Semi-orthogonal decompositions of moduli spaces

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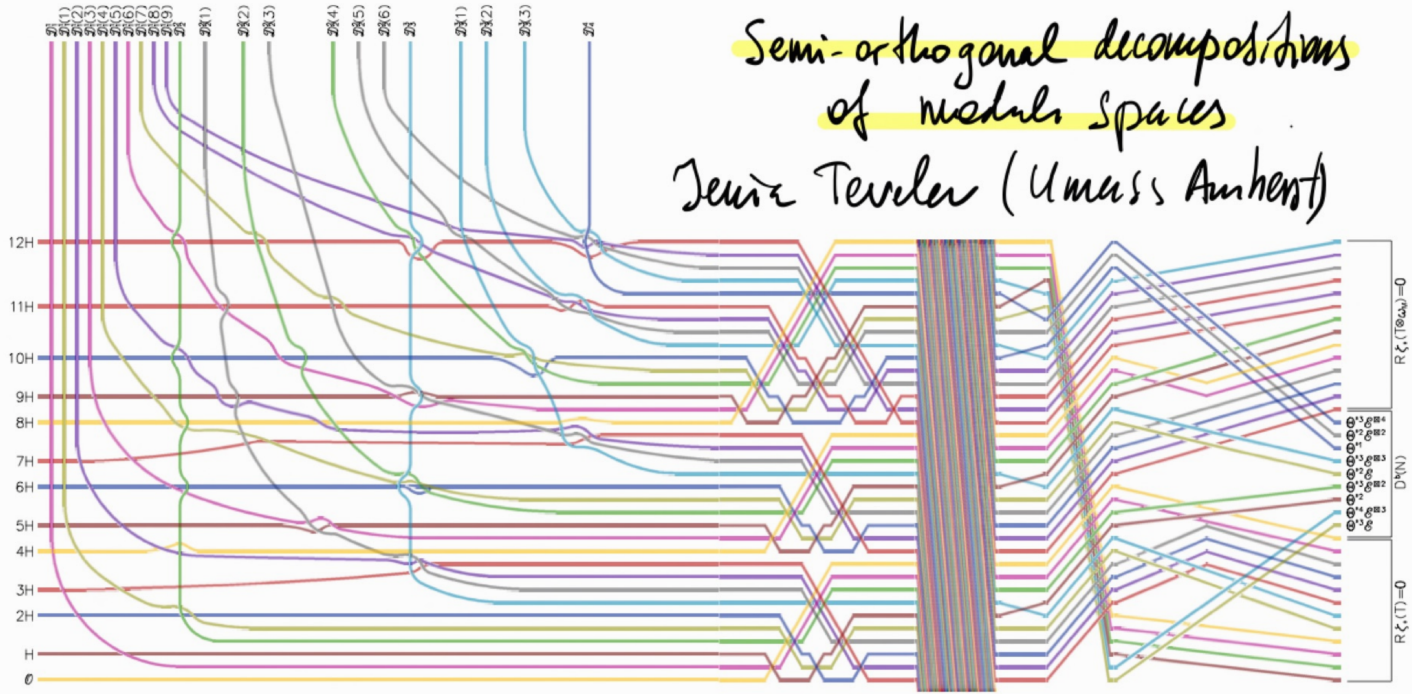

Sink-T., Noncommutative Resolution of $SL_2(\mathbb{C})$: arXiv: 2405.08891

T., Braid and Phantom : arXiv: 2304.01825

T.-Torres, BGMN Conjecture via Stable Pairs: to appear in Duke Math J.

- C smooth projective curve of genus $g \geq 2$
- $SU_C(2,\Lambda)$ moduli space of semi-stable rank 2 vector bundles F on C with $\det F = \Lambda$
- Fano variety, $\dim = 3g-3$, $\mathrm{Pic} = \mathbb{Z} = \langle \Theta \rangle$
- $SU_C(2,\Lambda) \cong SU_C(2, \Lambda \otimes M^{\otimes 2}) \Rightarrow$ 2 cases only

  $F \longleftrightarrow F \otimes M$
- $\deg \Lambda = 2g-1$ odd

  $SU_C(2,\Lambda)$ smooth, birational to $\mathbb{P}\mathrm{Ext}^1(\Lambda, \mathcal{O}) = \mathbb{P}^{3g-3}$
- $\deg \Lambda = 2g$ even

  coarse moduli of a symmetric stack

  $SU_C(2,\Lambda)$ is Gorenstein, has rational singularities

  rationality unknown (for $g \geq 3$)

  $\mathbb{P}\mathrm{Ext}^1(\Lambda, \mathcal{O}) = \mathbb{P}^{3g-2} \dashrightarrow SU_C(2,\Lambda)$ generic fibers $\mathbb{P}^1$

Main Theorem

odd degree $D^b(SU_C(2,\Lambda))$ has SOD with blocks $D^b(\mathrm{Sym}^k C)$

2 blocks for $k < g-1$, 1 block for $k = g-1$

even degree $\mathcal{D}$ = Păduraru–Špenko–Van den Bergh NCR of $D^b(SU_C(2,\Lambda))$

$\mathcal{D}$ has SOD with blocks $D^b(\mathrm{Sym}^k C)$, k even

4 blocks for $k < g-1$, 2 blocks for $k = g-1$ (if g is odd)

- g is even $\Rightarrow$ $\mathcal{D}$ is strongly crepant (in the sense of Kuznetsov) and categorifies $IH^\bullet(SU_C(2,\Lambda),\mathbb{C})$
- $D^b(\mathrm{Sym}^k C)$ indecomposable $k \le g-1$ (Lin) $\Rightarrow$ SOD is maximal

Tensor vector bundles

- $\mathcal{N}$ moduli stack of semi-stable vector bundles of rank 2, $\det = \Lambda$ (generic inertia group $\mathbb{G}_m$)
- $\mathbb{N}$ rigidified stack (trivial generic inertia)
  $SU_C(2,\Lambda) \cong \mathbb{N}$ if Λ is odd, its coarse moduli if Λ is even
- $\mathcal{F}$ universal vector bundle on $C \times \mathcal{N}$
- $\tau : C^k \times \mathcal{N} \longrightarrow \mathrm{Sym}^k C \times \mathcal{N}$ quotient by S_k (flat!)
  $\mathcal{F}^{\boxtimes k} = \tau_*^{S_k}(\pi_1^* \mathcal{F} \otimes \ldots \otimes \pi_k^* \mathcal{F})$ vector bundle on $\mathrm{Sym}^k C \times \mathcal{N}$ of rank 2^k

Family $\mathcal{F}^{\otimes k}_D$ of vector bundles on $\mathcal{N}$

- $D = p_1 + \dots + p_k \in \mathrm{Sym}^k C \Rightarrow \mathcal{F}^{\otimes k}_D := \mathcal{F}^{\otimes k}\big|_{\{D\}\times \mathcal{N}}$

  evaluation vector bundle on $\mathcal{N}$
- $p_i \neq p_j \Rightarrow \mathcal{F}^{\otimes k}_D \cong \mathcal{F}_{p_1} \otimes \cdots \otimes \mathcal{F}_{p_k}$
- In general, $\mathcal{F}^{\otimes k}_D$ is a deformation of $\mathcal{F}_{p_1} \otimes \cdots \otimes \mathcal{F}_{p_k}$
- Λ odd $\Rightarrow \mathcal{E}^{\otimes k} := (Z^{-1})^{\otimes k} \otimes \mathcal{F}^{\otimes k}$ descends to $\mathrm{Sym}^k C \times \mathcal{N}$
- Λ even, k even $\Rightarrow \mathcal{E}^{\otimes k} := (\Lambda^{-1})^{\otimes k/2} \otimes \mathcal{F}^{\otimes k}$ — " — " — " —
- Here $\Lambda = \det \mathcal{F}_p$ "weight 2" and Z is a certain "weight 1" l.b. (exists only when Λ is odd)

Fourier Mukai functors in the main theorem are

$$\Phi_{\mathcal{E}^{\boxtimes k} \otimes \Theta^{?}} : D^b(\mathrm{Sym}^k C) \to D^b\left(\begin{array}{ll} SU_C(2,\Lambda) & - \Lambda \text{ odd} \\ \mathcal{D} & - \Lambda \text{ even} \end{array} \right)$$

appropriate powers of Θ line bundle to guarantee semiorthogonality

The functor is fully-faithful for $k \leq g-1$:

$$\begin{array}{ccc} D \in \mathrm{Sym}^k C & \xrightarrow{\Phi_{\mathcal{E}^{\boxtimes k}}} & \mathcal{E}^{\boxtimes k}_D \\ \text{skyscraper sheaf } k(D) & & \text{evaluation bundle} \end{array}$$

$$\Lambda^i \mathbb{C}^k = \mathrm{Ext}^i(k(D), k(D)) \cong \mathrm{Ext}^i(\mathcal{E}^{\boxtimes k}_D, \mathcal{E}^{\boxtimes k}_D)$$

$$D \neq D' \quad \underbrace{\mathrm{Ext}^i(k(D), k(D'))}_{= 0} \cong \mathrm{Ext}^i(\mathcal{E}^{\boxtimes k}_D, \mathcal{E}^{\boxtimes k}_{D'})$$

Consequences for $k=1$

- $\mathrm{Hom}(\mathcal{E}_p, \mathcal{E}_p) = \mathbb{C}$ $\Rightarrow$ simple vector bundle on $SU_C(2,\Lambda)$
- $\mathrm{Ext}^1(\mathcal{E}_p, \mathcal{E}_p) = \mathbb{C}$ $\Rightarrow$ 1-dim infinitesimal deformations
- $\mathrm{Ext}^2(\mathcal{E}_p, \mathcal{E}_p) = 0$ $\Rightarrow$ unobstructed deformations
- $\mathrm{Hom}(\mathcal{E}_p, \mathcal{E}_q) = 0$ $p \neq q$ $\Rightarrow$ $\mathcal{E}_p$ not isomorphic to $\mathcal{E}_q$
  $\Rightarrow$ $\mathcal{E}_p$ moves with $p \in C$
- Semi-orthogonality of blocks $\Rightarrow \mathrm{Hom}(\Theta, \mathcal{E}_p) = 0$
  $$\langle \underset{D^b(C)}{\underset{\wr\|}{\mathcal{E}}}, \underset{D^b(pt)}{\underset{\wr\|}{\Theta}} \rangle$$
  $\Rightarrow$ $\mathcal{E}_p$ is a stable vector bundle

Two-ray game conjecture

$\exists$ compatible SODs

Smooth Fanos

$$\begin{tikzcd} & L \\ M \arrow[ur, dashed, "\pi_L"] \arrow[dr, dashed, "\pi_N"'] & \\ & N \end{tikzcd}$$

extremal contractions

$$D^b(L) = \langle \mathcal{A}_1 \ldots \mathcal{A}_s \rangle$$

$$D^b(M) = \langle \mathcal{A}_1 \ldots \mathcal{A}_s \; X_1 \ldots X_r \rangle$$

braid mutation

$$D^b(M) = \langle \mathcal{B}_1 \ldots \mathcal{B}_t \; Y_1 \ldots Y_e \rangle$$

$$D^b(N) = \langle \mathcal{B}_1 \ldots \mathcal{B}_t \rangle$$

Example

$g = 2$

Λ odd

Mori-Mukai 2-19

$\mathrm{Bl}_C \mathbb{P}^3 \longrightarrow \mathbb{P}^3 \hookleftarrow C$ (deg 5)

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$\mathrm{Bl}_{\mathbb{P}^1} Q_1 \cap Q_2 \longrightarrow Q_1 \cap Q_2 \hookleftarrow \mathbb{P}^1$ (line)

$Q_1 \cap Q_2 \subset \mathbb{P}^5$

Moduli Spaces of Stable Pairs (Thaddeus)

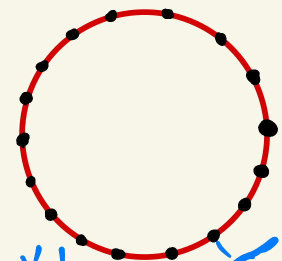
- $\deg \Lambda = 2g-1$
- Fano variety, $\mathrm{Pic} = \mathbb{Z}^2$

$$M = \left\{ (F, s) \ : \ \begin{array}{l} F \in SU_C(2,\Lambda) \\ s \in H^0(C,F) \setminus \{0\} \end{array} \right\}$$

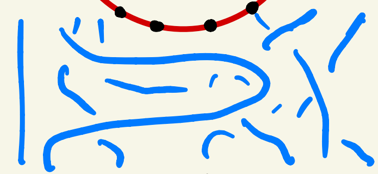
standard flips

$$\begin{tikzcd}
M_1 \simeq Bl_C \mathbb{P}^{3g-3} \arrow[r] & \mathbb{P}^{3g-3} \\
M_2 \arrow[u, dashed] & \\
M_{g-2} \arrow[u, dashed] & \\
Z \arrow[r, hook] \arrow[d] & M \arrow[u, dashed] \arrow[uuu, dashed] \arrow[d] \\
BN \arrow[r, hook] & SU_C(2,\Lambda)
\end{tikzcd}$$

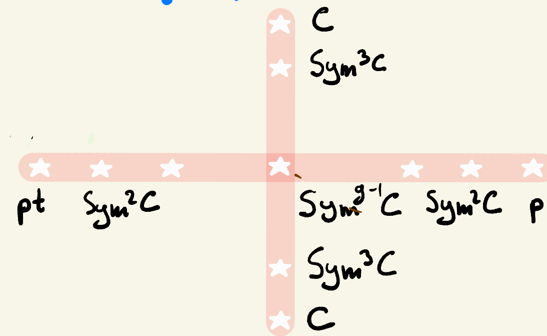
$$(F,s) \mapsto F$$



cyclotomic quantum spectrum



Braid?



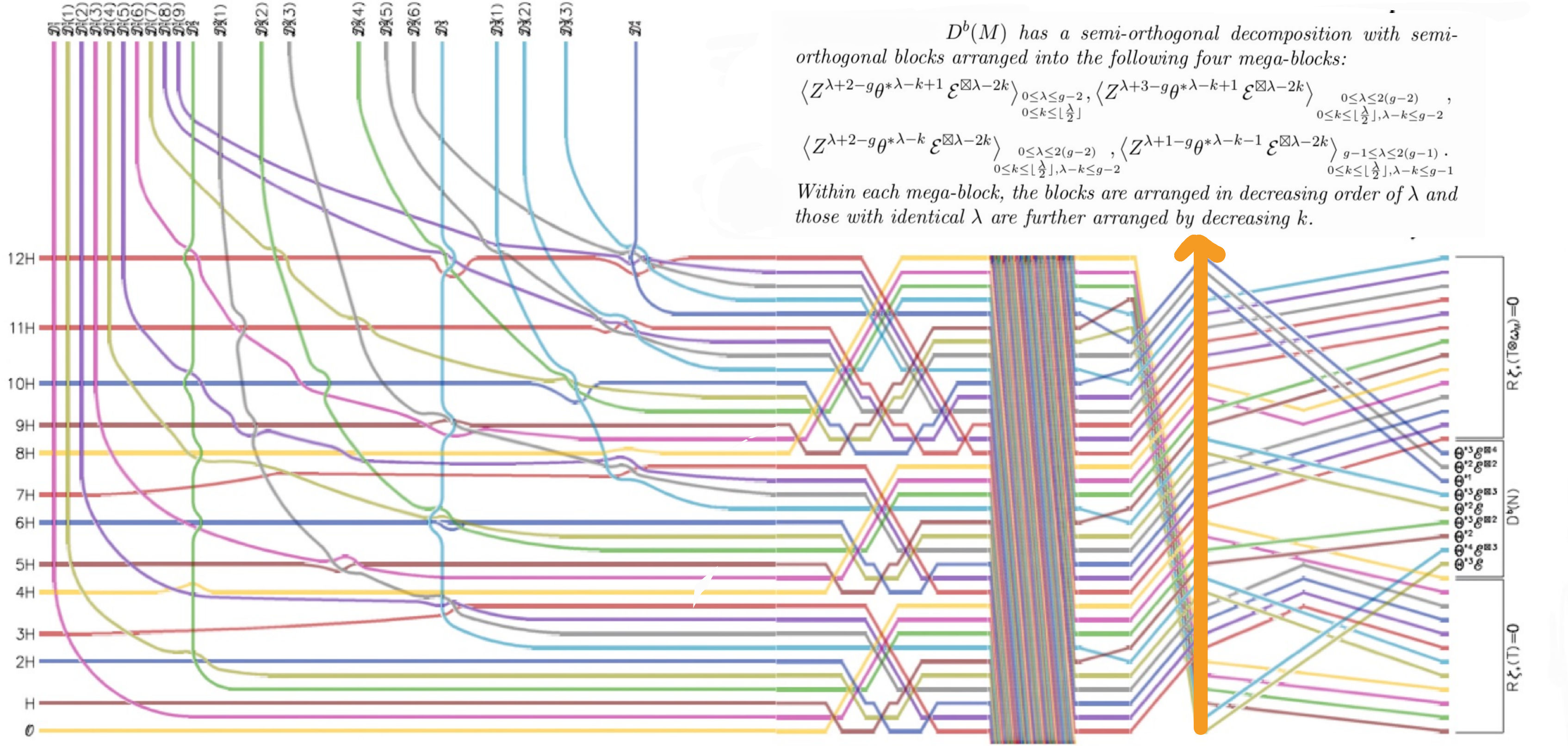
del Baño quantum spectrum

$D^b(M)$ has a semi-orthogonal decomposition with semi-orthogonal blocks arranged into the following four mega-blocks:

$$\left\langle Z^{\lambda+2-g}\theta^{*\lambda-k+1}\,\mathcal{E}^{\boxtimes\lambda-2k}\right\rangle_{\substack{0\le\lambda\le g-2\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor}},\ \left\langle Z^{\lambda+3-g}\theta^{*\lambda-k+1}\,\mathcal{E}^{\boxtimes\lambda-2k}\right\rangle_{\substack{0\le\lambda\le 2(g-2)\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-2}},$$

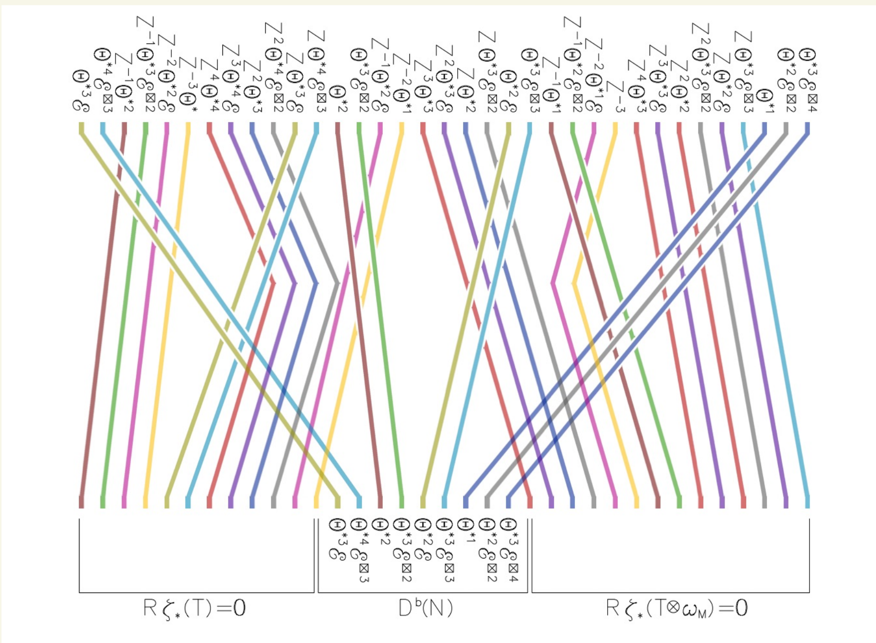
$$\left\langle Z^{\lambda+2-g}\theta^{*\lambda-k}\,\mathcal{E}^{\boxtimes\lambda-2k}\right\rangle_{\substack{0\le\lambda\le 2(g-2)\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-2}},\left\langle Z^{\lambda+1-g}\theta^{*\lambda-k-1}\,\mathcal{E}^{\boxtimes\lambda-2k}\right\rangle_{\substack{g-1\le\lambda\le 2(g-1)\\ 0\le k\le\lfloor\frac{\lambda}{2}\rfloor,\lambda-k\le g-1}}.$$

Within each mega-block, the blocks are arranged in decreasing order of λ and those with identical λ are further arranged by decreasing k.



Plain Weave

- Blocks with trivial power of Z are pulled back from N
  - correct number!
- Other blocks are mutated into $\{X \in D^b(M) : R\zeta_* X = 0\}$



$$\zeta : M \to SU_C(2,\Lambda)$$
$$(F,s) \mapsto F$$

"DG blow-up"

$$\begin{array}{ccc} M & \hookrightarrow & \mathbb{P}(A) \\ & & \downarrow \\ & & SU_C(2,\Lambda) \end{array}$$

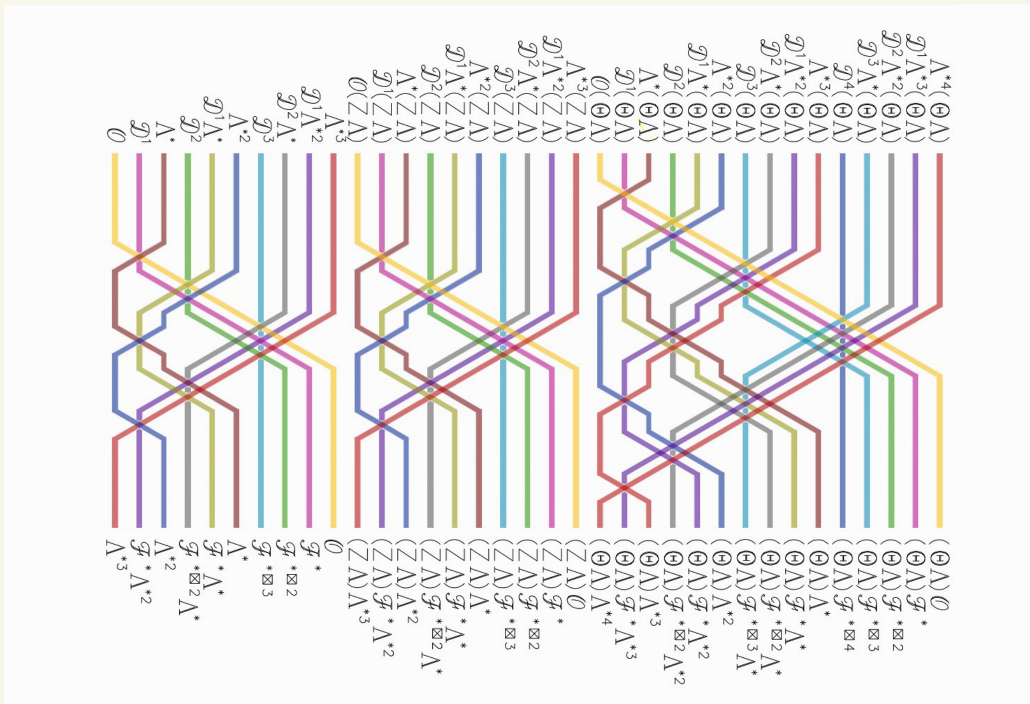
zero locus of a section of a vector bundle

Cross Warp

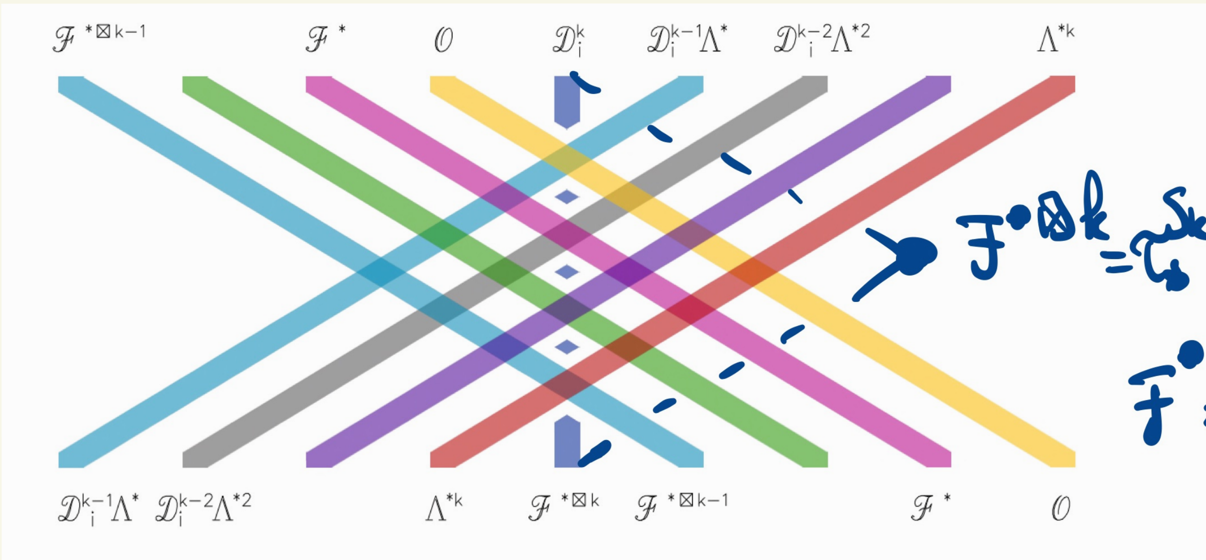
$$D^\alpha = \{ (D, F, s) : s|_D = 0 \} \subset \mathrm{Sym}^\alpha C \times M$$

Theorem $D^b(M)$ has an SOD $\left\langle \left\langle \Lambda^{*j} \otimes \mathcal{D}^k \right\rangle_{\substack{j+k\le g-2,\\ j,k\ge 0}},\ \left\langle (Z\,\Lambda)\Lambda^{*j} \otimes \mathcal{D}^k \right\rangle_{\substack{j+k\le g-2,\\ j,k\ge 0}},\ \left\langle (\theta\,\Lambda)\Lambda^{*j} \otimes\otimes\, \mathcal{D}^k \right\rangle_{\substack{j+k\le g-1\\ j,k\ge 0}} \right\rangle$.

within each megablock, the blocks are arranged by $j+k$ (increasing order) and, for fixed $j+k$, by j (increasing order)



Cross-Warp Mutation

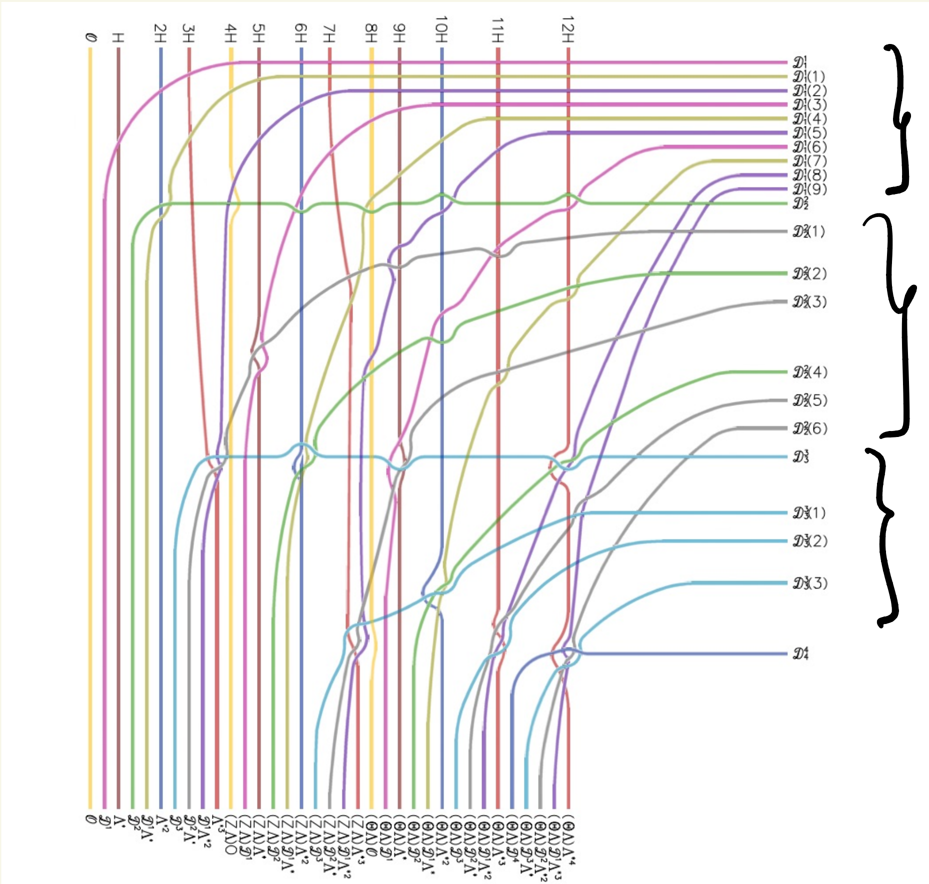


Lady vanishes

$$\mathcal{F}^{\bullet\boxtimes k} = \pi_*^{S_k}(\pi_1^*\mathcal{F}^\bullet \otimes \dots \otimes \pi_k^*\mathcal{F}^\bullet \otimes \mathrm{sgn})$$

$$\mathcal{F}^\bullet = [\mathcal{F}^* \xrightarrow[\Sigma]{} \mathcal{O}] \in D^b(C \times M_i)$$

Farey Twill



mutation in $D^b(M_1)$

mutation in $D^b(M_2)$

mutation in $D^b(M_3)$

mutation in $D^b(M_4)$

Variation of Moduli $\quad \sigma \in \mathbb{R}_{>0}$

- $\mathcal{M}(\sigma) = \{(F, s) : s \in H^0(C, F),\ \mathrm{rk}\, F = 2,\ \det F = \Lambda,\ s \neq 0\}$

  $\forall$ line subbundle $L \subset F$, $\deg L \leq \begin{cases} \frac{2g-1}{2} - \sigma & s \in H^0(L) \\ \frac{2g-1}{2} + \sigma & s \notin H^0(L) \end{cases}$

- No strictly semistable pairs between the walls $\Rightarrow$ $\mathcal{M}(\sigma) \subset \mathcal{M}$ open substack of the stack of all pairs
- Carries a universal pair $(\mathcal{F}, \Sigma)$
- variation of σ

$$\begin{tikzcd}
E \arrow[r, phantom, "\subset"] \arrow[d, "p"'] & \mathrm{Bl}_C \mathbb{P}^{3g-3} \arrow[r, phantom, "\cong"] & M_1 \arrow[d] & M_2 \arrow[l, dashed] & \cdots \arrow[l, dashed] & M_{g-1} = M \arrow[l, dashed] \arrow[d, "\xi"'] \\
C \arrow[rr, hook, "|\omega_C \otimes \Lambda|"] & & \mathbb{P}^{3g-3} & & & SU_C(2, \Lambda)
\end{tikzcd}$$

- For pairs $(F, s) \in M_i$, $s \in H^0(F)$ has at most i zeros.

Windows Theorem

$$
\begin{tikzcd}[column sep=small]
 & & & \mathrm{PGL} \curvearrowright X & & & \\
 & & X /\!\!/_{+} \mathrm{PGL} \arrow[d, phantom, "\parallel"] & \mathrm{VGIT} & X /\!\!/_{-} \mathrm{PGL} \arrow[d, phantom, "\parallel"] & & \\
\mathbb{P}W_i^- \arrow[r, phantom, "\subset"] \arrow[drrr, "\mathbb{P}^{i-1}\text{-bundle}"'] & & M_{i-1} \arrow[dr] & & M_i \arrow[dl] \arrow[r, phantom, "\supset"] & & \mathbb{P}W_i^+ \arrow[dlll, "\mathbb{P}^{3g-3-2i}\text{-bundle}"] \\
 & & & \text{singular } \overline{M}_{i-1} \arrow[d, phantom, "\vee"] & & & \\
 & & & \mathrm{Sym}^i C & & &
\end{tikzcd}
$$

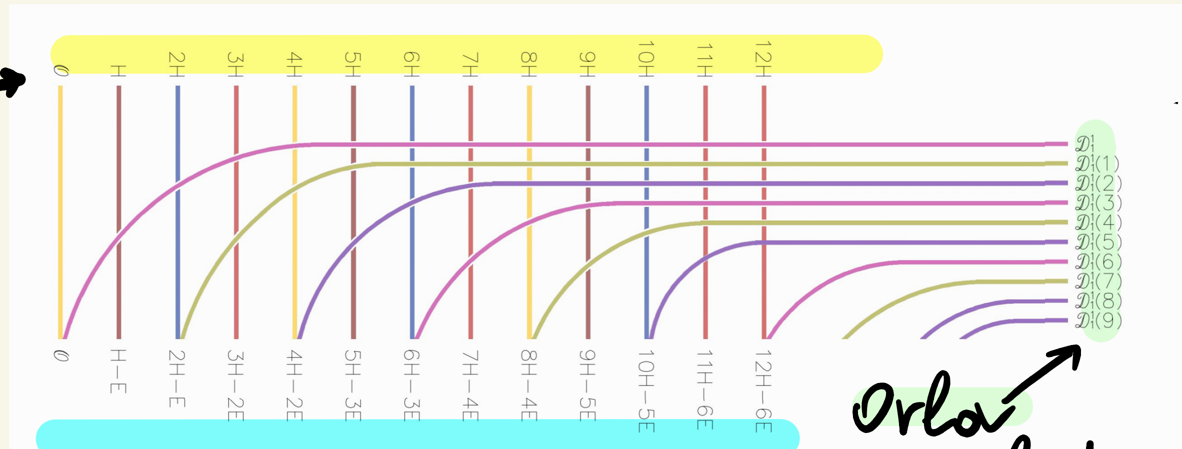
$B \in D^b_{\mathrm{PGL}}(X)$ and $-(3g-3-2i) \leq \mathrm{weights}(B) \leq i-1$

$$R\Gamma(M_i, B) \overset{\Downarrow}{=} R\Gamma^{\mathrm{PGL}}(X, B) \overset{\Downarrow}{=} R\Gamma(M_{i-1}, B)$$

Start weaving the twill!

$D^b(\mathbb{P}^{3g-3})$

Beilinson

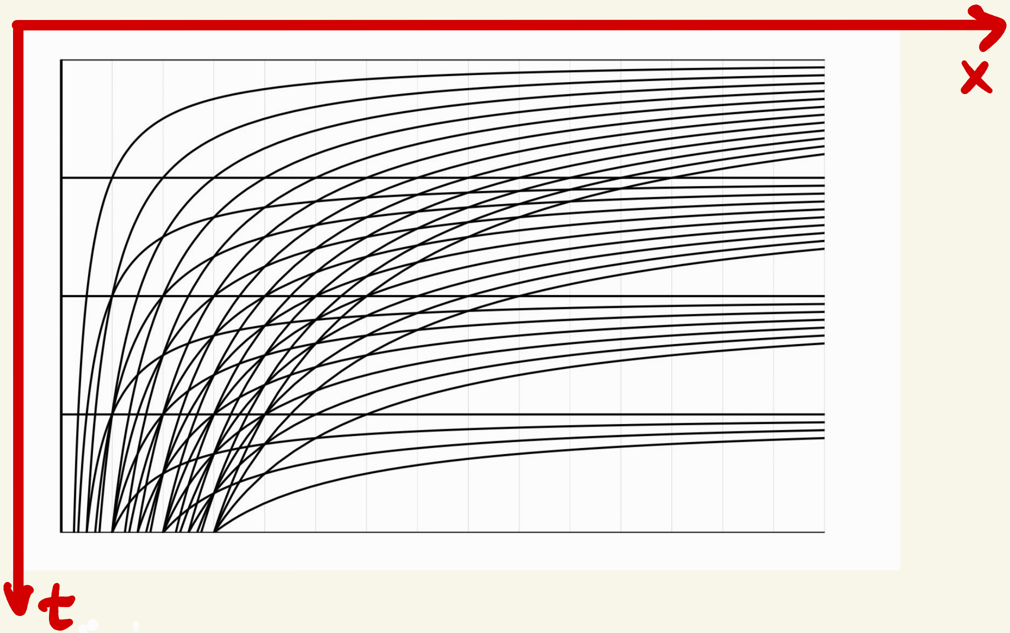


Orlov block

Mutated line bundles have weights in $[0,1)$

$\Rightarrow$ move unchanged to $D^b(M_2)$

Accurate trajectories of the Farey Twill:



$$x_{k,s}(t) = \frac{s}{t-k}$$

- Moving blocks $\langle \mathcal{D}_t^{k,s} \rangle \subset D^b(M_{\lfloor t \rfloor})$
- $\mathcal{D}^t_{k,s} = \mathcal{D}^k_{\lfloor t \rfloor} \otimes L^t_{k,s}$ line bundle that depends on t, k, s

Meeting trajectories

- $\frac{s}{t-k} = \frac{s'}{t-k'} \notin \mathbb{Z}$ then $\langle \mathcal{D}_t^{k,s} \rangle, \langle \mathcal{D}_t^{k',s'} \rangle$ are perpendicular and don't change
- $\frac{s}{t-k} = \frac{s'}{t-k'} \in \mathbb{Z}$ then Farey Twill Mutation