Beauty and the Beast

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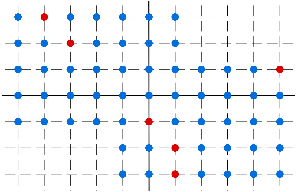
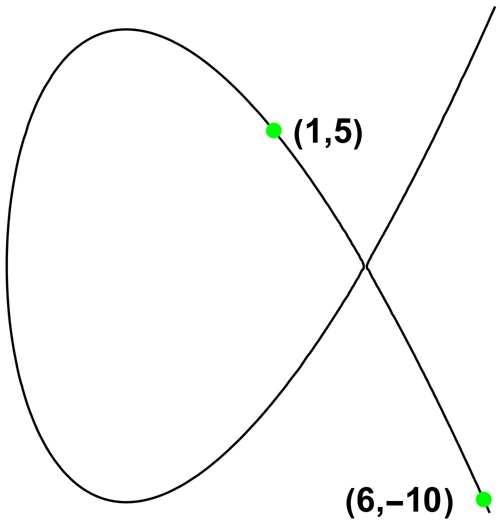
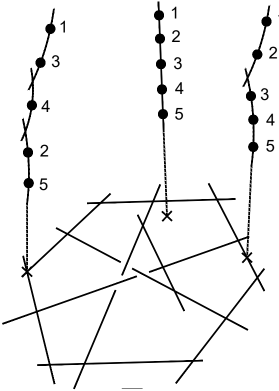


Elliptic curve $y^2 + y = x^3 - x^2 - 24x + 54$



Moduli space of stable rational curves



- $M_{0,n} = \left\{ {p_1,\ldots,p_n \in \mathbb{P}^1 \atop p_i \neq p_j} \right\} / \mathrm{PGL}_2$
- $M_{0,3} = \mathrm{pt}$ (send $p_1, p_2, p_3 \to 0, 1, \infty$)
- $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ via cross-ratio
- $\overline{M}_{0,4} = \mathbb{P}^1$
- $\overline{M}_{0,n}$ functorial compactification
- $\overline{M}_{0,5} = \mathbf{dP}_5$ (del Pezzo)
- $\overline{M}_{0,6} = \mathrm{Bl}_{10}\, \mathbf{S}_3$ (log Fano)

Effective cone of $\overline{M}_{0,n}$

- (Kapranov models) $\overline{M}_{0,n} = \ldots \mathrm{Bl}_{\binom{n-1}{3}} \mathrm{Bl}_{\binom{n-1}{2}} \mathrm{Bl}_{n-1} \mathbb{P}^{n-3}$
- Every boundary divisor is contracted by a Kapranov map $\Rightarrow$ generates an extremal ray of $\overline{\mathrm{Eff}}\,\overline{M}_{0,n}$
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- $\overline{\mathrm{Eff}}\ \overline{M}_{0,n}$ has many generators, *hypertree divisors*, contractible by birational contractions (Castravet–T)

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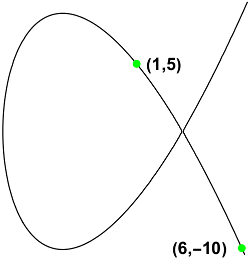
- $\overline{\mathrm{Eff}}\ \overline{M}_{0,n}$ has many generators, *hypertree divisors*, contractible by birational contractions (Castravet–T)
- And has more generators (Opie, based on Chen–Coskun)
- And more generators (Doran–Giansiracusa–Jensen), …

Beauty and the beast

Theorem

$\overline{\mathrm{Eff}}\ \overline{M}_{0,n}$ *is not polyhedral for* $n \geq 13$ *under any of the two conditions:*

- $(1,5)$ *has infinite order in the group law of the elliptic curve* C
- $(6,-10),\ 2(6,-10),\ 3(6,-10)$ *are not in the subgroup* $\langle(1,5)\rangle$



$$\overline{M}_{0,n}$$

Characteristic 0

Corollary

$\overline{\mathrm{Eff}}\,\overline{M}_{0,n}$ is not polyhedral for $n \geq 13$ in characteristic 0

Proof.

By `https://www.lmfdb.org/EllipticCurve/Q/997/a/1`, the Mordell–Weil group $C(\mathbb{Q}) = \mathbb{Z} \oplus \mathbb{Z}$ and is generated by $(6, -10)$ and $(1, 5)$. Therefore, $(1, 5)$ is not torsion $\square$

Characteristic p

Corollary

$\overline{\mathrm{Eff}}\,\overline{M}_{0,n}$ is not polyhedral for $n \geq 13$ in characteristic

$$p = 7, 11, 41, 67, 173, 307, 317, 347, 467, 503, 523,\\ 571, 593, 631, 677, 733, 809, 811, 827, 907, 937, \ldots$$

Proof.

```
C := EllipticCurve([0, -1, 1, -24, 54]);
PolyP:=[]; p:=2;
while p lt 1000 do
  if not IsDivisibleBy(Conductor(C), p) then
   Check:=true;
   Cp:=ChangeRing(C, GF(p));
   P:=Cp![6,-10,1]; Q:=Cp![1,5,1];
   for i:=0 to Order(Q)-1 do for j:=1 to 3 do
     if j*P eq i*Q then Check:=false; end if;
   end for; end for;
   if Check then PolyP:=Append(PolyP, p); end
if;
  end if;
  p:=NextPrime(p);
end while;
PolyPrimes;
```

□

"Euclid's Theorem"

Corollary

There exist infinitely many primes p *such that* $\overline{\mathrm{Eff}}\,\overline{M}_{0,n}$ *is not polyhedral for* $n \geq 13$ *in characteristic* p

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Proof.

We would like to apply a theorem of Tom Weston:

Let A be an abelian variety over a number field F such that $\mathrm{End}_F A$ is commutative. Take an element $x \in A(F)$ and a subgroup $\Sigma \subset A(F)$. If $\mathrm{red}_v x \in \mathrm{red}_v \Sigma$ for almost all places v of F then $x \in \Sigma + A(F)_{\mathrm{tors}}$.

"$\mathrm{End}_F A$ is commutative" is automatic for elliptic curves $/\mathbb{Q}$.
In fact, in our case C does not have CM $\Leftrightarrow \mathrm{End}_{\bar{\mathbb{Q}}} C = \mathbb{Z}$.
By Weston's theorem, since $6(6, -10) \notin \langle(1, 5)\rangle$ over $\mathbb{Q}$, the same is true for their reduction mod p for infinitely many p. $\square$

“Dirichlet’s theorem”

Corollary

There exist a positive density of primes p such that $\overline{\mathrm{Eff}}\ \overline{M}_{0,n}$ is not polyhedral for $n \geq 13$ in characteristic p

- We want to prove positive density of primes such that $(6,-10),\ 2(6,-10),\ 3(6,-10)$ are not in $\langle(1,5)\rangle \subset C(\mathbb{F}_p)$

“Dirichlet’s theorem”

Corollary

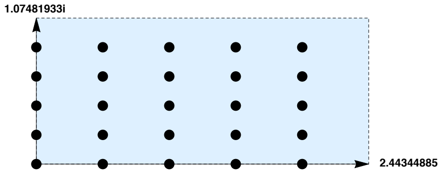
There exist a positive density of primes p such that $\overline{\mathrm{Eff}}\,\overline{M}_{0,n}$ is not polyhedral for $n \geq 13$ in characteristic p

- We want to prove positive density of primes such that $(6,-10),\ 2(6,-10),\ 3(6,-10)$ are not in $\langle(1,5)\rangle \subset C(\mathbb{F}_p)$
- Instead, we will fix *another* prime $q \neq 2,3$ and prove positive density of primes such that q divides the index of $\langle(1,5)\rangle \subset C(\mathbb{F}_p)$ but not the index of $\langle(6,-10)\rangle \subset C(\mathbb{F}_p)$

"Dirichlet's theorem"

Background on Galois representations.

- Let $C[q] \subset C(\bar{\mathbb{Q}})$ be the q-torsion, $C[q] \simeq (\mathbb{Z}/q\mathbb{Z})^2$



- Let $K = \mathbb{Q}(C[q])$. Then $\mathsf{Gal}(K/\mathbb{Q}) \subseteq \mathsf{GL}_2(\mathbb{Z}/q\mathbb{Z})$
- Equality for almost all q iff C has no CM (Deuring, Serre)
- $x \in C(\mathbb{Q})$, $K_x = K(\frac{x}{q})$, $\mathsf{Gal}(K_x/\mathbb{Q}) \subseteq \mathsf{GL}_2(\mathbb{Z}/q\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})^2$
- For almost all primes p, we have a Frobenius element $\sigma_p \in \mathsf{Gal}(K_x/\mathbb{Q})$ (well-defined up to conjugacy)
- We can view σ_p as a pair $(\gamma_p, \tau_p) \in \mathsf{GL}_2(\mathbb{Z}/q\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})^2$

"Dirichlet's theorem"

Corollary

There exist a positive density of primes p such that $\overline{\mathrm{Eff}}\,\overline{M}_{0,n}$ is not polyhedral for $n \geq 13$ in characteristic p.

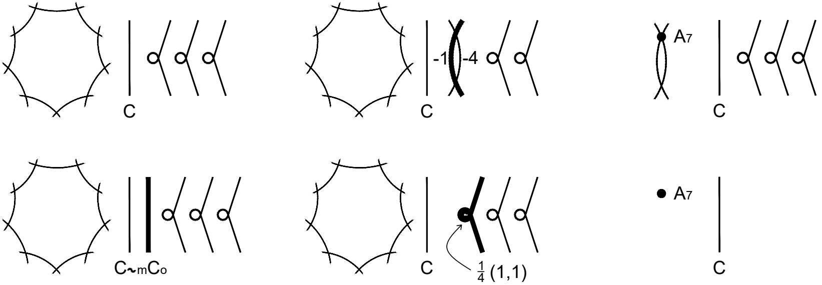
Proof.

- (Lang–Trotter) q divides the index of $\langle \mathrm{red}_p(x) \rangle \subset C(\mathbb{F}_p)$ iff the Frobenius element $\sigma_p = (\gamma_p, \tau_p)$ is as follows
  - $\gamma_p = \mathrm{Id}$ or
  - $\gamma_p \neq \mathrm{Id}$, γ_p has eigenvalue 1, and $\tau_p \in \mathrm{Im}(\gamma_p - \mathrm{Id})$.
- Let $x = (1, 5)$, $y = (6, -10)$
- $\mathrm{Gal}(K(\frac{x}{q}, \frac{y}{q})/\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Z}/q\mathbb{Z}) \ltimes ((\mathbb{Z}/q\mathbb{Z})^2 \oplus (\mathbb{Z}/q\mathbb{Z})^2)$
- By Chebotarev density theorem, for a set of primes p of positive density, q divides the index of $\langle \mathrm{red}_p\, x \rangle \subset C(\mathbb{F}_p)$ but not the index of $\langle \mathrm{red}_p\, y \rangle$ $\square$

Definition

A rational elliptic pair $C \subset X$

- a projective rational log terminal surface X
- a smooth genus 1 curve C such that $C^2 = 0$
- C is disjoint from singularities of X



Order of an elliptic pair

- Restriction map res : $C^\perp \to \mathrm{Pic}^0(C), \quad L \mapsto L|_C$
- C is contained in $C^\perp \subset \mathrm{Cl}(X)$
- $e(C,X)$ is the order of $\mathrm{res}(C)$ in $\mathrm{Pic}^0(C)$.

Lemma

- $e(C,X) < \infty \Leftrightarrow C$ *is a (multiple) fiber of an elliptic fibration*
- $e(C,X) = \infty$, $\rho(X) \geq 3 \Rightarrow \overline{\mathrm{Eff}}X$ *is not polyhedral*

Proof.

Observation (Nikulin): $\overline{\mathrm{Eff}}X$ is polyhedral, $\rho(X) \geq 3 \Rightarrow$

- $\overline{\mathrm{Eff}}X$ is generated by negative curves
- every irreducible curve with $C^2 = 0$ is contained in the interior of a facet, in particular its multiple moves $\square$

Minimal elliptic pairs

Definition

An elliptic pair (C, X) is called *minimal* if there are no smooth rational curves $E \subset X$ such that $K \cdot E < 0$ and $C \cdot E = 0$

Theorem

For an elliptic pair (C, X)*, there exists a minimal elliptic pair* (C, Y) *and a morphism* $\pi : X \to Y$*, an isomorphism in a neighborhood of* C

Theorem

(C, Y) *is minimal,* $e(C, Y) = \infty \Rightarrow Y$ *has Du Val singularities*

Proof.

- (C, Y) is minimal $\Leftrightarrow K + C \sim rC$, $r \in \mathbb{Q}$.
- $\mathcal{O}(K + C)|_C \simeq \mathcal{O}_C$ but $\mathcal{O}(C)|C$ has infinite order
- $r = 0 \Rightarrow K = -C$ is Cartier $\Rightarrow Y$ is Du Val $\square$

Minimal elliptic pairs with du Val singularities

Definition

Since $K \cdot C = 0$, we can define $\mathrm{Cl}_0(X) = C^{\perp}/\langle K \rangle$, reduced restriction map $\overline{\mathrm{res}} : \mathrm{Cl}_0(X) \to \mathrm{Pic}^0(C)/\langle \mathrm{res}(K) \rangle$

Theorem

Let (C, Y) be an elliptic pair such that Y has Du Val singularities. Let Z be the minimal resolution of Y.

- (C, Y) *is minimal* $\Leftrightarrow$ (C, Z) *is minimal* $\Leftrightarrow$ $\rho(Z) = 10$

In this case $\mathrm{Cl}_0(Z) \simeq \mathbb{E}_8$. *Suppose* $\mathrm{e}(C, X) < \infty$

- $\overline{\mathrm{Eff}}(Y)$ *is polyhedral* $\Leftrightarrow$ $\overline{\mathrm{Eff}}(Z)$ *is polyhedral* $\Leftrightarrow$ $\mathrm{Ker}(\overline{\mathrm{res}})$ *contains* 8 *linearly independent roots of* $\mathbb{E}_8$.

Synthesis

Let (C, Y) be the minimal model of (C, X). Then

- $e(C, X) = \infty \Rightarrow Y$ is Du Val, not polyhedral (if $\rho \geq 3$)
- $e(C, X) < \infty$ and Y is Du Val $\Rightarrow$ polyhedrality criterion

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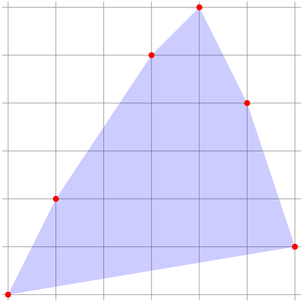
Problem

- *Suppose* $C, X, \mathrm{Cl}(X)$ *are defined over* $\mathbb{Q}$*,* $e(C, X) = \infty$
- $X \to Y$ *extends to the morphism of integral models* $\mathcal{X} \to \mathcal{Y}$ *over* $\mathrm{Spec}\,\mathbb{Z}$ *(outside of finitely many primes of bad reduction)*
- Y *is Du Val* $\Rightarrow Y_p$ *is Du Val*
- $e(C_p, X_p) < \infty$*. Study distribution of "polyhedral" primes*

Blown-up toric surface X



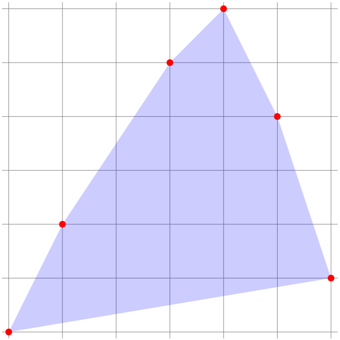
Definition

- Fix a lattice polygon $\Delta \subset \mathbb{Z}^2$
- S is a projective toric surface with an ample divisor D
- $X = \mathrm{Bl}_e S$ with exceptional divisor E

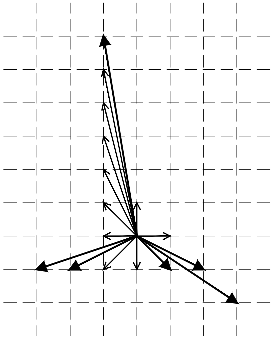
Elliptic pair (C, X)

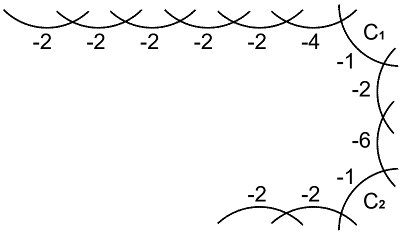
One can find $C \in |D - 6E|$ such that (C, X) is an elliptic pair:

$$x^4y^6 + 6x^5y^4 - 2x^4y^5 - 14x^5y^3 - 17x^4y^4 - 4x^3y^5 + x^6y + 11x^5y^2$$
$$+38x^4y^3 + 26x^3y^4 - 9x^5y - 27x^4y^2 - 34x^3y^3 + 22x^4y + 16x^3y^2$$
$$-10x^2y^3 - 24x^3y + 10x^2y^2 + 15x^2y + 5xy^2 - 11xy + 1 = 0$$



Minimal model (C, Y)





Y has an A_7 singularity and Picard number $\rho = 3$

Beautiful curve

- C is isomorphic to $y^2 + y = x^3 - x^2 - 24x + 54$
- $\mathrm{res}(C) = -(1, 5)$
- No elliptic fibration, no polyhedrality in characteristic 0

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- Let Z be the minimal resolution of Y
- $\mathbb{E}_8 \simeq \mathrm{Cl}_0(Z) \xrightarrow{\pi} \mathrm{Cl}_0(Y) \simeq \mathbb{Z} \xrightarrow{\overline{\mathrm{res}}} \mathrm{Pic}^0(C)/\langle(1,5)\rangle$
- Polyhedrality in characteristic $p \Leftrightarrow$
  there exists a root $\beta \in \mathbb{E}_8$ such that $\pi(\beta) \neq 0$ and $\overline{\mathrm{res}}(\beta) = 0$

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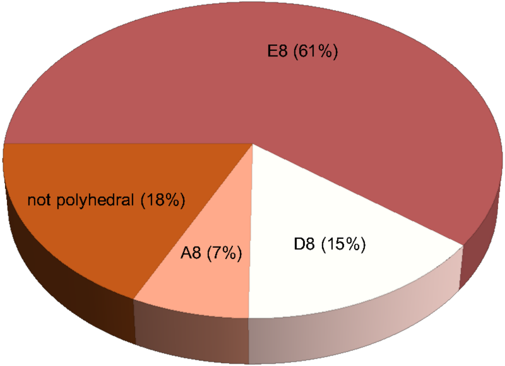
- Polyhedrality in characteristic $p \Leftrightarrow$
  there exists a root $\beta \in \mathbb{E}_8$ such that $\pi(\beta) \neq 0$ and $\overline{\mathrm{res}}(\beta) = 0$
- $\mathrm{Cl}_0(Y) = \mathbb{E}_8/\mathbb{A}_7, \quad \mathfrak{e}_8 = \bigoplus_{\bar{\beta} \in \mathrm{Cl}_0(Y)} (\mathfrak{e}_8)_{\bar{\beta}}$
- $\begin{matrix} \mathbb{C}^8 & \Lambda^2\mathbb{C}^8 & \Lambda^3\mathbb{C}^8 & \mathfrak{gl}_8 & \Lambda^3\mathbb{C}^8 & \Lambda^2\mathbb{C}^8 & \mathbb{C}^8 \\ 8 & 28 & 56 & 64 & 56 & 28 & 8 \end{matrix}$
- Polyhedrality $\Leftrightarrow (6,-10),\ 2(6,-10),\ 3(6,-10) \notin \langle(1,5)\rangle$

Distribution of primes



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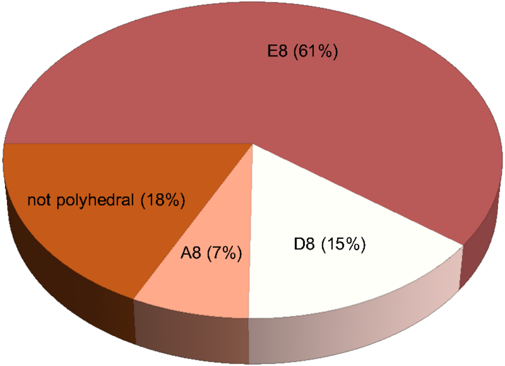


Remark (Lang–Trotter conjecture)

The point $(1, 5)$ *generates* $C(\mathbb{F}_p)$ *for* 44% *of primes*

Rational contractions

Definition

A *rational contraction* $X \dashrightarrow Y$ of projective $\mathbb{Q}$-factorial varieties is a rational map that can be decomposed into a sequence of

- surjective morphisms of $\mathbb{Q}$-factorial varieties
- small $\mathbb{Q}$-factorial modifications

Theorem

If X has any of these properties then Y does as well:

- *Mori Dream Space*
- *polyhedral effective cone*
- *effective cone with rational slopes*
- *every nef divisor on every rational contraction is semi-ample*

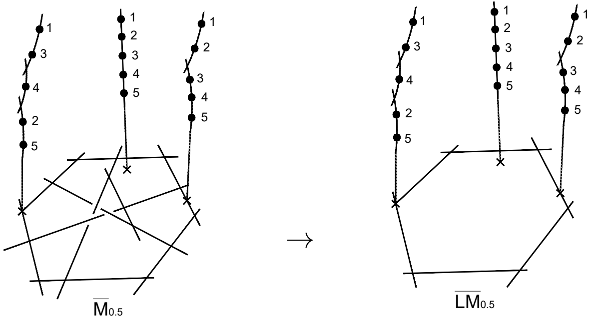
Philosophy (Fulton)

$\overline{M}_{0,n}$ is like a toric variety

Not true but can be salvaged in two different ways

Definition

$LM_{0,n}$ is the Hassett moduli space of stable rational curves with n marked points with weights $1, \varepsilon, \ldots, \varepsilon, 1$.



trees of $\mathbb{P}^1$'s   chains of $\mathbb{P}^1$'s

Universal toric variety

Theorem

For any projective toric variety S, there exists a toric rational contraction $LM_{0,n} \dashrightarrow S$ for $n \gg 0$.

Universal blown-up toric variety

Theorem

For any projective toric variety S, there exists a rational contraction $\mathrm{Bl}_e LM_{0,n} \dashrightarrow \mathrm{Bl}_e S$ *for* $n \gg 0$.

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Theorem (Castravet–T, 2015)

There are rational contractions $\mathrm{Bl}_e LM_{0,n+1} \dashrightarrow \overline{M}_{0,n} \to \mathrm{Bl}_e LM_{0,n}$

Corollary (Castravet–T, 2015)

In characteristic 0, $\overline{M}_{0,n}$ is not a MDS for $n \gg 0$, in fact it has a rational contraction $\mathrm{Bl}_e \mathbb{P}(a, b, c)$ for some a, b, c, which has a nef but not semi-ample divisor (Goto–Nishida–Watanabe).

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Remark

This argument can't work in characteristic p, where, by Artin's contractibility criterion, a nef divisor on $\mathrm{Bl}_e \mathbb{P}(a, b, c)$ is semi-ample.

Corollary

$\overline{M}_{0,n}$ is not a MDS for $n \gg 13$ in characteristic p for a positive density of primes p. It admits a rational contraction to an elliptic pair (X, C) with a non-polyhedral effective cone, where $X = \mathrm{Bl}_e S$