KURANISHI SPACES FOR ANALYTIC DGLA'S

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Start with a geometric object X. Its moduli "space" should also be some kind of a geometric object. Optimistically, expect a complex space or a scheme (with a marked point). Pessimistically, study a formal neighborhood of a marked point.

More precisely, we study "deformation functors"

$$Art \to Sets,$$

where Art is a category of local Artin rings. Usually $A \in Art$ maps to an isomorphism class of objects in $C/\operatorname{Spec} A$ that specialize to X at the closed point of $\operatorname{Spec} A$, where C is some category of geometric objects with a notion of "families parametrized by a scheme", i.e. a category fibered over the category of schemes.

0.1. Definition. A *dgla* (over $\mathbb{C}$) is a graded $\mathbb{C}$-algebra $L = \oplus_{i\geq 0} L_i$ with a differential d and a bracket $[\cdot,\cdot]$ such that

- $[L_i, L_j] \subset L_{i+j}$;
- $dL_i \subset L_{i+1}$;
- $[\cdot,\cdot]$ is supercommutative;
- graded Jacobi identity;
- d is a graded derivation of $[\cdot,\cdot]$.

0.2. Example. If X is a cpx manifold, then one can take

(1) $L = A^{0,\bullet}(T_X)$, $d = \bar{\partial}$

(2) If $P \to X$ is a principle G-bundle with a flat connection ∇ then one can take $L = A^{\bullet}(ad\,P)$, $d = \nabla$, where $ad\,P := P \times_G \mathfrak{g}$.

0.3. Conjecture (Folklore, Deligne). *In characteristic* 0*, every deformation problem comes from dgla.*

Today I will describe an actual cpx space which proprepresents the deformation functor arising from the dgla with "Hodge decomposition".

prorepresents = represents in a bigger category

0.4. Definition. If X is a compact cpx manifold then a deformation of X is the following data:

(1) $p: \mathcal{X} \to S$ proper, submersive.

(2) S is connected, with a marked point $0 \in S$.

(3) $X \simeq \mathcal{X}_0$ (the fiber of p at 0)

Take a tangent sequence

$$0 \to T_f \to T_{\mathcal{X}} \to p^* T_S \to 0$$

and restrict to the special fiber:

$$0 \to T_X \to T_{\mathcal{X}}|_X \to T_{S,0} \otimes \mathcal{O}_X \to 0$$

Take the connecting homomorphism in the corresponding long exact sequence:

$$\kappa: T_{S,0} \to H^1(X, T_X).$$

This is a *Kodaira–Spencer map.*

We want to think about κ as being the differential of the map

$$(s \in S) \mapsto (\text{cpx structure of the fiber } \mathcal{X}_s).$$

Note: all fibers are diffeomerphic to X (Ehressmann) but of course not always isomorphic. Think about the cpx structure on X as $I \in H^0(End(T_{X,\mathbb{C}}))$ such that $I^2 = -\mathrm{Id}$, I is integrable. One-parameter family I_s of cpx structures gives a 1-parameter family of splittings $T_{X,\mathbb{C}} = T_s^{1,0} \oplus T_s^{0,1}$. If s is small then $T_s^{0,1}$ is a graph of a function from $T^{0,1}$ to $T^{1,0}$, i.e. an element $\varphi_s \in A^{0,1}(T_X)$. A calculation shows that integrability of I_s is equivalent to *Maurer–Cartan equations*

$$\bar{\partial}\varphi_s + \frac{1}{2}[\varphi_s, \varphi_s] = 0$$

in $A^{0,2}(T_X)$. Expanding $\varphi = \sum_i \varphi_i s^i$ in powers of s gives a family of equations

$$\bar{\partial}\varphi_1 = 0, \quad \bar{\partial}\varphi_2 + \frac{1}{2}[\varphi_1, \varphi_1] = 0, \quad \ldots$$

So we get an element $[\varphi_1] \in H^1(X, T_X)$. This is the image of the Kodaira–Spencer map. To lift this element to the actual deformation, you need (at the very least) $[\varphi_1, \varphi_1] = 0$ in $H^2(X, T_X)$. This is an example of an obstruction.

One has notions of versal, semiuniversal (miniversal), and universal deformations. Universal: any family is a pull-back in a unique way. Versal: any family is a pullback. Semiuniversal: any family is a pullback and identity on tangent spaces (infinitesimal deformations).

0.5. Theorem (Kuranishi). *In the category of cpx spaces, every compact cpx manifold admits a semiuniversal deformation. Its base is called a Kuranishi space and it is analytic subspace of* $H^1(X, T_X)$.

The Maurer–Cartan equations make sense in arbitrary dgla L:

$$d\zeta + \frac{1}{2}[\zeta, \zeta] = 0, \quad \zeta \in L.$$

We can define a functor

$$D_L : Art \to Groupoids$$

(and therefore also $Art \to Sets$ by taking the set of isomorphism classes in the groupoid) as follows. For any $(A, m) \in Art$, let the objects in $D_L(A)$ be

$$Ob\, D_L(A) = \{\xi \in L_1 \otimes m \mid d\xi + \frac{1}{2}[\xi, \xi] = 0\}$$

and let the morphisms in $D_L(A)$ be induced by the action of the group

$$Mor\, D_L(A) = G(A) = \exp(L_1 \otimes m)$$

(note that exponents are well-defined). The multiplication in this group can be defined formally using the Baker–Campbell–Hausdorff formula

$$\log e^X e^Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \ldots$$