{B,b} + I.$$

Since the fiber X_b is reduced, we have

$$J = \mathcal{O}_{X,x}\pi^*\mathfrak{m}_{B,b} = \pi^*\mathcal{O}_{B,b}\pi^*\mathfrak{m}_{B,b} + I\pi^*\mathfrak{m}_{B,b} = \pi^*\mathfrak{m}_{B,b} + IJ.$$

Thus if $\alpha \in I \cap J$ then, modulo IJ, we can assume that $\alpha = \pi^*\beta$. But then

$$\beta = \sigma^*\pi^*\beta = \sigma^*\alpha = 0$$

because α vanishes along the section. Thus $\alpha = \pi^*\beta = 0$ as well. □

§6.5. **Flat and smooth morphisms.** Let's finally define smooth morphisms.

6.5.1. DEFINITION. A morphism $f : X \to Y$ of algebraic varieties is called *flat* if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module for every point $x \in X$ and $y = f(x)$.

6.5.2. DEFINITION. A morphism $f : X \to Y$ of algebraic varieties is called *smooth* if it is flat, has reduced fibers, and every fiber is non-singular.

Let $X \to B$ be an elliptic fibration, i.e. a smooth proper morphism with a section $A = \sigma(B)$ such that all fibers are elliptic curves. We would like to write down Weierstrass equation of X, possibly after shrinking B. When B is a point, the argument will reduce to the previous calculation with linear systems

$$\mathcal{L}(E, kP) = H^0(E, \mathcal{O}_E(kP)).$$

Recall that if $\mathcal{F}$ is a sheaf of Abelian groups on an algebraic variety X then we can define higher cohomology groups $H^k(X, \mathcal{F})$ in addition to the group $H^0(X, \mathcal{F})$ of global sections. There are several important facts to know about these groups.

6.5.3. THEOREM. *For every short exact sequence of sheaves*

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0,$$

we have a long exact sequence of cohomology groups

$$\ldots \to H^k(X, \mathcal{F}) \to H^k(X, \mathcal{F}') \to H^k(X, \mathcal{F}'') \to H^{k+1}(X, \mathcal{F}) \to \ldots,$$

functorial with respect to commutative diagrams of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G}'' & \longrightarrow & 0 \end{array}$$

6.5.4. DEFINITION. A sheaf $\mathcal{F}$ on an algebraic variety X is called *locally free of rank r* if every point has a neighborhood U such that $\mathcal{F}|_U \simeq \mathcal{O}_U^{\oplus r}$. Equivalently, $\mathcal{F}$ is a sheaf of sections of some vector bundle $\pi : F \to X$, i.e.

$$\mathcal{F}(U) = \{s : X \to F \mid \pi \circ s = \mathrm{Id}_X\}.$$

For example, a locally free sheaf of rank 1 is nothing but an invertible sheaf, a sheaf of sections of a line bundle.

6.5.5. THEOREM. *Let X be a projective algebraic variety and let $\mathcal{F}$ be a locally free sheaf on X. Then all cohomology groups $H^k(X, \mathcal{F})$ are finite-dimensional vector spaces which vanish for $k > \dim X$*[15]*. Their dimensions are denoted by $h^k(X, \mathcal{F})$.*

If X is a smooth projective curve[16] then we also have

6.5.6. THEOREM (Serre duality). *Let D be any divisor on X and let K be the canonical divisor. Then we have duality of vector spaces*

$$H^i(X, \mathcal{O}_X(D)) \simeq H^{1-i}(X, \mathcal{O}_X(K - D))^*.$$

[15] More generally, one can reach the same conclusion if $\mathcal{F}$ is a *coherent sheaf*.

[16] In fact much more generally but we won't need that.

In particular,

$$i(D) = \dim K(D) = h^0(K - D) = h^1(D)$$

and we can rewrite Riemann–Roch theorem in a (less useful) form

$$h^0(D) - h^1(D) = 1 - g + \deg D.$$

Now let $\pi : X \to B$ be an elliptic fibration with section $A = \sigma(B)$ and consider invertible sheaves $\mathcal{O}_X(kA)$ for $k \geq 0$. We would like to understand the space of global sections $H^0(X, \mathcal{O}_X(kA))$, but since we need a freedom to shrink B, it is better to study the push-forward $\pi_*\mathcal{O}_X(kA)$. Its sections over an open subset $U \subset B$ is the space of global sections

$$H^0(\pi^{-1}(U), \mathcal{O}_X(kA)).$$

Recall that this is the space of rational functions on X regular at any point $x \in \pi^{-1}(U) \setminus A$ and that can be written as g/f^k at any point $x \in A$, where g is regular and f is a local defining equation of the Cartier divisor A.

§6.6. **Pushforwards and derived pushforwards.** If $\mathcal{F}$ is a sheaf of Abelian groups on an algebraic variety X and $\pi : X \to B$ is a morphism then one can define derived pushforward sheaves $R^k\pi_*\mathcal{F}$ on B in addition to the push-forward $R^0\pi_*\mathcal{F} := \pi_*\mathcal{F}$. When B is a point, $\pi_*\mathcal{F} = H^0(X, \mathcal{F})$ and $R^k\pi_*\mathcal{F} = H^0(X, \mathcal{F})$. The analogue of Theorem 6.5.3 is the following:

6.6.1. THEOREM. *For every short exact sequence of sheaves*

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0,$$

we have a long exact sequence of derived push-forwards

$$\ldots \to R^k\pi_*\mathcal{F} \to R^k\pi_*\mathcal{F}' \to R^k\pi_*\mathcal{F}'' \to R^{k+1}\pi_*\mathcal{F} \to \ldots,$$

functorial with respect to commutative diagrams of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G}'' & \longrightarrow & 0
\end{array}$$

6.6.2. REMARK. Suppose $\mathcal{F}$ is a locally free sheaf on X and let $\pi : X \to B$ be a morphism. Notice that $\pi_*\mathcal{F}$ is not just a sheaf of Abelian groups but a sheaf of $\mathcal{O}_B$-modules. Indeed, for every open subset $U \subset B$,

$$\pi_*\mathcal{F}(U) = \mathcal{F}(\pi^{-1}U)$$

is an $\mathcal{O}_X(\pi^{-1}U)$-module but we have a homomorhism of rings

$$f^* : \mathcal{O}_B(U) \to \mathcal{O}_X(\pi^{-1}U)$$

which makes its a $\mathcal{O}_B(U)$-module as well. In particular, the stalk $(\pi_*\mathcal{F})_b$ at $b \in B$ is a $\mathcal{O}_{B,b}$-module. Derived pushforwards are also sheaves of $\mathcal{O}_B$-modules, generalizing the fact that cohomology groups are vector spaces.

Here's the analogue of Theorem 6.5.5:

6.6.3. THEOREM. *Let $\pi : X \to B$ be a proper morphism of algebraic varieties and let $\mathcal{F}$ be a locally free*[17] *sheaf on X. Then all derived push-forwards $R^k\pi_*\mathcal{F}$ are sheaves of finitely generated $\mathcal{O}_B$-modules.*

§6.7. **Cohomology and base change.**

6.7.1. DEFINITION. Take any vector bundle $E \to X$ and a subvariety $Z \subset X$. The *restriction* $E|_Z$ of E is a vector bundle on Z. If $\mathcal{F}$ is a (locally free) sheaf of sections of E then the sheaf of sections $\mathcal{F}|_Z$ of $E|_Z$ can be described as follows: if $U \subset X$ is an affine open set then

$$\mathcal{F}|_Z(U \cap Z) = \mathcal{F}(U)/\mathcal{I}_Z(A)\mathcal{F}(U).$$

In other words, we have an exact sequence of sheaves on X

$$0 \to \mathcal{I}_Z \otimes \mathcal{F} \to \mathcal{F} \to i_*\mathcal{F}_Z \to 0, \tag{4}$$

obtained by tensoring the short exact sequence (2) with $\mathcal{F}$.

6.7.2. EXAMPLE. Suppose $\pi : X \to B$ is a morphism with a section $A = \sigma(B)$ such that all fibers are reduced curves. We can apply the previous definition either to the fibers of π or to the section.

The restriction of an invertible sheaf $\mathcal{O}_X(kA)$ (its local sections are rational functions on X with poles of order at most k along A) to every fiber is the sheaf

$$\mathcal{O}_{X_b}(kA) = \mathcal{O}_{X_b}(kP)$$

(its local sections are rational functions on X_b with poles of order at most k at $P = \sigma(b)$.)

Another very useful short exact sequence for us will be the sequence (4) with $Z = A$. It goes as follows:

$$0 \to \mathcal{O}_X((k-1)A) \to \mathcal{O}_X(kA) \xrightarrow{\psi} i_*\mathcal{O}_A(kA) \to 0. \tag{5}$$

What is the meaning of the last map ψ?

6.7.3. CLAIM. *At any point $x \in A$, a local section α of $\mathcal{O}_X(kA)$ looks like g/f^k, where f is a local equation of A. Then*

$$\psi(g/f^k) = g|_A.$$

We call $\psi(\alpha)$ the *principal part* of α.

6.7.4. DEFINITION. We would like to compare $\pi_*\mathcal{F}$, which is a sheaf on B, with the vector space of global sections $H^0(X_b, \mathcal{F}|_{X_b})$. For every neighborhood $b \in U$, we have a restriction homomorphism

$$\pi_*\mathcal{F}(U) = \mathcal{F}(\pi^{-1}U) \to H^0(X_b, \mathcal{F}|_{X_b}).$$

These homomorphisms commute with further restrictions $b \in V \subset U$, and therefore give a homomorphism from the stalk of the push-forward

$$(\pi_*\mathcal{F})_b \to H^0(X_b, \mathcal{F}|_{X_b}).$$

This stalk $(\pi_*\mathcal{F})_b$ is a module over the local ring $\mathcal{O}_{B,b}$. If $f \in \mathfrak{m}_{B,b}$ then every section in $f(\pi_*\mathcal{F})_b$ restricts to X_b trivially. Thus we have a *canonical homomorphism*

$$i_b^0 : (\pi_*\mathcal{F})_b \otimes \mathbb{C} \to H^0(X_b, \mathcal{F}|_{X_b}),$$

[17] Or, more generally, a coherent sheaf.

where $\mathbb{C} \simeq \mathcal{O}_{B,b}/\mathfrak{m}_{B,b}$. More generally, we have *canonical homomorphisms*

$$i_b^k : (R^k \pi_* \mathcal{F})_b \otimes \mathbb{C} \to H^k(X_b, \mathcal{F}|_{X_b}).$$

An ideal situation would be if $R^k\pi_*\mathcal{F}$ were a locally free sheaf of sections of some vector bundle F^k n B. Then $(R^k\pi_*\mathcal{F})_b \otimes \mathbb{C}$ would be identified with the fiber of F^k at $b \in B$. The canonical homomorphism would give a linear map from the fiber of F^k to $H^k(X_b, \mathcal{F}|_{X_b})$. If that linear map were an isomorphism, we would be able to interpret $R^k\pi_*\mathcal{F}$ as a sheaf of sections of a vector bundle with fibers given by cohomologies of fibers $H^k(X_b, \mathcal{F}|_{X_b})$. This is not always the case (for example, dimensions of these cohomology groups can jump in special fibers), but there is a powerful cohomology and base change theorem, which gives a necessary condition.

6.7.5. THEOREM. *Let $\pi : X \to B$ be a proper flat morphism of algebraic varieties with reduced fibers and let $\mathcal{F}$ be an invertible or locally free sheaf on X.*

(1) *If i_b^k is surjective for some k and $b \in B$ then it is bijective.*
(2) *Suppose (1) is satisfied. Then $R^k\pi_*\mathcal{F}$ is locally free in a neighborhood of b if and only if i_b^{k-1} is surjective.*
(3) *If $H^{k+1}(X_b, \mathcal{F}|_{X_b}) = 0$ then i_b^k is an isomorphism.*

The following corollary is the most often used form of cohomology and base change:

6.7.6. COROLLARY. *If $H^1(X_b, \mathcal{F}|_{X_b}) = 0$ for some $b \in B$ then $\pi_*\mathcal{F}$ is locally free in a neighborhood of b and i_b^0 is an isomorphism.*

§6.8. **Riemann-Roch analysis.** Consider an elliptic fibration $\pi : X \to B$ with a section $A = \sigma(B)$. The key players will be invertible sheaves $\mathcal{O}_X(kA)$ for $k \geq 0$ and their push-forwards

$$\mathcal{F}_k := \pi_*\mathcal{O}_X(kA).$$

6.8.1. LEMMA. *$\mathcal{F}_k$ is a locally free sheaf of rank k for every $k \geq 1$. Also,*

$$\mathcal{F}_0 = \pi_*\mathcal{O}_X \simeq \mathcal{O}_B$$

(a canonical isomorphism via pull-back π^*).

Pushing an exact sequence (5) forward to B gives a long exact sequence

$$0 \to \mathcal{F}_{k-1} \to \mathcal{F}_k \xrightarrow{\psi} \mathcal{L}^{\otimes k} \to R^1\pi_*\mathcal{O}_X((k-1)A),$$

where

$$\mathcal{L} := \pi_* i_* \mathcal{O}_A(A) \simeq \sigma^*\mathcal{O}_X(A)$$

is an invertible sheaf and ψ is the principal parts map.

6.8.2. LEMMA. *For $k \geq 2$, this gives a short exact sequence*

$$0 \to \mathcal{F}_{k-1} \to \mathcal{F}_k \to \mathcal{L}^{\otimes k} \to 0$$

For $k = 1$, we get is an isomorphism $\mathcal{F}_0 \simeq \mathcal{F}_1 \simeq \mathcal{O}_B$.

Let $B = \operatorname{Spec} R$. Then

$$H^0(B, \mathcal{F}_0) = H^0(B, \mathcal{F}_1) = R.$$

Since $\mathcal{L}$ is invertible, we can shrink B to a smaller affine neighborhood so that $\mathcal{L} \simeq \mathcal{O}_B$. We are going to fix a trivialization of $\mathcal{L}$. Everything that follows will be determined up to making a different choice of trivialization, i.e. up to multiplying it by an invertible function $\lambda \in R^*$.

Shrink B further to get short exact sequences

$$0 \to \mathcal{F}_{k-1}(B) \to \mathcal{F}_k(B) \to \mathcal{L}^{\otimes k}(B) \to 0$$

for every $2 \le k \le 6$, or, equivalently, given our trivialization of $\mathcal{L}$,

$$0 \to \mathcal{F}_{k-1}(B) \to \mathcal{F}_k(B) \xrightarrow{\psi_k} R \to 0,$$

where ψ is the principal parts map.

The following easy lemma is left as a homework exercise.

6.8.3. LEMMA. *If $0 \to M \to N \to K \to 0$ is an exact sequence of R-modules and M and K are free then N is also free.*

By induction, in our case we see that $\mathcal{F}_k(B)$ is a free R-module of rank k for every $2 \le k \le 6$. Its generators can be obtained by choosing any generators in $\mathcal{F}_{k-1}(B)$ along with any element which maps to 1 by ψ_k.

Thus $\mathcal{F}_2(B)$ is a free R-module generated by $1 \in \mathcal{F}_1(B) = R$ and some element x such that $\psi_2(x) = 1$. Also, $\mathcal{F}_3(B)$ is a free R-module generated by $1, x \in \mathcal{F}_2(B)$ and some y such that $\psi_3(y) = 2$. An annoying renormalization "2" is here by historical reasons. As functions on X, x has a pole of order 2 and y has a pole of order 3 along A. Notice that x and y are not uniquely determined: we can add to x any R-multiple of $1 \in \mathcal{F}_1(B) = R$ and we can add to y any linear combination of 1 and $x \in \mathcal{F}_2(B)$.

Arguing as in the case of a single elliptic curve, we see that

6.8.4. CLAIM. $1, x, y, x^2, xy, x^3$ *freely generate $\mathcal{F}_6(B)$.*

But y^2 is also a section of $\mathcal{F}_6(B)$, thus it can be expressed as a linear combination of these generators (with coefficients in R). Since $\psi_6(y^2) = 4$ and $\psi_6(x^3) = 1$, and ψ_6 sends other generators to 0, the linear combination will look like

$$y^2 = 4x^3 + a + bx + cy + dx^2 + exy,$$

where $a, b, c, d, e \in R$. By completing the square with y and then the cube with x, we can bring this expression into the Weierstrass form

$$y^2 = 4x^3 - g_2x - g_3, \tag{6}$$

where $g_2, g_3 \in R$. This eliminates any ambiguities in choices of x and y.

Functions x and y map $X \setminus A$ to $\mathbb{A}^2_{x,y} \times B$, and the image lies on a hypersurface given by equation (6). Projectivizing, we get a morphism

$$X \to \mathbb{P}^2_{x,y,z} \times B,$$

and the image lies on a hypersurface $\mathcal{E}$ with equation

$$y^2z = 4x^3 - g_2xz^2 - g_3z^3.$$

It remains to show that the induced morphism $\alpha : X \to \mathcal{E}$ is an isomorphism. It restricts to an isomorphism on every fiber $X_b \to \mathcal{E}_b$, in particular it is bijective. Let's use one of the versions of Zariski main theorem:

6.8.5. THEOREM. *A morphism of algebraic varieties $X \to \mathcal{E}$ with finite fibers can be factored as an open embedding $i : X \hookrightarrow U$ and a finite morphism $g : U \to \mathcal{E}$.*

In our case, U must be equal to X. This follows from the following basic property of proper morphisms:

6.8.6. LEMMA. *Let $f : X \to B$ be a proper morphism and suppose it can be factored as $X \to U \to B$. Then the image of X in U is closed.*

Thus $X \to \mathcal{E}$ is a finite morphism. To show that it is an isomorphism, it is enough to check that the map of local rings

$$\alpha^* : \mathcal{O}_{\mathcal{E},\alpha(p)} \to \mathcal{O}_{X,p}$$

is surjective (and hence an isomorphism) for every $p \in X$. Since we know that this is true on every fiber, we have

$$\mathcal{O}_{X,p} = \alpha^*(\mathcal{O}_{\mathcal{E},\alpha(p)}) + \mathfrak{m}_{B,b}\mathcal{O}_{X,p} = \alpha^*(\mathcal{O}_{\mathcal{E},\alpha(p)}) + \alpha^*(\mathfrak{m}_{\mathcal{E},\alpha(p)})\mathcal{O}_{X,p}.$$

Thus we can finish by Nakayama's lemma which applies, because $X \to \mathcal{E}$ is finite, and therefore $\mathcal{O}_{X,p}$ is a finitely generated $\mathcal{O}_{\mathcal{E},\alpha(p)}$-module.

§6.9. **Homework 3.**

Problem 1. (3 points) Let $\mathcal{M}$ be the set of isomorphism (=conjugacy) classes of invertible complex 3×3 matrices. (a) Describe $\mathcal{M}$ as a set. (b) Let's define the following moduli problem: a family over a variety X is a 3×3 matrix $A(x)$ with coefficients in $\mathcal{O}(X)$ such that $\det A(x) \in \mathcal{O}^*(X)$, i.e. $A(x)$ is invertible for any $x \in X$. Describe the corresponding moduli functor. (c) Show that this moduli functor has no coarse moduli space.

Problem 2. (1 point) Show that a coarse moduli space (of any moduli functor) is unique (if exists) up to an isomorphism. Show that a fine moduli space is always also a coarse moduli space.

Problem 3. (1 point) In Definition 5.4.3, explain how to pullback sections.

Problem 4. (2 points) Let f be a rational function on an algebraic curve C such that all zeros of f have multiplicity divisible by 3 and all zeros of $f - 1728$ have multiplicities divisible by 2. Show that $C \setminus \{f = \infty\}$ carries an elliptic fibration with j-invariant f.

Problem 5. (2 points) Let E be an elliptic curve and consider the trivial family $\mathbb{P}^1 \times E$ over $\mathbb{P}^1$. Now take two copies of this algebraic surface and glue them along $\{0\} \times E$ by identifying E with E by identity and along $\{\infty\} \times E$ by identifying E with E via a non-trivial involution. This gives an elliptic fibration over a reducible curve obtained by gluing two copies of $\mathbb{P}^1$ along 0 and ∞. Show that all elliptic curves in this family are isomorphic but the family is not trivial.

Problem 6. (3 points) Let (C, P) be an elliptic curve. Let $\Gamma \subset C$ be the ramification locus of $\phi_{|2P|}$. (a) Show that $\Gamma \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is precisely the 2-torsion subgroup in the group structure on C. (b) A level 2 structure on (C, P) is a choice of a basis $\{Q_1, Q_2\} \in \Gamma$ (considered as a $\mathbb{Z}_2$-vector space). Based on Theorem 5.4.5, describe families of elliptic curves with level 2 structure. Define a moduli functor of elliptic curves with a level 2 structure. Show that $\mathbb{P}^1_\lambda \setminus \{0, 1, \infty\}$ carries a family of elliptic curves with a level 2 structure such that every curve with a level 2 structure appears (uniquely) as one of the fibers.

Problem 7. (2 points) Is $\mathbb{P}^1_\lambda \setminus \{0, 1, \infty\}$ a fine moduli space for the moduli functor of the previous problem?

Problem 8. (3 points) Consider the family of cubic curves

$$C_a = \{x^3 + y^3 + z^3 + axyz = 0\} \subset \mathbb{P}^2$$

parametrized by $a \in \mathbb{A}^1$. (a) Find all a such that C_a is smooth and find its inflection points. (b) Compute j as a function on a and find all a such that C_a has a special automorphism group.

Problem 9. (3 points) Let (C, P) be an elliptic curve equipped with a morphism $C \to C$ of degree 2. By analyzing the branch locus ϕ_{2P}, show that the j-invariant of C has only 3 possible values and find these values.

Problem 10. (2 points) Let X be an algebraic variety. Recall that it is called normal if it can be covered by affine open sets $X = \cup_i U_i$ such that every $\mathcal{O}(U_i)$ is integrally closed in its field of fractions. Show that X is normal if and only if every birational finite map $Y \to X$ is an isomorphism.

Problem 11. (2 points) Let $f : X \to Y$ be a finite morphism of algebraic varieties and let $g : X \to Z$ be an arbitrary morphism. Show that $f \times g : X \to Y \times Z$ is a finite morphism of algebraic varieties.

Problem 12. (1 point). Prove Claim 6.7.3

Problem 13. (1 point). Prove Corollary 6.7.6

Problem 14. (2 points). Prove Lemma 6.8.1

Problem 15. (2 points). Prove Lemma 6.8.2

Problem 16. (2 points). Prove Lemma 6.8.3.

Problem 17. (2 points). Prove Claim 6.8.4.

§7. **Invariants of finite groups**

In the second half of the course we are going to study invariant theory and orbit spaces more systematically. We will start with a finite group G acting linearly[18] on a vector space V and discuss the quotient morphism $\pi : V \to V/G$ to the orbit space (or the quotient space) V/G. There are several reasons to isolate this case:

- The quotient space V/G is typically singular. Singularities of this form (called *quotient* or *orbifold* singularities) form are very common.
- Globally, moduli spaces can often be constructed as quotients X/G of algebraic varieties by reductive group actions. Most of the results generalize to this set-up but new subtleties arise.
- Locally, near some point p, moduli spaces can often be modeled on the quotient V/G, where V is a vector space (a *versal deformation space* of the geometric object that corresponds to p) and G is an automorphism group of this object, which is often finite.

[18] Recall that a linear action is given by a homomorphism $G \to \mathrm{GL}(V)$. In this case we also say that V is a *representation* of G.

§7.1. **First examples.**

7.1.1. EXAMPLE. Let $G = S_n$ be a symmetric group acting on $\mathbb{C}^n$ by permuting the coordinates. Recall that our recipe for computing the orbit space calls for computing the ring of invariants

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n}.$$

By the classical *theorem on symmetric functions*, this ring of S_n-invariant polynomials is generated by *elementary symmetric polynomials*

$$\sigma_1 = x_1 + \ldots + x_n,$$

$$\ldots$$

$$\sigma_k = \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k},$$

$$\ldots$$

$$\sigma_n = x_1 \ldots x_n.$$

Thus the candidate for the quotient map is

$$\pi : \mathbb{A}^n \to \mathbb{A}^n, \quad (x_1, \ldots, x_n) \mapsto (\sigma_1, \ldots, \sigma_n).$$

This map is surjective and its fibers are the S_n-orbits. Indeed, we can recover $x_1, \ldots, x_n$ (up to permutation) from $\sigma_1, \ldots, \sigma_n$ because they are roots of the polynomial $T^n - \sigma_1 T^{n-1} + \ldots + (-1)^n \sigma_n = 0$.

7.1.2. EXAMPLE. A linear action of a finite group G on $\mathbb{C}$ is given by a *character*, a homomorphism $G \to \mathbb{C}^*$. Its image is a subgroup μ_d of d-th roots of unity. Let ζ be the primitive d-th root of 1. Thus

$$\mathbb{C}[x]^G = \mathbb{C}[x]^{\mu_d} = \mathbb{C}[x^d].$$

It is clear that x^d separates orbits because every non-zero orbit has d elements $x, \zeta x, \ldots, \zeta^{d-1}x$. The quotient morphism in this case is just

$$\pi : \mathbb{A}^1 \to \mathbb{A}^1, \quad x \mapsto x^d.$$

7.1.3. EXAMPLE. Let $\mathbb{Z}^2$ act on $\mathbb{A}^2$ by $(x, y) \mapsto (-x, -y)$. Invariant polynomials are just polynomials of even degree, and so

$$\mathbb{C}[x, y]^{\mathbb{Z}_2} = \mathbb{C}[x^2, y^2, xy].$$

The quotient morphism is

$$\pi : \mathbb{A}^2 \to \mathbb{A}^3, \quad (x, y) \mapsto (x^2, y^2, xy).$$

It is clear that invariants separate orbits. It is also clear that the quotient map is surjective onto the quadratic cone

$$(uv = w^2) \subset \mathbb{A}^3.$$

The quadratic cone is the simplest *du Val singularity* called A_1.

§7.2. **Quotient singularity $\frac{1}{r}(1,a)$ and continued fractions.** Computing the algebra of invariants can be quite complicated but things are much easier if the group is Abelian. Let's look at an amusing example of a *cyclic quotient singularity* $\frac{1}{r}(1,a)$. It is defined as follows: consider the action of μ_r on $\mathbb{C}^2$, where the primitive generator $\zeta \in \mu_r$ acts via the matrix

$$\begin{bmatrix} \zeta & 0 \\ 0 & \zeta^a \end{bmatrix}$$

How to compute the algebra of invariants $\mathbb{C}[x,y]^{\mu_r}$? Notice that the group acts on monomials diagonally as follows:

$$\zeta \cdot x^i y^j = \zeta^{-i-ja} x^i y^j.$$

So a monomial $x^i y^j$ is contained in $\mathbb{C}[x,y]^{\mu_r}$ if and only if

$$i + ja \equiv 0 \mod r.$$

7.2.1. EXAMPLE. Consider $\frac{1}{r}(1, r-1)$. Notice that this is the only case when $\mu_r \subset \mathrm{SL}_2$. The condition on invariant monomials is that

$$i \equiv j \mod r$$

(draw). We have

$$\mathbb{C}[x,y]^{\mu_r} = \mathbb{C}[x^r, xy, y^r] = \mathbb{C}[U,V,W]/(V^r - UW).$$

We see that the singularity $\frac{1}{r}(1, r-1)$, also known as the A_{r-1}-singularity, is a hypersurface in $\mathbb{A}^3$ given by the equation $V^r = UW$.

7.2.2. EXAMPLE. Consider $\frac{1}{r}(1,1)$. The condition on invariant monomials is

$$i + j \equiv 0 \mod r$$

(draw). We have

$$\mathbb{C}[x,y]^{\mu_r} = \mathbb{C}[x^r, x^{r-1}y, x^{r-2}y^2, \ldots, y^r].$$

The quotient morphism in this case is

$$\mathbb{A}^2 \to \mathbb{A}^{r+1}, \quad (x,y) \mapsto (x^r, x^{r-1}y, x^{r-2}y^2, \ldots, y^r).$$

The singularity $\frac{1}{r}(1,1)$ is a cone over the *rational normal curve*

$$[x^r : x^{r-1}y : x^{r-2}y^2 : \ldots : y^r] \subset \mathbb{P}^{r-1}.$$

7.2.3. DEFINITION. Let $r > b > 0$ be coprime integers. The following expression is called the Hirzebruch–Jung continued fraction:

$$r/b = a_1 - \cfrac{1}{a_2 - \frac{1}{a_3 - \ldots}} = [a_1, a_2, ..., a_k].$$

Hirzebruch–Young continued fractions are similar to ordinary continued fractions but have minuses instead of pluses. For example,

$$5/1 = [5],$$

$$5/4 = [2,2,2,2],$$

$$5/2 = [3,2].$$

Here's the result:

7.2.4. THEOREM. *Suppose μ_r acts on $\mathbb{A}^2$ with weights 1 and a, where $(a,r)=1$. Let $\frac{r}{r-a}=[a_1,a_2,...,a_k]$ be the Hirzebruch–Jung continued fraction expansion*[19]*. Then $\mathbb{C}[x,y]^{\mu_r}$ is generated by*

$$f_0 = x^r,\ f_1 = x^{r-a}y,\ f_2,\ \ldots,\ f_k,\ f_{k+1} = y^r,$$

where the monomials f_i are uniquely determined by the following equations:

$$f_{i-1}f_{i+1} = f_i^{a_i} \quad \text{for} \quad i = 1,\ldots,k. \tag{7.2.5}$$

7.2.6. EXAMPLE. An example is in Figure 2.

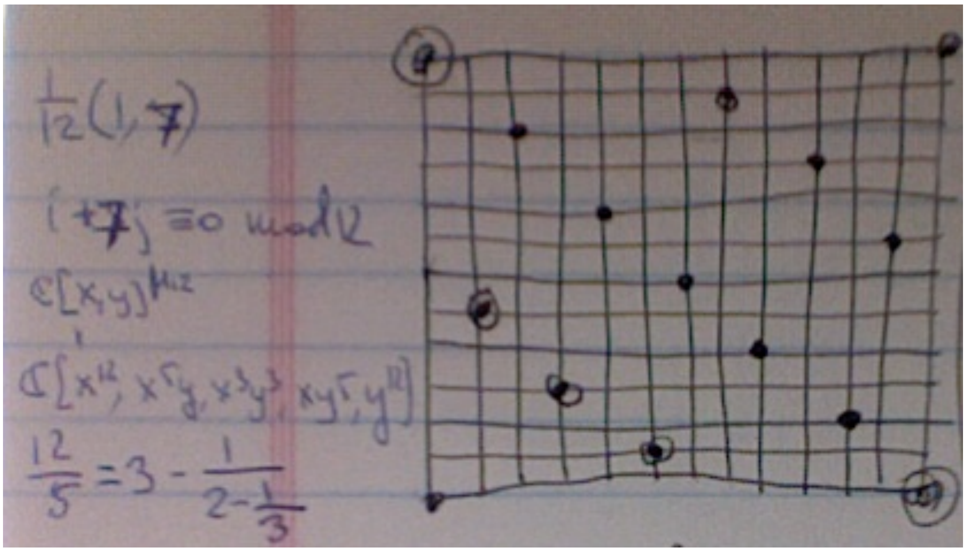


FIGURE 2

7.2.7. REMARK. We see that the codimension of $\mathbb{A}^2/\!\!/\Gamma$ in the ambient affine space $\mathbb{A}^{k+2}$ is equal to the length of the Hirzebruch–Young continued fraction. This is a good measure of the complexity of the singularity. From this perspective, $\frac{1}{r}(1,1)$ (the cone over a rational normal curve) is the most complicated singularity: the Hirzebruch–Young continued fraction

$$r/(r-1) = [2,2,2,\ldots,2] \quad (r-1 \text{ times})$$

uses the smallest possible denominators. It is analogous to the standard continued fraction of the ratio of two consecutive Fibonacci numbers, which has only 1's as denominators.

Proof of Theorem 7.2.4. Invariant monomials in $\mathbb{C}[x,y]^{\mu_r}$ are indexed by vectors of the first quadrant

$$\{(i,j)\mid i,j\geq 0\}\subset\mathbb{Z}^2$$

which are in the lattice

$$L = \{(i,j)\,|\, i+aj\equiv 0 \mod r\}\subset\mathbb{Z}^2.$$

[19]Notice that we are expanding $r/(r-a)$ and not r/a.

This intersection is a semigroup and we have to find its generators. We note for future use that L contains the sublattice $r\mathbb{Z}^2$ and can be described as the lattice generated by

$$\begin{bmatrix}0\\r\end{bmatrix}, \quad \begin{bmatrix}r-a\\1\end{bmatrix}, \quad \begin{bmatrix}r\\0\end{bmatrix}.$$

The semigroup of invariant monomials is generated by $\begin{bmatrix}r\\0\end{bmatrix}$, $\begin{bmatrix}0\\r\end{bmatrix}$, and by the monomials inside the square $\{(i,j)\,|\,0<i,j<r\}$, which are precisely the monomials

$$((r-a)j \mod r, \quad j), \quad j=1,\dots,r-1.$$

Of course many of these monomials are unnecessary. The first monomial in the square that we actually need is $\begin{bmatrix}r-a\\1\end{bmatrix}$. Now take multiples of $\begin{bmatrix}r-a\\1\end{bmatrix}$. The next generator will occur when $(r-a)j$ goes over r, i.e. when

$$j=\lceil\frac{r}{r-a}\rceil=a_1$$

(in the Hirzebruch–Young continued fraction expansion for $\frac{r}{r-a}$). Since

$$(r-a)a_1 \mod r=(r-a)a_1-r,$$

the next generator is

$$\begin{bmatrix}(r-a)a_1-r\\a_1\end{bmatrix}.$$

Notice that so far this confirms our formula (7.2.5). We are interested in the remaining generators of L inside the $r\times r$ square. Notice that they all lie above the line spanned by $\begin{bmatrix}r-a\\1\end{bmatrix}$. So we can restate our problem: find generators of the semigroup obtained by intersecting L with points lying in the first quadrant and above the line spanned by $\begin{bmatrix}r-a\\1\end{bmatrix}$.

Next we notice that

$$\begin{bmatrix}r\\0\end{bmatrix}=a_1\begin{bmatrix}r-a\\1\end{bmatrix}-\begin{bmatrix}(r-a)a_1-r\\a_1\end{bmatrix}.$$

It follows that lattice L is also spanned by $\begin{bmatrix}0\\r\end{bmatrix}$, $\begin{bmatrix}(r-a)a_1-r\\a_1\end{bmatrix}$, and $\begin{bmatrix}r-a\\1\end{bmatrix}$. We are interested in generators of the semigroup obtained by intersecting this lattice with the "angle" spanned by vectors $\begin{bmatrix}0\\r\end{bmatrix}$ and $\begin{bmatrix}r-a\\1\end{bmatrix}$.

Consider the linear transformation $\psi:\mathbb{R}^2\to\mathbb{R}^2$ such that

$$\psi\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\-\frac{1}{r}\end{bmatrix}, \quad \psi\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\\frac{r-a}{r}\end{bmatrix}.$$

Then we compute

$$\psi\begin{bmatrix}0\\r\end{bmatrix}=\begin{bmatrix}0\\r-a\end{bmatrix}, \quad \psi\begin{bmatrix}r-a\\1\end{bmatrix}=\begin{bmatrix}r-a\\0\end{bmatrix}, \quad \psi\begin{bmatrix}(r-a)a_1-r\\a_1\end{bmatrix}=\begin{bmatrix}(r-a)a_1-r\\1\end{bmatrix}.$$

So we get the same situation as before with a smaller lattice. Notice that if

$$\frac{r}{r-a} = a_1 - \frac{1}{q}$$

then

$$q = \frac{r-a}{(r-a)a_1 - r},$$

so we will recover all denominators in the Hirzebruch–Jung continued fraction as we proceed inductively. $\square$

§7.3. **Finite generation.**

7.3.1. THEOREM. *Let G be a finite group acting linearly on a vector space V. Then the algebra of invariants $\mathcal{O}(V)^G$ is finitely generated.*

7.3.2. REMARK. Since G acts on V, it also acts on the polynomial algebra $k[V] = \mathcal{O}(V)$. The right way to do it is as follows: if $f \in k[V]$ then

$$(g \cdot f)(x) = f(g^{-1}x).$$

This is how the action on functions is defined: if you try g instead of g^{-1}, the group action axiom will be violated.

We split the proof into Lemma 7.3.4 and Lemma 7.3.5. The second Lemma will be reused later to prove finite generation for reductive groups.

7.3.3. DEFINITION. Let G be a group acting on a $\mathbb{C}$-algebra A by automorphisms. A linear map

$$R:\ A \to A^G$$

is called a *Reynolds operator* if

- $R(1) = 1$ and
- $R(fg) = fR(g)$ for any $f \in A^G$ and $g \in A$.

In particular, the Reynolds operator is a *projector* onto A^G:

$$R(f) = R(f \cdot 1) = fR(1) = f \quad \text{for every} \quad f \in A^G.$$

7.3.4. LEMMA. *The Reynolds operator $R:\ A \to A^G$ exists if G is a finite group.*

Proof. We define the Reynolds operator R as an averaging operator:

$$R(a) = \frac{1}{|G|}\sum_{g\in G} g \cdot a.$$

It is clear that both axioms of the Reynolds operator are satisfied. This works over any field as soon as its characteristic does not divide $|G|$. $\square$

7.3.5. LEMMA. *Let G be a group acting linearly on a vector space V and possessing a Reynolds operator $R:\ \mathcal{O}(V) \to \mathcal{O}(V)^G$. Then $\mathcal{O}(V)^G$ is finitely generated.*

Proof. This ingenious argument belongs to Hilbert. First of all, the action of G on $\mathcal{O}(V)$ preserves degrees of polynomials. So $\mathcal{O}(V)^G$ is a graded subalgebra of $\mathcal{O}(V)$. Let $I \subset \mathcal{O}(V)$ be the ideal generated by homogeneous invariant polynomials $f \in \mathcal{O}(V)^G$ of *positive* degree. By the Hilbert's basis theorem (proved in the same paper as the argument we are discussing), I is finitely generated by homogeneous invariant polynomials $f_1, \ldots, f_r$ of positive degrees. We claim that the same polynomials generate $\mathcal{O}(V)^G$ as an algebra, i.e. any $f \in \mathcal{O}(V)^G$ is a polynomial in $f_1, \ldots, f_r$. Without loss of generality, we can assume that f is homogeneous and argue by induction on its degree. We have

$$f = \sum_{i=1}^{r} a_i f_i,$$

where $a_i \in \mathcal{O}(V)$. Now apply the Reynolds operator:

$$f = R(f) = \sum_{i=1}^{r} R(a_i) f_i.$$

Each $R(a_i)$ is an invariant polynomial, and if we let b_i be its homogeneous part of degree $\deg f - \deg f_i$, then we still have

$$f = \sum_{i=1}^{r} b_i f_i.$$

By inductive assumption, each b_i is a polynomial in $f_1, \ldots, f_r$. This shows the claim. □

§7.4. **Properties of quotients.**

7.4.1. DEFINITION. An algebraic group G is a group and an algebraic variety such that both the multiplication map $G \times G \to G$, $(g, g') \mapsto gg'$ and the inverse map $G \to G$, $g \mapsto g^{-1}$ are morphisms of algebraic varieties. Likewise, an action of an algebraic group on an algebraic variety is called algebraic if an action map $G \times X \to X$ is a morphism. All actions we consider will be algebraic.

7.4.2. DEFINITION. Let G be an algebraic group acting algebraically on an affine variety X. Suppose $\mathcal{O}(X)^G$ is finitely generated. The *categorical quotient $X/\!\!/G$* is defined as an affine variety such that

$$\mathcal{O}(X/\!\!/G) = \mathcal{O}(X)^G$$

and the quotient morphism $\pi : X \to X/\!\!/G$ is defined as a morphism such that the pull-back of regular functions given by inclusion

$$\pi^* : \mathcal{O}(X)^G \subset \mathcal{O}(X).$$

Concretely, choose a system of generators $f_1, \ldots, f_r$ of $\mathcal{O}(X)^G$ and write

$$\mathbb{C}[z_1, \ldots, z_r]/I \simeq \mathcal{O}(V)^G, \quad z_i \mapsto f_i.$$

We define $X/\!\!/G$ as an affine subvariety in $\mathbb{A}^r$ given by the ideal I and let

$$\pi : X \to X/\!\!/G \hookrightarrow \mathbb{A}^r, \quad v \mapsto f_1(v), \ldots, f_r(v).$$

A different system of generators gives an isomorphic affine variety.

For example, we can define the quotient of a vector space by a finite group action, due to the finite generation theorem. To show that this definition is reasonable, let's check two things:

- Fibers of π are exactly the orbits, i.e. any two orbits are separated by polynomial invariants.
- All points of $V/\!\!/G$ correspond to orbits, i.e. π is surjective.

7.4.3. THEOREM. *Any two G-orbits are separated by invariant polynomials.*

Proof. The proof relies significantly on finiteness of the group. Take two orbits, $S_1, S_2 \subset V$. Since they are finite, there exists a polynomial $f \in \mathcal{O}(V)$ such that $f|_{S_1} = 0$ and $f|_{S_2} = 1$. Then the average

$$F = R(f) = \frac{1}{|G|} \sum_{g \in G} g \cdot f$$

is an invariant polynomial but we still have $F|_{S_1} = 0$ and $F|_{S_2} = 1$. $\square$

Now surjectivity:

7.4.4. THEOREM. *Let G be an algebraic group acting on an affine variety X and possessing a Reynolds operator $\mathcal{O}(X) \to \mathcal{O}(X)^G$. Suppose $\mathcal{O}(X)^G$ is finitely generated. Then the quotient map $\pi : X \to X/\!\!/G$ is surjective.*

7.4.5. LEMMA. *A regular map $\pi : X \to Y$ of affine varieties is surjective if and only if $\mathcal{O}(X)\pi^*(\mathfrak{n}) \neq \mathcal{O}(X)$ for any maximal ideal $\mathfrak{n} \subset \mathcal{O}(Y)$.*

Proof. For every point $y \in Y$ (i.e. a maximal ideal $\mathfrak{n} \subset \mathcal{O}(Y)$) we have to show existence of a point $x \in X$ (i.e. a maximal ideal $\mathfrak{m} \subset \mathcal{O}(X)$) such that $f(x) = y$, equivalently $(\pi^*)^{-1}(\mathfrak{m}) = \mathfrak{n}$. So we have to show that there exists a maximal ideal $\mathfrak{m} \subset \mathcal{O}(X)$ that contains $\pi^*(\mathfrak{n})$. The image of an ideal under homomorphism is not necessarily an ideal, so the actual condition is that the ideal $\mathcal{O}(X)\pi^*(\mathfrak{n})$ is a proper ideal. $\square$

Proof of Theorem 7.4.4. Let $\mathfrak{n} \subset \mathcal{O}(X)^G$ be a maximal ideal. We have to show that

$$\mathcal{O}(X)\mathfrak{n} \neq \mathcal{O}(X)$$

(recall that a pull-back of functions for the quotient map $\pi : X \to X/\!\!/G$ is just the inclusion $\mathcal{O}(X)^G \subset \mathcal{O}(X)$). Arguing by contradiction, suppose that $\mathcal{O}(X)\mathfrak{n} = \mathcal{O}(X)$. Then we have

$$\sum a_i f_i = 1,$$

where $a_i \in \mathcal{O}(X)$ and $f_i \in \mathfrak{n}$. Applying the Reynolds operator, we see that

$$\sum b_i f_i = 1,$$

where $b_i \in \mathcal{O}(X)^G$. But $\mathfrak{n}$ is a proper ideal of $\mathcal{O}(X)^G$, contradiction. $\square$

This argument only uses the existence of a Reynolds operator, but for finite groups we can do a little bit better:

7.4.6. LEMMA. *Let G be a finite group acting on an affine variety X. Then $\mathcal{O}(X)$ is integral over $\mathcal{O}(X)^G$. In other words, the quotient morphism $\pi : V \to V/\!\!/G$ is finite (and in particular surjective) for finite groups.*

Proof. Indeed, any element $f \in \mathcal{O}(X)$ is a root of the monic polynomial

$$\prod_{g \in G} (T - g \cdot f).$$

Coefficients of this polynomial are in $\mathcal{O}(X)^G$ (by Vieta formulas). $\square$

7.4.7. REMARK. This argument relies on surjectivity of finite morphisms. Interestingly, this fact can be demonstrated in the spirit of Theorem 7.4.4! Indeed, suppose $\pi : X \to Y$ is a finite morphism of affine varieties and Y is normal. Algebraically, the inclusion $\mathcal{O}(Y) \hookrightarrow \mathcal{O}(X)$ is integral and $\mathcal{O}(Y)$ is integrally closed in its field of fractions $k(Y)$. The extension of fields $k(Y) \hookrightarrow k(X)$ in this case is finite, and hence we have a $k(Y)$-linear trace map $\mathrm{Tr} : k(X) \to k(Y)$ which sends every element of $k(X)$ to the first coefficient of its minimal polynomial (with a minus sign), i.e. to the sum of roots of the minimal polynomial. We claim that

$$\mathrm{Tr}(\mathcal{O}(X)) = \mathcal{O}(Y).$$

Indeed, all roots of the minimal polynomial of $\alpha \in \mathcal{O}(X)$ are integral over $\mathcal{O}(Y)$, hence $\mathrm{Tr}(\alpha)$ is also integral over $\mathcal{O}(Y)$, but it also belongs to $k(Y)$. Since $\mathcal{O}(Y)$ is integrally closed, in fact $\mathrm{Tr}(\alpha) \in \mathcal{O}(Y)$. Notice that the trace map has all the properties of the Reynolds operator: it is a $\mathcal{O}(Y)$-linear projector onto $\mathcal{O}(Y)$. Hence the argument from the proof of Theorem 7.4.4 applies.

7.4.8. REMARK. In Remark 7.4.7, the extension $k(Y) \hookrightarrow k(X)$ doesn't have to be a Galois extension.

§7.5. **Chevalley–Shephard–Todd theorem.** When is the algebra of invariants a polynomial algebra? The answer is pretty but hard to prove:

7.5.1. THEOREM. *Let G be a finite group acting linearly and faithfully on $\mathbb{C}^n$. The algebra of invariants $\mathbb{C}[x_1, \ldots, x_n]^G$ is a polynomial algebra if and only if G is generated by pseudo-reflections, i.e. by elements $g \in G$ such that the subspace of fixed points $\{v \in \mathbb{C}^n \mid gv = v\}$ has codimension* 1.

In other words, g is a pseudo-reflection if and only if its matrix is some basis is equal to $\mathrm{diag}[\zeta, 1, 1, \ldots, 1]$, where ζ is a root of unity. If $\zeta = -1$ then g is called a *reflection*. For example, if S_n acts on $\mathbb{C}^n$ by permuting coordinates then any transposition (ij) acts as a reflection with the mirror $x_i = x_j$. Further examples of groups generated by reflections are *Weyl groups* of *root systems*. On the other hand, the standard action of μ_d on $\mathbb{C}$ is generated by a pseudo-reflection $z \mapsto \zeta z$, which is not a reflection (for $d > 2$) but the algebra of invariants is still polynomial. The action of $\mathbb{Z}_2$ on $\mathbb{C}^2$ by ± 1 is not a pseudoreflection (a fixed subspace has codimension 2).

Groups generated by pseudo-reflections were classified by Shephard and Todd. There is one infinite family which depends on 3 integer parameters (and includes S_n) and 34 exceptional cases.

Sketch of the proof in one direction. Suppose $\mathbb{C}[x_1, \ldots, x_n]^G$ is a polynomial algebra. Then the quotient morphism is the morphism $\pi : X \to Y$, where both X and Y are isomorphic to $\mathbb{C}^n$. By Theorem 7.4.3, π separates orbits. Let $U \subset X$ be the complement of the union of subspaces of fixed points of all elements of G that are <u>not</u> pseudo-reflections. Then U is G-invariant and its complement has codimension at least 2. Let $V = \pi(U) \subset Y$.

We can endow complex algebraic varieties X and Y with Euclidean rather than Zariski topology, i.e. view them both as a complex manifold $\mathbb{C}^n$. Since the complements of V and U in $\mathbb{C}^n$ have codimension at least 2,

$$\pi_1(U) = \pi_1(V) = 0.$$

Then it is clear that the action of G on U can not be free: a simply connected manifold does not admit a non-trivial covering space! In our case, having a fixed point in U is equivalent to being a pseudoreflection, because fixed points of all other elements were removed by construction of U.

To finish the proof, we can use a lemma from geometric group theory:

7.5.2. LEMMA. *Let Γ be a discrete group of homeomorphisms of a linearly connected Hausdorff topological space U. Suppose the quotient space U/Γ is simply connected. Then Γ is generated by elements having a fixed point in U.*

It is instructive to re-examine a quotient $\mathbb{C}^2/\mu_2$ by the action $(x, y) \mapsto (-x, -y)$ which is not generated by a pseudo-refection. In this case the quotient is a quadratic cone $Y = (AB = C^2) \subset \mathbb{A}^3$ and the map

$$\mathbb{C}^2 \setminus (0,0) \to Y \setminus (0,0,0)$$

is the universal covering space. Thus $\pi_1(Y) = \pi_1(Y \setminus (0,0,0)) = \mu_2$. □

§7.6. **E_8-singularity.**

7.6.1. DEFINITION. Let Γ be a finite subgroup of SL_2. The quotient singularity $\mathbb{C}^2/\Gamma$ is called *du Val* or *ADE* or *canonical* singularity.

Three of the finite subgroups of SO_3 are groups of rotations of platonic solids. For example, A_5 is a group of rotations of the icosahedron (or dodecahedron). Thus A_5 acts on the circumscribed sphere of the icosahedron. This action is obviously *conformal* (preserves oriented angles), and so if we think about the sphere S^2 as the Riemann sphere $\mathbb{P}^1$ (by stereographic projection), we get an embedding $A_5 \subset \mathrm{PSL}_2$ (since it is proved in complex analysis that conformal maps are holomorphic). The preimage of A_5 in SL_2 is called the *binary icosahedral group* Γ. It has $2 \times 60 = 120$ elements. The orbit space $\mathbb{C}^2/\Gamma$ is called an E_8 du Val singularity.

In order to describe the E_8 singularity, we have to compute $\mathbb{C}[x, y]^\Gamma$. There is a miraculously simple way to write down some invariants using three special orbits of A_5 on $\mathbb{P}^1$:

- 20 vertices of the icosahedron,
- 12 midpoints of faces (vertices of the dual dodecahedron),
- and 30 midpoints of the edges.

Let f_{12}, f_{20}, and f_{30} be polynomials in x, y (homogeneous coordinates on $\mathbb{P}^1$) that factor into linear forms that correspond to these special points. These polynomials are defined uniquely up to a scalar multiple. We claim that these polynomials are invariant. Since Γ permutes their roots, they are clearly *semi-invariant*, i.e. any $\gamma \in \Gamma$ can only multiply them by a scalar, which will be a character of Γ. Since they all have even degree, the element $-1 \in \Gamma$ does not change these polynomials. But $\Gamma/\{\pm 1\} \simeq A_5$ is a simple group, hence has no characters at all, hence the claim.

7.6.2. THEOREM. $\mathbb{C}[x, y]^\Gamma = \mathbb{C}[f_{12}, f_{20}, f_{30}] \simeq \mathbb{C}[U, V, W]/(U^5 + V^3 + W^2)$.

Proof. Let's try to prove this using as few explicit calculations as possible. The key is to analyze a chain of algebras

$$\mathbb{C}[x, y] \supset \mathbb{C}[x, y]^\Gamma \supset \mathbb{C}[f_{12}, f_{20}, f_{30}] \supset \mathbb{C}[f_{12}, f_{20}].$$

7.6.3. CLAIM. $\mathbb{C}[f_{12}, f_{20}] \subset \mathbb{C}[x,y]$ *(and hence all other inclusions in the chain) is an integral extension.*

Proof. In other words, a regular map

$$\mathbb{A}^2 \to \mathbb{A}^2, \quad (x,y) \mapsto (f_{12}, f_{20}) \tag{7.6.4}$$

is a finite morphism. Since f_{12} and f_{20} have no common zeros in $\mathbb{P}^1$, Nullstellensatz implies that

$$\sqrt{(f_{12}, f_{20})} = (x,y),$$

i.e. $x^n, y^n \in (f_{12}, f_{20})$ for some large n. Thus $\mathbb{C}[x,y]$ is finitely generated by monomials $x^i y^j$ for $i, j < n$ as a module over $\mathbb{C}[f_{12}, f_{20}]$. ☐

Now let's consider the corresponding chain of fraction fields

$$\mathbb{C}(x,y) \supset \mathrm{Quot}\left(\mathbb{C}[x,y]^\Gamma\right) \supset \mathbb{C}(f_{12}, f_{20}, f_{30}) \supset \mathbb{C}(f_{12}, f_{20}). \tag{7.6.5}$$

Here are some basic definitions, and a fact.

7.6.6. DEFINITION. Let $f : X \to Y$ be a dominant map of algebraic varieties of the same dimension. It induces an embedding of fields

$$f^* : \mathbb{C}(Y) \subset \mathbb{C}(X).$$

We define *the degree* of f as follows[20]:

$$\deg f = [\mathbb{C}(X) : \mathbb{C}(Y)].$$

7.6.7. DEFINITION. An affine variety X is called *normal* if $\mathcal{O}(X)$ is integrally closed in $\mathbb{C}(X)$. More generally, an algebraic variety is called normal if every local ring $\mathcal{O}_{X,x}$ is integrally closed in $\mathbb{C}(X)$.

7.6.8. THEOREM. *Let $f : X \to Y$ be a finite map of algebraic varieties. Suppose that Y is normal. Then any fiber $f^{-1}(y)$ has at most $\deg f$ points. Let*

$$U = \{y \in Y \quad | \quad f^{-1}(y) \text{ has exactly } \deg f \text{ points } \}.$$

Then U is open and non-empty.

Let's see how to apply this theorem in our situation. First of all, any UFD is integrally closed, hence $\mathbb{C}[x,y]$ and $\mathbb{C}[f_{12}, f_{20}]$ are integrally closed.

Secondly, $\mathbb{C}[x,y]^\Gamma$ is integrally closed. Indeed, if $f \in \mathrm{Quot}\,\mathbb{C}[x,y]^\Gamma$ is integral over $\mathbb{C}[x,y]^\Gamma$ then it is also integral over $\mathbb{C}[x,y]$, but the latter is integrally closed, hence $f \in \mathbb{C}[x,y]$, and so $f \in \mathbb{C}[x,y]^\Gamma$.

It follows that $[\mathbb{C}(x,y) : \mathrm{Quot}\,\mathbb{C}[x,y]^\Gamma] = 120$ because fibers of the quotient morphism are orbits and the general orbit has 120 points.[21] The fibers of the map (7.6.4) are level curves of f_{12} and f_{20}, and therefore contain at most 240 points by Bezout theorem. One can show geometrically that general fibers contain exactly 240 points or argue as follows: if this is not the case then we can conclude from (7.6.5) that $\mathrm{Quot}\,\mathbb{C}[x,y]^\Gamma = \mathbb{C}(f_{12}, f_{20})$ and therefore $f_{30} \in \mathbb{C}(f_{12}, f_{20})$. But f_{30} is integral over $\mathbb{C}[f_{12}, f_{20}]$, and the latter

[20] The dimension is equal to the transcendence degree of the field of functions, so $\mathbb{C}(X)/\mathbb{C}(Y)$ is an algebraic extension, hence finite (because $\mathbb{C}(X)$ is finitely generated).

[21] if we knew that the second field is $\mathbb{C}(x,y)^\Gamma$, the formula follows from Galois theory.

is integrally closed, so $f_{30} \in \mathbb{C}[f_{12}, f_{20}]$. But this can't be true because of the degrees! So in fact we have

$$\mathrm{Quot}\,\mathbb{C}[x,y]^\Gamma = \mathbb{C}(f_{12}, f_{20}, f_{30}) \quad \text{and} \quad [\mathbb{C}(f_{12}, f_{20}, f_{30}) : \mathbb{C}(f_{12}, f_{20})] = 2.$$

The latter formula implies that the minimal polynomial of f_{30} over $\mathbb{C}(f_{12}, f_{20})$ has degree 2. The second root of this polynomial satisfies the same integral dependence as f_{20}, and therefore all coefficients of the minimal polynomial are integral over $\mathbb{C}[f_{12}, f_{20}]$, by Vieta formulas. But this ring is integrally closed, and therefore all coefficients of the minimal polynomial are in fact in $\mathbb{C}[f_{12}, f_{20}]$. So we have an integral dependence equation of the form $f_{30}^2 + af_{30} + b = 0$, where $a, b \in \mathbb{C}[f_{12}, f_{20}]$. Looking at the degrees, there is only one way to accomplish this (modulo multiplying f_{12}, f_{20}, and f_{30} by scalars), namely

$$f_{12}^5 + f_{20}^3 + f_{30}^2 = 0.$$

Note that we can't have, say, $f_{20}^3 + f_{30}^2 = 0$ because these polynomials are coprime.

It remains to prove that $\mathbb{C}[x,y]^\Gamma = \mathbb{C}[f_{12}, f_{20}, f_{30}]$. Since they have the same quotient field, it is enough to show that the latter algebra is integrally closed, and this follows from the following extremely useful theorem that we are not going to prove, see [**?**, page 198]. $\square$

7.6.9. THEOREM. *Let $X \subset \mathbb{A}^n$ be an irreducible affine hypersurface*[22] *such that its singular locus has codimension at least 2. Then X is normal.*

For example, a surface $S \subset \mathbb{A}^3$ with isolated singularities is normal. It is important that S is a surface in $\mathbb{A}^3$, it is easy to construct examples of non-normal surfaces with isolated singularities in $\mathbb{A}^4$.

Proof of Theorem 7.6.8. Let $y \in Y$ and choose a function $a \in \mathcal{O}(X)$ that takes different values on points in $f^{-1}(y)$. The minimal polynomial $F(T)$ of a over $\mathbb{C}(Y)$ has degree at most $\deg f$. Since Y is normal, all coefficients of the minimal polynomial are in fact in $\mathcal{O}(Y)$. Thus $f^{-1}(y)$ has at most n points. Since we are in characteristic 0, the extension $\mathbb{C}(X)/\mathbb{C}(Y)$ is separable, and hence has a primitive element. Let $a \in \mathcal{O}(X)$ be an element such that its minimal polynomial (=integral dependence polynomial) has degree n:

$$F(T) = T^n + b_1 T^{n-1} + \ldots + b_n, \quad b_i \in \mathcal{O}(Y).$$

Let $D \in \mathcal{O}(Y)$ be the discriminant of $F(T)$ and let $U = \{y \in Y \,|\, D \neq 0$ be the corresponding principal open set. We claim that f has exactly n different fibers over any point of U. Indeed, the inclusion $\mathcal{O}(Y)[a] \subset \mathcal{O}(X)$ is integral, hence induces a finite map, hence induces a surjective map. But over a point $y \in Y$, the fiber of

$$\mathrm{MaxSpec}\,\mathcal{O}(Y)[a] = \{(y,t) \in Y \times \mathbb{A}^1 \mid t^n + b_1(y)t^{n-1} + \ldots + b_n(y) = 0\}$$

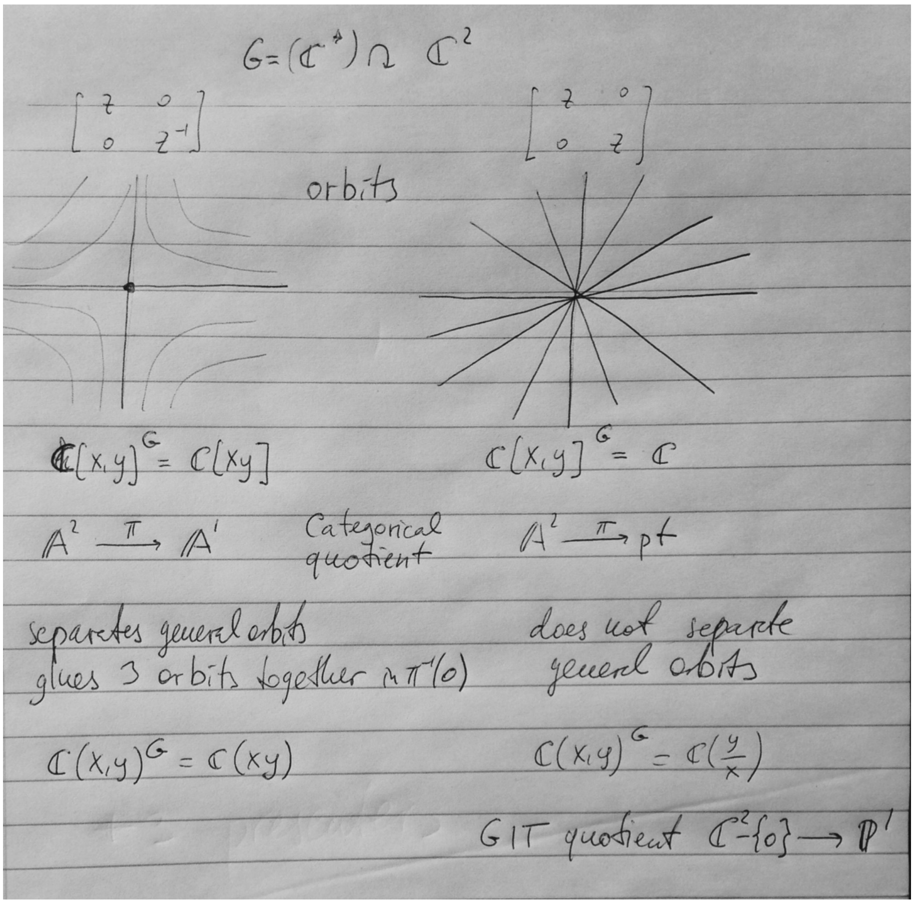
is just given by the roots of the minimal polynomial, and hence consists of n points. Thus the fiber $f^{-1}(y)$ also has n points. $\square$

[22]More generally, X can be a complete intersection, or Cohen–Macaulay, or just satisfy Serre's property S_2

§8. **Quotients by algebraic groups**

Let G be an algebraic group acting algebraically on an algebraic variety X, i.e. the action map $G \times X \to X$ is a morphism of algebraic varieties.

8.0.10. EXAMPLE.



§8.1. **Properties of actions.** We will use without proof various properties of algebraic actions (some of them are homework exercises).

8.1.1. THEOREM. *For every $x \in X$, the stabilizer G_x is a (Zariski) closed subgroup of G and the orbit Gx is a locally closed (i.e. open in its Zariski closure) subset of X. We have the formula*

$$\dim Gx + \dim G_x = \dim G.$$

The structure of an orbit as an algebraic variety is completely determined by the stabilizer:

8.1.2. THEOREM. *For every Zariski closed subgroup $H \subset G$, there exists a transitive action of G on an algebraic variety X such that H is the stabilizer of a point $x \in X$. This variety, called the homogeneous space G/H, is unique up to an isomorphism. Furthermore, suppose an algebraic group acts transitively on algebraic varieties X and Y and $G_x \subset G_y$ for some points $x \in X$, $y \in Y$. Then there exists a unique morphism $X \to Y$ sending x to y and commuting with the G-action.*

§8.2. **Linear algebraic groups.** We have a theorem of Chevalley:

8.2.1. THEOREM. *Let G be an algebraic group. The following are equivalent:*

(1) *G is an affine algebraic variety.*
(2) *G is isomorphic to a (closed) subgroup of $GL_n(\mathbb{C})$ for some n.*

A group satisfying these properties is called a linear algebraic group.

Examples:

- $\mathrm{GL}_n = D(\det) \subset \mathrm{Mat}_n$; $\mathrm{SL}_n = V(\det -1) \subset \mathrm{GL}_n$,
- *the maximal torus* of GL_n (diagonal matrices in GL_n),
- *the Borel subgroup* of GL_n (upper-triangular matrices in GL_n),
- O_n, SO_n, Sp_n,
- finite groups.

Non-examples:

- $\mathrm{SL}_2(\mathbb{Z})$ and other infinite discrete groups.
- $\mathrm{SU}_n \subset \mathrm{SL}_n(\mathbb{C})$ and other infinite compact linear Lie groups.

8.2.2. DEFINITION. A finite-dimensional representation of a linear algebraic group is called *algebraic* (or regular) if the corresponding homomorphism $G \to \mathrm{GL}(V)$ is a regular morphism. An arbitrary representation V of a linear algebraic group is called *algebraic* (or regular) if every vector $v \in V$ is contained in a finite-dimensional algebraic representation.

8.2.3. LEMMA. *If a linear algebraic group G acts on an affine variety X then an induced representation of G in the algebra of regular functions $\mathcal{O}(X)$ is algebraic.*

Proof. The action $\alpha : G \times X \to X$ is given by homomorphism of $\mathbb{C}$-algebras.

$$\alpha^* : \mathcal{O}(X) \to \mathcal{O}(G \times X) = \mathcal{O}(G) \otimes \mathcal{O}(X).$$

Let $f \in \mathcal{O}(X)$. Then

$$\alpha^*(f) = \sum h_i \otimes f_i, \quad h_i \in \mathcal{O}(G), \quad f_i \in \mathcal{O}(X).$$

We claim that the G-orbit of f in $\mathcal{O}(X)$ is contained in a finite-dimensional subspace, namely the linear span of f_i's. Indeed

$$(g \cdot f)(x) = f(g^{-1}x) = \alpha^*(f)(g^{-1}, x) = \sum h_i(g^{-1}) f_i(x).$$

It follows that the linear span of the G-orbit of f is finite-dimensional, and obviously preserved by G. The formula above also shows that the matrix of every $g \in G$ acting on this vector space depends polynomially on g, i.e. this finite-dimensional representation is algebraic. □

§8.3. **Reductive groups.**

8.3.1. THEOREM. *An algebraic group G is called reductive if it satisfies any of the following equivalent conditions:*

(1) *Every algebraic representation V of G is completely reducible, i.e. is a direct sum of finite-dimensional irreducible representations.*
(2) *Every algebraic representation V of G admits a unique G-equivariant linear projector $\pi_V : V \to V^G$.*
(3) *For every surjective linear map $A : V \to W$ of algebraic G-representations, the induced map $V^G \to W^G$ is also surjective.*

Proof. $(1) \Rightarrow (2)$. We claim that we have a unique decomposition

$$V = V^G \oplus V_0,$$

where V_0 is the sum of all non-trivial irreducible subrepresentations. The projector $V \to V_G$ then should be the projector along V_0.

Indeed, decompose V into irreducible representations:

$$V = \bigoplus_{\alpha \in I} V_\alpha.$$

Let $J \subset I$ be the subset of indices such that V_α is trivial. Let $U \subset V$ be an irreducible subrepresentation. By Schur's lemma, its projection on every V_α is either an isomorphism or a zero map. It follows that U is contained in either $\bigoplus\limits_{\alpha \in J} V_\alpha$ or $\bigoplus\limits_{\alpha \notin J} V_\alpha$. Thus $\bigoplus\limits_{\alpha \in J} V_\alpha = V^G$ and $\bigoplus\limits_{\alpha \notin J} V_\alpha = V_0$

$(2) \Rightarrow (3)$. It is enough to prove this for finite-dimensional V and W. Suppose the induced map $V^G \to W^G$ is not onto. Choose $w \in W^G$ not in the image of V^G and choose any projector $W^G \to \langle w \rangle$ that annihilates the image of V^G. Then the composition

$$V \xrightarrow{A} W \xrightarrow{\pi_W} W^G \to \langle w \rangle = \mathbb{C}$$

is a surjective G-invariant linear map $f : V \to \mathbb{C}$ that annihilates V^G. After dualizing, we have a G-invariant vector $f \in V^*$ which is annihilated by all G-invariant linear functions on V^*. However, this is nonsense: we can easily construct a G-invariant linear function on V^* which does not annihilate f by composing a G-invariant projector $V^* \to (V^*)^G$ (which exists by (2)) with any projector $(V^*)^G \to \langle f \rangle$.

$(3) \Rightarrow (1)$. We will check this if V is finite-dimensional and leave the general case to the homework. It is enough to show that any sub-representation $W \subset V$ has an invariant complement. Here we get sneaky and apply (3) to the restriction map of G-representations

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(W, W).$$

The G-invariant lift of $\mathrm{Id} \in \mathrm{Hom}(W, W)$ gives a G-invariant projector $V \to W$ and its kernel is a G-invariant complement of W. □

§8.4. **Unitary trick.** Let's show that $\mathrm{SL}_n(\mathbb{C})$ and $\mathbb{C}^*$ are reductive.

8.4.1. LEMMA.

- *The circle $S^1 = \{|z| = 1\}$ is a Zariski dense subgroup of $\mathbb{C}^*$.*
- *The special unitary group SU_n is a Zariski dense subgroup of $\mathrm{SL}_n(\mathbb{C})$.*

Proof. The first claim is clear because S^1 is infinite.

Let f be a regular function on $\mathrm{SL}_n(\mathbb{C})$ that vanishes on SU_n. We have to show that f is identically 0. In fact any function holomorphic in a neighborhood U of the identity $\mathrm{Id} \in \mathrm{SL}_n(\mathbb{C})$ will vanish in U if it vanishes on $U \cap \mathrm{SU}_n$. Indeed, consider the exponential map

$$\exp : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}), \quad A \mapsto \exp(A) = \mathrm{Id} + A + \frac{A^2}{2} + \dots$$

This map is biholomorphic in a neighborhood of 0 (the inverse map is given by matrix $\log$) and (locally) identifies $SL_n(\mathbb{C})$ with a complex vector subspace $\mathfrak{sl}_n$ of complex matrices with trace 0 and SU_n with a real subspace $\mathfrak{su}_n$ of skew-Hermitian matrices such that $A + \bar{A}^t = 0$.

So $g = f(\exp(A))$ is a holomorphic function near the origin which vanishes on $\mathfrak{su}_n$. But since $\mathfrak{su}_n + i\mathfrak{su}_n = \mathfrak{sl}_n$, this function vanishes identically by Cauchy–Riemann equations. □

8.4.2. THEOREM. *SL_n and $\mathbb{C}^*$ are reductive groups.*

Proof. We use a *unitary trick* of Weyl (and Hurwitz). We let $G = \mathrm{SL}_n$ (resp. $\mathbb{C}^*$) and $K = \mathrm{SU}_n$ (resp. S^1). An algebraic representation V of G induces a continuous representation of K. Any sub-representation of K in V is a sub-representation for G by Lemma 8.4.1. So it is enough to show that every continuous complex representation of K is completely reducible.[23]

8.4.3. CLAIM. *V has an K-invariant positive-definite Hermitian form h.*

Given the claim, every complex sub-representation $U \subset V$ has a complementary sub-representation, the orthogonal complement $U^\perp$ with respect to h. We can keep decomposing V into pieces until each piece is irreducible. To prove the claim, we need the lemma:

8.4.4. LEMMA. *Let $S \subset \mathbb{R}_n$ is a convex subset preserved by a compact subgroup K of the group of affine transformations (i.e. compositions of linear transformations and translations) of $\mathbb{R}^n$. Then K has a fixed point on S.*

Proof. We can assume that S is both convex and compact. Indeed, since K is compact, any K-orbit in S is compact (being the image of K under a continuous map). The convex hull S' of this K-orbit is a compact, convex, and G-invariant subset of S. A fixed point in S' will be a fixed point in S.

We can also assume that S spans $\mathbb{R}^n$. Indeed, take the minimal affine subspace $\mathbb{R}^k$ containing S. Since K preserves S, it also preserves $\mathbb{R}^k$ and acts there by affine transformations.

Now let p be the center of mass of S with coordinates

$$p_i = \frac{\int_S x_i \, dV}{\int_S dV}$$

(here dV is the standard measure on $\mathbb{R}^n$). Since S is convex, the Riemann sum definition of the integral shows that $p \in S$ and that p is preserved by any affine transformation of $\mathbb{R}^n$ that preserves S. So, p is fixed by K. □

Back to the claim, consider the action of K on the real vector space of all Hermitian forms $(\cdot, \cdot)$ on V. Let S be the subset of positive-definite forms. The action of K preserves the set S, which is convex because every positive

[23] One approach is to construct an equivariant projector $V \to V^K$ for every finite-dimensional continuous representation. Just like in the case of finite groups, one can take any projector $p : V \to V^K$ and then take its average

$$\pi(v) = \frac{\int_K p(gv) \, d\mu}{\int_K d\mu}.$$

Here μ should be an equivariant measure on K (called *Haar measure*). We follow a different approach which doesn't use Haar measure.

linear combination of positive-definite Hermitian forms is positive-definite. So we are done by the Lemma. □

§8.5. **Geometric Quotient.** Let X be an algebraic variety with an algebraic action of an algebraic group G. The most obvious way to construct the quotient $Y = X/G$ would be

- As a set, Y should be the set of G-orbits on X.
- Topology on Y should be the quotient topology.
- For every open subset $U \subset Y$, functions in $\mathcal{O}_Y(U)$ embed, via pull-back, into G-invariant functions in $\mathcal{O}_X(\pi^{-1}U)$. And vice versa, G-invariant functions in $\mathcal{O}_X(\pi^{-1}U)$ should descend to functions on U. Thus we can ask for
$$\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}U)^G,$$
i.e. $\mathcal{O}_Y = (\pi_*\mathcal{O}_X)^G$.

In addition, note that π has to be an open map. Indeed, if $V \subset X$ is open then $\pi^{-1}(\pi(V))$ is the union of G-translates of V, hence open. Therefore $\pi(V)$ is open by definition of the quotient topology. The same argument shows that π is closed if G is a finite group. In general, it shows that the image of a G-invariant closed set is closed.

Of course $(Y, \mathcal{O}_Y)$ defined this way does not always exist as an algebraic variety. If it does, we say that $Y = X/G$ is a *geometric quotient*. One can also characterize the geometric quotient as follows:

8.5.1. LEMMA. *Let X be an algebraic variety with an algebraic action of an algebraic group G. A morphism $\pi : X \to Y$ is a geometric quotient if and only if the following properties are satisfied:*

(1) *π is surjective and the fibers are precisely the orbits;*
(2) *π is open;*
(3) *$\mathcal{O}_Y = (\pi_*\mathcal{O}_X)^G$.*

The geometric quotient rarely exists. For example, it doesn't exist in both cases of Example 8.0.10 (unless we remove some orbits).

§8.6. **Categorical quotient.**

8.6.1. DEFINITION. Let G be an algebraic group acting on an algebraic variety X. A morphism $\pi : X \to Y$ is called a *categorical quotient* if

- π is constant along G-orbits and
- any morphism $\pi' : X \to Z$ constant along G-orbits factors through Y.

If it exists, the categorical quotient is clearly unique up to an isomorphism.

Mumford proved the following theorem, some parts of which we will prove and other leave as an exercise.

8.6.2. THEOREM. *Let G be a reductive group acting on an affine variety X. Then*

- *There exists a categorical quotient $X/\!\!/G$, namely* $\operatorname{MaxSpec} \mathcal{O}(X)^G$.
- *$X/\!\!/G$ is a geometric quotient if and only if all orbits are closed (for example if G is a finite group).*

We start by proving finite generation of the algebra of invariants.

8.6.3. THEOREM. *Consider an algebraic action of a reductive group G on an affine variety X (for example an algebraic representation of G in a vector space V). Then*

- *There exists a unique G-equivariant Reynolds operator $\mathcal{O}(X) \to \mathcal{O}(X)^G$.*
- *The algebra $\mathcal{O}(X)^G$ is finitely generated.*
- *The quotient morphism $\pi : X \to X/\!\!/G$ is surjective.*

Proof. By Lemma 8.2.3, the induced action of G on the algebra of regular functions $\mathcal{O}(X)$ is algebraic. By Theorem 8.3.1, there exists a unique G-invariant projector $R : \mathcal{O}(X) \to \mathcal{O}(X)^G$.

8.6.4. CLAIM. *R is the Reynolds operator, i.e. $R(fg) = fR(g)$ for $f \in \mathcal{O}(X)^G$.*

Indeed, $R(fg) = fR(g)$ for every $g \in k[X]^G$ and both sides vanish if $g \in k[X]_0$ because $k[X]_0$ is preserved by multiplication with $f \in \mathcal{O}(X)^G$.

Next we show that $\mathcal{O}(X)^G$ is finitely generated. The algebra $\mathcal{O}(X)$ is finitely generated by a finite-dimensional subspace $V \subset \mathcal{O}(X)$. Let $S^\bullet(V)$ be the symmetric algebra of V. The homomorphism $S^\bullet(V) \to \mathcal{O}(X)$ is surjective, and therefore the homomorphism $S^\bullet(V)^G \to \mathcal{O}(X)^G$ is also surjective by one of the characterizations of reductive groups (Theorem 8.3.1). So it suffices to prove that the algebra $S^\bullet(V)^G$ is finitely generated. But

$$S^\bullet(V) = \mathcal{O}(V^*)$$

and G acts on V^* linearly. So the theorem follows from Lemma 7.3.5.

Surjectivity of the quotient map follows from Theorem 7.4.4. $\square$

Although the categorical quotient does not necessarily separate all orbits, it does separate closed orbits:

8.6.5. THEOREM. *Let G be a reductive group acting on an affine variety X. Take two orbits O and O'. Then their closure are disjoint ($\bar{O} \cap \bar{O}' = \emptyset$) if and only if they are separated by invariants. In particular, every fiber of $\pi : X \to X/\!\!/G$ contains exactly one closed orbit.*

Proof. One direction is clear because every G-invariant regular function is constant along every orbit and its closure. For an opposite direction, let $I, I' \subset \mathcal{O}(X)$ be the ideals of $\bar{O}, \bar{O}'$. By Nullstellensatz,

$$I + I' = \mathcal{O}(X),$$

i.e. we can write $1 = f + g$, where $f \in I$ and $g \in I'$.

Now apply the Reynolds operator: $R(f) + R(g) = 1$.

8.6.6. CLAIM. *$R(f) \in I$ (and similarly $R(g) \in I'$).*

Given the claim, $R(f)$ is an invariant function which is equal to 0 on $\bar{O}$ and 1 on $\bar{O}'$, i.e. they are separated by invariants.

To prove the Claim, it suffices to demonstrate a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(X) & \longrightarrow & \mathcal{O}(\bar{O}) \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ \mathcal{O}(X)^G & \longrightarrow & \mathcal{O}(\bar{O})^G \end{array},$$

which we leave as an exercise.

Finally, why is there a closed orbit in each fiber of π? Take some orbit O. By Theorem 8.1.1, O is open in its closure $\bar{O}$. If it's closed, we are done. If not, pick an orbit in $\bar{O} \setminus O$, which will have smaller dimension, etc. $\square$

8.6.7. THEOREM. *Let G be a reductive group acting on an affine variety X. Let $U \subset X/\!\!/G$ be an open subset (not necessarily affine). Then*

$$\mathcal{O}_{X/\!\!/G}(U) = \mathcal{O}_X(\pi^{-1}(U))^G,$$

i.e. $\mathcal{O}_{X/\!\!/G} = (\pi_*\mathcal{O}_X)^G$. Moreover, $X/\!\!/G$ is a categorical quotient.

Proof. This is clear for principal affine open sets $D(f) \subset X/\!\!/G$ for $f \in k[X]^G$:

$$k[U] = (k[X]^G)_f = (k[X]_f)^G = k[\pi^{-1}(U)]^G.$$

But this implies the general case because principal affine open sets form the basis of Zariski topology.

Now let's show that $X/\!\!/G$ is a categorical quotient. Suppose we have a morphism $f : X \to Z$ constant on G-orbits, i.e. equivariant if we endow Z with trivial action. We have to show that f factors through π. This is clear set-theoretically: since f is constant on G-orbits and every fiber of π contains a unique closed orbit, every fiber of π is mapped by f to the same point of Z. On the level of varieties, this is clear if Z is affine: in this case f is determined by $f^* : k[Z] \to k[X]$, which has to factor through $k[X]^G$.

In general, take an affine chart $U \subset Z$. Let $V = f^{-1}(U)$. Since f is constant on G-orbits and every fiber of π contains a unique closed orbit, we have $\pi^{-1}(\pi(V)) = V$, in particular $\pi(V)$ is open. Note that V is not necessarily affine but the morphism to an affine variety U is uniquely determined by the pull-back $k[U] \to k[V]$ which has to factor through $k[V]^G$, which is equal to $k[\pi(V)]$ by $\mathcal{O}_{X/\!\!/G} = (\pi_*\mathcal{O}_X)^G$. Thus we have a morphism $\pi(V) \to U \subset Z$ which glue and give a morphism $X/\!\!/G \to Z$. $\square$

How to compute the algebra of invariants? Here's one technique and an example.

8.6.8. LEMMA. *Let G be a group acting linearly on a vector space V. Let $L \subset V$ be a linear subspace. Let*

$$Z = \{g \in G \,|\, g|_L = Id|_L\}, \quad N = \{g \in G \,|\, g(L) \subset L\}, \quad \text{and } W = N/Z.$$

Then we have a natural homomorphism $\pi : k[V]^G \to k[L]^W$, which is injective if $\overline{G \cdot L} = V$.

8.6.9. EXAMPLE. Let $G = \mathrm{SO}_n(\mathbb{C})$ be an orthogonal group preserving a quadratic form $f = x_1^2 + \ldots + x_n^2$. We claim that $\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[f]$. Indeed, we can apply Lemma 8.6.8 to $L = \mathbb{C}e_1$).

§8.7. **Homework 4.**

Problem 1. (2 point) Let $F : \mathbb{A}^n \to \mathbb{A}^n$ be a morphism given by homogeneous polynomials $f_1, \ldots, f_n$ such that $V(f_1, \ldots, f_n) = \{0\}$. Show that F is a finite morphism.

Problem 2. (1 point) Let G be a group acting by automorphisms on a normal affine variety X. Show that the algebra of invariants $\mathcal{O}(X)^G$ is integrally closed.

Problem 3. (2 points) For the cyclic quotient singularity $\frac{1}{7}(1,3)$, compute generators of $\mathbb{C}[x,y]^{\mu_7}$ and compute generators of the ideal of $\mathbb{A}^2/\mu_7$ as an affine subvariety of $\mathbb{A}^?$ (use computer algebra for the second part).

Problem 4. (2 points). (a) Let G be a finite group acting linearly on a vector space V. Show that $\mathbb{C}(V)^G$ (the field of invariant rational functions) is equal to the quotient field of $k[V]^G$. (b) Show that (a) can fail for an infinite group.

Problem 5. (3 points) Let G be a finite group acting linearly on a vector space V. Show that the algebra of invariants $\mathcal{O}(V)^G$ is generated by polynomials of degree at most $|G|$.

Problem 6. (2 points) An affine variety X is called a *cone* if its coordinate algebra $R = k[X]$ admits a grading $R = \sum_{k\geq 0} R_k$ such that $R_0 = k$. Show that the cone X is non-singular if and only if R is isomorphic to a polynomial algebra.

Problem 7. (1 point) Let $G \to \mathrm{GL}(V)$ be a representation of a reductive group and let $\pi : V \to V//G$ be the quotient. Show that the following properties are equivalent (2 points):

- $V//G$ is non-singular at $\pi(0)$.
- $V//G$ is non-singular.
- $\mathcal{O}(V)^G$ is a polynomial algebra.

Problem 8. (3 points) Let $G = \mathrm{GL}_n$ be a general linear group acting on Mat_n by conjugation. (a) Let $L \subset \mathrm{Mat}_n$ be the space of diagonal matrices. Show that $\overline{G \cdot L} = \mathrm{Mat}_n$. (b) Show that $k[\mathrm{Mat}_n]^G$ is generated by coefficients of the characteristic polynomial.

Problem 9. (2 point) (a) In the previous problem, find all fibers of the quotient morphism $\pi : \mathrm{Mat}_n \to \mathrm{Mat}_n /\!\!/ G$ that contain only one orbit. (b) Find a closed orbit in very fiber of π.

Problem 10. (2 points) Prove Theorem 8.1.1

Problem 11. (2 points) Prove Lemma 8.5.1

Problem 12. (2 points) (a) Let G be a reductive group and let $\phi : X \to Y$ be an equivariant map of affine varieties with G-action. Show that we have a commutative diagram of morphisms

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X/\!\!/G & \longrightarrow & Y/\!\!/G \end{array}$$

(b) Let G be a reductive group acting on an affine variety X. Let $Y \subset G$ be a G-invariant closed subset. Show that $\pi_X(Y)$ is closed in $X/\!\!/G$ and isomorphic to $Y/\!\!/G$.

Problem 13. (3 points) Let V_4 be the space of degree 4 polynomials in 2 variables. Show that $k[V_4]^{\mathrm{SL}_2}$ is a polynomial algebra generated by invariants of degrees 2 and 3. Hint: apply Lemma 8.6.8.

Problem 14. (3 points) Consider the action of SL_2 on homogeneous polynomials in x and y of degree 6 written as follows:

$$\zeta_0 x^6 + 6\zeta_1 x^5 y + 15\zeta_2 x^4 y^2 + 20\zeta_3 x^3 y^3 + 15\zeta_4 x^2 y^4 + 6\zeta_5 x y^5 + \zeta_6 y^6.$$

Show that the function

$$\det \begin{bmatrix} \zeta_0 & \zeta_1 & \zeta_2 & \zeta_3 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\ \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 \\ \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 \end{bmatrix}$$

belongs to the algebra of invariants $k[\zeta_0, \zeta_1, \ldots, \zeta_6]^{\mathrm{SL}_2}$.

§9. GIT quotients and stability

§9.1. **Weighted projective space.** Fix positive integers $a_0, \ldots, a_n$ (weights) and consider the action of $\mathbb{C}^*$ on $\mathbb{A}^{n+1}$ defined as follows:

$$t \cdot (x_0, \ldots, x_n) = (t^{a_0} x_0, \ldots, t^{a_n} x_n) \quad \text{for every} \quad t \in \mathbb{C}^*.$$

The *weighted projective space* is the geometric quotient

$$\mathbb{P}(a_0, \ldots, a_n) := (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$

which we are going to construct. For now we define it as the set of orbits. For example, $\mathbb{P}(1, \ldots, 1) = \mathbb{P}^n$.

9.1.1. EXAMPLE. Recall that any elliptic curve has a Weierstrass equation

$$y^2 = 4x^3 - g_2 x - g_3, \quad \Delta = g_2^3 - 27 g_3^2 \neq 0$$

and coefficients g_2 and g_3 are defined up to admissible transformations

$$g_2 \mapsto t^4 g_2, \quad g_3 \mapsto t^6 g_3.$$

So the moduli space of elliptic curves is $\mathbb{P}(4, 6)$ with one point removed (which corresponds to the $\mathbb{C}^*$-orbit $\{\Delta = 0\}$ of singular cubics). Of course

$$\mathbb{P}(4, 6)_{[g_2 : g_3]} \simeq \mathbb{P}^1_{[j:1]},$$

where $j = 1728 g_2^3/\Delta$ by the usual formula (5.2.7). But thinking about $M_{1,1}$ as $\mathbb{P}(4, 6)$ has its advantages. For example, one can refine the structure of an algebraic variety $\mathbb{P}(4, 6)$ to construct an *algebraic stack* $\mathcal{P}(4, 6)$, which in appropriate sense "represents" the functor of elliptic fibrations $\mathcal{M}_{1,1}$.

9.1.2. EXAMPLE. Let's construct $\mathbb{P}(1, 1, 2)$ by hand. Take the map

$$\pi : \mathbb{A}^3_{x,y,z} \setminus \{0\} \to \mathbb{P}^3_{[A:B:C:D]}, \quad (x, y, z) \mapsto [x^2 : xy : y^2 : z].$$

It is easy to see that it separates orbits, i.e. $\pi(x, y, z) = (x', y', z')$ if and only if there exists $t \in \mathbb{C}^*$ such that

$$x' = tx, \; y' = ty, \; z' = t^2 z.$$

The image is a quadratic cone $AB = C^2$ in $\mathbb{P}^3$.

Now let's discuss the construction of $\mathbb{P}(a_0, \ldots, a_n)$ in general. The ring of invariants

$$\mathbb{C}[x_0, \ldots, x_n]^{\mathbb{C}^*} \simeq \mathbb{C}$$

and the categorical quotient $\mathbb{A}^{n+1}//\mathbb{C}^*$ is a point. This agrees with the fact that there is only one closed orbit – the origin. We will remove it. Another idea is that there are many *rational invariant functions* $\mathbb{C}(x_0, \ldots, x_n)^{\mathbb{C}^*}$, which are constant along G-orbits wherever they are defined.

Concretely, we cover $\mathbb{A}^{n+1} \setminus \{0\}$ by principal affine open sets

$$D(x_i) = (x_i \neq 0) \simeq \mathbb{A}^n$$

and take the quotients

$$D_{x_i} = D(x_i) /\!\!/ \mathbb{C}^*.$$

All $\mathbb{C}^*$-orbits in $D(x_i)$ are closed, and therefore $\pi_i : D(x_i) \to D_{x_i}$ is a geometric quotient by Theorem 8.6.2. Then we would like to glue these quotients to obtain the quotient

$$\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}(a_0, \ldots, a_n)$$

just like in the definition of the standard projective space. Before doing that, let's try to understand D_{x_i} better. Notice that

$$\mathcal{O}(D(x_i)) = \mathbb{C}\left[x_0, \ldots, x_n, \frac{1}{x_i}\right] \subset \mathbb{C}(x_0, \ldots, x_n)$$

and that $D(x_i)$ is an affine variety. To compute its categorical quotient, we take the algebra of invariants and its spectrum:

$$\mathcal{O}(D_{x_i}) = \mathcal{O}(D(x_i))^{\mathbb{C}^*} = \left\{ \frac{p}{x_i^k} \quad | \quad p \in \mathbb{C}[x_0, \ldots, x_n], \quad \deg p = k a_i \right\},$$

fractions of degree 0 (here and after the degree $\deg$ is our weighted degree). There are two cases: if $a_i = 1$ then we just have

$$\mathcal{O}(D_{x_i}) = \mathbb{C}\left[\frac{x_0}{x_i^{a_0}}, \ldots, \frac{x_n}{x_i^{a_n}}\right] \simeq \mathbb{C}[y_1, \ldots, y_n].$$

The chart $D_{x_i} \simeq \mathbb{A}^n$, just like for the standard $\mathbb{P}^n$. In the general case, let's restrict discussion to the weighted projective plane $\mathbb{P}(a_0, a_1, a_2)$.

9.1.3. CLAIM. *D_{x_0} is the cyclic quotient singularity $\frac{1}{a_0}(a_1, a_2)$.*

Proof. First an informal argument. If $a_0 = 1$ then we can kill the $\mathbb{C}^*$-action by setting $x_0 = 1$ and identify $D(x_0) /\!\!/ \mathbb{C}^*$ with $\mathbb{A}^2$. In the weighted case $a_0 > 1$ setting $x_i = 1$ does not eliminate the $\mathbb{C}^*$-action but reduces it to the action of a subgroup $\mu_{a_i} \subset \mathbb{C}^*$. Thus

$$D_{x_0} = D(x_0)/\mathbb{C}^* \simeq \mathbb{A}^2/\mu_{a_0} = \operatorname{MaxSpec} \mathbb{C}[x_1, x_2]^{\mu_{a_0}},$$

where μ_{a_0} acts with weights a_1, a_2.

More formally, we would like to have an element of weight 1, which can be achieved by considering a cyclic field extension

$$\mathbb{C}(x_0, x_1, x_2) \subset \mathbb{C}(z_0, x_1, x_2),$$

where $x_0 = z_0^{a_0}$. Then we have

$$\mathcal{O}(D_{x_i}) = \left\{ \frac{p}{z_0^{a_0 k}} \quad | \quad p \in \mathbb{C}[x_0, x_1, x_2], \quad \deg p = k a_0 \right\} =$$

$$\left\{ \sum_{i,j} a_{ij} \left(\frac{x_1}{z_0^{a_1}} \right)^i \left(\frac{x_2}{z_0^{a_2}} \right)^j \quad | \quad a_1 i + a_2 j \equiv 0 \mod a_0 \right\} \subset \mathbb{C}\left[\frac{x_1}{z_0^{a_1}}, \frac{x_2}{z_0^{a_2}} \right].$$

We get a subalgebra in $\mathbb{C}[y_1, y_2]$ spanned by all monomials $y_1^i y_2^j$ such that $a_1 i + a_2 j \equiv 0 \mod a_0$, the algebra of functions of a CQS $\frac{1}{a_0}(a_1, a_2)$. $\square$

9.1.4. EXAMPLE. A projective quadratic cone $\mathbb{P}(1, 1, 2)$ is covered by two copies of $\mathbb{A}^2$ and $\frac{1}{2}(1, 1)$, which is isomorphic to an affine quadratic cone.

§9.2. **Projective spectrum.** Instead of showing how to glue charts D_{x_i} to get the weighted projective space, we give a more general construction.

Let R be any finitely generated graded integral domain with $R_0 = \mathbb{C}$. We define an algebraic action of $\mathbb{C}^*$ on R by setting that $t \cdot f = t^n f$ for every $f \in R$ of degree n. Then $\mathbb{C}^*$ acts on R by automorphisms of this algebra, and therefore defines an action on

$$X = \operatorname{MaxSpec} R.$$

One of the points is special: the linear span R_+ of elements of positive degree is a maximal ideal and hence defines the point, which we call $0 \in X$.

We are going to define an algebraic variety, the *projective spectrum* of R, which in our special case of a graded polynomial algebra is $\mathbb{P}(a_0, \ldots, a_n)$. As a set, we define

$$Y = \operatorname{Proj} R$$

to be the set of non-zero $\mathbb{C}^*$-orbits on X[24].

9.2.1. REMARK. A geometric way to see the action of $\mathbb{C}^*$ on X is to choose homogeneous generators for R (say of degrees $a_0, \ldots, a_n$) and to use them to embed X into $\mathbb{A}^{n+1}$. The group $\mathbb{C}^*$ then acts on $\mathbb{A}^{n+1}$: an element $t \in \mathbb{C}^*$ sends $x_i \mapsto t^{a_i} x_i$. This action preserves X. The special point $0 \in X$ is just the origin $0 \in \mathbb{A}^n$. This also shows that Y is a subvariety of $\mathbb{P}(a_0, \ldots, a_n)$.

Rational functions on Y are defined as $\mathbb{C}^*$-invariant rational functions on X, i.e. ratios of polynomials of the same degree:

$$\mathbb{C}(Y) = (\operatorname{Quot} R)_0,$$

where the subscript means that we are only taking fractions of degree 0. We call a function regular at some point if it has a presentation as a fraction with a denominator non-vanishing at this point. It is clear that Y is covered by affine charts D_f for each homogeneous element $f \in R$ of positive degree, where

$$\mathcal{O}(D_f) = \mathcal{O}(D(f))^{\mathbb{C}^*} = R[1/f]_0.$$

What is the gluing? Given D_f and D_g, notice that

$$D_f \cap D_g = D_{fg},$$

[24]Technically speaking, it's better to call it MaxProj because we are not going to enhance it to an algebraic scheme.

is a principal open subset in both D_f (where it is a complement of a vanishing set of a regular function $\frac{g^{\deg f}}{f^{\deg g}}$) and D_g (where we use $\frac{f^{\deg g}}{g^{\deg f}}$). Formally speaking, we have to check that in $\mathbb{C}(Y)$ we have

$$R[1/fg]_0 = R[1/f]_0 \left[\frac{f^{\deg g}}{g^{\deg f}}\right], \tag{9.2.2}$$

which we leave it as an exercise.

9.2.3. REMARK. We can recast the structure of a graded algebra in the language of semi-invariants. Every homogeneous element $x \in R_d$ is a semi-invariant for the $\mathbb{C}^*$-action of weight d (i.e. $t \in \mathbb{C}^*$ acts by multiplying this element by t^d). Invariant rational functions are rational functions of *degree* 0 (with respect to the grading):

$$(\mathrm{Quot}\, R)_0 \subset \mathrm{Quot}\, R.$$

§9.3. **Abstract algebraic varieties.**

9.3.1. DEFINITION. An *algebraic pre-variety* X is given by the following data:

- A finitely generated field extension K of $\mathbb{C}$. This will be a field of rational functions on X.
- Topological space X.
- For each open subset $U \subset X$ we need a subalgebra $\mathcal{O}_X(U) \subset K$. It should satisfy the condition
$$\mathcal{O}_X\left(\bigcup_{i\in I} U_i\right) = \bigcap_{i\in I} \mathcal{O}_X(U_i).$$
$\mathcal{O}_X$ is called the *structure sheaf*.
- Finally, X should admit a finite cover $\{U_i\}$ such that each U_i (with an induced topology) is an affine variety (with Zariski topology) with function field K and for each open subset $V \subset U_i$, $\mathcal{O}_X(V) \subset K$ is the algebra of rational functions regular on V.

An algebraic variety is a separated algebraic pre-variety, i.e. the diagonal morphism $X \to X \times X$ has a closed image.

Algebraic pre-varieties can be constructed by gluing affine varieties. Suppose A and B are affine varieties with the same function field K. In addition, we are given another affine variety C and open immersions

$$i_A : C \hookrightarrow A \quad \text{and} \quad i_B : C \hookrightarrow B.$$

Then we define the topological space $X = A \cup_C B$ by identifying points $i_A(x)$ with $i_B(x)$ for any $x \in C$ and by declaring a subset $U \subset X$ open if $U \cap A$ and $U \cap B$ is open. Finally, we set

$$\mathcal{O}_X(U) = \mathcal{O}_A(U \cap A) \cap \mathcal{O}_B(U \cap B)$$

It is easy to generalize this to several affine charts: given affine varieties

$$U_0, \ldots, U_r$$

with the same function field and, for each pair U_i, U_j, affine open subsets

$$U_{ij} \subset U_i, \quad U_{ji} \subset U_j$$

and an isomorphism
$$\phi_{ij} : U_{ij} \to U_{ji}.$$
satisfying

- $\phi_{ij} = \phi_{ji}^{-1}$,
- $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and
- $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$,

we can glue $X = U_0 \cup \ldots \cup U_r$.

9.3.2. LEMMA. $\operatorname{Proj} R$ *is an algebraic pre-variety.*

Proof. We have $K = (\operatorname{Quot} R)_0$. For any homogeneous $f \in R$ we have an affine variety
$$D_f = \operatorname{MaxSpec} R[1/f]_0.$$
To get a finite atlas, take only homogeneous generators of R. To see the gluing condition, notice that D_{fg} is a principal open subset in both D_f and D_g. The compatibility conditions on triple overlaps are of set-theoretic nature, and are clearly satisfied. □

§9.4. **Veronese embedding.** We now have two models for $\mathbb{P}(1,1,2)$: as a weighted projective plane defined by charts and as a quadratic cone in $\mathbb{P}^3$. What is the relationship between these models? We are going to show that in fact every $\operatorname{Proj} R$ is a projective variety (and in particular separated).

9.4.1. DEFINITION. If R is a graded ring then its subring $R^{(d)} = \sum_{d|n} R_n$ is called the d-th Veronese subring.

For example, for $\mathbb{P}(1,1,2)$ the second Veronese subring is generated by x^2, xy, y^2, and z, subject to a single quadratic relation. So
$$R^{(2)} = \mathbb{C}[A,B,C,D]/(AC - B^2),$$
which explains why $\operatorname{Proj} R^{(2)}$ is a quadratic cone in $\mathbb{P}^3$. The basic fact is:

9.4.2. LEMMA. $\operatorname{Proj} R = \operatorname{Proj} R^{(d)}$ *for any d.*

Proof. First of all, we have $(\operatorname{Quot} R)_0 = (\operatorname{Quot} R^{(d)})_0$. Indeed, any fraction $a/b \in (\operatorname{Quot} R)_0$ can be written as $ab^{d-1}/b^d \in (\operatorname{Quot} R^{(d)})_0$.

Let $f_1, \ldots, f_r$ be homogeneous generators of R We claim that
$$\operatorname{Proj} R = \bigcup_{i=1}^{r} D_{f_i} \quad \text{and} \quad \operatorname{Proj} R^{(d)} = \bigcup_{i=1}^{r} D_{f_i^d}$$
The first claim is clear. For the second claim, note that $f_1^d, \ldots, f_r^d \in R^{(d)}$ do not necessarily generate $R^{(d)}$. However, if all f_i^d vanish at some point $p \in \operatorname{Proj} R^{(d)}$ then every function in the radical of the ideal $(f_i^d) \subset R^{(d)}$ vanishes at p as well. We claim that this radical is equal to $R_+^{(d)}$. Indeed, let $g \in R^{(d)}$ be a homogeneous element of positive degree. It can be expressed as a polynomial in $f_1, \ldots, f_r$, and therefore a sufficiently high power of g belongs to the ideal $(f_i^d) \subset R^{(d)}$.

The basic local calculation we need is that charts D_{f_i} of $\operatorname{Proj} R$ and $D_{f_i^d}$ of $\operatorname{Proj} R^{(d)}$ can be identified, i.e. that
$$R^{(d)}[1/f^d]_{(0)} \simeq R[1/f]_{(0)}$$

for any homogeneous element f of R. But indeed, as soon as $dj > i$ we have

$$\frac{g}{f^i} = \frac{f^{dj-i}g}{f^{dj}}.$$

So $\operatorname{Proj} R$ and $\operatorname{Proj} R^d$ have the same charts glued in the same way. $\square$

9.4.3. LEMMA. *For a sufficiently large d, $R^{(d)}$ is generated by R_d.*

Proof. Let $a_1, \ldots, a_r$ be degrees of homogeneous generators $f_1, \ldots, f_r$ of R. Let $a = \mathrm{l.\,c.\,m.}(a_1, \ldots, a_r)$ and let $d = ra$. We claim that this d works, which we leave as a homework exercise. $\square$

9.4.4. COROLLARY. $\operatorname{Proj} R$ *is a projective variety.*

Proof. By Lemmas 9.4.2 and 9.4.3 we can assume without loss of generality that R is generated by R_1 by passing to the Veronese subalgebra.. Then $R = \mathbb{C}[y_0, \ldots, y_N]/I$, where I is a homogeneous ideal. We claim that

$$\operatorname{Proj} R = V(I) \subset \mathbb{P}^N.$$

Indeed, the description and gluing of affine charts $V(I) \cap \mathbb{A}^N_i$, $i = 0, \ldots, N$ matches the description of charts D_{x_i} of $\operatorname{Proj} R$. $\square$

§9.5. **GIT quotients.** So far we discussed quotients of affine varieties and examples of gluing them. How about quotients of projective varieties?

9.5.1. EXAMPLE. Here is a preview: what is the quotient of $\mathbb{P}^2$ by the action of the symmetric group S_3 that acts by permuting x_1, x_2, x_3?

We can realize $\mathbb{P}^2$ as the quotient of $\mathbb{A}^3 \setminus \{0\}$ by the action of $\mathbb{C}^*$ which commutes with the action of S_3. Let's change the order of taking quotients: first take the quotient of $\mathbb{A}^3$ by the action of S_3:

$$\operatorname{MaxSpec} \mathbb{C}[x_1, x_2, x_3]^{S_3} = \operatorname{MaxSpec} \mathbb{C}[\sigma_1, \sigma_2, \sigma_3] = \mathbb{A}^3,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the elementary symmetric functions. Next take the quotient of $\mathbb{A}^3 \setminus \{0\}$ by the action of $\mathbb{C}^*$ but notice that $\sigma_1, \sigma_2, \sigma_3$ have weights $1, 2, 3$ for the $\mathbb{C}^*$ action! So the quotient morphism is

$$\pi : \mathbb{P}^2 \to \mathbb{P}(1, 2, 3),$$

$$[x_1 : x_2 : x_3] \mapsto [x_1 + x_2 + x_3 : x_1x_2 + x_2x_3 + x_1x_3 : x_1x_2x_3].$$

The quotient space is the weighted projective plane $\mathbb{P}(1, 2, 3)$ with two du Val singularities,

$$\frac{1}{2}(1, 3) = \frac{1}{2}(1, 1) = A_1 \quad \text{and} \quad \frac{1}{3}(1, 2) = A_2.$$

More systematically, the procedure is as follows. Let G be a reductive group acting on a projective variety X. We are going to make two choices:

- Write $X = \operatorname{Proj} R$, where R is a finitely generated graded algebra (choice of polarization).
- Find an action of G on R that preserves the grading (if it exists) and which gives back our action on X (choice of linearization).

Then we can form the GIT quotient

$$X /\!\!/_{\mathrm{GIT}} G = \operatorname{Proj} R^G.$$

9.5.2. EXAMPLE. In the example above, $\mathbb{P}^2 = \operatorname{Proj}\mathbb{C}[x_1, x_2, x_3]$ and we lift the action of S_3 to a standard action on $\mathbb{C}^3$ on $\mathbb{C}[x_1, x_2, x_3]$. Then

$$\mathbb{P}^2/\!\!/_{\mathrm{GIT}}S_3 = \operatorname{Proj}\mathbb{C}[x_1, x_2, x_3]^{S_3} =$$

$$= \operatorname{Proj}\mathbb{C}[x_1 + x_2 + x_3, x_1x_2 + x_2x_3 + x_1x_3, x_1x_2x_3] = \mathbb{P}(1, 2, 3).$$

9.5.3. DEFINITION. A *polarization* is a choice of an ample divisor D on X, equivalently an ample line bunde $L = \mathcal{O}(D)$. The coordinate algebra of D is the graded algebra

$$R = \bigoplus_{n\geq 0} H^0(X, \mathcal{O}(nD)) = \bigoplus_{n\geq 0} H^0(X, L^{\otimes n}). \tag{9.5.4}$$

A *linearization* is an action of G on the total space of the line bundle L such that, for every point of $x \in X$ and $g \in G$, g takes the fiber L_x to the fiber L_{gx} and the induced map $L_x \to L_{gx}$ is linear. A line bundle with a linearization is called a *linearized (or equivariant) line bundle*. Notice that G acts naturally on the space of global sections of any linearized line bundle.

9.5.5. EXAMPLE. Consider the action of $\mathrm{GL}(V)$ on the vector space V and induced action on $\mathbb{P}(V)$. We claim that the tautological line bundle $\mathcal{O}(-1)$ admits a natural linearization. Indeed, by the definition its total space L embeds into the product $\mathbb{P}(V) \times V$ and the natural action of $\mathrm{GL}(V)$ on this product preserves L. More geometrically, we can also identify L with the blow-up of V at the origin. This also shows that $\mathcal{O}(-1)$ is linearized for every algebraic subgroup of $\mathrm{GL}(V)$ (for example $\mathrm{SL}(V)$).

9.5.6. DEFINITION. A tensor product of linearized line bundles and a dual of a linearized line bundle are naturally linearized. Thus linearized line bundles (modulo isomorphism) form a group with respect to $\otimes$ denoted by

$$\mathrm{Pic}_G(X).$$

In particular, every tensor power $L^{\otimes n}$ is linearized. Thus G acts naturally on the graded algebra (9.5.4). The induced action on $\operatorname{Proj} R$ gives back the original action of G on X.

9.5.7. REMARK. In practice one can pass to the Veronese subalgebra of R by substituting an ample divisor D for nD for some integer n. Thus we can assume that R is generated by R_1. Then $\operatorname{Proj} R$ embeds equivariantly into the projective space $\mathbb{P}^N = \operatorname{Proj}(\operatorname{Sym}^\bullet R_1)$. The GIT quotient $\operatorname{Proj} R^G$ will then embed in the GIT quotient $\operatorname{Proj}(\operatorname{Sym}^\bullet R_1)^G$ of $\mathbb{P}^N$.

The case of a finite group action, as in the example $\mathbb{P}^2 \to \mathbb{P}(1, 2, 3)$ above, is special. In general, the GIT quotient $\pi : X \dashrightarrow X/\!\!/_{\mathrm{GIT}}G$ is not defined on the whole X: some orbits have to be removed. More generally, if $S \subset R$ is a finitely generated graded subalgebra then the map $\operatorname{Proj} R \dashrightarrow \operatorname{Proj} S$ is only a rational map. Where is it not regular? We always have a map $\operatorname{Spec} R \to \operatorname{Spec} S$ but recall that points of $\operatorname{Proj} R$ correspond to non-zero $\mathbb{C}^*$-orbits in $\operatorname{Spec} R$. So the map $\operatorname{Proj} R \dashrightarrow \operatorname{Proj} S$ is not defined at orbits that map to the zero orbit of $\operatorname{Spec} S$, i.e. at points where every function $f \in S$ of positive degree vanishes. In our case, this leads to the following

9.5.8. DEFINITION. A point $x \in X$ is called *unstable* if all functions in R^G of positive degree vanish at x. Let $X_{us} \subset X$ be the locus of unstable points. Not unstable points are called *semistable*. Let $X_{ss} = X \setminus X_{us}$ be their locus.

9.5.9. DEFINITION. The GIT quotient is the map

$$\pi : X_{ss} \to X/\!\!/_{\mathrm{GIT}} G := \operatorname{Proj} R^G$$

induced by the inclusion $R^G \subset R$. Concretely, choose homogeneous generators $f_1, \ldots, f_r$ for R^G. Let $D_{f_i} \subset \operatorname{Proj} R^G$ (resp. $\tilde{D}_{f_i} \subset \operatorname{Proj} R$) be the corresponding principal affine open sets. Then

$$\bigcup_{i=1}^{r} \tilde{D}_{f_i} = X_{ss} \quad \text{and} \quad \bigcup_{i=1}^{r} D_{f_i} = \operatorname{Proj} R^G.$$

We have

$$\mathcal{O}(\tilde{D}_{f_i}) = R\left[\frac{1}{f_i}\right]_0 \quad \text{and} \quad \mathcal{O}(D_{f_i}) = R^G\left[\frac{1}{f_i}\right]_0,$$

where the subscript 0 denotes elements of degree 0. The map $\pi : \tilde{D}_{f_i} \to D_{f_i}$ is dual to the inclusion of algebras

$$R^G\left[\frac{1}{f_i}\right]_0 \hookrightarrow R\left[\frac{1}{f_i}\right]_0.$$

9.5.10. EXAMPLE. Let $X = \mathbb{P}^3 = \operatorname{Proj} \mathbb{C}[x_1, x_2, y_1, y_2]$. Let's take the action of $G = \mathbb{C}^*$ on X induced by the following action on $\mathbb{C}^4$: every $t \in \mathbb{C}^*$ acts by

$$(x_1, x_2, y_1, y_2) \mapsto (tx_1, tx_2, t^{-1}y_1, t^{-1}y_2).$$

The fixed points of this action are the two lines

$$L_1 = \{x_1 = x_2 = 0\}, \qquad L_2 = \{y_1 = y_2 = 0\}.$$

Every other orbit is isomorphic to $\mathbb{C}^*$ and its closure is isomorphic to $\mathbb{P}^1$, with one point on L_1 and another on L_2. Geometrically, orbit closures are just lines connecting a point of L_1 to a point of L_2. It is clear that

$$\mathbb{C}[x_1, x_2, y_1, y_2]^G = \mathbb{C}[x_1y_1, x_1y_2, x_2y_1, x_2y_2] = \mathbb{C}[A, B, C, D]/(AD - BC).$$

So the GIT quotient is the quadric

$$Q = \operatorname{Proj} \mathbb{C}[x_1, x_2, y_1, y_2]^G = \{AD - BC = 0\} \subset \mathbb{P}^3.$$

The quotient map $\pi : X \dashrightarrow Q$ sends

$$[x_1 : x_2 : y_1 : y_2] \mapsto [x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2].$$

It is undefined at points of the unstable locus

$$X_{us} = \{x_1y_1 = x_1y_2 = x_2y_1 = x_2y_2 = 0\} = L_1 \cup L_2.$$

If we use an isomorphism $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with homogeneous coordinates $[x_1 : x_2]$ and $[y_1 : y_2]$ then the quotient map is simply

$$\mathbb{P}^3 \setminus (L_1 \cup L_2) \to \mathbb{P}^1 \times \mathbb{P}^1,$$

$$[x_1 : x_2 : y_1 : y_2] \mapsto ([x_1 : x_2], [y_1 : y_2]).$$

§9.6. **More on polarization and linearization.** The map forgetting a linearization gives a homomorphism

$$\mathrm{Pic}_G(X) \to \mathrm{Pic}(X).$$

We would like to understand its kernel and image (uniqueness and existence of linearizations).

9.6.1. LEMMA. *Suppose $L \to X$ is a linearized line bundle with the action*

$$G \times L \to L, \quad (g, l) \to g \cdot l.$$

Then every other linearization of L is obtained as follows: choose a homomorphism $\chi : G \to \mathbb{C}^*$ (a character) and define a new action

$$(g, l) \to g \star l = \chi(g)(g \cdot l).$$

Proof. It is clear that $g \star l$ is indeed a linearization. Now suppose that we have two linearizations, $g \cdot l$ and $g \star l$. We have a function $\chi : G \times L \to \mathbb{C}^*$ defined as follows:

$$g \star l = \chi(g, l)(g \cdot l)$$

and it is clear that $\chi(g, l)$ is a character of G for any fixed l. The claim is that characters of G can not deform, i.e. $\chi(g, l)$ is locally constant in l and so in fact only depends on g.

Characters of G are the same as characters of the abelianization $G/[G, G]$, a commutative linear algebraic group. These come in several flavors:

- Algebraic tori $(\mathbb{C}^*)^n$. Their characters are given by

  $$(z_1, \ldots, z_n) \mapsto z_1^{k_1} \ldots z_n^{k_n}$$

  for a fixed vector $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ – see Lemma 9.6.2 below.
- Vector groups $\mathbb{C}^n$. They don't have non-trivial characters, in fact $\mathbb{C}^n$ obviously doesn't have any non-constant invertible functions!
- Finite abelian group A. Their characters form a "Pontryagin dual group" $\hat{A}$, non-canonically isomorphic to A.

It turns out that every commutative linear algebraic group is either a product $A \times (\mathbb{C}^*)^n \times \mathbb{C}^k$ of groups as above or its quotient by a finite subgroup $\Gamma \in A \times (\mathbb{C}^*)^n$. In particular, every function $\chi(g, l)$ is locally constant with respect to the second argument. □

9.6.2. LEMMA. *An algebraic torus T is a linearly reductive group. In fact, every algebraic representation of T is diagonalizable and is isomorphic to a direct sum of one dimensional irreducible representations given by algebraic characters $\chi : T \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$. Any character has a form*

$$(z_1, \ldots, z_n) \to z_1^{m_1} \ldots z_n^{m_n}$$

for some vector $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$.

Proof. This can be proved using unitary trick like in Theorem 8.4.2 - the analogue of S^1 (or SU_n) is the real torus $(S^1)^n \subset (\mathbb{C}^*)^n$. Just for fun, let's give a different proof. We have

$$\mathcal{O}(T) = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}],$$

the algebra of Laurent polynomials. Let $\mu \subset \mathbb{C}^*$ be the subgroup of all roots of unity. Being infinite, it is Zariski dense in $\mathbb{C}^*$. And in fact, $\mu^n \subset (\mathbb{C}^*)^n$ (the subgroup of all torsion elements) is also Zariski dense. Indeed, if

$$f(z_1, \ldots, z_n) = \sum_i g_i(z_1, \ldots, z_{n-1}) z_n^i$$

vanishes on μ^n then all functions g_i must vanish on μ^{n-1}, hence they are identically zero by inductive assumption.

Every representation $T \to \mathrm{GL}(V)$ restricts to a representation $\mu^n \to \mathrm{GL}(V)$. The image consists of commuting matrices of finite order, hence can be simultaneously diagonalizable. But then the image of T is diagonalizable in the same basis since $\mu^n \subset (\mathbb{C}^*)^n$ is Zariski dense.

For the description of one-dimensional representations, notice that a character $T \to \mathbb{C}^*$ is a non-vanishing regular function on T. We can write it as a Laurent monomial multiplied by a polynomial $f(z_1, \ldots, z_n)$ which does not vanish in $(\mathbb{C}^*)^n$. Therefore, its vanishing locus in $\mathbb{A}^n$ is a union of coordinate hyperplanes. By factoriality of the ring of polynomials (and Nullstellensatz), it follows that f is a monomial multiplied by a constant. Since $f(1, \ldots, 1) = 1$, this constant is equal to 1. $\square$

Let's address existence of linearizations. For simplicity, let $L = \mathcal{O}(D)$ be a very ample line bundle on X. If L admits a linearization then G acts on $H^0(X, L)$ and on the projectivization of this vector, the linear system $|D|$. In particular, for every $g \in G$, the divisor g^*D is linearly equivalent to D. Equivalently, the image of $\operatorname{Pic}_G X$ in $\operatorname{Pic} X$ is contained in $(\operatorname{Pic} X)^G$, the subgroup of G-invariant divisor classes.

9.6.3. EXAMPLE. If S_2 acts on $\mathbb{P}^1 \times \mathbb{P}^1$ by permuting two factors then the only polarizations that can be S_2-linearized are $\mathcal{O}(d, d)$ for some d.

9.6.4. EXAMPLE. If G is a connected linear algebraic group (like SL_n) then $(\operatorname{Pic} X)^G = \operatorname{Pic} X$. Indeed, the map

$$G \to \operatorname{Pic} X, \quad g \mapsto [g^*L]$$

must be a constant map. Indeed, $\operatorname{Pic} X$ contains an abelian variety (a projective algebraic group like an elliptic curve) such that $\operatorname{Pic} X / \operatorname{Pic}_0(X)$ is discrete. Therefore the image of G belongs to one of the cosets, which is isomorphic to an abelian variety. Abelian varieties do not contain any rational curves but G is covered by them, so the map $G \to \operatorname{Pic} X$ must be constant. Of course this is only a sketch of the proof, one needs to check that various maps are in fact regular, etc.

Suppose $L \in (\operatorname{Pic} X)^G$. Then G acts on the projective space $|D| = \mathbb{P}(V)$ and G is linearized if and only if this action can be lifted to the action on V. Indeed, in this case $\mathcal{O}(-1)$ and therefore $\mathcal{O}(1)$ on $\mathbb{P}(V)$ are G-linearized but L is just a pull-back of $\mathcal{O}(1)$. So we reduce to a classical problem of representation theory: given a homomorphism $G \to \mathrm{PGL}(V)$, when it can be lifted to a homomorphism $G \to \mathrm{GL}(V)$? This is not always the case can be done if G is simply-connected in complex topology, for example if $G = \mathrm{SL}_n$. Indeed, let $\tilde{G}$ be the preimage of G in $\mathrm{SL}(V)$. Since the kernel of the homomorphism $\mathrm{SL}(V) \to \mathrm{PGL}(V)$ is a finite group, $\tilde{G}$ is a finite cover of G. Since G is simply-connected, the connected component of $\tilde{G}$ is isomorphic to G and thus the action lifts.

§9.7. **More general GIT.** The weighted projective space is the "quotient" of the affine variety given by Proj of the algebra of semi-invariants and the GIT "quotient" of a projective variety is given by Proj of the algebra of invariants of the homogenous coordinate algebra. What's the relation?

In fact is a more general GIT construction which covers both cases. Instead of giving full details, let's explain the basic idea. Suppose G acts on an affine variety X. Regular functions on X can be identified with global sections of the trivial line bundle $L = X \times \mathbb{C}$. We have

$$H^0(X, L^{\otimes n}) = \mathcal{O}(X)$$

for every n and assembling these pieces into a graded algebra gives

$$\bigoplus_{n \geq 0} H^0(X, L^{\otimes n}) = \mathcal{O}(X) \oplus \mathcal{O}(X) \oplus \mathcal{O}(X) \oplus \ldots \simeq \mathcal{O}(X) \otimes \mathbb{C}[t]$$

with the grading by powers of t. Note however that the degree 0 part is no longer $\mathbb{C}$! From the general theory of projective spectrum, $\mathrm{Proj}\, R$ is not a projective variety if $R_0 \neq \mathbb{C}$ but rather admits a projective morphism to $\mathrm{Spec}\, R_0$, which in our case is the identity map:

$$\mathrm{Proj}\,(\mathcal{O}(X) \otimes \mathbb{C}[t]) = \mathrm{Spec}\, \mathcal{O}(X) \times \mathrm{Proj}\, \mathbb{C}[t] = \mathrm{Spec}\, \mathcal{O}(X) = X.$$

The trivial line bundle has an obvious linearization (given by the trivial action on $\mathbb{C}$). Tensor powers of L are then also trivially linearized and

$$H^0(X, L^{\otimes n})^G = \mathcal{O}(X)^G.$$

Assembling these spaces into a graded algebra gives

$$\bigoplus_{n \geq 0} H^0(X, L^{\otimes n})^G = \mathcal{O}(X)^G \oplus \mathcal{O}(X)^G \oplus \mathcal{O}(X)^G \oplus \ldots \simeq \mathcal{O}(X)^G \otimes \mathbb{C}[t]$$

and

$$\mathrm{Proj}\,\left(\mathcal{O}(X)^G \otimes \mathbb{C}[t]\right) = \mathrm{Spec}\, \mathcal{O}(X)^G \times \mathrm{Proj}\, \mathbb{C}[t] = \mathrm{Spec}\, \mathcal{O}(X)^G = X /\!\!/ G.$$

In this case the GIT gives the Mumford's categorical quotient.

Let's change the linearization by a non-trivial character $\chi : G \to \mathbb{C}^*$. Then

$$H^0(X, L)^G = \{f : X \to \mathbb{C} \mid f(gx) = \chi(g) f(x)\},$$

semi-invariants of weight χ^{-1}. It follows that

$$R = \bigoplus_{n \geq 0} H^0(X, L^{\otimes n})^G$$

is a graded subalgebra of $\mathcal{O}(X)$ of semi-invariants of weights $1, \chi^{-1}, \chi^{-2}, \ldots$.

Note that the degree 0 part is the algebra of invariants $\mathcal{O}(X)^G$. Thus $X /\!\!/_{\mathrm{GIT}} G = \mathrm{Proj}\, R$ admits a projective morphism to $X//G = \mathrm{Spec}\, \mathcal{O}(X)^G$, which is a point only if $\mathcal{O}(X)^G = \mathbb{C}$. This was the case for the weighted projective space and also in the case of the Grassmannian $G(k, n)$, which in this language is the GIT quotient of $\mathbb{A}^{kn} = \mathrm{Mat}(k, n)$ by GL_k with polarization given by the trivial line bundle and linearization given by the determinant. The unstable locus is the set of matrices of rank less than k.

§9.8. **GIT moduli space of n points in $\mathbb{P}^1$.** Consider the action of $G = \mathrm{PGL}_2$ on $X = (\mathbb{P}^1)^n$. We have $\operatorname{Pic} X \simeq \mathbb{Z}^n$ and polarizations are line bundles $L = \mathcal{O}(d_1, \ldots, d_n)$ such that $d_i > 0$ for every i. We identify the space of global sections $H^0(X, L)$ with the space of polynomial in $2n$ variables, which we arrange in a $2 \times n$ matrix

$$\begin{bmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \end{bmatrix},$$

of degree d_i in variables in the i-th column (homogeneous coordinates on the i-th copy of $\mathbb{P}^1$). The space of global sections of all line bundles (known as the Cox ring) of X is nothing but

$$\tilde{R} = \bigoplus_{d_1,\ldots,d_n>0} H^0(X, \mathcal{O}(d_1, \ldots, d_n)) = \mathcal{O}(\mathrm{Mat}_{2,n}).$$

The coordinate ring of $L = \mathcal{O}(d_1, \ldots, d_n)$ is the following subring :

$$R = \bigoplus_{kd_1,\ldots,kd_n} H^0(X, \mathcal{O}(kd_1, \ldots, kd_n)) \subset \tilde{R}.$$

A PGL_2-linearization of L induces a unique SL_2-linearization and the matrix $-\mathrm{Id}$ will act by $(-1)^{d_1+\ldots+d_n}$. Since $\mathrm{PGL}_2 = \mathrm{SL}_2 / \langle \pm \mathrm{Id} \rangle$, a PGL_2-linearization exists if and only if $d_1 + \ldots + d_n$ is even. We have

$$R^{\mathrm{PGL}_2} = R^{\mathrm{SL}_2} \subset \tilde{R}^{\mathrm{SL}_2} = \mathcal{O}(\mathrm{Mat}_{2,n})^{\mathrm{SL}_2}.$$

By the first fundamental theorem of invariant theory, this ring is generated by 2×2 Plücker minors

$$\mathcal{O}(\mathrm{Mat}_{2,n})^{\mathrm{SL}_2} = \mathbb{C}[\Delta_{ij}]/(\text{Plücker relations}). \tag{9.8.1}$$

Thus we have proved the following theorem:

9.8.2. THEOREM. *The GIT quotient of $(\mathbb{P}^1)^n$ by PGL_2 with respect to polarization $\mathcal{O}(d_1, \ldots, d_n)$ is the projective spectrum of a subring R of the ring*

$$\mathbb{C}[\Delta_{ij}]/(\text{Plücker relations})$$

of polynomials of multidegree $(kd_1, \ldots, kd_n)$ in variables $(x_1, y_2), \ldots, (x_n, y_n)$.

9.8.3. REMARK. Note that (9.8.1) is the coordinate ring of the Grassmannian

$$R(G(2,n)) = \bigoplus_{k \geq 0} H^0(G(2,n), \mathcal{L}^{\otimes k}),$$

where $\mathcal{L}$ is the Plücker polarization. Note that we have a torus $T = (\mathbb{C}^*)^n$ acting on $G(2,n)$ and $\mathcal{L}$ has a natural linearization coming from the action of T on $\mathbb{C}^n$, which can be changed by a character $z_1^{d_1} \ldots z_n^{d_n}$. We leave it to the exercises to prove *Gelfand–Macpherson correspondence*: GIT quotients of $(\mathbb{P}^1)^n$ by PGL_2 (with respect to different polarizations) are isomorphic to GIT quotients of $G(2,n)$ by $(\mathbb{C}^*)^n$ (with respect to different linearizations).

As an example, suppose $n = 4$ and consider the polarization $\mathcal{O}(1,1,1,1)$. Using the Plücker relation

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23},$$

we can rewrite any polynomial of multidegree (d_1, d_2, d_3, d_4) as a polynomial in all minors Δ_{ij} except Δ_{13}. Denoting by d_{ij} the degree with respect to Δ_{ij}, we see that

$$d_1 + d_3 = 2d_{13} + d_{12} + d_{14} + d_{23} + d_{34} \geq d_{14} + d_{12} + d_{23} + d_{34} = d_2 + d_4$$

with equality only if $d_{13} = 0$. Therefore every invariant polynomial of multidegree (d, d, d, d) is a polynomial in $\Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23}$. Moreover, since

$$d_1 = d_{12} + d_{14} = d_{12} + d_{23} = d_2$$

and $d = d_1 = d_2$, in fact we have $d_{14} = d_{23}$ and analogously $d_{12} = d_{34}$, i.e.every invariant polynomial of multidegree (d, d, d, d) is in fact a polynomial in $\Delta_{14}\Delta_{23}$ and $\Delta_{12}\Delta_{34}$. In short,

$$R^G = \bigoplus_{d,d,d,d} H^0(X, \mathcal{O}(d, d, d, d))^G = k[\Delta_{14}\Delta_{23}, \Delta_{12}\Delta_{34}].$$

Therefore, the GIT quotient is

$$\operatorname{Proj} k[\Delta_{14}\Delta_{23}, \Delta_{12}\Delta_{34}] = \mathbb{P}^1$$

and the quotient is given by

$$\begin{bmatrix} x_1 & \dots & x_4 \\ y_1 & \dots & y_4 \end{bmatrix} \mapsto [\Delta_{14}\Delta_{23}, \Delta_{12}\Delta_{34}].$$

More concretely, representing a point in $(z_1, z_2, z_3, z_4) \in (\mathbb{P}^1)^4$ by a matrix

$$\begin{bmatrix} 1 & \dots & 1 \\ z_1 & \dots & z_4 \end{bmatrix} \mapsto \frac{(z_4 - z_1)(z_3 - z_2)}{(z_2 - z_1)(z_4 - z_3)},$$

the cross-ratio.

§10. **Hilbert–Mumford criterion**

§10.1. **Case of a linear action.** Computing invariants explicitly is a daunting task that can be achieved only in a few special cases. Therefore it is important to have an efficient algorithm to describe semi-stable orbits instead of relying on Definition 9.5.8. We first explain it in a special case when a reductive group G acts on the projective space $\mathbb{P}(V)$ and the action is induced by a linear action of G on V.

10.1.1. DEFINITION. Let G be a connected reductive group. An algebraic subgroup $T \subset G$ is called a *maximal torus* if $T \simeq (\mathbb{C}^*)^n$ and T is maximal by inclusion among algebraic subtori of G.

10.1.2. THEOREM. *All maximal tori in G are conjugate.*

10.1.3. EXAMPLE. We won't prove this theorem but the result is clear for $G = \mathrm{GL}_n$. Indeed, any algebraic torus $T \subset G$ acts on $\mathbb{C}^n$ hence is diagonalizable in some basis of $\mathbb{C}^n$. Equivalently, after conjugation by a change of basis matrix, T is contained in a subgroup

$$\operatorname{diag}(z_1, \dots, z_n).$$

It follows that this subgroup is a maximal torus and every other maximal torus is conjugate to it. The same argument applies for SL_n: every maximal torus is conjugate to the standard torus

$$\operatorname{diag}(z_1, \dots, z_n), \quad \text{where } z_1 \dots z_n = 1.$$

The Hilbert–Mumford criterion consists of two parts: reduction from G to T and analysis of stability for torus actions. First the reduction part.

10.1.4. THEOREM. *Let $G \to \mathrm{GL}(V)$ be an algebraic finite-dimensional representation of a reductive group. Let $T \subset G$ be a maximal torus, and let $v \in V$. The following are equivalent:*

(1) *v is unstable, i.e. every homogeneous polynomial $f \in \mathcal{O}(V)^G$ of positive degree vanishes at v.*
(2) *The G-orbit of v contains 0 in its closure.*
(3) *There exists $u \in Gv$ such that the T-orbit Tu contains 0 in its closure.*

Proof. It is clear that (3)⇒(2)⇒(1). Theorem 8.6.5 shows that (1)⇒(2).

We will explain a difficult implication (2)⇒(3) only for $G = \mathrm{GL}_n$. Recall that every matrix $A \in \mathrm{GL}_n$ has a polar decomposition

$$A = UP,$$

where U is a unitary matrix and P is a positive-definite Hermitian matrix. Let $K := \mathrm{U}_n$ be the unitary group. By the spectral theorem, P has an orthonormal basis of eigenvectors, so we can write

$$P = U' D (U')^{-1},$$

where $U' \in K$ and $D \in T$ is a diagonal matrix. Combining these facts, we get a useful *Cartan decomposition*

$$G = KTK.$$

Analogous decomposition holds in any connected complex reductive group, where K is its maximal connected compact subgroup (which is defined uniquely up to conjugation). For example, $K = \mathrm{SU}_n$ for $G = \mathrm{SL}_n$.

By hypothesis, $0 \in \overline{Gv}$ (Zariski closure). Since $\overline{Gv}$ contains Gv as a Zariski open subset, in fact we also have $0 \in \overline{Gv}$ (closure in the Euclidean topology). Since K is compact, this implies that

$$0 \in \overline{TKv}$$

(closure in the Euclidean topology). Consider the quotient map

$$\pi_T : V \to V/\!\!/T$$

and let

$$O = \pi_T(0).$$

As any morphism of complex algebraic varieties, π_T is continuous in Euclidean topology (since polynomials are continuous functions), so we have

$$O \in \pi_T(\overline{TKv}) \quad \Rightarrow \quad O \in \overline{\pi_T(TKv)} = \overline{\pi_T(Kv)}$$

(closure in the Euclidean topology). But by compactness,

$$\overline{\pi_T(Kv)} = \pi_T(Kv),$$

and so there exists $g \in K$ such that $\pi_T(gv) = O$, i.e. $0 \in \overline{Tu}$ for $u = gv$. $\square$

§10.2. **Unstable locus of torus linear actions.** Let $T = (\mathbb{C}^*)^n$ be an algebraic torus and let $M \simeq \mathbb{Z}^n$ be its lattice of characters. For every $m = (m_1, \dots, m_n) \in M$, the corresponding character is

$$\chi_m : T \to \mathbb{C}^*, \quad z = (z_1, \dots, z_n) \mapsto z^m := z_1^{m_1} \dots z_n^{m_n}.$$

Consider any finite-dimensional representation $T = (\mathbb{C}^*)^n \to \mathrm{GL}(V)$. Let

$$V = \bigoplus_{m \in M} V_m$$

be the decomposition of V into T-eigenspaces.

10.2.1. DEFINITION. For any $u \in V$, let $u = \sum u_m$ be the decomposition of u into T-eigenvectors. Then

$$NP(u) = \text{Convex Hull}\{m \in \mathbb{Z}^n \mid u_m \neq 0\}$$

is called the *Newton polytope* of u.

10.2.2. THEOREM. *Let $(\mathbb{C}^*)^n \to \mathrm{GL}(V)$ be a finite-dimensional algebraic representation. Let $v \in V$. TFAE:*

(1) *v is unstable.*
(2) *$0 \notin NP(v)$.*

Proof. There are two "dual" ways to study convexity: using positive linear combinations and using supporting hyperplanes. We need the following lemma, known as Farkas Lemma, Gordan Theorem, etc.

10.2.3. LEMMA. *Let $S \subset \mathbb{R}^n$ be a convex hull of lattice points $m_1, \dots, m_k \in \mathbb{Z}^n$. Then*

- *$0 \notin S$ if and only if there exists $u \in \mathbb{Z}^n$ such that $u \cdot m_i > 0$ for every i.*
- *$0 \in S$ if and only if there exist integers $\alpha_1, \dots, \alpha_k \geq 0$ such that*

$$0 = \sum \alpha_i m_i \quad \text{and} \quad \sum \alpha_i > 0.$$

Now we can prove the Theorem. If $0 \notin NP(v)$ then by Lemma we can choose a vector $u = (u_1, \dots, u_n) \in \mathbb{Z}^n$ such that $u \cdot v_i > 0$ for any i. Consider a subgroup $\chi(t) = (t^{u_1}, \dots, t^{u_n}) \subset T$. Then we have

$$\chi(t) \cdot v = \sum_{m \in \mathbb{Z}^n} \chi(t) \cdot v_m = \sum_m t^{m \cdot u} v_m.$$

Therefore $0 \in \overline{Tv}$ because

$$\lim_{t \to 0} \chi(t) \cdot v = 0.$$

On the other hand, let's suppose that $0 \in NP(v)$. By Lemma 10.2.3, we can choose integers $\alpha_m \geq 0$ (indexed by m such that $v_m \neq 0$), not all of them equal to 0, such that

$$0 = \sum_{v_m \neq 0} \alpha_m m$$

Set $\alpha_m = 0$ if $v_m = 0$. Choose coordinates f_m on V dual to the basis of eigenvectors for the $(\mathbb{C}^*)^n$ action. Consider the function

$$I = \prod f_m^{\alpha_m}.$$

Then $I(v) \neq 0$ and I is T-invariant:

$$I(t \cdot v) = \prod f_m^{\alpha_m}(t \cdot v) = \prod f_m^{\alpha_m}(t \cdot v_m) = \prod f_m^{\alpha_m}(t^m v_m)$$
$$= t^{\sum \alpha_m m} \prod f_m^{\alpha_m}(v_m) = \prod f_m^{\alpha_m}(v) = I(v),$$

where notation t^m stands for $z_1^{m_1} \ldots z_n^{m_n}$ for $t = (z_1, \ldots, z_n)$ and $m = (m_1, \ldots, m_n)$. So 0 is not in the closure of Tv. $\square$

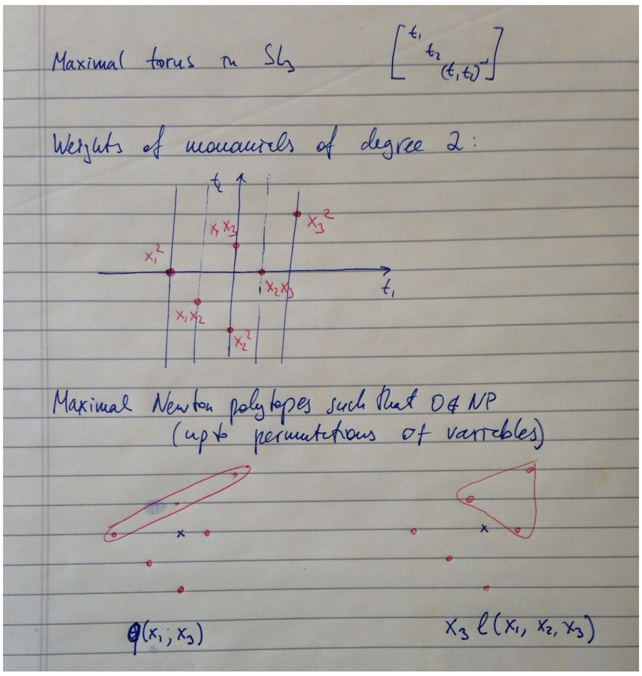
10.2.4. EXAMPLE. Consider the action of SL_2 on the space $V_d = \mathbb{C}[x, y]_d$ of binary forms of degree d. The maximal torus T in SL_2 consists of diagonal matrices

$$\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$$

and the weights of the monomials are

$$z^{-d}, z^{-d+2}, \ldots, z^d.$$

The unstable locus S for the T-action is the union of two components: S_+ (resp. S_-) is a linear span of monomials $x^m y^{d-m}$ (resp. $x^{d-m} y^m$) for $m > d/2$. By the Hilbert–Mumford criterion, the unstable locus for the SL_2-action is $\mathrm{SL}_2 \cdot S$. Concretely, it is the set of binary forms with a multiple root of multiplicity $m > d/2$.



10.2.5. EXAMPLE. Consider the action of SL_3 on degree 2 polynomials in three variables. The following analysis illustrates that the unstable locus consists of reducible conics. In fact $\mathcal{O}(V_2)^{\mathrm{SL}_3}$ is generated by a single invariant, namely the discriminant.

§10.3. **General case.** Now consider any action of a reductive group G on a projective variety X with a G-linearized ample line bundle $\mathcal{L}$.

10.3.1. DEFINITION. A 1-parameter subgroup of G is a homomorphism

$$\lambda : \mathbb{C}^* \to G, \quad t \mapsto \lambda(t).$$

Suppose $x \in X$ is a λ-fixed point. Since $\mathcal{L}$ is linearized, λ then acts on the fiber $\mathcal{L}|_x$ over x by a character

$$\lambda(t) \cdot \mathcal{L}|_x = t^w \mathcal{L}|_x.$$

The number w is called the weight of λ at x, denoted by $\mathrm{wt}_\lambda \mathcal{L}|_x$. Let Z_λ be the union of points such that $\mathrm{wt}_\lambda < 0$. Let Σ_Λ be the union of points $x \in X$ such that

$$\lim_{t\to 0} \lambda(t)x \in Z_\lambda.$$

Finally, let

$$S_\lambda = G \cdot \Sigma_\lambda.$$

10.3.2. THEOREM. *The unstable locus X_{us} is the union of strata S_λ for all one-parameter subgroups λ, which can be taken up to conjugacy.*

10.3.3. EXAMPLE. Let $G = \mathrm{PGL}_2$ acting on $X = (\mathbb{P}^1)^n$ with a G-linearized ample line bundle $\mathcal{L} = \mathcal{O}(d_1, \ldots, d_n)$, $d_i > 0$ for every i and $\sum d_i$ even.

Up to conjugation, a 1-parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{PGL}_2$ has the form

$$\lambda(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.$$

Its fixed points in $\mathbb{P}^1$ are $0 = [0:1]$ and $\infty = [1:0]$. We have

$$\mathrm{wt}_\lambda \mathcal{O}_{\mathbb{P}^1}(1)_{|p} = \begin{cases} 1 & \text{if} \quad p = 0 \\ -1 & \text{if} \quad p = \infty \end{cases}$$

The λ-fixed points in $(\mathbb{P}^1)^n$ have the following description: fix a subset $K \subseteq \{1, \ldots, n\}$ and consider

$$z_K = (p_1, \ldots, p_n), \text{ with } p_i = \left\{ \begin{array}{ll} 0 & \text{if } i \notin K \\ \infty & \text{if } i \in K \end{array} \right\}.$$

It follows that

$$\mathrm{wt}_\lambda \mathcal{L}_{|z_K} = \sum_{i \in K^c} d_i - \sum_{i \in K} d_i.$$

The weight is negative if

$$\sum_{i \in K} d_i > \sum_{i \in K^c} d_i. \tag{10.3.4}$$

Take any point $p = (p_1, \ldots, p_n)$. Then

$$\lim_{t\to 0} \lambda(t)p = z_K$$

if and only of $p_i = \infty$ for $i \in K$ and $p_i \neq \infty$ for $i \in K^c$. Thus

$$\Sigma_K = (p_1, \ldots, p_n), \text{ with } p_i \left\{ \begin{array}{ll} \neq \infty & \text{if } i \notin K \\ = \infty & \text{if } i \in K \end{array} \right\}.$$

Let Δ_K be a diagonal in $(\mathbb{P}^1)^n$ consisting of points $(p_1, \ldots, p_n)$ such that for all $i \in K$ the points p_i are equal. Let $S_K \subset \overline{\Delta}_K$ be a locally closed diagonal (the complement to the union of other diagonals). Then

$$S_K = G \cdot \Sigma_K.$$

The unstable locus is the union of diagonals Δ_K such that (10.3.4) holds.

If the polarization is symmetric ($d_1 = \ldots = d_n$) the condition just means that more than half of the n points are equal.

Proof of Theorem 10.3.2. We only sketch the proof and leave details to exercises. By considering a high tensor power of $\mathcal{L}$ (which doesn't change neither the unstable locus nor the sign of weights of one-parameter subgroups), we can assume without loss of generality that $\mathcal{L}$ is very ample that the coordinate algebra R of $\mathcal{L}$ is generated by R_1. Choose generators $r_0, \ldots, r_n$. Then we have a surjection

$$\mathbb{C}[x_0, \ldots, x_n] \to R$$

which induces a G-equivariant embedding

$$X \hookrightarrow \mathbb{P}^n.$$

Since G is reductive, we also have a surjection

$$\mathbb{C}[x_0, \ldots, x_n]^G \to R^G,$$

which shows that the unstable locus of in X is the intersection of X with the unstable locus in $\mathbb{P}^n$. It remains to interpret the definition of Σ_λ in terms of Lemma 10.2.3, which we leave as an exercise. □

§10.4. **Stability of smooth hypersurfaces.**

10.4.1. DEFINITION. For a reductive group G acting on an affine variety X, we call a point of X *stable* if its orbit is closed and its stabilizer is finite.

10.4.2. THEOREM. *Let $\pi : X \to X/\!\!/G = \operatorname{MaxSpec} \mathcal{O}(X)^G$ be the quotient for an action of the reductive group on an affine variety. Let*

$$X^s \subset X$$

be the set of stable points and let

$$Z \subset X$$

be the subset of points such that G_x is not finite. Then Z is closed, X^s is open and X_s is the complement of $\pi^{-1}(\pi(Z))$. The quotient π induces a $1-1$ bijection between G-orbits in X^s and points in $\pi(X^s)$.

Proof. Consider the map

$$G \times X \to X \times X, \qquad (g, x) \mapsto (gx, x).$$

Let $\tilde{Z}$ be the preimage of the diagonal. It is closed. But

$$Z = \{x \in X \mid \dim(\pi_2|_{\tilde{Z}})^{-1}(x) > 0\}.$$

Thus Z is closed by semi-continuity of dimension of fibers. Since Z is clearly G-invariant, $\pi(Z)$ is also closed (a HW exercise).

Now suppose $x \in \pi^{-1}(\pi(Z))$. Then the fiber of π through x contains a closed orbit with a positive-dimensional stabilizer. Thus either Gx is not closed or G_x is positive-dimensional. In any case x is not stable.

If $x \notin \pi^{-1}(\pi(Z))$ then G_x is finite. If Gx is not closed then a closed orbit in the closure of Gx also does not belong to $\pi^{-1}(\pi(Z))$, which is a contradiction. So in fact x is stable. □

We would like to prove a classical theorem of Matsumura, Monsky and Mumford that smooth hypersurfaces are GIT stable. Specifically, consider the representation of SL_{n+1} in the vector space

$$V_{n,d} = \mathrm{Sym}^d(\mathbb{C}^{n+1})^*$$

which parametrizes polynomials of degree d in $n+1$ variables. Let

$$U_{n,d} \subset V_{n,d}$$

be the locus of polynomials F such that the corresponding hypersurface $V(F) \subset \mathbb{P}^n$ is smooth. Let $\mathcal{D}_{n,d}$ be the complement, the *discriminant set*.

10.4.3. THEOREM. *$\mathcal{D}_{n,d}$ is an irreducible hypersurface. Its defining equation $D_{n,d}$ (called the discriminant) belongs to $\mathcal{O}(V_{n,d})^{\mathrm{SL}_{n+1}}$.*

Proof. The proof is by *dimension count*. Consider the *incidence subset*

$$Z \subset \mathbb{P}(V^{n,d}) \times \mathbb{P}^n$$

of pairs (F, z) such that $z \in \mathrm{Sing}(F = 0)$. This is a closed subset defined by vanishing of partial derivatives of F. All fibers of the projection of Z onto $\mathbb{P}^n$ are projective spaces of dimension $\dim \mathbb{P}(V_{n,d}) - n - 1$ (*why?*) . Therefore Z is irreducible (in fact smooth) and

$$\dim Z = \dim \mathbb{P}(V_{n,d}) - 1$$

by the theorem on dimension of fibers. Notice that the projectivization of $\mathcal{D}_{n,d}$ is the image of Z. Thus $\mathcal{D}_{n,d}$ is irreducible and to count its dimension it suffices to show that a general hypersurface singular at $z \in \mathbb{P}^n$ is singular only there: this would imply that the first projection

$$Z \to \mathcal{D}_{n,d}$$

is birational (giving resolution of singularities of the discriminant locus). But this is easy: just take the cone over any smooth hypersurface $S \subset \mathbb{P}^{n-1}$. The only singular point of a cone is the vertex. □

10.4.4. THEOREM (Matsumura–Monsky–Mumford). *Every smooth hypersurface of degree $d \geq 3$ is stable.*

Proof. The main point is that every point $F \in U_{n,d}$ has a finite stabilizer in SL_{n+1}. Given that, we claim that F is stable, i.e. its orbit is in fact closed. Indeed, if F' is the closure of the SL_{n+1}-orbit of F then F' has positive-dimensional stabilizer. On the other hand, the discriminant of F' has to be equal to the discriminant of F, so it's not equal to 0. Thus $(F' = 0)$ is also smooth, contradiction. For a simple proof that the stabilizer is finite, see my book *Projective duality and homogeneous spaces*. □

As the main application, we can now consider the GIT quotient

$$\mathbb{P}(V_{n,d})/\!\!/_{GIT}\,\mathrm{SL}_{n+1},$$

which by what we have just proved compactifies the principal open subset

$$(\mathbb{P}(V_{n,d})\setminus\mathcal{D}_{n,d})/\!\!/\,\mathrm{SL}_{n+1},$$

the moduli space of non-singular hypersurfaces.

What kind of moduli spaces one can get this way?

10.4.5. EXAMPLE. Let $n=1$, $d=2g+2\geq 6$. The GIT quotient compactifies the moduli space of unordered d-tuples of distinct points in $\mathbb{P}^1$. A double cover of $\mathbb{P}^1$ ramified at these points is a hyperelliptic curve of genus g by the Riemann-Hurwitz formula. One can show that every hyperelliptic curve of genus $g>1$ can be written as a double cover of $\mathbb{P}^1$ uniquely. Thus the GIT quotient in this case is the compactification of H_g, the coarse moduli space of hyperelliptic curves.

10.4.6. EXAMPLE. Let $n=2$, $d=4$. The GIT quotient compactifies the moduli space of smooth quartic curves in $\mathbb{P}^2$, or equivalently $M_3\setminus H_3$, the coarse moduli space of non-hyperelliptic curves of genus 3.

10.4.7. EXAMPLE. Let $n=2$, $d=6$. The GIT quotient compactifies the moduli space of smooth sextic curves in $\mathbb{P}^2$. A double cover of $\mathbb{P}^2$ branched along a sextic is a polarized K3 surface. The GIT quotient compactifies one of the components in the moduli space of polarized $K3$ surfaces, of dimension

$$\binom{6+2}{2}-1-8=19.$$

10.4.8. EXAMPLE. Let $n=3$, $d=4$. The GIT quotient compactifies the moduli space of smooth quartic surfaces, another component in the moduli space of $K3$ surfaces, of dimension

$$\binom{4+3}{3}-1-15=19.$$

§10.5. **Homework 5.**

Problem 1. (1 point) Check (9.2.2).

Problem 2. (2 points) Finish the proof of Lemma 9.4.3.

Problem 3. (2 points) A weighted projective space $\mathbb{P}(a_0,\ldots,a_n)$ is called well-formed if no n of the weights $a_0,\ldots,a_n$ have a common factor. For example, $\mathbb{P}(1,1,3)$ is well-formed but $\mathbb{P}(2,2,3)$ is not. Consider the polynomial ring $R=\mathbb{C}[x_0,\ldots,x_n]$, where x_i has weight a_i. (a) Let $d=\gcd(a_0,\ldots,a_n)$. Show that $R^{(d)}=R$ and $\mathbb{P}(a_0,\ldots,a_n)\simeq\mathbb{P}(a_0/d,\ldots,a_n/d)$. (b) Let $d=\gcd(a_1,\ldots,a_n)$ and suppose that $(a_0,d)=1$. Compute $R^{(d)}$ and show that $\mathbb{P}(a_0,\ldots,a_n)\simeq\mathbb{P}(a_0,a_1/d\ldots,a_n/d)$. Conclude that every weighted projective space is isomorphic to a well-formed one.

Problem 4. (2 points) Compute $\operatorname{Proj}\mathbb{C}[x,y,z]/(x^5+y^3+z^2)$. Here x has weight 12, y has weight 20, and z has weight 30.

Problem 5. (2 points) Let $C \subset \mathbb{P}^d$ be a rational normal curve of degree d, let $\hat{C} \subset \mathbb{A}^{d+1}$ be the affine cone over it, and let $\bar{C} \subset \mathbb{P}^{d+1}$ be its projective closure. Show that $\bar{C}$ is isomorphic to $\mathbb{P}(1,1,d)$.

Problem 6. (2 points) Let $P \in E$ be a pointed elliptic curve. Consider the graded algebra $R = \bigoplus_{k \geq 0} H^0(E, \mathcal{O}(kP))$. Find all d such that the Veronese subalgebra $R^{(d)}$ is generated by R_d.

Problem 7. (2 points) Let G be a reductive group acting on a projective variety X with a G-linearized ample line bundle L. Let X^{ss} be the set of semistable points. Show that the GIT quotient $X/\!\!/_{GIT}G$ is a categorical quotient of X^{ss} by G in the sense of Mumford. Moreover, $X/\!\!/_{GIT}G$ is a geometric quotient if all G-orbits in X^{ss} are closed.

In problems 8–11, we consider linear actions of algebraic groups with a standard linearization.

Problem 8. (2 points) Use the Hilbert–Mumford criterion to show that a degree 3 polynomial in 3 variables in $\mathrm{Sym}^3 \mathbb{C}^3$ is semistable for the action of SL_3 if and only if the corresponding cubic curve is smooth or nodal.

Problem 9. (2 points) Use the Hilbert–Mumford criterion to describe the unstable locus for the action of GL_3 on $\mathrm{Mat}_{3,3}$ by conjugation. Then compute generators of the algebra of invariant polynomials and their common zero locus. Why is the answer the same?

Problem 10. (3 points) Show that a degree 4 polynomial in 3 variables in $\mathrm{Sym}^4 \mathbb{C}^3$ is semistable for the action of SL_3 if and only if the corresponding quartic curve in $\mathbb{P}^2$ has no triple points[25] and is not the union of the plane cubic and an inflectional tangent line.

Problem 11. (3 points) Show that a degree 3 polynomial in 4 variables in in $\mathrm{Sym}^3 \mathbb{C}^4$ is semistable for the action of SL_4 if and only if all points of the corresponding cubic surface $S \subset \mathbb{P}^3$ are either smooth, or ordinary double points, or double points p such that (after a linear change of variables) the quadratic part of $F(x,y,z,1)$ is xy and the line $x = y = 0$ is not contained in S. Show that (after a holomorphic change of variables) the last singularities are A_2 singularities.

Problem 12. (2 points) In Example 9.5.10, describe all possible polarizations and linearizations for the action of G on X. For every choice, describe the unstable locus, the GIT quotient and the quotient map.

Problem 13. (2 points) Describe unstable locus for the action of $(\mathbb{C}^*)^n$ on $G(2,n)$. Use Plücker polarization and symmetric linearization.

Problem 14. (2 points) Prove Gelfand–Macpherson correspondence of Remark 9.8.3.

[25] A hypersurface $F(x_0, \dots, x_n) \subset \mathbb{P}^n$ has a point of multiplicity d at $p \in \mathbb{P}^n$ if the following holds. Change coordinates so that $p = [0 : \dots : 0 : 1]$. Then $F(x_0, \dots, x_{n-1}, 1)$ should have no terms of degree less than d. A point of multiplicity 2 (resp. 3) is called a *double* (resp. *triple*) point. A point p is called an *ordinary double point* (or a *node*) if p is a double point and the quadratic part of $F(x_0, \dots, x_{n-1}, 1)$ is non-degenerate.

Problem 15. (2 points) Prove using theorem 9.8.2 (but without using the Hilbert–Mumford stability criterion) that the unstable locus of the GIT quotient of $(\mathbb{P}^1)^n$ by PGL_2 with respect to a symmetric polarization $\mathcal{O}(d,\ldots,d)$ consists of n-tuples of points such that more than half of them are equal.

Problem 16. (3 points) Prove using theorem 9.8.2 that the GIT quotient of $(\mathbb{P}^1)^n$ by PGL_2 with respect to the symmetric polarization $\mathcal{O}(1,\ldots,1)$ is a cubic hypersurface in $\mathbb{P}^4$. What is its equation? Use computer if necessary.

Problem 17. (2 points) Describe the unstable locus for the action of PGL_3 on $(\mathbb{P}^2)^5$. Use "symmetric" polarization $\mathcal{O}(d,d,d,d,d)$.

Problem 18. (2 points) Finish the proof of Theorem 10.3.2.

§11. **Genus 2 curves.**

Our next goal is to study the moduli space M_2 of algebraic curves of genus 2. Incidentally, this will also give us the moduli space A_2 of *principally polarized Abelian surfaces*, talgebraic surfaces isomorphic to $\mathbb{C}^2/\Lambda$, where $\Lambda \simeq \mathbb{Z}^4$ is a lattice. So Abelian surfaces are naturally Abelian groups just like elliptic curves. We will see that M_2 embeds in A_2 as an open subset (via the *Jacobian* construction) and the complement $A_2 \setminus M_2$ parametrizes split Abelian surfaces of the form $E_1 \times E_2$, where E_1 and E_2 are elliptic curves. The map $M_g \hookrightarrow A_g$ can be constructed in any genus (its injectivity is called the *Torelli theorem*) but the dimensions are vastly different:

$$\dim M_g = 3g-3 \quad \text{and} \quad \dim A_g = \frac{g(g+1)}{2}.$$

The characterization of M_g as a sublocus of A_g is called the *Shottky problem*.

§11.1. **Genus 2 curves: analysis of the canonical ring.**

Let's start with a basic Riemann–Roch analysis of a genus 2 curve C. We fix a canonical divisor K. We have

$$\deg K = 2\times g - 2 = 2 \quad \text{and} \quad l(K) = g = 2.$$

So we can assume that

$$K \geq 0$$

is an effective divisor. by Riemann–Roch, for any point $P \in C$,

$$l(K-P) - l(K-(K-P)) = 1 - 2 + \deg(K-P) = 0.$$

Since $l(P) = 1$ (otherwise C is isomorphic to $\mathbb{P}^1$), we have $l(K-P) = 1$. So $|K|$ has no fixed part, and therefore gives a degree 2 map

$$\phi_{|K|} : C \to \mathbb{P}^1.$$

By Riemann–Hurwitz, it has 6 ramification points called *Weierstrass points*. We also see that C admits an involution permuting two branches of $\phi_{|2K|}$. It is called the *hyperelliptic involution*.

Now consider $|3K|$. By Riemann–Roch, we have $l(3K) = 5$ and $l(3K - P - Q) = 3$ for any points $P, Q \in C$. It follows that $|3K|$ is very ample and gives an embedding

$$C \hookrightarrow \mathbb{P}^4.$$

To get a bit more, we observe that most of geometry of C is nicely encoded in the *canonical ring*

$$R(K) = \bigoplus_{n=0}^{\infty} \mathcal{L}(nK).$$

We can give a more general definition:

11.1.1. DEFINITION. Let $D \geq 0$ be an effective divisor on a curve C. Its graded algebra is defined as follows:

$$R(D) = \bigoplus_{n=0}^{\infty} \mathcal{L}(nD).$$

This is a graded algebra: notice that if $f \in \mathcal{L}(aD)$ and $g \in \mathcal{L}(bD)$ then

$$(fg) + (a+b)D = (f) + aD + (g) + bD \geq 0,$$

so $fg \in \mathcal{L}(a+b)D$.

11.1.2. REMARK. We have only defined divisors on curves in this class, but in principle it is no harder to defined a graded algebra of any divisor on an algebraic variety of any dimension. The canonical ring $R(K)$ of a smooth variety of dimension n was a subject of a really exciting research in the last 30 years which culminated in the proof of a very important theorem of Siu and Birkar–Cascini–Hacon–McKernan: $R(K)$ is a finitely generated algebra. This does not sound like much, but it allows us to define $\operatorname{Proj} R(K)$, the so-called canonical model of X. It is easy to see that it depends only on the field of rational functions $\mathbb{C}(X)$. In the curve case, C is uniquely determined by its field of functions, by in dimension > 1 it is easy to modify a variety without changing its field of rational functions (e.g. by blow-ups). So it is very handy to have this canonical model of the field of rational functions. There exists a sophisticated algorithm, called the Minimal Model Program, which (still conjecturally) allows one to construct the canonical model by performing a sequence of basic "surgeries" on X called divisorial contractions and flips.

We can compute the Hilbert function of $R(K)$ by Riemann–Roch:

$$h_n(R(K)) = l(nK) = \begin{cases} 1 & \text{if} \quad n = 0 \\ 2 & \text{if} \quad n = 1 \\ 3 & \text{if} \quad n = 2 \\ 5 & \text{if} \quad n = 3 \\ 2n-1 & \text{if} \quad n \geq 2. \end{cases}$$

Let's work out the generators. $\mathcal{L}(0) = \mathbb{C}$ is generated by 1. This is a unity in $R(K)$. Let x_1, x_2 be generators of $\mathcal{L}(K)$. One delicate point here is that we can (and will) take x_1 to be $1 \in \mathbb{C}(C)$, but it should not be confused with a previous 1 because it lives in a different degree in $R(K)$! In other words, $R(K)$ contains a graded polynomial subalgebra $\mathbb{C}[x_1]$, where any power x_1^n is equal to 1 as a rational function on C.

Any other element of first degree has pole of order 2 at K (because if it has a pole of order 1, it would give an isomorphism $C \simeq \mathbb{P}^1$.

A subalgebra $S = \mathbb{C}[x_1, x_2]$ of R is also a polynomial subalgebra: if we have some homogeneous relation $f(x_1, x_2)$ of degree d then we have

$$f(x_1, x_2) = \prod_{i=1}^{d}(\alpha_i x_1 + \beta_i x_2) = 0 \quad \text{in} \quad \mathbb{C}(C),$$

which implies that $\alpha_i x_1 + \beta_i x_2 = 0$ for some i, i.e. that x_1 and x_2 are not linearly independent, contradiction.

The Hilbert function of S is

$$h_n(S) = \begin{cases} 1 & \text{if} \quad n = 0 \\ 2 & \text{if} \quad n = 1 \\ 3 & \text{if} \quad n = 2 \\ 4 & \text{if} \quad n = 3 \\ n & \text{if} \quad n \geq 2. \end{cases}$$

So the next generator we need for $R(K)$ is a generator y in degree 3.

What happens in degree 4? We need 7 elements and we have 7 elements

$$x_1^4,\ x_1^3 x_2,\ x_1^2 x_2^3,\ x_1 x_2^3,\ x_2^4, \quad y x_1, y x_2.$$

We claim that they are indeed linearly independent, and in fact we claim:

11.1.3. LEMMA. *There is no linear relation in $\mathbb{C}(C)$ of the form*

$$y f_k(x_1, x_2) = f_{k+3}(x_1, x_2),$$

where the lower index is the degree. In particular, $R(K)$ is generated by x_1, x_2, y.

Proof. Suppose the linear relation of the form above exists. Then y, as a rational function on C, is a rational function $f(x_1, x_2)$. One can show that this is impossible either by an elementary analysis of possible positions of roots of y and this rational function $f(x_1, x_2)$ or by simply invoking the fact that as we already know $3K$ is very ample, and in particular functions in $|3K|$ separate points of C. But if y is a rational function in x_1 and x_2 then y takes the same values on two points from each fiber of $\phi_{|2K|}$. □

It follows that

11.1.4. LEMMA. *$R(K)$ is isomorphic to a polynomial algebra in x_1, x_2, y modulo a relation*

$$y^2 = f_6(x_1, x_2),$$

where f_6 is a polynomial of degree 6.

Proof. We already know that $R(K)$ is generated by x_1, x_2, y, and that $y \notin \mathbb{C}(x_1, x_2)$. It follows that y^2, $y\mathbb{C}[x_1, x_2]_3$, and $\mathbb{C}[x_1, x_2]_6$ are linearly dependent in $R(K)_6$ and this gives the only relation in $R(K)$:

$$y^2 = y f_3(x_1, x_2) + f_6(x_1, x_2).$$

We can make a change of variables $y' = y - \frac{1}{2} f_3$ to complete the square, which brings the relation in the required form. □

§11.2. **Graded algebra of an ample divisor.** Now let's interpret these algebraic results geometrically. The basic fact is:

11.2.1. LEMMA. *If D is an ample divisor on a curve C then* $\operatorname{Proj} R(D) = C$.

Proof. If D is very ample and $R(D)$ is generated by $R(D)_1$ then $R(D)$ is isomorphic to a a polynomial algebra in $x_0, \ldots, x_N \in \mathcal{L}(D)$ modulo the relations that they satisfy, i.e. $R(D) = \mathbb{C}[x_0, \ldots, x_N]/I$, where I is a homogeneous ideal of $C \subset \mathbb{P}^N$. So in this case clearly $\operatorname{Proj} R(D) = C$. In general, if D is ample then kD is very ample for some $k > 0$. Also, we know by Lemma 9.4.3 that the Veronese subalgebra $R(lD) = R(D)^{(l)}$ is generated by its first graded piece for some $l > 0$. So klD is a very ample divisor and $R(klD) = R^{(kl)}$ is generated by its first graded piece. Then we have $\operatorname{Proj} R(D) = \operatorname{Proj} R(klD) = C$. We are not using here that C is a curve, so if you know your divisors in higher dimension, everything works just as nicely. □

As a corollary, we have

11.2.2. COROLLARY. *Let C be a genus 2 curve. Then $R(K)$ induces an embedding*

$$C \subset \mathbb{P}(1,1,3)$$

and the image is defined by an equation

$$y^2 = f_6(x_1, x_2). \tag{11.2.3}$$

The embedding misses a singularity of $\mathbb{P}(1,1,3)$ (where $x_1 = x_2 = 0$, $y = 1$). In the remaining two charts of $\mathbb{P}(1,1,3)$, the curve is given by equations

$$y^2 = f_6(1, x_2) \quad \text{and} \quad y^2 = f_6(x_1, 1).$$

The projection onto $\mathbb{P}^1_{[x_1:x_2]}$ is a bicanonical map $\phi_{|2K|}$ and roots of f_6 are branch points of this $2:1$ cover. In particular, f_6 has no multiple roots and any equation of the form (11.2.3) *defines a genus 2 curve.*

The tricanonical embedding $C \subset \mathbb{P}^4$ factors through the Veronese embedding

$$\mathbb{P}(1,1,3) \hookrightarrow \mathbb{P}^4, \quad (x_1, x_2, x_3, y) \mapsto [x_1^3 : x_1^2 x_2 : x_1 x_2^2 : x_2^3 : y],$$

where the image is a projectivized cone over a rational normal curve.

This sets up a bijection between curves of genus 2 and unordered 6-tuples of distinct points $p_1, \ldots, p_6 \in \mathbb{P}^1$ modulo PGL_2. We are going to use this to construct M_2. The classical way of thinking about 6 unordered points in $\mathbb{P}^1$ is to identify them with roots of a binary form $f_6(x_1, x_2)$ of degree 6. Let V_6 be a vector space of all such forms and let $D \subset \mathbb{P}(V_6)$ be the discriminant hypersurface (which parameterizes binary sextics with multiple roots). Thus we have (set-theoretically):

$$M_2 = (\mathbb{P}(V_6) \setminus D)/\operatorname{PGL}_2 .$$

§11.3. **Classical invariant theory of a binary sextic.** We have to describe the algebra $R = \mathcal{O}(V_6)^{\mathrm{SL}_2}$ of SL_2-invariant polynomial functions for the linear action of SL_2 on V_6. The classical convention for normalizing the coefficients of a binary form is to divide coefficients by the binomial coefficients:

$$f_6 = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6.$$

Explicit generators for R were written down in the 19-th century by Clebsch, Cayley, and Salmon. We are not going to prove that they indeed generate the algebra of invariants but let's discuss them to see how beautiful the answer is. Let $p_1, \ldots, p_6$ denote the roots of the dehomogenized form $f_6(x, 1)$ and write (ij) as a shorthand for $p_i - p_j$. Then we have the following generators (draw some graphs):

$$I_2 = a^2 \sum_{\text{fifteen}} (12)^2(34)^2(56)^2$$

$$I_4 = a^4 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2$$

$$I_6 = a^6 \sum_{\text{sixty}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2$$

$$D = I_{10} = a^{10} \prod_{i<j} (ij)^2$$

$$I_{15} = a^{15} \sum_{\text{fifteen}} ((14)(36)(52) - (16)(32)(54)).$$

Here the summations are chosen to make the expressions S_6-invariant. In particular, they can all be expressed as polynomials in $\mathbb{C}[a, b, c, d, e, f, g]$, for example

$$I_2 = -240(ag - 6bf + 15ce - 10d^2). \tag{11.3.1}$$

Here is the main theorem:

11.3.2. THEOREM. *The algebra $R = \mathcal{O}(V_6)^{\mathrm{SL}_2}$ is generated by invariants I_2, I_4, I_6, I_{10}, and I_{15}. The subscript is the degree. Here $D = I_{10}$ is the discriminant which vanishes iff the binary form has a multiple root. The unique irreducible relation among the invariants is*

$$I_{15}^2 = G(I_2, I_4, I_6, I_{10}).$$

Now we use our strategy to construct M_2:

- Compute $V_6 /\!/ \mathrm{SL}_2 = \operatorname{MaxSpec} R$ first. By 19-th century, this is
$$\mathbb{C}[I_2, I_4, I_6, I_{10}, I_{15}]/(I_{15}^2 = G(I_2, I_4, I_6, I_{10})).$$
- Now quotient the result by $\mathbb{C}^*$, i.e. compute $\operatorname{Proj} R$. Here we have a magical simplification: $\operatorname{Proj} R = \operatorname{Proj} R^{(2)}$ but the latter is generated by I_2, I_4, I_6, I_{10}, and I_{15}^2. Since I_{15}^2 is a polynomial in other invariants, in fact we have
$$\operatorname{Proj} R^{(2)} = \operatorname{Proj} \mathbb{C}[I_2, I_4, I_6, I_{10}] = \mathbb{P}(2, 4, 6, 10) = \mathbb{P}(1, 2, 3, 5).$$
- To get M_2, remove a hypersurface $D = 0$, i.e. take the chart $D_{I_{10}}$ of $\mathbb{P}(1, 2, 3, 5)$. This finally gives
$$M_2 = \mathbb{A}^3/\mu_5,$$
where μ_5 acts with weights $1, 2, 3$.

- One can show that $\mathbb{C}[A,B,C]^{\mu_5}$ has 8 generators. So as an affine variety, we have

$$M_2 = (\mathbb{P}(V_6) \setminus D)/\operatorname{PGL}_2 \hookrightarrow \mathbb{A}^8,$$
$$\{y^2 = f(x)\} \mapsto \left(\frac{I_2^5}{I_{10}}, \frac{I_2^3 I_4}{I_{10}}, \frac{I_2 I_4^2}{I_{10}}, \frac{I_4^5}{I_{10}^2}, \frac{I_2^2 I_6}{I_{10}}, \frac{I_2 I_6^3}{I_{10}^2}, \frac{I_6^5}{I_{10}^3}, \frac{I_4 I_6}{I_{10}}\right).$$

This of course leaves more questions then gives answers:

(1) How do we know that points of M_2 correspond to isomorphism classes of genus 2 curves? In other words, why is it true that our quotient morphism
$$\mathbb{P}(V_6) \setminus D \to \mathbb{A}^3/\mu_5$$
is surjective and separates PGL_2-orbits? It is of course very easy to give examples of quotients by infinite group actions that do not separate orbits.
(2) Can one prove the finite generation of the algebra of invariants and separation of orbits by the quotient morphism without actually computing the algebra of invariants?
(3) Is M_2 a coarse moduli space (and what is a family of genus 2 curves)?
(4) Our explicit description of M_2 as $\mathbb{A}^3/\mu_5$ shows that it is singular. Which genus 2 curves contribute to singularities?
(5) Our construction gives not only M_2 but also its compactification by $\operatorname{Proj} R$. Can we describe the boundary $\operatorname{Proj} R \setminus M_2$?
(6) Are there other approaches to the construction of M_2?

Let's summarize where we stand. We want to construct M_2 as an orbit space for
$$\mathrm{SL}_2 \quad \text{acting on} \quad \mathbb{P}(V_6) \setminus D.$$
We use our standard approach using invariants. The classical invariant theory tells us that $\mathcal{O}(V_6)^{\mathrm{SL}_2}$ is generated by $I_2, I_4, I_6, I_{10} = D$, and I_{15} with a single quadratic relation $I_{15}^2 = g(I_2, I_4.I_6, I_{10})$.

So our natural candidate for the quotient is $\operatorname{Proj} \mathcal{O}(V_6)^{\mathrm{SL}_2}$, and the quotient map is
$$f \mapsto [I_2(f) : \ldots : I_{15}(f)] \in \mathbb{P}(2,4,6,10,15).$$
Here we got lucky: since $\operatorname{Proj} R = \operatorname{Proj} R^{(2)}$, we can also write the quotient map as
$$f \mapsto [I_2(f) : \ldots : I_{10}(f)] \in \mathbb{P}(2,4,6,10) = \mathbb{P}(1,2,3,5).$$
Since there are no relations between $I_2, \ldots, I_{10}$ we actually expect the quotient to be $\mathbb{P}(1,2,3,5)$.

If we throw away the vanishing locus of the discriminant, we get the affine chart
$$\{D \neq 0\} \subset \mathbb{P}(1,2,3,5).$$
So our hope is that
$$M_2 = \mathbb{A}^3/\mu_5,$$
where μ_5 acts with weights $1,2,3$. We've seen that if we want to embed this cyclic quotient singularity in the affine space, we need at least $\mathbb{A}^8$.

Of course this construction alone does not guarantee that each point of $\mathbb{A}^3/\mu_5$ corresponds to a genus 2 curve and that different points correspond to different curves: this is something we are trying to work out in general.

Surjectivity of the quotient map implies

11.3.3. COROLLARY. *Any point of* $\mathbb{A}^3/\mu_5$ *represents a genus* 2 *curve.*

Now we can finally describe M_2:

11.3.4. THEOREM. *There are natural bijections (described previously) between*

(1) *isomorphism classes of genus* 2 *curves;*
(2) SL_2 *orbits in* $\mathbb{P}(V_6) \setminus D$*;*
(3) *points in* $\mathbb{A}^3/\mu_5$ *acting with weights* $1, 2, 3$.

Proof. The only thing left to check is that all SL_2 orbits in $\mathbb{P}(V_6) \setminus D$ are closed. But this is easy: for any orbit O and any orbit $O' \neq O$ in its closure, $\dim O' < \dim O$. However, all SL_2 orbits in $\mathbb{P}(V_6) \setminus D$ have the same dimension 3, because the stabilizer can be identified with a group of projective transformations of $\mathbb{P}^1$ permuting roots of the binary sextic, which is a finite group if all roots are distinct (or even if there are at least three distinct roots). $\square$

This gives a pretty decent picture of the quotient $\mathbb{P}(V_6)/\mathrm{SL}_2$, at least in the chart $D \neq 0$, which is the chart we mostly care about. To see what's going on in other charts, let's experiment with generators I_2, I_4, I_6, I_{10} (defined in §11.3). Simple combinatorics shows that (do it):

- if $f \in V_6$ has a root of multiplicity 4 then f is unstable.
- if $f \in V_6$ has a root of multiplicity 3 then all basic invariants vanish except (potentially) I_2.

So we should expect the following theorem:

11.3.5. THEOREM. *Points of* $\mathbb{P}(V_6)/\mathrm{SL}_2 = \mathbb{P}(1, 2, 3, 5)$ *correspond bijectively to* GL_2*-orbits of degree* 6 *polynomials with at most a double root (there can be several of them) plus an extra point* $[1 : 0 : 0 : 0]$*, which has the following description. All polynomials with a triple root (but no fourtuple root) map to this point in the quotient. The corresponding orbits form a one-parameter family (draw it), with a closed orbit that corresponds to the polynomial* x^3y^3.

To prove this theorem, it is enough to check the following facts:

(1) Any unstable form f has a fourtuple root (or worse). In other words, semistable forms are the forms that have at most triple roots.
(2) A semistable form $f \in \mathbb{P}(V_6)$ has a finite stabilizer unless $f = x^3y^3$ (this is clear: this is the only semistable form with two roots).
(3) Any semistable form f without triple roots has a closed orbit *in the semistable locus in* $\mathbb{P}(V_6)$, and hence in any principal open subset $D_+(I)$ it belongs to, where I is one of the basic invariants. Notice that we *do not* expect f to have a closed orbit in the whole $\mathbb{P}(V_6)$, in fact one can show that there is only one closed orbit there, namely the orbit of x^6.
(4) If f has a triple root then it has the orbit of x^3y^3 in its closure. Indeed, suppose $f = x^3g$, where $g = y^3 + ay^2x + byx^2 + cx^3$ is a cubic

form (it has to start with y^3, otherwise f has a fourtuple root). Let's act on f by a matrix $\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$. We get $x^3y^3 + at^2y^2x + bt^4yx^5 + ct^6x^6$. So as $t \to 0$, we get x^3y^3 in the limit.

§11.4. **Homework 6.**

Problem 1. Show that I_2 (see (11.3.1)) is indeed an SL_2-invariant polynomial. (2 points)

Problem 2. Using the fact that $M_2 = \mathbb{A}^3/\mu_5$, where μ_5 acts with weights $1, 2, 3$, construct M_2 as an affine subvariety of $\mathbb{A}^8$ (1 point).

Problem 3. Show that any genus 2 curve C can be obtained as follows. Start with a line $l \subset \mathbb{P}^3$. Then one can find a quadric surface Q and a cubic surface S containing l such that $Q \cap S = l \cup C$ (2 points).

Problem 4. Assuming that $M_2 = \mathbb{A}^3/\mu_5$ set-theoretically, define families of curves of genus 2 (analogously to families of elliptic curves), and show that M_2 is a coarse moduli space (2 points).

Problem 5. Assuming the previous problem, show that M_2 is not a fine moduli space (2 points).

Problem 6. Find a genus 2 curve C such that $\mathrm{Aut}\, C$ contains $\mathbb{Z}_5$ and confirm (or disprove) my suspicion that this curve gives a unique singular point of M_2 (2 points).

Problem 7. An algebraic curve is called bielliptic if it admits a $2:1$ morphism $C \to E$ onto an elliptic curve; the covering transformation is called a bielliptic involution. Let C be a genus 2 curve. (a) Show that if C is bielliptic then its bielliptic involution commutes with its hyperelliptic involution. (b) Show that C is bielliptic if and only if the branch locus $p_1, \ldots, p_6 \in \mathbb{P}^1$ of its bi-canonical map has the following property: there exists a $2:1$ morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(p_1) = f(p_2)$, $f(p_3) = f(p_4)$, and $f(p_5) = f(p_6)$. (c) Show that (b) is equivalent to the following: if we realize $\mathbb{P}^1$ as a conic in $\mathbb{P}^1$ then lines $\overline{p_1p_2}$, $\overline{p_3p_4}$, and $\overline{p_5p_6}$ all pass through a point (3 points).

§12. **Jacobians and periods**

So far we have focussed on constructing moduli spaces using GIT, but there exists a different approach using variations of Hodge structures. I will try to explain the most classical aspect of this theory, namely the map

$$M_g \to A_g, \quad C \mapsto \mathrm{Jac}\, C.$$

Injectivity of this map is the classical Torelli theorem.

§12.1. **Albanese torus.** Let X be a smooth projective variety. We are going to integrate in this section, so we will mostly think of X as a complex manifold. Recall that we have the first homology group

$$H_1(X, \mathbb{Z}).$$

We think about it in the most naive way, as a group generated by smooth oriented loops $\gamma : S^1 \hookrightarrow X$ modulo relations $\gamma_1 + \ldots + \gamma_r = 0$ if loops

$\gamma_1, \ldots, \gamma_r$ bound a smooth oriented surface in X (with an induced orientation on loops). We then have a first cohomology group

$$H^1(X, \mathbb{C}) = \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}).$$

This group can also be computed using de Rham cohomology

$$H^1_{dR}(X, \mathbb{C}) = \frac{\{\text{complex-valued 1-forms } \omega = \sum f_i\, dx_i \text{ such that } d\omega = 0\}}{\{\text{exact forms } \omega = df\}}.$$

Pairing between loops and 1-forms is given by integration

$$\int_\gamma \omega,$$

which is well-defined by Green's theorem. The fact that X is a smooth projective variety has important consequences for the structure of cohomology, most notably one has Hodge decomposition, which in degree one reads

$$H^1_{dR}(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1},$$

where $H^{1,0} = H^0(X, \Omega^1)$ is the (finite-dimensional!) vector space of holomorphic 1-forms, and $H^{0,1} = \overline{H^{1,0}}$ is the space of anti-holomorphic 1 forms. Integration gives pairing between $H_1(X, \mathbb{Z})$ (modulo torsion) and $H^0(X, \Omega^1)$, and we claim that this pairing is non-degenerate. Indeed, if this is not the case then $\int_\gamma \omega = 0$ for some fixed non-trivial cohomology class $\gamma \in H_1$ (modulo torsion) and for any holomorphic 1-form ω. But then of course we also have $\int_\gamma \bar{\omega} = 0$, which contradicts the fact that pairing between $H_1(X, \mathbb{Z})$ (modulo torsion) and $H^1(X, \mathbb{C})$ is non-degenerate.

It follows that we have a complex torus

$$\mathrm{Alb}(X) = \frac{H^0(X, \Omega^1)^*}{H_1(X, \mathbb{Z})/\mathrm{Torsion}} = V/\Lambda = \mathbb{C}^q/\mathbb{Z}^{2q}$$

called the Albanese torus of X. Λ is called the *period lattice* and

$$q = \dim H^0(X, \Omega^1)$$

is called the *irregularity* of X. If we fix a point $p_0 \in X$, then we have a holomorphic Abel–Jacobi map

$$\mu : X \to \mathrm{Alb}(X), \quad p \mapsto \int_{p_0}^p \bullet$$

The dependence on the path of integration is killed by taking the quotient by periods. Moreover, for any 0-cycle $\sum a_i p_i$ (a formal combination of points with integer multiplicities) such that $\sum a_i = 0$, we can define $\mu(\sum a_i p_i)$ by breaking $\sum a_i p_i = \sum (q_i - r_i)$ and defining

$$\mu(\sum a_i p_i) = \sum \int_{r_i}^{q_i} \bullet.$$

Again, any ambiguity in paths of integration and breaking the sum into differences disappears after we take the quotient by periods.

When $\dim X > 1$, we often have $q = 0$ (for example if $\pi_1(X) = 0$ or at least $H_1(X, \mathbb{C}) = 0$), but for curves $q = g$, the genus, and some of the most beautiful geometry of algebraic curves is revealed by the Abel–Jacobi map.

§12.2. **Jacobian.** Let C be a compact Riemann surface (= an algebraic curve). The Albanese torus in this case is known as the *Jacobian*

$$\mathrm{Jac}(C) = \frac{H^0(C,K)^*}{H_1(C,\mathbb{Z})} = V/\Lambda = \mathbb{C}^g/\mathbb{Z}^{2g}$$

The first homology lattice $H_1(C,\mathbb{Z})$ has a non-degenerate skew-symmetric intersection pairing $\gamma \cdot \gamma'$, which can be computed by first deforming loops γ and γ' a little bit to make all intersections transversal and then computing the number of intersection points, where each point comes with $+$ or $-$ depending on orientation of γ and γ' at this point. In the standard basis of α and β cycles (draw), the intersection pairing has a matrix

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

$H^1_{dR}(C,\mathbb{C})$ also has a non-degenerate skew-symmetric pairing given by

$$\int_C \omega \wedge \omega'.$$

We can transfer this pairing to the dual vector space $H_1(C,\mathbb{C})$ and then restrict to $H_1(C,\mathbb{Z})$. It should come at no surprise that this restriction agrees with the intersection pairing defined above. To see this concretely, let's work in the standard basis

$$\delta_1,\ldots,\delta_{2g} = \alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g$$

of α and β cycles. We work in the model where the Riemann surface is obtained by gluing the $4g$ gon Δ with sides given by

$$\alpha_1,\beta_1,\alpha_1^{-1},\beta_1^{-1},\alpha_2,\ldots$$

Fix a point p_0 in the interior of Δ and define a function $\pi(p) = \int_{p_0}^p \omega$ (integral along the straight segment). Since ω is closed, the Green's formula shows that for any point $p \in \alpha_i$, and the corresponding point $q \in \alpha_i^{-1}$, we have

$$\pi(q) - \pi(p) = \int_{\beta_i} \omega.$$

For any point $p \in \beta_i$ and the corresponding point $q \in \beta_i^{-1}$, we have

$$\pi(q) - \pi(p) = \int_{\alpha_i^{-1}} \omega = -\int_{\alpha_i} \omega.$$

Then we have

$$\begin{aligned}
\int_C \omega \wedge \omega' &= \int_\Delta d\pi \wedge \omega' = \int_\Delta d(\pi\omega') \qquad \text{(because } \omega' \text{ is closed)} \\
&= \int_{\partial\Delta} \pi\omega' \qquad \text{(by Green's formula)} \\
&= \sum \int_{\alpha_i \cup \alpha_i^{-1}} \pi\omega' + \sum \int_{\beta_i \cup \beta_i^{-1}} \pi\omega' = \\
&= -\sum \int_{\beta_i} \omega \int_{\alpha_i} \omega' + \sum \int_{\alpha_i} \omega \int_{\beta_i} \omega',
\end{aligned}$$

which is exactly the pairing dual to the intersection pairing[26].

Specializing to holomorphic 1-forms gives *Riemann bilinear relations*

12.2.1. PROPOSITION. *Let ω and ω' be holomorphic* 1*-forms. Then*

$$\sum\left(\int_{\alpha_i}\omega\int_{\beta_i}\omega'-\int_{\beta_i}\omega\int_{\alpha_i}\omega'\right)=\int_C\omega\wedge\omega'=0,$$

and

$$\sum\left(\int_{\alpha_i}\omega\int_{\beta_i}\bar{\omega}'-\int_{\beta_i}\omega\int_{\alpha_i}\bar{\omega}'\right)=\int_C\omega\wedge\bar{\omega}'.$$

We define a Hermitian form H on $H^0(C,K)$ by formula

$$i\int_C\omega\wedge\bar{\omega}'$$

(notice an annoying i in front) and we transfer it to the Hermitian form on $V:=H^0(C,K)^*$, which we will also denote by H. The imaginary part $\operatorname{Im}H$ is then a real-valued skew-symmetric form on $H^0(C,K)$ (and on V). We can view V and $H^0(C,K)$ as dual *real* vector spaces using the pairing $\operatorname{Re}v(\omega)$. Simple manipulations of Riemann bilinear relations give

$$\sum\left(\operatorname{Re}\int_{\alpha_i}\omega\right)\left(\operatorname{Re}\int_{\beta_i}\bar{\omega}'\right)-\left(\operatorname{Re}\int_{\beta_i}\omega\right)\left(\operatorname{Re}\int_{\alpha_i}\bar{\omega}'\right)=\operatorname{Im}i\int_C\omega\wedge\bar{\omega}',$$

i.e. we have

12.2.2. COROLLARY. *The restriction of* $\operatorname{Im}H$ *on* $\Lambda:=H_1(C,\mathbb{Z})$ *is the standard intersection pairing.*

The classical way to encode Riemann's bilinear identities is to choose a basis $\omega_1,\dots,\omega_g$ of $H^0(C,K)$ and consider the *period matrix*

$$\Omega=\begin{bmatrix}\int_{\alpha_1}\omega_1 & \dots & \int_{\alpha_g}\omega_1 & \int_{\beta_1}\omega_1 & \dots & \int_{\beta_g}\omega_1\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \int_{\alpha_1}\omega_g & \dots & \int_{\alpha_g}\omega_g & \int_{\beta_1}\omega_g & \dots & \int_{\beta_g}\omega_g\end{bmatrix}$$

Since H is positive-definite, the first minor $g\times g$ of this matrix is non-degenerate, and so in fact there exists a unique basis $\{w_i\}$ such that

$$\Omega=[\mathrm{Id}\,|\,Z],$$

where Z is a $g\times g$ matrix. Riemann's bilinear identities then imply that

$$Z=Z^t\quad\text{and }\operatorname{Im}Z\text{ is positive-definite.}$$

12.2.3. DEFINITION. The Siegel upper-half space S_g is the space of symmetric $g\times g$ complex matrices Z such that $\operatorname{Im}Z$ is positive-definite.

To summarize our discussion above, we have the following

[26]Note that a general rule for computing dual pairing in coordinates is the following: if B is a non-degenerate bilinear form on V, choose bases $\{e_i\}$ and $\{\tilde{e}_i\}$ of V such that $B(e_i,\tilde{e}_j)=\delta_{ij}$. Then the dual pairing on V^* is given by $B^*(f,f')=\sum f(e_i)f'(\tilde{e}_i)$. It does not depend on the choice of bases. In our example, the first basis of $H_1(C,\mathbb{Z})$ is given by cycles $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$, and the second basis is then given by $\beta_1,\dots,\beta_g,-\alpha_1,\dots,-\alpha_g$.

12.2.4. COROLLARY. *Let C be a genus g Riemann surface. Let $\mathrm{Jac}(C) = V/\Lambda$ be its Jacobian. Then $V = H^0(C, K)^*$ carries a Hermitian form $H = i\int_C \omega \wedge \bar{\omega}'$, and $\mathrm{Im}\, H$ restricts to the intersection pairing on $\Lambda = H_1(C, \mathbb{Z})$. Any choice of symplectic basis $\{\delta_i\} = \{\alpha_i\} \cup \{\beta_i\}$ in Λ determines a unique matrix in S_g.*

Different choices of a symplectic basis are related by the action of the symplectic group $\mathrm{Sp}(2g, Z)$. So we have a map

$$M_g \to A_g := S_g / \mathrm{Sp}(2g, \mathbb{Z}).$$

It turns out that A_g is itself a moduli space.

§12.3. **Abelian varieties.**

12.3.1. DEFINITION. A complex torus V/Λ is called an *Abelian variety* if carries a structure of a projective algebraic variety, i.e. there exists a holomorphic embedding $V/\Lambda \hookrightarrow \mathbb{P}^N$.

One has the following theorem of Lefschetz:

12.3.2. THEOREM. *A complex torus is projective if and only if there exists a Hermitian form H on V (called polarization) such that $\mathrm{Im}\, H$ restricts to an integral skew-symmetric form on Λ.*

It is easy to classify integral skew-symmetric forms Q on $\mathbb{Z}^{2g}$:

12.3.3. LEMMA. *There exist uniquely defined positive integers $\delta_1|\delta_2|\ldots|\delta_g$ such that the matrix of Q in some $\mathbb{Z}$-basis is*

$$\begin{bmatrix} & & & & -\delta_1 & & & \\ & & & & & -\delta_2 & & \\ & & & & & & \ddots & \\ & & & & & & & -\delta_g \\ \delta_1 & & & & & & & \\ & \delta_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \delta_g & & & & \end{bmatrix}.$$

Proof. For each $\lambda \in \Lambda = \mathbb{Z}^{2g}$, let d_λ be the positive generator of the principal ideal $\{Q(\lambda, \bullet)\} \subset \mathbb{Z}$. Let $\delta_1 = \min(d_\lambda)$, take $\lambda_1, \lambda_{g+1} \in \Lambda$ such that $Q(\lambda, \lambda_{g+1}) = \delta_1$. Those are the first two vectors in the basis. For any $\lambda \in \Lambda$, we know that δ_1 divides $Q(\lambda, \lambda_1)$ and $Q(\lambda, \lambda_{g+1})$, and therefore

$$\lambda + \frac{Q(\lambda, \lambda_1)}{\delta_1}\lambda_{g+1} + \frac{Q(\lambda, \lambda_{g+1})}{\delta_1}\lambda_1 \in \langle \lambda_1, \lambda_{g+1}\rangle_{\mathbb{Z}}^{\perp}.$$

Now we proceed by induction by constructing a basis in $\langle \lambda_1, \lambda_{g+1}\rangle_{\mathbb{Z}}^{\perp}$. □

12.3.4. DEFINITION. A polarization H is called *principal* if we have

$$\delta_1 = \ldots = \delta_g = 1$$

in the canonical form above. An Abelian variety V/Λ endowed with a principal polarization is called a *principally polarized Abelian variety*.

So we have

12.3.5. COROLLARY. *A_g parametrizes principally polarized Abelian varieties.*

In fact A_g has a natural structure of an algebraic variety. One can define families of Abelian varieties in such a way that A_g is a coarse moduli space.

§12.4. **Abel's Theorem.** Returning to the Abel–Jacobi map, we have the following fundamental

12.4.1. THEOREM (Abel's theorem). *The Abel–Jacobi map $AJ : \mathrm{Div}^0(C) \to \mathrm{Jac}(C)$ induces a bijection*

$$\mu : \mathrm{Pic}^0(C) \simeq \mathrm{Jac}(C).$$

The proof consists of three steps:

(1) $AJ(f) = 0$ for any rational function $f \in k(C)$, hence AJ induces μ.
(2) μ is injective.
(3) μ is surjective.

For the first step, we consider a holomorphic map $\mathbb{P}^1_{[\lambda:\mu]}$ given by

$$[\lambda, \mu] \mapsto AJ(\lambda f + \mu).$$

It suffices to show that this map is constant. We claim that any holomorphic map $r : \mathbb{P}^1 \to V/\Lambda$ is constant. It suffices to show that $dr = 0$ at any point. But the cotangent space to V/Λ at any point is generated by global holomorphic forms $dz_1, \ldots, dz_g$ (where $z_1, \ldots, z_g$ are coordinates in V. A pull-back of any of them to $\mathbb{P}^1$ is a global 1-form on $\mathbb{P}^1$, but $K_{\mathbb{P}^1} = -2[\infty]$, hence the only global holomorphic form is zero. Thus $dr^*(dz_i) = 0$ for any i, i.e. $dr = 0$.

§12.5. **Differentials of the third kind.** To show injectivity of μ, we have to check that if $D = \sum a_i p_i \in \mathrm{Div}^0$ and $\mu(D) = 0$ then $D = (f)$. If f exists then

$$\nu = \frac{1}{2\pi i} d \log(f) = \frac{1}{2\pi i} \frac{df}{f}$$

has only simple poles, these poles are at p_i's and $\mathrm{Res}_{p_i} \nu = a_i$ (*why?*) . Moreover, since branches of $\log$ differ by integer multiples of $2\pi i$, any period

$$\int_\gamma \nu \in \mathbb{Z}$$

for any closed loop γ. And it is easy to see that if ν with these properties exists then we can define

$$f(p) = \exp(2\pi i \int_{p_0}^{p} \nu).$$

This will be a single-valued meromorphic (hence rational) function with $(f) = D$. So let's construct ν. Holomorphic 1-forms on C with simple poles are classically known as *differentials of the third kind*. They belong to the linear system $H^0(C, K + p_1 + \ldots + p_r)$. Notice that we have an exact sequence

$$0 \to H^0(C, K) \to H^0(C, K + p_1 + \ldots + p_r) \xrightarrow{\psi} \mathbb{C}^r,$$

where ψ is given by taking residues. by Riemann–Roch, dimensions of the linear systems are g and $g + r - 1$. By a theorem on the sum of residues, the image of ψ lands in the hyperplane $\sum a_i = 0$. It follows that ψ is surjective onto this hyperplane, i.e. we can find a differential η of the third kind with any prescribed residues (as long as they add up to zero). The game now is to make periods of η integral by adding to η a holomorphic form (which of course would not change the residues). Since the first $g \times g$ minor of the period matrix is non-degenerate, we can arrange that A-periods of η are equal to 0.

Now, for any holomorphic 1-form ω, arguing as in the proof of Prop. 12.2.1, we have the following identity:

$$\sum\left(\int_{\alpha_i}\omega\int_{\beta_i}\eta-\int_{\beta_i}\omega\int_{\alpha_i}\eta\right)=\sum_{i=1}^{r}a_i\pi(p_i)=\sum_{i=1}^{r}a_i\int_{p_0}^{p_i}\omega.$$

Indeed, we can remove small disks around each p_i to make η holomorphic in their complement, and then compute $\int\omega\wedge\eta$ by Green's theorem as in Prop. 12.2.1. This gives

$$\sum\int_{\alpha_i}\omega\int_{\beta_i}\eta=\sum_{i=1}^{r}a_i\int_{p_0}^{p_i}\omega.$$

Since $\mu(D)=0$, we can write the RHS as $\int_\gamma\omega$, where $\gamma=\sum m_i\delta_i$ is an integral linear combination of periods. Applying this to the normalized basis of holomorphic 1-forms gives

$$\int_{\beta_i}\eta=\int_\gamma\omega_i$$

Now let

$$\eta'=\eta-\sum_{k=1}^{g}m_{g+k}\omega_k.$$

Then we have

$$\int_{\alpha_i}\eta'=-m_{g+i}$$

and

$$\int_{\beta_i}\eta'=\int_\gamma\omega_i-\sum m_{g+k}\int_{\beta_i}\omega_k=$$

$$\sum m_k\int_{\alpha_i}\omega_i+\sum m_{g+k}\int_{\beta_k}\omega_i-\sum m_{g+k}\int_{\beta_k}\omega_k=m_i$$

§12.6. **Summation maps.** To show surjectivity, we are going to look at the summation maps

$$C^d\to\operatorname{Pic}^d\to\operatorname{Jac}(C),\qquad(p_1,\ldots,p_d)\mapsto\mu(p_1+\ldots+p_d-dp_0),$$

where $p_0\in C$ is a fixed point. It is more natural to define

$$\operatorname{Sym}^d C=C^d/S_d,$$

and think about summation maps as maps

$$\phi_d:\ \operatorname{Sym}^d C\to\operatorname{Jac}C.$$

It is not hard to endow $\mathrm{Sym}^d C$ with a structure of a complex manifold in such a way that ϕ_d is a holomorphic map[27]. We endow $\mathrm{Sym}^d C$ with a quotient topology for the map $\pi : C^d \to \mathrm{Sym}^d C$, and then define complex charts as follows: at a point $(p_1, \ldots, p_d)$, choose disjoint holomorphic neighborhoods U_i's of p_i's (if $p_i = p_j$ then choose the same neighborhood $U_i = U_j$). Let z_i's be local coordinates. Then local coordinates on $\pi(U_1 \times \ldots \times U_d)$ can be computed a follows: for each group of equal points p_i, $i \in I$, use elementary symmetric functions in z_i. $i \in I$ instead of z_i's themselves.

The main point is absolutely obvious

12.6.1. LEMMA. *For $D \in \mathrm{Pic}^d$, $\mu^{-1}(D) = |D|$. Fibers of μ are projective spaces.*

To show that μ is surjective it suffices to show that ϕ_d is surjective. Since this is a proper map of complex manifolds of the same dimension, it suffices to check that a general fiber is a point. In view of the previous Lemma this boils down to showing that if $(p_1, \ldots, p_g) \in \mathrm{Sym}^g$ is sufficiently general then $H^0(C, p_1 + \ldots + p_g) = 1$. For inductive purposes, lets show that

12.6.2. LEMMA. *For any $k \leq g$, and sufficiently general points $p_1, \ldots, p_k \in C$, we have $H^0(C, p_1 + \ldots + p_k) = 1$.*

Proof. By Riemann-Roch, we can show instead that

$$H^0(C, K - p_1 - \ldots - p_k) = g - k$$

for $k \leq g$ and for a sufficiently general choice of points. Choose an effective canonical divisor K and choose points p_i away from it. Then we have an exact sequence

$$0 \to \mathcal{L}(K - p_1 - \ldots - p_k) \to \mathcal{L}(K - p_1 - \ldots - p_{k-1}) \to \mathbb{C},$$

where the last map is the evaluation map at the point p_k. It follows that either $|K - p_1 - \ldots - p_k| = |K - p_1 - \ldots - p_{k-1}|$ or dimensions of these two projective spaces differ by 1, the latter happens if one of the functions in $\mathcal{L}(K - p_1 - \ldots - p_{k-1})$ does not vanish at p_k. So just choose p_k to be a point where one of these functions does not vanish. $\square$

12.6.3. COROLLARY. *We can identify Pic^0 and Jac by means of μ.*

§12.7. **Theta-divisor.**

12.7.1. COROLLARY. *The image of ϕ_{d-1} i a hypersurface Θ in $\mathrm{Jac}\, C$.*

12.7.2. DEFINITION. Θ is called the *theta-divisor*.

12.7.3. EXAMPLE. If $g = 1$, not much is going on: $C = \mathrm{Jac}\, C$. If $g = 2$, we have $\phi_1 : C \hookrightarrow \mathrm{Jac}\, C$: the curve itself is a theta-divisor! The map ϕ_2 is a bit more interesting: if $h^0(C, p+q) > 1$ then $p + q \in |K|$ by Riemann–Roch. In other words, p and q are permuted by the hyperelliptic involution and these pairs (p, q) are parametrized by $\mathbb{P}^1$ as fibers of the $2:1$ map $\phi_{|K|} : C \to \mathbb{P}^1$. So ϕ_2 is an isomorphism outside of $K \in \mathrm{Pic}^2$, but $\phi_2^{-1}(K) \simeq \mathbb{P}^1$. Since both $\mathrm{Sym}^2 C$ and $\mathrm{Jac}\, C$ are smooth surfaces, this implies that ϕ_2 is a blow-up of the point.

[27]It is also not hard to show that $\mathrm{Sym}^d C$ is a projective algebraic variety. Since $\mathrm{Jac}\, C$ is projective by Lefschetz theorem, it follows (by GAGA) that ϕ_d is actually a regular map.

12.7.4. EXAMPLE. In genus 3, something even more interesting happens. Notice that ϕ_2 fails to be an isomorphism only if C carries a pencil of degree 2, i.e. if C is hyperelliptic. In this case ϕ_2 again contracts a curve $E \simeq \mathbb{P}^1$, but this time it is not a blow-up of a smooth point. To see this, I am going to use adjunction formula. Let $\tilde{E} \subset C \times C$ be the preimage. Then $\tilde{E}$ parametrizes points (p,q) in the hyperelliptic involution, i.e. $\tilde{E} \simeq C$ but not a diagonally embedded one. We can write a holomorphic 2-form on $C \times C$ as a wedge product $\mathrm{pr}_1^*(\omega) \wedge \mathrm{pr}_2^*(\omega)$, where ω is a holomorphic 1-form on C. Since $\deg K_C = 2$, the canonical divisor K on $C \times C$ can be chosen as a union of 4 vertical and 4 horizontal rulings. This $K \cdot \tilde{E} = 8$, but

$$(K + \tilde{E}) \cdot \tilde{E} = 2g(\tilde{E}) - 2 = 4$$

by adjunction, which implies that $\tilde{E} \cdot \tilde{E} = -4$. Under the $2:1$ map $C \times C \to \mathrm{Sym}^2 C$, $\tilde{E}$ $2:1$ covers our $E \simeq \mathbb{P}^1$. So $E^2 = -2$. This implies that the image of ϕ_2 has a simple quadratic singularity at $\phi_2(E)$. So the Abel-Jacobi map will distinguish between hyperelliptic and non-hyperelliptic genus 3 curves by appearance of a singular point in the theta-divisor.

§12.8. **Homework 7.**

Problem 1. Generalizing the action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper-half plane, give formulas for the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on S_g (1 point).

Problem 2. In the proof of Lemma 12.3.3, show that indeed we have

$$\delta_1 | \delta_2 | \ldots | \delta_g$$

(1 point).

Problem 3. Show that $\mathrm{Sym}^d \mathbb{P}^1 = \mathbb{P}^d$ (1 point).

Problem 4. Let C be an algebraic curve. Define $\mathrm{Sym}^d C$ as an algebraic variety (1 point).

Problem 5. Show that if $\phi_1(C) \subset \mathrm{Jac}\, C$ is symmetric (i.e. $\phi_1(C) = -\phi_1(C)$) then C is hyperelliptic. Is the converse true? (1 point).

Problem 6. Show that either the canonical map $\phi_{|K|}$ is an embedding or C is hyperelliptic. (1 point).

Problem 7. Let C be a non-hyperellptic curve. C is called trigonal if it admits a $3:1$ map $C \to \mathbb{P}^1$. (a) Show that C is trigonal if and only if its canonical embedding $\phi_{|K|}$ has a trisecant, i.e. a line intersecting it in (at least) three points. (b) Show that if C is trigonal then its canonical embedding is not cut out by quadrics[28]. (2 points).

Problem 8. Show that the secant lines of a rational normal curve in $\mathbb{P}^n$ are are parametrized by the surface in the Grassmannian $G(2, n+1)$ and that this surface is isomorphic to $\mathbb{P}^2$ (2 points).

Problem 9. Consider two conics $C_1, C_2 \subset \mathbb{P}^2$ which intersect at 4 distinct points. Let $E \subset C_1 \times C_2$ be a curve that parametrizes pairs (x, y) such that the line L_{xy} connecting x and y is tangent to C_2 at y. (a) Show that E is an elliptic curve. (b) Consider the map $t : E \to E$ defined as follows: send (x, y) to (x', y'), where x' is the second point of intersection of L_{xy} with C_1

[28] This is practically if and only if statement by Petri's theorem.

and $L_{x'y'}$ is the second tangent line to C_2 through x'. Show that t is a translation map (with respect to the group law on the elliptic curve). (c) Show that if there exists a 7-gon inscribed in C_1 and circumscribed around C_2 then there exist infinitely many such 7-gons, more precisely there is one through each point of C_1 (3 points).

Problem 10. Let C be a hyperelliptic curve and let $R = \{p_0, \ldots, p_{2g+1}\}$ be the branch points of the $2 : 1$ map $C \to \mathbb{P}^1$. We choose p_0 as the base point for summation maps $\phi_d : \mathrm{Sym}^d \to \mathrm{Jac}$. For any subset $S \subset R$, let $\alpha(S) = \phi_{|S|}(S)$. (a) Show that $\alpha_S \in \mathrm{Jac}[2]$ (the 2-torsion part). (b) Show that $\alpha_S = \alpha_{S^c}$. (c) Show that α gives a bijection between subsets of B_g of even cardinality defined upto $S \leftrightarrow S^c$ and points of $\mathrm{Jac}[2]$ (3 points).

Problem 11. A divisor D on C is called a theta-characteristic if $2D \sim K$. A theta-characteristic is called vanishing if $h^0(D)$ is even and positive. Show that a curve of genus 2 has no vanishing theta characteristics but a curve of genus 3 has a vanishing theta characteristic if and only if it is a hyperelliptic curve (1 point).

Problem 12. Show that a nonsingular plane curve of degree 5 does not have a vanishing theta characteristic (3 points).

Problem 13. Let $E = \{y^2 = 4x^3 - g_2 x - g_3$ be an elliptic curve with real coefficients g_2, g_3. Compute periods to show that $E \simeq \mathbb{C}/\Lambda$, where either $\Lambda = \mathbb{Z} + \tau i \mathbb{Z}$ or $\Lambda = \mathbb{Z} + \tau(1+i)\mathbb{Z}$ (with real τ) depending on the number of real roots of the equation $4x^3 - g_2 x - g_3 = 0$ (3 points).

Problem 14. Consider a (non-compact!) curve $C = \mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}$. Since $\mathbb{P}^1$ has no holomorphic 1-forms, lets consider instead differentials of the third kind and define

$$V := H^0(\mathbb{P}^1, K + p_1 + \ldots + p_r)^*.$$

Show that $\Lambda := H_1(C, \mathbb{Z}) = \mathbb{Z}^{r-1}$, define periods, integration pairing, and the "Jacobian" V/Λ. Show that $V/\Lambda \simeq (\mathbb{C}^*)^{r-1}$ and that C embeds in $V/\Lambda \simeq (\mathbb{C}^*)^{r-1}$ by the Abel-Jacobi map (2 points).

Problem 15. Let C be an algebraic curve with a fixed point p_0 and consider the Abel–Jacobi map $\phi = \phi_1 : C \to \mathrm{Jac}$. For any point $p \in C$, we have a subspace $d\phi(T_p C) \subset T_{\phi(p)} \mathrm{Jac}$. By applying a translation by $\phi(p)$, we can identify $T_{\phi(p)} \mathrm{Jac}$ with $T_0 \mathrm{Jac} \simeq \mathbb{C}^g$. Combining these maps together gives a map $C \to \mathbb{P}^{g-1}$, $p \mapsto d\phi(T_p C)$. Show that this map is nothing but the canonical map $\phi_{|K|}$ (2 points).

Problem 16. Let F, G be homogeneous polynomials in $\mathbb{C}[x, y, z]$. Suppose that curves $F = 0$ and $G = 0$ intersect transversally at the set of points Γ. (a) Show that associated primes of (F, G) are the homogeneous ideals $I(p_i)$ of points $p_i \in \Gamma$. (b) Show that every primary ideal of (F, G) is radical by computing its localizations at p_i's (c) Conclude that $I(\Gamma) = (F, G)$, i.e. any homogeneous polynomial that vanishes at Γ is a linear combination $AF + BG$ (2 points).