ARTIN'S APPROXIMATION THEOREM

SUKHENDU MEHROTRA

Basic question: Let F be (some moduli) functor $(Sch/S)^0 \to Sets$. Can we assert representability of F based on local properties of F, e.g. if its deformation theory is "good"?

Let A be a Noetherian ring, $m \subset A$ an ideal (will be a maximal ideal, but does not matter). Let $\hat{A}$ be a completion. Let $F : (A - Alg) \to Sets$ be a functor. Let $c \in \mathbb{N}$.

Q1: can $\bar{\xi} \in F(\hat{A})$ be approximated by some $\xi \in F(A)$ modulo m^c?

0.1. Definition. A functor $F : (A - Alg) \to Sets$ is called of finite presentation (FP) if it commutes with colimits:

$$\lim_{\to} F(B_i) \simeq F(\lim_{\to} B_i).$$

By Grothendieck, for representable functors this is equivalent to the usual definition of "finite presentation" (finitely many generators and relations).

Let B be an A-algebra. Then $B = \lim_{\to} B_i$, where all B_i's are algebras of finite presentation. Let $B_i \simeq A[Y]/(f(Y))$, where $Y = \{Y_1, \ldots, Y_N\}$, $f = (f_1, \ldots, f_m)$. Giving $\phi : B_i \to C$, for some A-algebra C is equivalent to giving a solution to $f(Y) = 0$ over C.

Since F is FP, for any $\xi \in F(B)$ there exists an i such that ξ is induced by ξ. Thus, given $\xi \in F(B)$, there exists a system $f(Y)$ such that to every solution $y_C \in C$ of $f(Y) = 0$, one has a functorial assignment of an object $\xi_{y_C} \in F(C)$.

Thus Q1 can be answered affirmatively if the following question can answered:

Let $(f_1, \ldots, f_m) \in A[Y]$, let $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \in \hat{A}$ be a solution to $f(Y) = 0$ in $\hat{A}$. Then, given natural c, does there exist a solution $y = (y_1, \ldots, y_m)$ in A such that $y_i = \bar{y}_i \mod m^c$?

0.2. Remark. Q2 admits an affirmative answer if $f(Y) = 0$ is a linear system and A is local, $m \neq A$, by the faithful flatness of the inclusion $A \subset \hat{A}$.

Thus it is natural to study Q2 etale locally for Henselian rings.

0.3. Definition. A is Henselian if given a solution $y^0 \in A/m$ to a system with the Jacobian not equal to 0, there exists a solution y in A which reduces to Y^0.

Let R be a field or an excellent DVR.

A is a henselization of a finite type R-algebra at a prime ideal.

$m \subset A$ a proper ideal.

0.4. Theorem. *Given a system $f(Y) = 0$ with coefficients in A, a solution $\bar{y}$ in $\hat{A}$, $c \in N$, there exists a solution to $f(Y) = 0$ over A that reduces to $\bar{y}$ modulo m^c.*

0.5. Corollary. *With above assumptions, for any (FP) functor, given $\bar{\xi} \in F(\hat{A})$, there exists $\xi \in F(A)$ such that $\xi = \bar{\xi} \mod m^c$.*

Application (R,A as above)

0.6. Theorem. *Let $S = \operatorname{Spec} A$, $f : X \to S$ proper morphism Then,*

$$\theta : H^1(X, GL(N)) \to \lim_{\leftarrow} H^1(X_n, GL(N))$$

is injective with dense image, where $X_n = X \times_S \operatorname{Spec}(A/m^{n+1})$

Proof. By Grothendieck's existence theorem $H^1(\hat{X}, GL(N)) \simeq \lim\limits_{\leftarrow} H^1(X_n, GL(N))$, where $\hat{X} = X \times_S \operatorname{Spec}(\hat{A})$.

(EGA IV.8) implies that $H^1(X \times_S \cdot, GL(N))$ is FP.

By approximation theorem, the image of θ is dense (just for the stupid direct limit topology).

For injectivity, if $\theta(L)$ is free with trivializing sections $\hat{s}_1, \ldots, \hat{s}_N$ then, by approximation theorem (H^0 is also FP) there exist sections $s_1, \ldots, s_N$ of L that approximate $\hat{s}_i$ modulo m. So by Nakayama Lemma, $s_1, \ldots, s_N$ trivialize L. □

Now: algebraization theorem.

Let S be a scheme locally of FP over a field or an excellent Dedekind domain. Let $F : (Sch/S)^0 \to Sets$ a functor.

Let $X = \operatorname{Spec} A \in Sch/S$.

0.7. DEFINITION. A formal deformation $\hat{A}, \xi_n \in F(A/m^{n+1})$ is said to be effective if there exists a deformation $\hat{A}, \bar{\xi} \in F(\hat{A})$ inducing ξ_n.

0.8. THEOREM. *Assume F is FP and $(\hat{A}, \bar{\xi})$ be an effective versal deformation. Let $k' = A/m$. Then there exists a scheme $X \in (Sch/S)$, a closed point $x \in X$ with residue field k', $\xi \in F(X)$ such that $\hat{\mathcal{O}}_{X,x} \simeq \hat{A}$ inducing $(\hat{A}, \xi_n)$. If $(\hat{A}, \xi_n)$ is universal then X is unique.*