VARIATIONS OF HODGE STRUCTURES - II

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$X \to S$ a family of smooth complex manifolds (submersion, proper map).

0.1. Theorem (Ehresmann). *f is a C^∞ fibration.*

Proof. Pick $s_0 \in S$. Think about S as a small disk. Let X_{s_0} is a fiber. Locally f is a projection. We can lift any vector field v from the base to the vector field $\hat{v}$ on X. Can start the lift at any point of X_{s_0}. By compactness we can do it in the neihgborhood of s_0. So the flow of $\hat{v}$ for $|s| < \delta$ gives the diffeomorphism $\phi_v : X_{s_0} \simeq X_s$ (that depends on the vector field v). □

The lift depends on many choices. Let $\hat{v}_1, \hat{v}_2$ be two lifts of the same vector field. Then $df(\hat{v}_1 - \hat{v}_2) = 0$. This gives the map $T_{s_0}(S) \to C^1(X_{s_0}, T^1_{X_{s_0}})$ (the connecting homomorphism in the relative tangent sequence) and the canonical Kodaira-Spencer map

$$K_v : T_s S \to H^1(X_{s_0}, T^1_{X_{s_0}}).$$

Another use of lifting these vector fields: given a curve $\gamma : [0,1] \to S$, $\gamma(0) = s_0$, we get a map $\phi^* : X_{s_0} \to X_{\gamma(1)}$ that depends on many choices. But in fact it is well-defined in cohomology:

$$[\phi^*]^{-1} : H^k(X_{\gamma(1)}, \mathbb{Z}) \to H^k(X_{s_0}, \mathbb{Z}).$$

It only depends on a homotopy class of γ, so gives a map

$$\pi_1(S, s_0) \to \mathrm{End}_{\mathbb{Z}}(H^k(X_{s_0}, \mathbb{Z})$$

(the *monodromy representation*).

This gives a locally constant sheaf of $\mathbb{Z}$-modules on Z. But a locally constant sheaf of $\mathbb{C}$-vector spaces is the same thing as a vector bundle with flat connection. The locally constant sheaf is $\mathcal{H}^k := R^k f_* \mathbb{C}$, and the connection is the *Gauss–Manin connection* ∇.

If X_s is a smooth projective variety (or just a compact Kähler manifold), the cohomology carries a lot of structure. We want to transport these structures along the Gauss–Manin connection.

0.2. Theorem (Hodge decomposition).

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \quad H^{p,q} = \overline{H^{q,p}}.$$

$H^{p,q}$ are elements of $H^k(X, \mathbb{C})$ that have a harmonic representative of bidegree (p,q) (harmonic relative to the Kähler metric, i.e. killed by the Laplacian).

$H^{p,q}$ is isomorphic to $H^q(X, \Omega^p)$ and also to $H^{p,q}_{\bar{\partial}}(X)$ (Dolbeault complex). In particular, $X^{k,0} \simeq H^0(X, \Omega^k)$. For example, if $\dim X = 1$ then

$$H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$$

and $H^{1,0}$ are "abelian differentials". The LHS is independent on the fiber but the decomposition on the RHS depends on the Kähler structure. Fix the basis $\omega_1, \ldots, \omega_g$ of $H^{1,0}$. Classically, fix the basis a_i, b_j in 1-homology $H_1(X, \mathbb{Z})$ and integrate ω_i's. This gives a matrix of periods $\Lambda = [\int_{a_i} \omega_j, \int_{b_k} \omega_j]$.

Griffiths (68): do the same thing for any X.

Let $\omega \in H^{1,1} \cap H^2(\mathbb{R})$, $n = \dim X$, $k = n - l$. If $\alpha, \beta \in H^k(X, \mathbb{C})$, consider

$$Q(\alpha, \beta) = (-1)^{k(k-1)/2} \int_X \alpha \cup \beta \cup \omega^l$$

Then morally $i^{p-q} Q(\alpha, \bar{\alpha}) > 0$ if $\alpha \in H^{p,q}$. (More precisely: α should also be in the primitive cohomology $\mathrm{Ker}(L_\omega^{l+1})$ (L_ω is a left multiplication by ω). Primitive cohomology if flat if, for example, the Kähler structure on fibers X_s comes from the Kähler structure on the total space X).

We have a decomposition of vector bundles $\mathcal{H}^k = \oplus_{p+q=k} \mathcal{H}^{p,q}$ but only as C^∞ vector bundles! E.g., take a 3-fold. Then

$$H^3 = H^{3,0}(+) \oplus H^{2,1}(-) \oplus H^{1,2}(+) \oplus H^{0,3}(-),$$

wjere $\pm$ is the signature of the Hermitian form. $H^{3,0} \oplus H^{1,2}$ is the "Weil's intermediate Jacobian" but it is not a holomorphic bundle!

Griffiths realized that we should look not at the decomposition but at the filtration associated to this decomposition

$$\mathcal{F}^p = \oplus_{a \geq p, a+b=k} \mathcal{H}^{a,b}.$$

0.3. Theorem (Griffiths). *$\mathcal{F}^p$ are holomorphic subbundles.*

We are looking for a classifying space for Hodge structures.

0.4. Definition. A Hodge structure is the following datum: $H = H_{\mathbb{Z}} \otimes \mathbb{C}$ (a fixed vector space), a form Q ("polarization"), k (the "weight"). A polarized HS of weight k is the decomposition $H = \oplus_{p+q=k} H^{p,q}$, $H^{p,q} = \overline{H^{q,p}}$, Q has parity $(-1)^k$ such that $Q(H^{p,q}, H^{p',q'}) = 0$ if $q' \neq p$ and $i^{p-q} Q(\alpha, \bar{\alpha}) > 0$ if $\alpha \in H^{p,q}$.

Now define $F^p = \oplus_{a \geq p} H^{a,b}$. Then

$$H = F^p \oplus \overline{F^{k-p+1}} \tag{1}$$

And conversely, consider the flag $F^k \subset F^{k-1} \subset \ldots \subset F^0 = H$. If (1) is satisfied then it is a Hodge structure with $H^{p,q} = F^p \cap \overline{F^{k-p}}$.

Define the (closed subset of) flag variety $D^\vee$ of flags with $Q(F^p, F^{k-p+1}) = 0$ and inside it the open subset D of polarized Hodge structures with numbers $h^{p,q}$.

Given a family $X \to S$, we have a map $S \to D/(\text{monodromy group})$ induced by the map that sends s to $[\phi^*]^{-1}(H^{p,q}(X_s))$. This is the general period map.

Differential of the period map. What is the tangent space to the space of flags $\mathcal{F}$? We have bundles $\mathcal{E}^p \to \mathcal{F}$ that to each flag associate F^p. Note: $\mathcal{E}_0$ is a constant bundle H. Then $T\mathcal{F} \simeq \oplus_p \mathrm{Hom}(\mathcal{E}_p, \mathcal{E}_0/\mathcal{E}_p)$ "with some compatibility condition" (should be lower-triangular matrices).

0.5. Theorem (Griffiths' Transversality). *In fact the image is in*

$$\oplus_p \mathrm{Hom}(H^{p,q}, H^{p-1,q+1}).$$

How do you prove this? $H^{p,q} = H^q(X_s, \Omega^p)$. In fact the corresponding component of the direct sum is just a cup product with the Kodaira-Spencer class of v (a vector from $T_{s_0}S$).