INTRODUCTION TO VARIATIONS OF HODGE STRUCTURES

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Here is the plan: let $\mathcal{X} \xrightarrow{f} S$ be a family of smooth projective varieties. Suppose f is submersive, proper, has connected fibers.

Ehressmann: f is C^∞-trivial, in particular all fibers are diffeomorphic. We think about this as a variation of fibers $X_s = f^{-1}(s)$.

Griffiths: construct a vector bundle over S, namely $R^k f_* \mathbb{C}$. The fiber over $s \in S$ is $H^k(X_s, \mathbb{C})$ We can also take integral coefficients and get a local system on S. So this vector bundle has a flat "Gauss–Manin" connection ∇ (can identify cohomology classes). So we can translate cohomology classes in a parallel way. This will give a monodromy action.

We should think about this as a linearization of the problem of classifying complex structures. There are many interesting structures on this vector bundle

Hodge decomposition: z_i – holomorphic coordinates on $X = X_{s_0}$. dz_i, $d\bar{z}_i$ – basis for forms. Then can decompose holomorphic forms: $A^k(X) = \oplus A^{p,q}(X)$ (how many dz_i, $\bar{d}z_i$). But $d = \partial + \bar{\partial}$.

One can represent cohomology classes by harmonic forms. If the manifold is Kähler (e.g. smooth projective) then (p, q) components of a harmonic form are also harmonic. This gives $H^k(X, \mathbb{C}) = \oplus H^{p,q}$.

Take a simple case: an elliptic curve. Then $H^1(X, \mathbb{C}) = \mathbb{C}^2 = H^{1,0} \oplus H^{0,1}$. $H^{1,0} \simeq H^0(X, K_X)$ is the space of holomorphic differentials, $H^{0,1} = \overline{H^{1,0}}$. The family is locally trivial. Fix the standard basis in homology $H_1(X, \mathbb{Z})$ and see how cohomology classes vary by integrating them over α, β. These are classical periods. And the family is completely determined by these periods.

Griffiths proposed to study how Hodge decomposition varies as a first step in understanding how the cpx structure varies. In general, there is a (period) map S to some flag variety that encodes how the Hodge decomposition varies.

One can study infinitesimal picture (in the formal neighborhood of a point)

But more interestingly, consider the family over a punctured disk Δ^* and try to see the "asymptotic" at the puncture. This will produce the map $\Gamma : \Delta \to \mathfrak{g}$ to some Lie algebra (logarithmic part of the Gauss–Manin flat connection).